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DEPARTMENT OF PHYSICS

III B.Sc Physics

Numerical Methods

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UNIT-IV

FINITE DIFFERENCES

Finite Differences

Let $y = f(x)$ be a given function of x and let $y_0, y_1, y_2, \ldots$ be the values of y corresponding to x_0, $x_0 + h$, $x_0 + 2h$, ... of the values of x. *i.e.*, $y_0 = f(x_0), y_1 = f(x_0 + h), y_2 = f(x_0 + 2h), \ldots, y_n = f(x_0 + nh)$. Here the independent variable (or argument), x proceeds at equally spaced intervals and 'h'(constant), the difference between two consecutive values of x is called the interval of differencing. Now $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the first differences of the function y and differences of the y_n values are denoted by

$$\Delta y_n = y_{n+1} - y_n \qquad [n = 0, 1, 2, \ldots]$$

Here 'Δ' acts as an operator called forward difference operator.

Thus
$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots \qquad \vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

The differences of these first differences are called second differences.

Thus
$$\Delta^2 (y_0) = \Delta (\Delta y_0) = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^2 (y_1) = \Delta (\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$$

and so on.

In general $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$ defines n^{th} differences where k and n are integers.

The difference table is a standard format for displaying finite differences and is explained in the following table called forward difference table.

x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Here each difference proves to be a combination of y values. For example,

$$\begin{aligned}
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0) \\
&= \{(y_3 - y_2) - (y_2 - y_1)\} - \{(y_2 - y_1) - (y_1 - y_0)\} \\
&= y_3 - 3y_2 + 3y_1 - y_0
\end{aligned}$$

❑ Backward Differences

We use another operator called the backward difference operator ∇ and is defined by

$$\nabla y_n = y_n - y_{n-1}$$

For $n = 0, 1, 2, \dots$ we get

$$\nabla y_0 = y_0 - y_{-1}$$

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1, \text{ and so on.}$$

The second backward difference is

$$\nabla^2 y_n = \nabla(\nabla y_n)$$

$$= \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$

$$= y_n - 2y_{n-1} + y_{n-2}$$

Similarly the third backward difference is

$$\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$$

$$= (y_n - 2y_{n-1} + y_{n-2}) - (y_{n-1} - 2y_{n-2} + y_{n-3})$$

$$= y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} \text{ and so on.}$$

☐ Central Differences

In the preceding two sections we have discussed Newton's Forward and Backward interpolation formulae which were suited for interpolation near the beginning and end values of the given data. If we want to find the values of x near the middle of the data we use central difference formula.

The central difference operator is denoted by δ and is defined by

$$\delta y_x = y_{x+h/2} - y_{x-h/2}$$

For example $y_{-1} - y_{-2} = \delta y_{-3/2}$ (taking $h = 1$)

$$y_2 - y_1 = \delta y_{3/2}$$

Let $y = y_0$ be the central ordinate corresponding to $x = x_0$. Let the other ordinates be y_r at $x = x_r$ ($r = \pm 1, \pm 2, \ldots$) so that the central difference table is as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
x_{-3}	y_{-3}				
		Δy_{-3} $(=\delta y_{-5/2})$			
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$ $(=\delta^2 y_{-2})$		
		Δy_{-2} $(=\delta y_{-3/2})$		$\Delta^3 y_{-3}$ $(=\delta^3 y_{-3/2})$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$ $(=\delta^2 y_{-1})$		$\Delta^4 y_{-3}$ $(=\delta^4 y_{-1})$
		Δy_{-1} $(=\delta y_{-1/2})$		$\Delta^3 y_{-2}$ $(=\delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1}$ $(=\delta^2 y_0)$		$\Delta^4 y_{-2}$ $(=\delta^4 y_0)$
		Δy_0 $(=\delta y_{1/2})$		$\Delta^3 y_{-1}$ $(=\delta^3 y_{1/2})$	
x_1	y_1		$\Delta^2 y_0$ $(=\delta^2 y_1)$		$\Delta^4 y_{-1}$ $(=\delta^4 y_1)$
		Δy_1 $(=\delta y_{3/2})$		$\Delta^3 y_0$ $(=\delta^3 y_{3/2})$	
x_2	y_2		$\Delta^2 y_1$ $(=\delta^2 y_2)$		
		Δy_2 $(=\delta y_{5/2})$			
x_3	y_3				

The most important central difference formula is **Stirling's.** We shall derive this by first deriving two other central difference formulas and then taking the mean of the latter in pairs.

Operators

Forward difference operator (Δ)

The forward difference operator 'Δ' is defined by

$$\Delta f(x) = f(x+h) - f(x)$$

where 'h' is a constant and is the difference between two arguments.

If we apply Δ on $f(x)$ twice, we get

$$\begin{aligned}
\Delta^2 f(x) &= \Delta[\Delta f(x)] \\
&= \Delta[f(x+h) - f(x)] \\
&= \Delta f(x+h) - \Delta f(x) \\
&= f(x+2h) - f(x+h) - f(x+h) + f(x) \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}$$

Similarly, we can find any higher order forward differences in terms of the entries.

Backward difference operator (∇)

The backward difference operator '∇' is defined by

$$\nabla f(x) = f(x) - f(x-h)$$

where 'h' is a constant. If we apply ∇ on $f(x)$ twice, we get

$$\begin{aligned}
\nabla^2 f(x) &= \nabla[\nabla f(x)] \\
&= \nabla[f(x) - f(x-h)] \\
&= \nabla f(x) - \nabla f(x-h) \\
&= \{f(x) - f(x-h)\} - \{f(x-h) - f(x-2h)\} \\
&= f(x) - 2f(x-h) + f(x-2h)
\end{aligned}$$

Similarly we can find any higher order backward differences in terms of the entries.

Central difference operator (δ)

The Central difference operator δ is defined by

$$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h}$$

$$\delta y_{\frac{1}{2}} = y_1 - y_0 \qquad \left[\text{Taking } x = \frac{1}{2} \text{ and } h = 1\right]$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1 \qquad \left[\text{Taking } x = \frac{3}{2} \text{ and } h = 1\right]$$

$$\delta y_{\frac{5}{2}} = y_3 - y_2 \text{ and so on.}$$

The higher order central difference operator can be defined by

$$\delta^2 y_x = \delta(\delta y_x) \; ; \; \delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}}$$

$$= \delta\left(y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}\right)$$

$$= \delta y_{x+\frac{1}{2}} - \delta y_{x-\frac{1}{2}}$$

$$= (y_{x+1} - y_x) - (y_x - y_{x-1})$$

$$= y_{x+1} - 2y_x + y_{x-1} \text{ and so on.}$$

$= y_x - y_{x-1}$

Shift:

❑ Interpolation

Consider the table

	x	x_0	x_1	x_2	$\cdots$	x_n
$y=$	$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

If the value of $f(y)$ is to be found at some point y in the interval $[x_0, x_n]$ and y is not one of the tabulated points, then the value of $f(y)$ is estimated by using the known values of $f(x)$ at the surrounding points. This process of computing the value of a function inside the given range is called **interpolation**. Simply interpolation means insertion or filling up intermediate terms of a series. If the point y lies outside the domain $[x_0, x_n]$ then the estimation of $f(y)$ is called **extrapolation.** In this chapter we will be mainly concerned with interpolation.

☐ Newton's Forward Interpolation Formula

We know that

$$\Delta y_0 = y_1 - y_0 \quad i.e., \; y_1 = y_0 + \Delta y_0 = (1+\Delta)\, y_0$$

$$\Delta y_1 = y_2 - y_1 \quad i.e., \; y_2 = y_1 + \Delta y_1 = (1+\Delta)\, y_1 = (1+\Delta)^2\, y_0$$

$$\Delta y_2 = y_3 - y_2 \quad i.e., \; y_3 = y_2 + \Delta y_2 = (1+\Delta)\, y_2 = (1+\Delta)^3\, y_0$$

In general, $y_n = (1+\Delta)^n\, y_0$

Expanding $(1+\Delta)^n$ by using Binomial theorem we have

$$y_n = \left\{1 + n\Delta + \frac{n\,(n-1)\,\Delta^2}{2!} + \frac{n\,(n-1)\,(n-2)\,\Delta^3}{3!} + \ldots\right\} y_0$$

$$y_n = f\,(x_0 + nh) = y_0 + n\,\Delta y_0 + \frac{n\,(n-1)}{2!}\,\Delta^2 y_0 + \frac{n\,(n-1)\,(n-2)}{3!}\,\Delta^3 y_0 + \ldots$$

This result is known as Gregory–Newton forward interpolation (or) **Newton's formula for equal intervals.**

• EXAMPLE 8 •

A function $f(x)$ is given by the following table. Find f (0.2) by a suitable formula.

x	0	1	2	3	4	5	6
$f(x)$	176	185	194	203	212	220	229

Solution

The difference table is as follows :

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0 (x_0)	176 (y_0)	Δy_0					
		9	$(\Delta^2 y_0)$				
1	185		0	$(\Delta^3 y_0)$			
		9		0	$(\Delta^4 y_0)$		
2	194		0		0	$(\Delta^5 y_0)$	
		9		0		-1	
3	203		0		-1		
		9		-1		4	$(\Delta^6 y_0)$
4	212		-1		3		5
		8		2			
5	220		1				
		9					
6	229						

Here $x_0 = 0$, $h = 1$, $y_0 = 176 = f(x_0)$

We have to find the value of f (0.2). By Newton's forward interpolation formula we have,

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \ldots$$

$$y(0.2) = ?$$

i.e., $\quad x_0 + nh = 0.2$

i.e., $\quad 0 + n \cdot 1 = 0.2 \quad$ *i.e.,* $n = 0.2$

$$\therefore y(0.2) = 176 + (0.2)(9) + \frac{(0.2)(0.2-1)}{2} \cdot 0 + \ldots$$

$$= 176 + 1.8$$

$$= 177.8$$

Hence $f(0.2) =$ 177.8.

The following table gives the population of a town during the last six census. Estimate, using Newton's interpolation formula, the increase in the population during the period 1946 to 1948.

Year	1911	1921	1931	1941	1951	1961
Population (in thousands)	12	13	20	27	39	52

Solution

The difference table is as follows :

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911(x_0)	12 (y_0)	Δy_0				
		1	$(\Delta^2 y_0)$			
1921	13		6	$(\Delta^3 y_0)$		
		7		–6	$(\Delta^4 y_0)$	
1931	20		0		11	$(\Delta^5 y_0)$
		7		5		–20
1941	27		5		–9	
		12		–4		
1951	39		1			
		13				
1961	52					

Here $x_0 = 1911, h = 10, y_0 = 12$

By Newton's formula we have

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y(1946) = ?$$

i.e., $x_0 + nh = 1946$

i.e., $1911 + n \cdot 10 = 1946$ *i.e.*, $n = 3.5$

$$\therefore y(1946) = 12 + (3.5)(1) + \frac{(3.5)(3.5-1)}{2} \times 6$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)}{6} \times (-6)$$

$$+ \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{24} \times 11$$

$$+ \frac{(3.5)(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{120} \times (-20)$$

$$= 12 + 3.5 + 26.25 - 13.125 + 3.0078 + 0.5469$$

$$= 12 + 3.5 + 26.25 + 3.0078 + 0.5469 - 13.125$$

$$= 32.18$$

$\therefore$ The population in the year 1946 is 32.18.

To find the population in the year 1948 :

i.e., To find $y(1948)$.

$$\text{i.e.,} \qquad x_0 + nh = 1948$$

$$1911 + n \cdot 10 = 1948$$

$$n = 3.7$$

$$\therefore y(1948) = 12 + 3.7 + \frac{(3.7)(3.7-1)}{2} \times 6$$

$$+ \frac{3.7(3.7-1)(3.7-2)}{6} \times (-6)$$

$$+ \frac{3.7(3.7-1)(3.7-2)(3.7-3)}{24} \times 11$$

$$+ \frac{3.7(3.7-1)(3.7-2)(3.7-3)(3.7-4)}{120} \times (-20)$$

$$= 12 + 3.7 + 29.97 - 16.983 + 5.4487 + 0.5944$$

$$= 34.73$$

The population in the year 1948 is 34.73.

Increase in the population during the period 1946 to 1948.

$$= \text{Population in 1948} - \text{Population in 1946}$$

$$= 34.73 - 32.18$$

$$= \textbf{2.55 thousands}$$

Lagrange's Interpolation Formula for unequal intervals

Let $f(x_0)$, $f(x_1)$, ..., $f(x_n)$ be the values of the function $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$, not necessarily equally spaced.

Let $f(x)$ be a polynomial in x of degree n. Then we can represent $f(x)$ as

$$\begin{aligned} f(x) = {} & a_0 (x-x_1)(x-x_2)\ldots(x-x_n) \\ & + a_1 (x-x_0)(x-x_2)\ldots(x-x_n) + \ldots \\ & + a_n (x-x_0)(x-x_1)\ldots(x-x_{n-1}) \end{aligned} \quad \ldots (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

Now we have to determine the $(n+1)$ constants $a_0, a_1, \ldots a_n$.

Putting $x = x_0$ in (1), we get

$$f(x_0) = a_0 (x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

i.e.,

$$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \quad \ldots (2)$$

Putting $x = x_1$ in (1), we get

$$f(x_1) = a_1 (x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

i.e.,

$$a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \quad \ldots (3)$$

Similarly

$$a_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} \quad \ldots (4)$$

....

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} \quad \ldots (5)$$

Substituting (2), (3), (4), (5) in (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1)$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n)$$

If we denote $f(x_0), f(x_1), \dots f(x_n)$ by $y_0, y_1, \dots, y_n$ we get,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

which is **Lagrange's interpolation** formula.

Using Lagrange's interpolation formula, find the value corresponding $x = 10$ from the following table :

x	5	6	9	11
y	12	13	14	16

Solution

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here $x = 10, \quad x_0 = 5, \quad x_1 = 6, \quad x_2 = 9, \quad x_3 = 11$

$y_0 = 12, \quad y_1 = 13, \quad y_2 = 14 \quad y_3 = 16$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{4 \cdot 1 \cdot 1}{1 \cdot 4 \cdot 6}(12) - \frac{5 \cdot 1 \cdot 1}{1 \cdot 3 \cdot 5}(13) + \frac{5 \cdot 4 \cdot 1}{4 \cdot 3 \cdot 2}(14) + \frac{5 \cdot 4 \cdot 1}{6 \cdot 5 \cdot 2}(16)$$

$$f(10) = 14.63$$

EXAMPLE 2

Find the equation of the parabola passing through the points (0,0), (1,1) and (2,20) using Langrange's formula.

Solution

The given points can be arranged in the form of table as given below.

x	0	1	2
y	0	1	20

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

$y_0 = 0$, $y_1 = 1$, $y_2 = 20$

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2 \quad \dots (1)$$

Substituting the above values, we get

$$y(x) = \frac{(x-1)(x-2)}{(-1)(-2)} \times (0) + \frac{(x-0)(x-2)}{(1)(-1)} \times (1) + \frac{(x-0)(x-1)}{(2)(1)} \times (20)$$

$$= (-x)(x-2) + x(x-1)(10)$$

$$= -x^2 + 2x + (x^2 - x)(10)$$

$$= -x^2 + 2x + 10x^2 - 10x$$

$$= 9x^2 - 8x$$

The required equation of parabola is

$$\boxed{y = 9x^2 - 8x}$$

EXAMPLE 3

Using Lagrange's interpolation formula, find the equation of the cubic curve that passes through the points (− 1 , − 8) , (0 , 3) , (2 , 1) and (3,2).

Solution

The given data can be arranged in the form of table as given below.

x	-1	0	2	3
y	-8	3	1	2

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

Here $x_0 = -1, \quad x_1 = 0, \quad x_2 = 2, \quad x_3 = 3$

$y_0 = -8, \quad y_1 = 3, \quad y_2 = 1, \quad y_3 = 2.$

$$f(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)$$

$$+ \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)$$

$$+ \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(2)$$

$$= \frac{2}{3}(x^3 - 5x^2 + 6x) + \frac{1}{2}(x^3 - 4x^2 + x + 6) - \frac{1}{6}(x^3 - 2x^2 - 3x$$

$$+ \frac{1}{6}(x^3 - x^2 - 2x)$$

$$y = x^3\left(\frac{2}{3} + \frac{1}{2} - \frac{1}{6} + \frac{1}{6}\right) + x^2\left(\frac{-10}{3} - \frac{4}{2} + \frac{2}{6} - \frac{1}{6}\right)$$

$$+ x\left(\frac{12}{3} + \frac{1}{2} + \frac{3}{6} - \frac{2}{6}\right) + \frac{6}{2}$$

On simplification, we get

$$y = \frac{7}{6}x^3 - \frac{31}{6}x^2 + \frac{14}{3}x + 3$$

$$y = \frac{7x^3 - 31x^2 + 28x + 18}{6}$$

$$6y = 7x^3 - 31x^2 + 28x + 18$$

The required equation of the cubic curve is

$$\therefore 6y = 7x^3 - 31x^2 + 28x + 18$$

Bessel's formula :-

Gauss's forward interpolation formula is

$$y(x_0+nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_{-1} + \frac{(n+1)(n)(n-1)}{3!}\Delta^3 y_{-1} + \frac{(n+1)n(n-1)(n-2)}{4!}\Delta^4 y_{-2} + \cdots \rightarrow (1)$$

we know that

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \longrightarrow (2)$$

$$\Delta^2 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^3 y_{-2} = \Delta^3 y_{-1} - \Delta^4 y_{-2}$$

$$\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2} \longrightarrow (3)$$

and so on. Now (1) can be written as

$$y(x_0+nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\left[\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right] + \frac{(n+1)n(n-1)}{3!}\Delta^3 y_{-1} + \frac{(n+1)n(n-1)(n-2)}{4!}\left[\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right] + \cdots$$

$$= y_0 + n\Delta y_0 + \frac{1}{2}\cdot\frac{n(n-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{n(n-1)}{2!}\left(\Delta^2 y_0 - \Delta^3 y_{-1}\right) + \frac{n(n^2-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\cdot\frac{(n+1)(n)(n-1)(n-2)}{4!}\Delta^4 y_{-2}$$

$$+\frac{1}{2}\frac{n(n+1)(n-1)(n-2)}{4!}\left(\Delta^4 y_{-1}-\Delta^5 y_{-2}\right)+\cdots$$

Using (2) and (3) in (1)

$$= y_0+n\Delta y_0+n(n-1)\left[\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2!}\right]+\frac{n(n-1)}{2!}\left(\frac{n+1}{3}-\frac{1}{2}\right)\Delta^3 y_{-1}$$

$$+\frac{(n+1)n(n-1)(n-2)}{4!}\left[\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right]+\cdots$$

$$y(x_0+nh)=y_0+n\Delta y_0+\frac{n(n-1)}{2!}\left[\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right]$$

$$+\frac{n(n-1)(n-\frac{1}{2})}{3!}\Delta^3 y_{-1}$$

$$+\frac{(n+1)n(n-1)(n-2)}{4!}\left[\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right]+\cdots$$

This is Bessel's formula.

EXAMPLE 1

Using Bessel's formula find $f(25)$ given $f(20) = 2854$, $f(24) = 3162$, $f(28) = 3544$, $f(32) = 3992$.

Solution

Here take $x_0 = 24$, $h = 4$, The Bessel's formula is

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{n(n-1)\left(n-\frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{(n+1)n(n-1)(n-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots$$

$$y(25) = ?$$

i.e., $x_0 + nh = 25$

$$n(4) = 25 - x_0 = 25 - 24$$

$$\therefore n = \frac{1}{4} = 0.25$$

The difference table is as given below

x	$f(x)$	Δ	Δ^2	Δ^3
20 (x_{-1})	2854 (y_{-1})			
		308 Δy_{-1}		
24 (x_0)	3162 (y_0)		74 $\Delta^2 y_{-1}$	
		382 Δy_0		−8 $\Delta^3 y_{-1}$
28 (x_1)	3544 (y_1)		66 $\Delta^2 y_0$	
		448 Δy_1		
32 (x_2)	3992 (y_2)			

Using the above formula, we get,

$$y(25) = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74+66}{2}\right) + \frac{(0.25)(0.25-1)\left(0.25-\frac{1}{2}\right)}{3!}(-8)$$

$$= 3162 + 95.5 - 6.5625 - 0.0625$$

$$y(25) = \mathbf{3250.875}$$

EXAMPLE 2

Given that sin(0.1) = 0.0998, sin(0.2) = 0.1986, sin(0.3) = 0.2955 ; sin(0.4) = 0.3894 and sin(0.5) = 0.4794, find sin(0.35).

Solution

The difference table is as given below

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$0.1 (x_{-2})$	$0.0998 (y_{-2})$				
		$0.0988\ \Delta y_{-2}$			
$0.2 (x_{-1})$	$0.1986 (y_{-1})$		$-0.0019\ \Delta^2 y_{-2}$		
		$0.0969\ \Delta y_{-1}$		-0.0011	
$0.3 (x_0)$	$0.2955 (y_0)$		$-0.0030\ \Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	0.0002 $\Delta^4 y_{-2}$
		$0.0939\ \Delta y_0$			
$0.4 (x_1)$	$0.3894 (y_1)$		$-0.0039\ \Delta^2 y_0$	-0.0009	
		$0.0900\ \Delta y_1$		$\Delta^3 y_{-1}$	
$0.5 (x_2)$	$0.4794 (y_2)$				

Here take $x_0 = 0.3$; $y_0 = 0.2955$ and $h = 0.1$

We know that Bessel's formula is

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{n(n-1)\left(n - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(n+1)n(n-1)(n-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots \quad \ldots(1)$$

$$y(0.35) = ?$$

i.e., $x_0 + nh = 0.35$

$$n(0.1) = 0.35 - 0.3 \Rightarrow n = 0.5$$

Substituting the values available in the table and $n = 0.5$ in (1), we get

$$y(0.35) = 0.2955 + (0.5) \times (0.0939) + \frac{(0.5)(0.5-1)}{2!}\left(\frac{-0.0030 - 0.0039}{2}\right)$$

$$+ \frac{(0.5)(0.5-1)\left(0.5 - \frac{1}{2}\right)}{3!}(-0.0009) + \ldots$$

$$= 0.2955 + 0.04695 + 0.00043125 + \ldots$$

$$= 0.3429$$

Therefore sin (0.35) = 0.3429

NUMERICAL INTEGRATION

The term Numerical integration is the numerical evaluation of a definite integral

$$A = \int_a^b f(x)\, dx$$

where 'a' and 'b' are given constants and $f(x)$ is a function given analytically by a formula or empirically by a table of values. Geometrically, A is the area under the curve of $f(x)$ between the ordinates $x = a$ and $x = b$.

But in engineering problems we frequently come across the integrals whose integrand is an empirical function given by a table. In these cases we may use a numerical method for approximate integration. When we apply numerical integration to a function of a single variable, the process is sometimes called **mechanical quadrature;** when we apply numerical integration to the computation of a double integral involving a function of two independent variables it is called **mechanical cubature.**

The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrand by an interpolation formula and then integrating this formula between the given limits. Thus, to find the value of the definite integral $\int_a^b f(x)\, dx$ (or) $\int_a^b y\, dx$ we replace the function $f(x)$ (or y) by an interpolation formula, usually one involving differences, and then integrate this formula between the limits a and b. In this way we can derive quadrature formulae for the approximate integration of any function for which numerical values are known.

Of the many possible quadrature formulae, here we shall derive some of the simplest and most useful one.

QUADRATURE FORMULA FOR EQUIDISTANT ORDINATES

Consider the Newton's forward difference formula

$$y(x) = y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

This formula can also be written by replacing n by u as

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \quad \ldots (1)$$

Let $y = y(x) \ldots (2)$ be the given function.

Let us now integrate (2) over n equidistant intervals of width $h (= \Delta x)$.

$$\text{i.e.,} \int_{x_0}^{x_0+nh} y(x)\,dx = ?$$

$$\text{Let } x = x_0 + uh$$

$$\therefore dx = h\,du$$

When $x = x_0,\ u = 0$

$x = x_0 + nh,\ u = n$

$$\therefore \int_{x_0}^{x_0+nh} y(x)\,dx = h\int_0^n y(x_0 + uh)\,du$$

$$= h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\right) du$$

[by 1]

$$= h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \ldots\right) du$$

$$= h\left[(uy_0)_0^n + \left(\frac{u^2}{2}\Delta y_0\right)_0^n + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)_0^n \Delta^2 y_0 \right.$$

$$\left. + \frac{1}{6}\left(\frac{u^4}{4} - u^3 + u^2\right)_0^n \Delta^3 y_0 + \ldots\right]$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 \right.$$

$$\left. + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\right]$$

$$\therefore \int_{x_0}^{x_0+nh} y(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \dots\right] \quad \dots (A)$$

This gives the general **Quadrature Formula** for equidistant ordinates and is known as **Newton-Cote's formula**.

❑ TRAPEZOIDAL RULE

Putting $n = 1$ in (A), we get

$$\int_{x_0}^{x_0+h} y(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

(neglecting higher order differences)

$$= \frac{h}{2}\,[2y_0 + \Delta y_0] = \frac{h}{2}\,[y_0 + (y_0 + \Delta y_0)]$$

$$= \frac{h}{2}\,[y_0 + y_1] \quad \dots (1)$$

In the interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} y(x)\,dx = h\,[y_1 + \frac{1}{2}\,\Delta y_1]$$

$$= \frac{h}{2}\,[2y_1 + \Delta y_1] = \frac{h}{2}\,[y_1 + (y_1 + \Delta y_1)]$$

$$= \frac{h}{2}\,[y_1 + y_2] \quad \dots (2)$$

....

.... etc

$$\int_{x_0+(n-1)h}^{x_0+nh} y(x)\,dx = \frac{h}{2}\,[y_{n-1} + y_n] \quad \dots (3)$$

Adding (1), (2) and (3), we get

$$\boxed{\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{h}{2}\,[(y_0 + y_n) + 2\,(y_1 + y_2 + \dots + y_{n-1})] \quad \dots (B)}$$

This is called the **Trapezoidal Rule**.

> *Note :* **The trapezoidal rule is the simplest of the formulae for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval h.**

TRUNCATION ERROR IN THE TRAPEZOIDAL RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \dots \quad \dots (1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'' \dots$ etc are the values of y', y'', etc at $x = x_1$.

$$\therefore \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} \left[y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \dots \right] dx$$

$$= \left[y_1 x + \frac{(x - x_1)^2}{2!} y_1' + \frac{(x - x_1)^3}{3!} y_1'' + \dots \right]_{x_1}^{x_2}$$

$$= y_1 (x_2 - x_1) + \frac{(x_2 - x_1)^2}{2!} y_1' + \frac{(x_2 - x_1)^3}{3} y_1'' + \dots$$

$$= h y_1 + \frac{h^2}{2!} y_1' + \frac{h^3}{3!} y_1'' + \quad \dots (2)$$

where $h = x_2 - x_1$

Now, A_1 = area of the trapezium in the interval (x_1, x_2)

$$= \frac{1}{2} h (y_1 + y_2) \quad \dots (3)$$

Putting $x = x_2$ and $y = y_2$ in (1), we get

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \dots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \dots \quad \dots (4)$$

where $h = x_2 - x_1$

Substituting (4) in (3), we get

$$A_1 = \frac{h}{2}\left[2y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \dots\right]$$

$$= hy_1 + \frac{h^2}{2!}y_1' + \frac{h^3}{2 \times 2!}y_1'' + \dots \quad \cdots (5)$$

$(2) - (5) \Rightarrow$

$$\int_{x_1}^{x_2} y\,dx - A_1 = \left(\frac{1}{3!} - \frac{1}{2 \times 2!}\right) h^3 y_1'' + \dots$$

$$= \frac{-h^3}{12} y_1'' + \dots$$

i.e., Principal part of the error in (x_1, x_2)

$$= \frac{-h^3}{12} y_1''$$

Similarly principal part of the error in the interval (x_2, x_3)

$$= \frac{-h^3}{12} y_2'' \text{ and so on.}$$

Hence the total error $E = \dfrac{-h^3}{12}\left[y_1'' + y_2'' + \dots + y_n''\right]$

$$\therefore E < \frac{-nh^3}{12} y''(\xi)$$

Where $y''(\xi)$ is the largest of the n quantities $y_1'', y_2'', \dots, y_n''$.

i.e., $\quad E < \dfrac{-nh^3}{12} y''(\xi) = -\dfrac{(b-a)h^2}{12} y''(\xi) \quad [\because n = \dfrac{b-a}{h}]$

$\therefore$ **Error in the trapezoidal rule is of the order h^2.**

Example 1

Compute the value of the definite integral $\int_4^{5.2} \log_e x\,dx$ or $\int_4^{5.2} \ln x\,dx$ using trapezoidal rule.

Solution

Divide the interval of integration into six equal parts each of width 0.2 *i.e.*, $h = 0.2$. The values of the function $y = \ln x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$ln\ x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By **Trapezoidal rule,** we have

$$\int_4^{5.2} ln\, x\, dx = \frac{h}{2}\left[(y_0 + y_6) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}[(1.386294 + 1.648658) + 2\,(1.435084 + 1.481604 + 1.526056 + 1.568616 + 1.609437)]$$

$$= (0.1)\,[\,3.034952 + 15.241562\,]$$

$$\boxed{\int_4^{5.2} ln\, x\, dx = \mathbf{1.8276544}}$$

Example 2

Evaluate $\int_0^1 e^{-x^2}\, dx$ by dividing the range of integration into 4 equal parts using trapezoidal rule. *[Nov. '91, Nov. '89]*

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By **Trapezoidal rule** we have

$$\int_0^1 e^{-x^2}\, dx = \frac{h}{2}\left[\,(y_0 + y_4) + 2\,(y_1 + y_2 + y_3)\,\right]$$

$$= \frac{0.25}{2}[1.3678 + 2\,(2.2876)] = (0.125)\,(5.943)$$

$$\boxed{\int_0^1 e^{-x^2}\, dx = \mathbf{0.7428}}$$

Example 3

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ *, using Trapezoidal rule with* $h = 0.2$*.* *Hence determine the value of* π*.* [April '92]

Solution

Here $h = 0.2$. The values of the function $y = \frac{1}{1+x^2}$ for each point of subdivision are given below.

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1 y_0	0.9615 y_1	0.8621 y_2	0.7353 y_3	0.6098 y_4	0.5 y_5

By **Trapezoidal rule** we have,

$$\boxed{\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4) \right]}$$

$$= \frac{0.2}{2}\left[1.5 + 2(3.1687)\right] = (0.1)(7.8374)$$

$$\boxed{\int_0^1 \frac{dx}{1+x^2} = \mathbf{0.78374}}$$

We know that

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 = \frac{\pi}{4} \qquad \therefore \pi = 4\int_0^1 \frac{dx}{1+x^2}$$

$$= 4(0.78374) \qquad \text{[From Trapezoidal Rule]}$$

$$\boxed{\therefore \quad \pi = \mathbf{3.13496}}$$

Example 4

Using Trapezoidal rule evaluate $\int_{0.6}^{2} y\, dx$ *from the following table.*

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

Solution

Here $h = 0.2$

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Trapezoidal rule, we have

$$\boxed{\int_{0.6}^{2} y\,dx = \frac{h}{2}\left[(y_0 + y_7) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)\right]}$$

$$= \frac{0.2}{2}\left[13.68 + 2\,(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)\right]$$

$$= (0.1)\,[79.22]$$

$$\boxed{\int_{0.6}^{2} \boldsymbol{y\,dx = 7.922}}$$

❑ SIMPSON'S $\frac{1}{3}$ RULE

Putting $n = 2$ in the above relation (A) (Refer Pg. No. 3.32) and neglecting all differences above the second we get,

$$\int_{x_0}^{x_0+2h} y(x)\,dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = 2h\left[\frac{6y_0 + 6\Delta y_0 + \Delta^2 y_0}{6}\right]$$

$$= 2h\left[\frac{6y_0 + 6\,(y_1 - y_0) + y_2 - 2y_1 + y_0}{6}\right]$$

$$= \frac{h}{3}\,[\,y_0 + 4y_1 + y_2\,]$$

$$\therefore \int_{x_0}^{x_0+2h} y(x)\,dx = \frac{h}{3}\,[\,y_0 + 4y_1 + y_2\,] \qquad \ldots (1)$$

Similarly for the next two intervals $x_0 + 2h$ to $x_0 + 4h$ we get,

$$\int_{x_0+2h}^{x_0+4h} y(x)\,dx = \frac{h}{3}\,[\,y_2 + 4y_3 + y_4\,] \qquad \ldots (2)$$

In general,

$$\int_{x_0+(n-2)h}^{x_0+nh} y(x)\,dx = \frac{h}{3}\,[\,y_{n-2} + 4y_{n-1} + y_n\,] \qquad \cdots (3)$$

Adding all the above integrals (1), (2), (3) we get,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[y_0 + 4\,(y_1 + y_3 + \ldots) + 2\,(y_2 + y_4 + \ldots) + y_n]$$

$$= \frac{h}{3}\,[y_0 + y_n + 4\,(\text{sum of odd ordinates}) + 2\,(\text{sum of even ordinates})]$$

This is called Simpson's one third rule or **Simpson's $\frac{1}{3}$ rule.**

Note 1 : **When using this formula the student must bear in mind that the interval of integration must be divided into an even number of subintervals of width h.**

Note 2 : **Simpson's $\frac{1}{3}$ rule is also called a closed formula, since the end point y_0 and y_1 are also included in the formula.**

❑ SIMPSON'S THREE – EIGHTH RULE :

Putting $n = 3$ in (A) (Refer Pg. No. 3.32) and neglecting the higher order differences above the third we get

$$\int_{x_0}^{x_0+nh} y(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2\,(y_3 + y_6 + \ldots + y_{n-3})].$$

This is known as **Simpson's three - eighth rule.**

Note : **This rule can be applied only if the number of subintervals is a multiple of 3.**

❑ TRUNCATION ERROR IN SIMPSON'S RULE

The Taylor series expansion of $y = f(x)$ about $x = x_1$ is given by

$$y = y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \qquad \ldots (1)$$

where y_1 is the value of y at $x = x_1$ and $y_1', y_1'', \ldots$ *etc*. are the values of $y', y'', \ldots$ *etc*. at $x = x_1$.

Hence

$$\int_{x_1}^{x_3} y\,dx = \int_{x_1}^{x_3} \left[y_1 + \frac{(x - x_1)}{1!} y_1' + \frac{(x - x_1)^2}{2!} y_1'' + \ldots \right] dx$$

$$= \left[y_1 x + \frac{(x - x_1)^2}{2!} y_1' + \frac{(x - x_1)^3}{3!} y_1'' + \ldots \right]_{x_1}^{x_3}$$

$$= y_1 (x_3 - x_1) + \frac{(x_3 - x_1)^2}{2!} y_1' + \frac{(x_3 - x_1)^3}{3!} y_1'' + \ldots$$

$$= 2hy_1 + \frac{(2h)^2}{2!} y_1' + \frac{(2h)^3}{3!} y_1'' + \frac{(2h)^4}{4!} y_1''' + \frac{(2h)^5}{5!} y_1^{iv} + \ldots$$

$$[\because x_2 - x_1 = h; \therefore x_3 - x_1 = 2h]$$

$$= 2hy_1 + 2h^2 y_1' + \frac{4h^3}{3} y_1'' + \frac{2h^4}{3} y_1''' + \frac{4h^5}{15} y_1^{iv} + \ldots \qquad \ldots (2)$$

Now,Area A_1 =area over the first double strip by Simpson's $\frac{1}{3}$ rule

$$= \frac{1}{3} h (y_1 + 4y_2 + y_3) \qquad \ldots (3)$$

Putting $x = x_2$ and therefore $y = y_2$ in (1), we get,

$$y_2 = y_1 + \frac{(x_2 - x_1)}{1!} y_1' + \frac{(x_2 - x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \qquad \ldots (4)$$

where $h = x_2 - x_1$

Putting $x = x_3$ and therefore $y = y_3$ in (1), we get,

$$y_3 = y_1 + \frac{(x_3 - x_1)}{1!} y_1' + \frac{(x_3 - x_1)^2}{2!} y_1'' + \ldots$$

$$= y_1 + \frac{2h}{1!} y'_1 + \frac{4h^2}{2!} y_1'' + \frac{8h^3}{3!} y_1''' + \frac{16h^4}{4!} y_1^{iv} + \ldots \quad \ldots (5)$$

Substituting (4) and (5) in (3), we get

$$A = \frac{h}{3}\left[y_1 + 4\left\{y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots\right\}\right.$$

$$\left. + \left\{y_1 + \frac{2h}{1!} + \frac{4h^2}{2!} y_1'' + \frac{8h^3}{3!} y_1''' + \frac{16h^4}{4!} y_1^{iv} + \right\}\ldots\right] \quad \ldots (6)$$

Subtracting (2) and (6), we get

$$\int_{x_1}^{x_3} y\,dx - A_1 = \left[\frac{4h^5}{15} y_1^{iv} - \frac{5h^5}{18} y_1^{iv}\right] + \ldots$$

$\therefore$ The error in the interval (x_1, x_3),

$$= \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_1^{iv} = \left(\frac{24-25}{90}\right) h^5 y^{iv}$$

$$= \frac{-h^5}{90} y_1^{iv}$$

$\therefore$ The principal part of the error in (x_1, x_3)

$$= \frac{-h^5}{90} y_1^{iv}$$

Similarly, principal part of the error in the interval (x_3, x_5),

$$= \frac{-h^5}{90} y_3^{iv} \text{ and so on.}$$

Hence the total error E

$$= \frac{-h^5}{90} y_1^{iv} - \frac{h^5}{90} y_3^{iv} - \ldots - \frac{h^5}{90} y_{2n-1}^{iv}$$

$$= \frac{-h^5}{90}\left[y_1^{iv} + y_3^{iv} + \ldots + y_{2n-1}^{iv}\right]$$

$$= \frac{-nh^5}{90} y^{iv}(\xi)$$

where $y^{iv}(\xi)$ is the largest of the n quantities $y_1^{iv}, y_3^{iv}, \ldots y_{2n-1}^{iv}$.

i.e., $$E < \frac{-(b-a)}{2h} \cdot \frac{h^5}{90} \left[\because \frac{b-a}{2n} = h, \text{ i.e., } h = \frac{b-a}{2h}\right]$$

$$< -\frac{h^4}{180}(b-a)$$

$\therefore$ **Error in the Simpson's $\frac{1}{3}$ rule is of the order h^4.**

Example 1

Compute the value of the definite integral $\int_4^{5.2} \log_e x\, dx$ or $\int_4^{5.2} \ln x dx$ using Simpson's rule.

Solution

Divide the interval of integration into six equal parts each of width 0.2 *i.e.*, $h = 0.2$. The values of the function $y = \ln x$ are next calculated for each point of subdivision as given below.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\ln x$	1.386294	1.435084	1.481604	1.526056	1.568616	1.609437	1.648658
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By **Simpson's $\frac{1}{3}$ rule, we have**

$$\boxed{\int_4^{5.2} \ln x\, dx = \frac{h}{3}\,[(y_0 + y_6) + 2\,(y_2 + y_4) + 4(y_1 + y_3 + y_5)]}$$

$$= \frac{0.2}{3}\,[3.034952 + 2\,(3.050221) + 4\,(4.570577)]$$

$$\boxed{\int_4^{5.2} \ln x\, dx = \quad 1.827847}$$

Example 2

Evaluate $\int_0^1 e^{-x^2}\, dx$ by dividing the range of integration into 4 equal parts using Simpson's rule. [Nov. '91, Nov. '89]

Solution

Here the length of the interval is $h = \frac{1-0}{4} = 0.25$. The values of the function $y = e^{-x^2}$ for each point of subdivision are given below.

x	0	0.25	0.5	0.75	1
e^{-x^2}	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

By **Simpson's rule** we have

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_4)+2y_2+4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[1.3678+1.5576+6.6368\right]$$

$$\boxed{\int_0^1 e^{-x^2}\,dx = \mathbf{0.7468}}$$

Example 3

Find the value of $\int_0^{\pi/2}\sqrt{1-0.162\sin^2 x}\,dx$***, using Simpson's one third rule.***

Solution

Let us divide the interval of integration into 6 equal subintervals

$$\text{i.e., } h = \frac{\pi/2-0}{6} = \frac{\pi}{12} = 15°$$

The values of the function $y = \sqrt{1-0.162\sin^2 x}$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1.0000	0.9946	0.9795	0.9586	0.9373	0.9213	0.9154
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^{\pi/2} y\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$\int_0^{\pi/2} \sqrt{1-0.162 \sin^2 x}\, dx = \frac{\pi}{36} [(1.0000 + 0.9154) + 4(0.9946$$

$$+ 0.9586 + 0.9213) + 2(0.9795 + 0.9373)]$$

$$\boxed{\int_0^{\pi/2} \sqrt{1-0.162 \sin^2 x}\, dx = 1.5051}$$

Example 4

Find the value of log 2 $^{\frac{1}{3}}$ ***from*** $\int_0^1 \frac{x^2}{1+x^3}\, dx$ ***using Simpson's*** $\frac{1}{3}$ ***rule with*** $h = 0.25$***.*** *[April '91]*

Solution

Given $h = 0.25$. The values of the function $y = \frac{x^2}{1+x^3}$ for each point of subdivisions are given below.

x	0	0.25	0.5	0.75	1.0
$y = \frac{x^2}{1+x^3}$	0 y_0	0.06154 y_1	0.22222 y_2	0.39560 y_3	0.5000 y_4

By Simpson's $\frac{1}{3}$ rule, we have

$$\boxed{\int_0^1 \frac{x^2}{1+x^3}\, dx = \frac{h}{3} [(y_0 + y_4) + 2y_2 + 4 (y_1 + y_3)]}$$

$$= \frac{0.25}{3} [(0+0.5) + 2 (0.22222)$$

$$+ 4 (0.06154 + 0.39560)]$$

$$= \frac{0.25}{3} [0.5 + 0.44444 + 1.82856]$$

$$\boxed{\int_0^1 \frac{x^2}{1+x^3}\, dx = 0.231083}$$

We know that,

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}(\log 2 - \log 1) = \frac{1}{3}\log_e 2$$

$$\therefore \log 2^{\frac{1}{3}} = \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$\boxed{\log 2^{\frac{1}{3}} = 0.231083}$$

Example 5

When a train is moving at 30 metres per second steam is shut off and brakes are applied. The speed of the train (V) in metres per second after t seconds is given by

t	*0*	*5*	*10*	*15*	*20*	*25*	*30*	*35*	*40*
V	*30*	*24*	*19.5*	*16*	*13.6*	*11.7*	*10.0*	*8.5*	*7.0*

Using Simpson's rule determine the distance moved by the train in 40 secs.

Solution

We know that velocity is the rate of change displacement.

i.e., $$V = \frac{ds}{dt} \text{ or } ds = V\,dt$$

Here we have to find the total distance moved by the train in 40 secs.

$$\therefore S = \int_0^{40} V\,dt$$

The given table is

t	0	5	10	15	20	25	30	35	40
V	30 V_0	24 V_1	19.5 V_2	16 V_3	13.6 V_4	11.7 V_5	10.0 V_6	8.5 V_7	7.0 V_8

By Simpson's rule we have

$$S = \int_0^{40} V\,dt$$

$$= \frac{h}{3}[(V_0 + V_8) + 2(V_2 + V_4 + V_6) + 4(V_1 + V_3 + V_5 + V_7)]$$

$$= \frac{5}{3}[37 + 2(19.5 + 13.6 + 10.0) + 4(24 + 16 + 11.7 + 8.5)]$$

$$= \frac{5}{3}[37 + 86.2 + 240.8] = \textbf{606.66 metres.}$$

$\therefore$ **Distance moved by the train in 40 secs = 606.66 *m*.**

Example 6

Given $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$. Use Simpson's rule to find an approximate value of $\int_0^4 e^x\,dx$. Also compare your result with the exact value of the integral.

Solution *[AMIE, S '88]*

The given values can be arranged in the form of table as given below.

x	0	1	2	3	4
$y = e^x$	1 y_0	2.72 y_1	7.39 y_2	20.09 y_3	54.60 y_4

By Simpson's rule, we have

$$\boxed{\int_0^4 e^x\,dx = \frac{h}{3}[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)]}$$

$$= \frac{1}{3}[55.60 + 14.78 + 4(2.72 + 20.09)]$$

$$= \frac{1}{3}[70.38 + 91.24]$$

$$\boxed{\int_0^4 e^x\,dx = 53.8733}$$

Now by ordinary integration we get

$$\int_0^4 e^x\,dx = (e^x)_0^4 = e^4 - e^0 = 54.598 - 1$$

$$\boxed{\int_0^4 e^x\,dx = 53.598}$$

Example 7

A river is 80 feet wide. The depth 'd' in feet at a distance x feet from one bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross section of the river using Simpson's rule. [AMIE S' 76]

Solution

Here $h = 10$. The given table is

x	0	10	20	30	40	50	60	70	80
$y = d$	0 y_0	4 y_1	7 y_2	9 y_3	12 y_4	15 y_5	14 y_6	8 y_7	3 y_8

By **Simpson's $\frac{1}{3}$ rule**, we have

$$\text{Area of cross-section} = \int_0^{80} y\, dx$$

$$\boxed{= \frac{h}{3}\left[(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)\right]}$$

$$= \frac{10}{3}\left[3 + 2(33) + 4(36)\right]$$

$$\boxed{\textbf{Area of cross section} = \textbf{710 sq. feet.}}$$

Example 8

Evaluate $\int_{0.2}^{1.4} (\sin x - \ln x + e^x)\, dx$ *by Simpson's* $\frac{1}{3}$ *rule.*

Solution

Let us divide the interval of integration into twelve equal parts by taking $h = 0.1$. Now the table of values of the given function $y = \sin x - \ln x + e^x$ at each point of subdivision is as given below.

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	3.02951 y_0	2.84936 y_1	2.79754 y_2	2.82130 y_3	2.89754 y_4	3.01465 y_5	3.16604 y_6
x	0.9	1.0	1.1	1.2	1.3	1.4	
y	3.34830 y_7	3.55935 y_8	3.80007 y_9	4.06984 y_{10}	4.37050 y_{11}	4.70418 y_{12}	

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{h}{3}\left[(y_0 + y_{12}) + 2\,(y_2 + y_4 + y_6 + y_8 + y_{10}) + 4\,(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11})\right]$$

$$= \frac{0.1}{3}\,[7.73369 + 2\,(16.49077) + 4\,(20.20418)]$$

$$= 4.05106$$

$$\therefore \int_{0.2}^{1.4} (\sin x - \ln x + e^x)\,dx = 4.05106$$

Example 9

Use Simpson's $\frac{1}{3}$ rule to estimate the value of $\int_{1}^{5} f(x)\,dx$ given

x	1	2	3	4	5
$y = f(x)$	13	50	70	80	100
	y_0	y_1	y_2	y_3	y_4

Solution

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{1}^{5} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_4) + 2\,(y_2) + 4\,(y_1 + y_3)]$$

$$= \frac{1}{3}\,[\,(13 + 100) + 2\,(70) + 4\,(50 + 80)]$$

$$= \frac{1}{3}\,[\,113 + 140 + 520\,]$$

$$\int_{1}^{5} f(x)\,dx = 257.67$$

Example 10

Evaluate $\int_1^4 f(x)\,dx$ *from the following table by Simpson's* $\frac{3}{8}$ *rule.*

x	1	2	3	4
$y = f(x)$	1 y_0	8 y_1	27 y_2	64 y_3

Solution

By Simpson's $\frac{3}{8}$ rule, we have

$$\boxed{\int_1^4 f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]}$$

$$= \frac{3(1)}{8}\left[1 + 3(8) + 3(27) + 64\right]$$

$$= \frac{3}{8}\left[1 + 24 + 81 + 64\right] = \frac{3}{8}[170]$$

$$\boxed{\int_1^4 f(x)\,dx = 63.75}$$

Example 11

Evaluate $\int_0^{\pi/2} \sin x\,dx$*, using Simpson's* $\frac{3}{8}$ *rule.*

Solution

To use Simpson's $\frac{3}{8}$ rule the number of subintervals should be a multiple of 3. Hence we divide the interval of integration $\left(0, \frac{\pi}{2}\right)$ into 9 subintervals of width $\frac{\pi}{18}$. Let $y = \sin x$. The values of the function $y = \sin x$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
y	0 y_0	0.1736 y_1	0.3420 y_2	0.5000 y_3	0.6428 y_4	0.7660 y_5	0.8660 y_6	0.9397 y_7	0.9848 y_8	1 y_9

By **Simpson's $\frac{3}{8}$ rule**, we have

$$\boxed{\int_0^{\pi/2} y\,dx = \frac{3h}{8}\left[(y_0 + y_9) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) + 2(y_3 + y_6)\right]}$$

$$\therefore \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{48}\left[(0 + 1) + 3(0.1736 + 0.3420 + 0.6428 + 0.7660 + 0.9397 + 0.9848) + 2(0.5 + 0.8660)\right]$$

$$= \frac{\pi}{48}(15.2787)$$

$$\boxed{\int_0^{\pi/2} \sin x\,dx = 0.999988}$$

Checking : $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = 1.$

Example 12

The velocity V of a particle at distances from a point on its path is given by the table :

S (feet)	**0**	**10**	**20**	**30**	**40**	**50**	**60**
V (feet/sec)	**47**	**58**	**64**	**65**	**61**	**52**	**38**

Estimate the time taken to travel 60 feet by using Simpson's one-third rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

[AMIE S' 90]

Solution

We know that the rate of change of displacement is velocity.

$$\text{i.e.,} \quad \frac{ds}{dt} = V$$

$$\text{(or)} \quad ds = V\,dt$$

$$\text{i.e.,} \quad dt = \frac{1}{V}\,ds \qquad \ldots (1)$$

Here we want to find the time taken to travel 60 feet. Therefore integrate (1) from 0 to 60, we get $\int_0^{60} dt = \int_0^{60} \frac{1}{V}\,ds$

The time taken to travel 60 feet is

$$t = \int_0^{60} \frac{1}{V}\, ds = \int_0^{60} y\, dx$$

The given table can be written as given below.

$x\ (= s)$	0	10	20	30	40	50	60
$y = \frac{1}{V}$	0.02127 y_0	0.01723 y_1	0.01563 y_2	0.01538 y_3	0.01639 y_4	0.01923 y_5	0.0263 y_6

By **Simpson's one third rule,** we have

$$\boxed{\int_0^{60} y\, dx = \frac{h}{3}\ [(y_0 + y_6) + 2\,(y_2 + y_4) + 4\,(y_1 + y_3 + y_5)]}$$

$$= \frac{10}{3}\ [(0.02127 + 0.0263) + 2\,(0.01563 + 0.01639)$$

$$+ 4\,(0.01724 + 0.01538 + 0.01923)]$$

$$= \frac{10}{3}\ [0.04757 + 0.06404 + 0.2074] = 1.063 \text{ secs.}$$

Hence time taken to travel 60 feet is 1.063 secs.

By **Simposon's $\frac{3}{8}$ rule**

$$\boxed{\int_0^{60} y\, dx = \frac{3h}{8}\ [(y_0 + y_6) + 3\,(y_1 + y_2 + y_4 + y_5) + 2\,(y_3)]}$$

$$= \frac{3 \times 10}{8}\ [(0.02127 + 0.02630)$$

$$+ 3\,(0.01723 + 0.01563 + 0.01639 + 0.01923)$$

$$+ 2\,(0.01538)]$$

$$= 3.75\ [0.04757 + 0.20544 + 0.03076]$$

$$\boxed{\int_0^{60} y\, dx = 1.064 \text{ secs.}}$$

By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x \, dx$ by using Simpson's $\frac{1}{3}$ rule. Is it possible to evaluate the same by Simpson's $\frac{3}{8}^{th}$ rule. Justify your answer.

Solution

Here range = $\pi - 0 = \pi$

$$\therefore \qquad h = \frac{\pi}{10}$$

The values of the function $y = \sin x$ for each point of subdivisions are given below.

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
y	0 y_0	0.3090 y_1	0.5878 y_2	0.8090 y_3	0.9511 y_4	1.0 y_5	0.9511 y_6	0.8090 y_7	0.5878 y_8	0.3090 y_9	0 y_{10}

By **Simpson's $\frac{1}{3}$ rule**

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3}[(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$= \frac{\frac{\pi}{10}}{3}[(0 + 0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1.0 + 0.8090 + 0.3090)]$$

$$\int_0^{\pi} \sin x \, dx \doteq 2.00091$$

Note : Here we cannot use Simpson's $\frac{3}{8}^{th}$ rule since the subintervals is not a multiple of 3.

GAUSS QUADRATURE FORMULA

Carl Frederich Gauss approached the problem of numerical integration in a different way. Instead of finding the area under the given curve, he tried to evaluate the function at some points along with the abscissa. Here the values of abscissa are not equal. Then apply certain weight to the evaluated function.

Thus for Gauss two point formula,

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

$$= \omega_1 f(t_1) + \omega_2 f(t_2) \qquad \dots (1)$$

The function $f(t)$ is evaluated at t_1 and t_2. ω_1 and ω_2 are the weights given to the two functions.

The basic methodology is explained as given below for **Gauss two point formula.**

Weddle's rule

Put $n = 6$ in Newton-Cote's quadrature formula. Neglecting all differences of order higher than sixth, we get

$$\int_{x_0}^{x_0+6h} f(x)dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right].$$

We now replace the term $\frac{41}{140}\Delta^6 y_0$ by $\frac{42}{140}\Delta^6 y_0 = \frac{3}{10}\Delta^6 y_0$. As a result of this adjustment the error crept in is $\frac{h\Delta^6 y_0}{140}$ which is very small and negligible when h and $\Delta^6 y_0$ are small. Using $E - 1 = \Delta$ and replacing all differences in terms of $y's$ we get

$$\int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6].$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

(Assuming n is a multiple of 6).

Adding all these integrals we get

$$\boxed{\begin{aligned}\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}[&(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5)\\ &+(2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}) + \cdots\\ &+(2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)].\end{aligned}}$$

This equation is called *Weddle's rule*.

❑ GAUSSIAN QUADRATURE (3 POINT) FORMULA

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(t)\,dt$$

where the interval (a, b) is changed into $(-1, 1)$ by the transformation,

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

Then
$$\boxed{\begin{aligned} \int_{-1}^{1} f(t)\,dt &= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \\ \text{where} \quad A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \\ t_1 &= -0.7745 \\ t_2 &= 0 \\ t_3 &= 0.7745 \end{aligned}}$$

Example 1

Evaluate $\int_1^2 \frac{dx}{x}$ *using Gauss 3 - point formula.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$= \frac{3}{2} + \frac{t}{2}$$

i.e., $$x = \frac{3+t}{2}$$

$$\therefore I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} f(t)\,dt$$

$$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \quad \ldots (1)$$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots (2)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{3-0.7745} = 0.4493 \\ f(t_2) &= f(0) = \frac{1}{3} = 0.3333 \\ f(t_3) &= f(0.7745) = \frac{1}{3+0.7745} = 0.2649 \end{aligned}\right\} \quad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.4493) + 0.8888\,(0.3333) + (0.2649)\,(0.5555)$$

$$\boxed{\mathbf{I = 0.6929}}$$

Example 2

Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ *using the three point Gaussian Quadrature.*

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

where $a = 0.2$, $b = 1.5$

$$= \frac{1.7}{2} + \frac{1.3t}{2}$$

i.e., $$x = \frac{1.7 + 1.3\,t}{2} \Rightarrow dx = \frac{1.3\,dt}{2} = 0.65\,dt$$

$$\therefore \quad I = \int_{0.2}^{1.5} e^{-x^2}\,dx$$

$$= \int_{-1}^{1} -e^{\left(\frac{1.7+1.3\,t}{2}\right)^2} (0.65)\,dt$$

$$= 0.65 \int_{-1}^{1} -e^{\left(\frac{1.7+1.3\,t}{2}\right)^2} dt \quad \ldots (1)$$

$$I = 0.65[A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)] \quad \ldots (2)$$

$$\text{where } f(t) = e^{-\left(\frac{1.7 + 1.3\,t}{2}\right)^2}$$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \qquad \ldots (3)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = e^{-\left(\frac{1.7 + 1.3(-0.7745)}{2}\right)^2} \\ &= 0.8868 \\ f(t_2) &= f(0) = e^{-\left(\frac{1.7 + 1.3\,(0)}{2}\right)^2} \\ &= 0.48555 \\ f(t_3) &= f(0.7745) = e^{-\left(\frac{1.7 + 1.3\,(0.7745)}{2}\right)^2} \\ &= 0.16013 \end{aligned}\right\} \qquad \ldots (4)$$

Substituting (2) and (3) in (1), we get

$$I = 0.5555\,(0.8868) + 0.8888\,(0.4855) + (0.5555)(0.16013)$$

$$= 0.4926 + 0.4315 + 0.08895$$

$$\boxed{\mathbf{I = 1.01307}}$$

Example 3

Evaluate $\int_0^1 \frac{1}{1+t}\,dt$ by Gaussian quadrature formula.

Solution

Transform the variable from t to x by the transformation

$$t = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)x \qquad \ldots (1)$$

Here $a = 0$, $b = 1$.

$$\therefore \qquad t = \frac{1}{2} + \frac{x}{2} = \frac{x+1}{2} = dt = \frac{dx}{2}$$

when $t = 0$, $x = -1$

$t = 1$, $x = 1$

$$I = \int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx/2}{1+\left(\frac{x+1}{2}\right)}$$

$$= \int_{-1}^{1} \frac{dx}{2+x+1}$$

$$\boxed{I = A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)} \quad \ldots (2)$$

where $f(x) = \dfrac{1}{x+3}$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots (3)$$

$$\left.\begin{aligned} f(x_1) &= f(-0.7745) = \frac{1}{(-0.7745)+3} = 0.4493 \\ f(x_2) &= f(0) = \frac{1}{3} = 0.3333 \\ f(x_3) &= f(0.7745) = \frac{1}{0.7745+3} = 0.2649 \end{aligned}\right\} \quad \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$I = [0.5555 \times 0.4493 + 0.8888 \times 0.3333 + 0.5555 \times 0.2649]$$

$$= [0.2495 + 0.2962 + 0.14715]$$

$$\boxed{\mathbf{I = 0.69285}}$$

Example 4

Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2}$***, using Gauss 3 point formula.***

Solution

Transform the variable from x to t by the transformation

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$= \frac{1}{2} + \frac{t}{2} = \frac{t+1}{2}$$

i.e., $x = \dfrac{t+1}{2}$ when $x = 0, t = -1$

$x = 1, t = 1$

$$dx = \frac{dt}{2}$$

$$\therefore\ I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{1+\left(\frac{t+1}{2}\right)^2}\frac{dt}{2}$$

$$= 2\int_{-1}^{1} \frac{dt}{4+(t+1)^2}$$

$$\boxed{I = 2\{A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3)\}} \quad \ldots(1)$$

where $f(t) = \dfrac{1}{4+(t+1)^2}$

$$\left.\begin{aligned} A_1 = A_3 &= 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \quad \ldots(2)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{4+(-0.7745+1)^2} \\ &= 0.2468 \\ f(t_2) &= f(0) = \frac{1}{4+1} = 0.2 \\ f(t_3) &= f(0.7745) = \frac{1}{4+(0.7745+1)^2} \\ &= 0.13988 \end{aligned}\right\} \quad \ldots(3)$$

Substituting (2) and (3) in (1), we get

$$I = 2[0.5555(0.2468) + 0.8888(0.2) + 0.5555(0.13988)]$$

$$= 2[0.39256]$$

$$\boxed{\mathbf{I = 0.78512}}$$

Example 5

Evaluate $\int_1^2 \frac{dx}{1+x^3}$ using Gauss 3 point formula.

Solution

Transform the variable from x to t by the transformation

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

$$x = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2}$$

$$\therefore I = \int_1^2 \frac{dx}{1+x^3} = \int_{-1}^{1} \frac{1}{1+\frac{(3+t)^3}{8}} \frac{dt}{2}$$

$$= 4\int_{-1}^{1} \frac{1}{8+(3+t)^3}\, dt \qquad \ldots (1)$$

$$\therefore I = 4\int_{-1}^{1} \frac{1}{8+(3+t)^3}\, dt$$

$$= A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) \qquad \ldots (2)$$

where $f(t) = \dfrac{1}{(3+t)^3+8}$

$$\left.\begin{aligned} A_1 &= A_3 = 0.5555 \\ A_2 &= 0.8888 \end{aligned}\right\} \qquad \ldots (3)$$

$$\left.\begin{aligned} f(t_1) &= f(-0.7745) = \frac{1}{(3-0.7745)^3+8} = 0.0525 \\ f(t_2) &= f(0) = \frac{1}{35} = 0.0285 \\ f(t_3) &= f(0.7745) = \frac{1}{(3+0.7745)^3+8} = 0.0162 \end{aligned}\right\} \qquad \ldots (4)$$

Substituting (3) and (4) in (2), we get

$$I = 4\,[0.5555 \times 0.0525 + 0.8888 \times 0.0285 + 0.5555 \times 0.0162]$$

$$= 4\,[\,0.06349\,]$$

$$\boxed{\mathbf{I = 0.25396}}$$

EXERCISES

1. Applying Gauss's quadrature 3 point formula, evaluate $\int_5^{12} \frac{dx}{x}$.

[*Ans :* 0.25009]

2. Evaluate $\int_0^1 x\,dx$ by 3 point Gaussian formula. [*Ans :* 0.4999]

3. Evaluate by Gaussian 3 point formula $\int_0^{10} e^{\frac{1}{1+x^2}}\,dx$. [*Ans :* 11.986]

SCANNED BOOKS

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3. **S. ARUMUGAM**, Numerical Methods (Second Edition), Scitech Publications (INDIA) Pvt. Ltd. Chennai 600 017.