Mathematical Physics

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Unit – I

Vector Analysis

Concept of Vector and Scalar fields

In vector calculus and physics, a vector field is an assignment of a vector to each point in a subset of space. For instance, a vector field in the plane can be visualised as a collection of arrows with a given magnitude and direction, each attached to a point in the plane. A vector field is a vector each of whose components is a scalar field, that is, a function of our variables. We use any of the following notations for one: v (x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)) v (x, y, z) = v_1(x, y, z) I + v_2(x, y, z)j + v_3(x, y, z)k

Given a subset S in R^n, a vector field is represented by a vector-valued function V: S → R^n in standard Cartesian coordinates (x_1, …, x_n). If each component of V is continuous, then V is a continuous vector field, and more generally V is a C^k vector field if each component of V is k times continuously differentiable.

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Scalar field

A scalar field is a field for which there is a single number associated with every point in. space. We have seen that the temperature of the Earth's atmosphere at the surface is an. example of a scalar field. Let f be a scalar function, such that any point X is attached a real number. Ordered pair (f) is called a stationary scalar field, while function f is called the potential of this field.

For example, the temperature of all points in a room at a particular time t is a scalar field. The gradient of this field would then be a vector that pointed in the direction of greatest temperature increase. Its magnitude represents the magnitude of that increase.

Gradient of a vector field

Let $\vec{u}$ be an unit vector, and $f = f(x, y, z)$ a scalar function which gradient is then $\vec{\nabla} f$. Now, lets take their dot product:

$$\vec{\nabla} f \cdot \vec{u} = ||\vec{\nabla} f|| \cdot ||\vec{u}|| \cdot cos(\vec{\nabla} f, \vec{u})$$

Since $\vec{u}$ is a unit vector, the above can be written as:

$$\vec{\nabla} f \cdot \vec{u} = ||\vec{\nabla} f|| \cdot cos(\vec{\nabla} f, \vec{u})$$

You can think of this as projecting $\vec{\nabla} f$ on the direction of $\vec{u}$.

Now, if $\vec{u}$ is in the same direction as $\vec{\nabla} f$, then the projection has a maximum value:

$$\vec{\nabla} f \cdot \vec{u} = ||\vec{\nabla} f||$$

which is exactly the value of the change of the function f. You can see now that this quantity is the greatest in the direction of $\vec{u}$ which is in the same direction as $\vec{\nabla} f$.

$$\begin{aligned}
\frac{\partial f(x^2)}{\partial x^\alpha} &= \left(x^\mu \frac{\partial x_\mu}{\partial x^\alpha} + x_\mu \frac{\partial x^\mu}{\partial x^\alpha} \right) f'(x^2) \\
&= \left(x^\mu \eta_{\mu\nu} \frac{\partial x^\nu}{\partial x^\alpha} + x_\mu \frac{\partial x^\mu}{\partial x^\alpha} \right) f'(x^2) \\
&= \left(x^\mu \eta_{\mu\nu} \delta^\nu_\alpha + x_\mu \delta^\mu_\alpha \right) f'(x^2) \\
&= 2 x_\alpha f'(x^2) \ ,
\end{aligned}$$

Find the value of r and t using gradient of a vectorfield

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(x(t+\Delta t), y(t+\Delta t)) - (x(t), y(t))}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(x(t+\Delta t) - x(t), y(t+\Delta t) - y(t))}{\Delta t} \\
&= (\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}) \\
&= (x'(t), y'(t))
\end{aligned}$$

$$let\, \boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

$$\Rightarrow \boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t) \rangle - \langle f(t), g(t), h(t) \rangle}{\Delta t}$$

$$\Rightarrow \boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t) - f(t), g(t+\Delta t) - g(t), h(t+\Delta t) - h(t) \rangle}{\Delta t}$$

$$\Rightarrow \boldsymbol{r}'(t) = \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$\Rightarrow \boldsymbol{r}'(t) = \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$\Rightarrow \boldsymbol{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\Rightarrow \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

$$let\, \boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k})$$

$$\Rightarrow \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(f(t)\boldsymbol{i}) + \frac{d\boldsymbol{r}}{dt}(g(t)\boldsymbol{j}) + \frac{d\boldsymbol{r}}{dt}(h(t)\boldsymbol{k})$$

$$\Rightarrow \boldsymbol{r}'(t) = \left(f'(t)\boldsymbol{i} + f(t)\frac{d}{dt}\boldsymbol{i}\right) + \left(g'(t)\boldsymbol{j} + g(t)\frac{d}{dt}\boldsymbol{j}\right) + \left(h'(t)\boldsymbol{k} + h(t)\frac{d}{dt}\boldsymbol{k}\right)$$

$$\Rightarrow \boldsymbol{r}'(t) = (f'(t)\boldsymbol{i} + 0) + (g'(t)\boldsymbol{j} + 0) + (h'(t)\boldsymbol{k} + 0)$$

$$\Rightarrow \boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

Product rules

The similar relations between
calculation of derivatives and vector derivatives

(1)Sum rules

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$
$$\nabla\cdot(\vec{A}+\vec{B})=(\nabla\cdot\vec{A})+(\nabla\cdot\vec{B})$$
$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

(2)The rule for multiplication by a constant

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$
$$\nabla\cdot(k\vec{A})=k(\nabla\cdot\vec{A})$$
$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

Gradient of a scalar function

Gradient: differential change of a scalar

$$dT(x,y,z)=\frac{\partial T}{\partial x}dx+\frac{\partial T}{\partial y}dy+\frac{\partial T}{\partial z}dz,$$
$$=\frac{\partial T}{\partial x}\hat{x}\bullet d\bar{l}+\frac{\partial T}{\partial y}\hat{y}\bullet d\bar{l}+\frac{\partial T}{\partial z}\hat{z}\bullet d\bar{l}$$
$$=(\frac{\partial T}{\partial x}\hat{x}+\frac{\partial T}{\partial y}\hat{y}+\frac{\partial T}{\partial z}\hat{z})\bullet d\bar{l},$$
$$=\nabla T\bullet d\bar{l},$$

$$\nabla T\equiv(\frac{\partial T}{\partial x}\hat{x}+\frac{\partial T}{\partial y}\hat{y}+\frac{\partial T}{\partial z}\hat{z}),$$

$$\Delta T=\int dT(x,y,z)=\int\nabla T\bullet d\bar{l}$$

The direction of ∇T is along the maximum increase of T.

$$\begin{aligned}D_{\mathbf{v}}f(x,y)&=\nabla f(x,y)\cdot\mathbf{v}\\&=\left.\frac{\partial f}{\partial x}\right|_{(1,1)}v_x+\left.\frac{\partial f}{\partial y}\right|_{(1,1)}v_y\\&=1\cos(\theta)+1\sin(\theta)\\&=\cos(\theta)+\sin(\theta)\\&=(\sqrt{2})\left[\cos(\theta)\sin\left(\tfrac{\pi}{4}\right)+\sin(\theta)\cos\left(\tfrac{\pi}{4}\right)\right]\\&=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\text{ by the two angle formula}\end{aligned}$$

$$
\begin{aligned}
D_{\mathbf{v}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{v} \\
&= \left.\frac{\partial f}{\partial x}\right|_{(1,1)} v_x + \left.\frac{\partial f}{\partial y}\right|_{(1,1)} v_y \\
&= 1 \cos(\theta) + 1 \sin(\theta) \\
&= \cos(\theta) + \sin(\theta) \\
&= (\sqrt{2}) \left[\cos(\theta) \sin\left(\tfrac{\pi}{4}\right) + \sin(\theta) \cos\left(\tfrac{\pi}{4}\right)\right] \\
&= \sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) \text{ by the two angle formula}
\end{aligned}
$$

The gradient of a scalar function $f(x)$ with respect to a vector variable $x = (x_1, x_2, \ldots, x_n)$ is denoted by ∇f where ∇ denotes the vector differential operator del. By definition, the gradient is a vector field whose components are the partial derivatives of f:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

The form of the gradient depends on the coordinate system used.

For Cartesian Coordinates:

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

For Cylindrical Coordinates:

$$\nabla f(\rho, \theta, z) = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

(where θ is the azimuthal angle and z is the axial coordinate)

For Spherical Coordinates:

$$\nabla f(r, \theta, \varphi) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r \sin\theta}\frac{\partial f}{\partial \varphi}\right)$$

(where θ is the azimuthal angle and φ is the polar angle).

Divergence of a vectorfield

The divergence and curl of a vector field are two vector operators whose basic properties can be understood geometrically by viewing a vector field as the flow of a fluid or gas. Divergence is discussed on a companion page. The curl of a vector field captures the idea of how a fluid may rotate. A vector field is a vector function of position. We can have a constant vector field, meaning at each position the vector is the same. But in general a vector field can have an arbitrary value for the vector at every position.

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The curl of a vector field captures the idea of how a fluid may rotate.

$$\begin{aligned}\nabla \bullet G(x,y,z) &= \frac{\partial}{\partial x}(G_1) + \frac{\partial}{\partial y}(G_2) + \frac{\partial}{\partial z}(G_3) \\ &= \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(\ln(xy)) + \frac{\partial}{\partial z}(e^{xyz}) \\ &= e^x + \frac{\partial}{\partial y}((\ln x) + (\ln y)) + e^{xyz}\,\frac{\partial}{\partial z}(xyz) \\ &= e^x + \frac{1}{y} + e^{xyz}xy\end{aligned}$$

$$\boldsymbol{F} = F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{k} = \left\langle F_1, F_2, F_3 \right\rangle$$

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\text{Div}=\nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Find the value of e^x using divergencemethod

$$\begin{aligned}\nabla \bullet G(x,y,z) &= \frac{\partial}{\partial x}(G_1) + \frac{\partial}{\partial y}(G_2) + \frac{\partial}{\partial z}(G_3) \\ &= \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(\ln(xy)) + \frac{\partial}{\partial z}(e^{xyz}) \\ &= e^x + \frac{\partial}{\partial y}((\ln x) + (\ln y)) + e^{xyz}\,\frac{\partial}{\partial z}(xyz) \\ &= e^x + \frac{1}{y} + e^{xyz}xy\end{aligned}$$

Find the value of v using divergence method

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3$$

$$= \frac{\partial \left(\frac{2}{3}x\right)}{\partial x} + \frac{\partial \left(\frac{2}{3}y\right)}{\partial y} v_2 + \frac{\partial \left(\frac{2}{3}z\right)}{\partial z} v_3$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

$$div\ v = \nabla \cdot v = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] \cdot \left[iv_1 + jv_2 + kv_3 \right]$$

$$\nabla \cdot v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 v_1$$

$$+ \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (v_2 \sin\theta)$$

$$+ \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} v_3$$

$$\nabla \cdot \vec{v} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho v_1 + \frac{1}{\rho} \frac{\partial}{\partial \phi} v_2 + \frac{\partial}{\partial z} v_3$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \left(\frac{2}{3}\rho\right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} 0 + \frac{\partial}{\partial z} \left(\frac{2}{3}z\right)$$

$$= \frac{2/3}{\rho} \frac{\partial}{\partial \rho} \rho^2 + 0 + \frac{2}{3}$$

$$= 2\left(\frac{2}{3}\right) \frac{\not{\rho}}{\not{\rho}} + \frac{2}{3} = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

Curl of a vector field

In vector calculus, the curl is a vector operator that describes the infinitesimal circulation of a vector field in three-dimensional Euclidean space. ... The curl of a field is formally defined as the circulation density at each point of the field. A vector field whose curl is zero is called irrotational.

The curl is a three-dimensional vector, and each of its three components turns out to be a combination of derivatives of the vector field F. You can read about one can use the same spinning spheres to obtain insight into the components of the vector curl. Counter clockwise is defined as positive curl for the same reason the cross product is defined as it is (the right hand rule - the cross product of i and j is k). If r points in the x direction and F points in the y direction, then tau is in the positive z direction, by the definition of cross product.

Physically, it also doesn't make sense to apply curl and divergence of a scalar field. This means that any divergence is equal to its negation! Clearly, this makes no sense. If you use the common, intuitive fluid metaphor for fields, you can also visualise why the divergence of a scalar field doesn't make sense.

In vector calculus, the curl is a vector operator that describes the infinitesimal circulation of a vector field in three-dimensional Euclidean space. The curl at a point in the field is represented by a **vector** whose length and direction denote the magnitude and axis of the maximum circulation.

$$\nabla \times \underline{A} = \begin{vmatrix} \underline{a}_x & \underline{a}_y & \underline{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \underline{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \underline{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \underline{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Counter clockwise is defined as positive curl for the same reason the cross product is defined as it is (the right hand rule -- the cross product of i and j is k).

If r points in the x direction and F points in the y direction, then tau is in the positive z direction, by the definition of cross product.

In vector calculus, the curl is a vector operator that describes the infinitesimal circulation of a vector field in three-dimensional Euclidean space. The curl of a field is formally defined as the circulation density at each point of the field. A vector field whose curl is zero is called irrotational.

Find the value of F using curl function, using the values p, q and r

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & x-y-z & x^2+y^2+z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x-y-z)\right)i - \left(\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x+y+z)\right)j +$$
$$\left(\frac{\partial}{\partial x}(x-y-z) - \frac{\partial}{\partial y}(x+y+z)\right)k$$

$$= (2y+1)i - (2x-1)j + (1-1)k$$
$$= (2y+1)i + (1-2x)j + 0k$$
$$= (2y+1, 1-2x, 0)$$

Find the value of F using curl function, using the values p, q and

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y) & 4z & x^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)i - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2 - y)\right)j +$$

$$\left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y)\right)k$$

$$= (0-4)i - (2x-0)j + (0+1)k$$

$$= (-4)i - (2x)j + 1k$$

$$= (-4, -2x, 1)$$

Find the value of F using curl function, using the values p, q and

$$curlF = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+z & x-y-z & x^2+y^2+z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x-y-z)\right)i - \left(\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial z}(x+y+z)\right)j +$$

$$\left(\frac{\partial}{\partial x}(x-y-z) - \frac{\partial}{\partial y}(x+y+z)\right)k$$

$$= (2y+1)i - (2x-1)j + (1-1)k$$

$$= (2y+1)i + (1-2x)j + 0k$$

$$= (2y+1, 1-2x, 0)$$

Curl and divergence vector

We define the ***vector differential operator*** ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Del has meaning when it operates on a scalar function f to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Laplacian of a vector field

A Laplacian vector field in the plane satisfies the Cauchy–Riemann equations: it is holomorphic. Therefore, the potential of a Laplacian field satisfies Laplace's equation. n mathematics, the Laplace operator or Laplacian is a differential operator given by the divergence of the gradient of a function on Euclidean space. It is usually denoted by the symbols ∇ The Laplacian represents the flux density of the gradient flow of a function.

The Laplacian is a scalar operator. If it is applied to a scalar field, it generates a scalar field. = ∂2 ∂x2 (xy2 + z3) + ∂2 ∂y2 (xy2 + z3) + ∂2 ∂z2 (xy2 + z3) = ∂ ∂x (y2 + 0) + ∂ ∂y (2xy + 0) + ∂ ∂z (0 + 3z2) = 0+2x + 6z = 2x + 6z.

The Laplacian operator can be defined, not only as a differential operator, but also through its averaging properties. Such a definition lends geometric significance to the operator: a large Laplacian at a point reflects a "nonconformist" (i.e., different from average) character for the function

In mathematics, the Laplace transform, named after its inventor Pierre-Simon Laplace is an integral transform that converts a function of a real variable (often time) to a function of a complex variable. Advertisements. Laplacian Operator is also a derivative operator which is used to find edges in an image.

The major difference between Laplacian and other operators like Prewitt, Sobel, Robinson and Kirsch is that these all are first order derivative masks but Laplacian is a second order derivative mask.

Also the thresholder magnitude of Laplacian operator produces double edges. For these reasons, together with its inability to detect the edge direction, the Laplacian as such is not a good edge detection operator. A better utilization of it is to use its zero-crossing to detect the edge locations.

$$\nabla^2 \mathbf{v} = \begin{bmatrix} \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2} \\ \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{\partial^2 v_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2} \\ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} \end{bmatrix}.$$

Cartesian co-ordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Find the laplacian for the given values

Gradient at $X = [f(P) - f(P_1)]/r\,\delta\theta$

$$= \{[f(P) - f(P_2)] + [f(P_2 - f(P_1)]\}/r\,\delta\theta$$

$$= \{(\frac{\partial f}{\partial \theta})_X\,\delta\theta - \frac{\partial f}{\partial r}.\frac{r(\delta\theta)^2}{2}\}\Big/ r\delta\theta$$

$$= \frac{1}{r}(\frac{\partial f}{\partial \theta})_X \quad - \quad \frac{\partial f}{\partial r}.\frac{\delta\theta}{2}$$

Gradient at $Y = [f(P_4) - f(P)]/r\,\delta\theta$

$$= \frac{1}{r}(\frac{\partial f}{\partial \theta})_Y \quad + \quad \frac{\partial f}{\partial r}.\frac{\delta\theta}{2}$$

Then $\quad \partial^2 f/\partial s^2 = \lim_{\delta\theta\to 0}\{\frac{1}{r}[(\frac{\partial f}{\partial \theta})_Y - (\frac{\partial f}{\partial \theta})_X]\Big/ r\delta\theta + \frac{\partial f}{\partial r}.\delta\theta\Big/ r\,\delta\theta\}$

$$= \quad \frac{1}{r^2}.\frac{\partial^2 f}{\partial \theta^2} \quad + \quad \frac{1}{r}.\frac{\partial f}{\partial r}$$

So the mean curvature flow equation can be written as

$$\frac{\partial X}{\partial t} = \triangle_{g(t)} X. \tag{2.2}$$

At first sight it appears to be strictly parabolic. However, the Laplacian is taken in the induced metric which changes with $X(\cdot, t)$, and this adds extra terms to the symbol. In fact,

$$\begin{aligned}
\triangle X^{\alpha} &= g^{ij}\{\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X^\alpha}{\partial x^k}\} \\
&= g^{ij}\{\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \frac{1}{2} g^{kl}(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})\frac{\partial X^\alpha}{\partial x^k}\} \\
&= g^{ij}\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - \frac{1}{2} g^{ij} g^{kl} \frac{\partial X^\alpha}{\partial x^k}\cdot(\frac{\partial^2 X^{\beta}}{\partial x^i \partial x^j}\frac{\partial X^\beta}{\partial x^l} + \frac{\partial X^\beta}{\partial x^j}\frac{\partial^2 X^\beta}{\partial x^i \partial x^l} \\
&\quad + \frac{\partial^2 X^\beta}{\partial x^j \partial x^i}\frac{\partial X^\beta}{\partial x^l} + \frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\beta}{\partial x^j \partial x^l} - \frac{\partial^2 X^\beta}{\partial x^l \partial x^i}\frac{\partial X^\beta}{\partial x^j} - \frac{\partial X}{\partial x^i}\frac{\partial^2 X}{\partial x^l \partial x^j}) \\
&= g^{ij}\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - g^{ij} g^{kl}\frac{\partial X^\alpha}{\partial x^k}\frac{\partial X^\beta}{\partial x^l}\frac{\partial^2 X^\beta}{\partial x^i \partial x^j} \\
&= g^{ij}\{\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - g^{kl}\frac{\partial X^\alpha}{\partial x^k}\frac{\partial X^\beta}{\partial x^l}\frac{\partial^2 X^\beta}{\partial x^i \partial x^j}\}.
\end{aligned}$$

Hence the Laplacian operator is degenerate on the tangential directions. This just states the fact that the mean curvature flow is geometric one and is independent of the parametrization of the hypersurfaces.

Curl and divergence of a vector

$$
\begin{aligned}
\nabla(fg) &= f\nabla g + g\nabla f \\
\nabla(\vec{u}\cdot\vec{v}) &= \vec{u}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{u}) + (\vec{u}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{u} \\
\nabla\cdot(f\vec{v}) &= f(\nabla\cdot\vec{v}) + \vec{v}\cdot(\nabla f) \\
\nabla\cdot(\vec{u}\times\vec{v}) &= \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v}) \\
\nabla\times(f\vec{v}) &= (\nabla f)\times\vec{v} + f(\nabla\times\vec{v}) \\
\nabla\times(\vec{u}\times\vec{v}) &= \vec{u}\,(\nabla\cdot\vec{v}) - \vec{v}\,(\nabla\cdot\vec{u}) + (\vec{v}\cdot\nabla)\,\vec{u} - (\vec{u}\cdot\nabla)\,\vec{v}
\end{aligned}
$$

Laplacian in cylindrical co-ordinates

$$
\begin{aligned}
\vec{\nabla}^2\vec{A}(r,\varphi,z) = \hat{r}(\varphi)&\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial A_r}{\partial\varphi} - A_\varphi\right) - \frac{1}{r^2}\left(A_r + \frac{\partial A_\varphi}{\partial\varphi}\right) + \frac{\partial^2 A_r}{\partial z^2}\right] + \\
+\hat{\varphi}(\varphi)&\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial A_r}{\partial\varphi} - A_\varphi\right) + \frac{1}{r^2}\frac{\partial}{\partial\varphi}\left(A_r + \frac{\partial A_\varphi}{\partial\varphi}\right) + \frac{\partial^2 A_\varphi}{\partial z^2}\right] + \\
+\hat{z}&\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial\varphi^2} + \frac{\partial^2 A_z}{\partial z^2}\right]
\end{aligned}
$$

Recall that the vector Laplacian of a vector function $\underline{V}(x, y, z)$ is defined as

$$\nabla^2\underline{V} \equiv \nabla(\nabla\cdot\underline{V}) - \nabla\times(\nabla\times\underline{V}).$$

Prove that in rectangular coordinates the following identity holds:

$$\nabla^2\underline{V} = \hat{\underline{x}}\nabla^2 V_x + \hat{\underline{y}}\nabla^2 V_y + \hat{\underline{z}}\nabla^2 V_z .$$

In words, this identity says that the rectangular components of the vector Laplacian are equal to the scalar Laplacian of the corresponding components of the vector function. This identity only holds in rectangular coordinates, not in cylindrical or spherical coordinates.

(The vector Laplacian and the identity that you are proving above are both very useful when deriving the scalar Helmholtz equation for the components of the electric and magnetic fields, something we will be doing later.)

The major difference between Laplacian and other operators like Prewitt, Sobel, Robinson and Kirsch is that these all are first order derivative masks but Laplacian is a second order derivative mask.

Also the thresholder magnitude of Laplacian operator produces double edges. For these reasons, together with its inability to detect the edge direction, the Laplacian as such is not a good edge detection operator. A better utilization of it is to use its zero-crossing to detect the edge locations.

Unit – II

Tensors and Matrix Theory

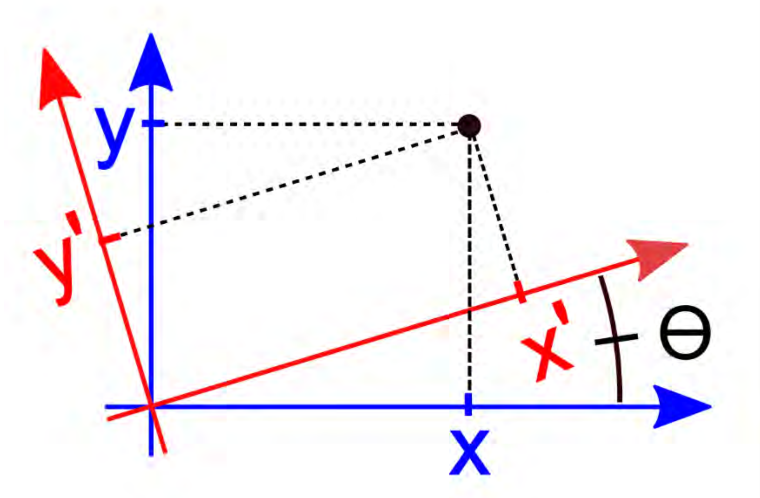
Tensors

In mathematics, a tensor is an algebraic object that describes a (multilinear) relationship between sets of algebraic objects related to a vector space. Objects that tensors may map between include vectors and scalars, and even other tensors. Tensors can take several different forms – for example: scalars and vectors (which are the simplest tensors), dual vectors, multilinear maps between vector spaces, and even some operations such as the dot product. Tensors are defined independent of any basis, although they are often referred to by their components in a basis related to a particular coordinate system.

Tensors are important in physics because they provide a concise mathematical framework for formulating and solving physics problems in areas such as mechanics (stress, elasticity, fluid mechanics, moment of inertia, electrodynamics (electro magnetictensor, Maxwell tensor, permittivity, magnetic susceptibility), or general relativity (stress–energy tensor, curvature tensor) and others. In applications, it is common to study situations in which a different tensor can occur at each point of an object; for example, the stress within an object may vary from one location to another. This leads to the concept of a tensor field. In some areas, tensor fields are so ubiquitous that they are often simply called "tensors".

Transformation of Tensors

Coordinate systems are essential for studying the equations of curves using the methods of analytic geometry. ... The process of making this change is called a **transformation of coordinates**. The solutions to many problems can be simplified by rotating the **coordinate** axes to obtain new axes through the same origin.



the introduction of a new set of mathematical coordinates that are stated distinct functions of the original coordinates. If a standard right-handed Cartesian coordinate system is used, with the x-axis to the right and the y-axis up, the rotation R(θ) is counter clockwise. If a left-handed Cartesian coordinate system is used, with x directed to the right but y directed down, R(θ) is clockwise.

contravariant tensor

A contravariant tensor is a tensor having specific transformation properties (cf., a covariant tensor). To examine the transformation properties of a contravariant tensor, first consider a tensor of rank 1. he valence of a tensor is the number of variant and covariant terms, and in Einstein notation, covariant components have lower indices, while contravariant components have upper indices. On a manifold, a tensor field will typically have multiple, upper and lower indices, where Einstein notation is widely used.

covariant tensor

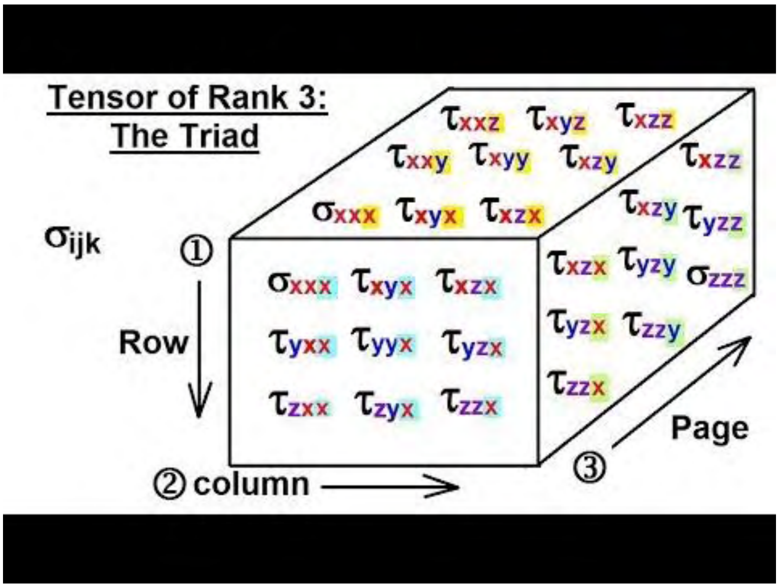
A covariant tensor, denoted with a lowered index (e.g.,) is a tensor having specific transformation properties. In general, these transformation properties differ from those of a contravariant tensor. If the transformation matrix of an index is the basis transformation itself, then the index is called covariant and is denoted with a lower index (subscript). so the tensor corresponding to the matrix of a linear operator has one covariant and one contravariant index: it is of type (1,1).

Mixed Tensors

In tensor analysis, a mixed tensor is a tensor which is neither strictly covariant nor strictly contravariant; at least one of the indices of a mixed tensor will be a subscript (covariant) and at least one of the indices will be a superscript (contravariant). In mathematics, a tensor is an algebraic object that describes a (multilinear) relationship between sets of algebraic objects related to a vector space. Objects that tensors may map between include vectors and scalars, and even other tensors.

Rank of a Tensor

A tensor is a multilinear transformation that maps a set of vector spaces to another vector space. A data tensor is a collection of multivariate observations organized into a M-way array.



In mathematics, a tensor is an algebraic object that describes a (multilinear) relationship between sets of algebraic objects related to a vector space. ... This leads to the concept of a tensor field. In some areas, tensor fields are so ubiquitous that they are often simply called "tensors". So a third-order tensor has three indices. You can imagine it like a many matrices that is not a rectangle on the paper, but a cuboid in the room. The matrix is a special case of a second-order ("two-dimensional") tensor. The common interpretation of tensors is as multilinear functional.

If it's not a physical quantity, then it's usually called a matrix. The vast majority of engineering tensors are symmetric. One common quantity that is not symmetric, and not referred to as a tensor, is a rotation matrix. Tensors are in fact any physical quantity that can be represented by a scalar, vector, or matrix.

The only difference is that tensor is the generalized form of scalars and vectors. Means scalars and vectors are the special cases of tensor quantities. Scalar is a tensor of rank 0 and vector is a tensor of rank 1. There is no such major difference in vectors and tensors.

The reason tensors are useful is because every multilinear (i.e., separately linear in each variable) map from the Cartesian product of several vector spaces to another vector space T can be extended in a unique way to a linear map from the tensor product of those spaces to T, and, conversely, every linear map.

IRREDUCIBLE $\mathbf{T}_l$: $\mathbf{T}_{lm}$		REDUCIBLE	IRREDUCIBLE $\mathbf{T}_L$: $\mathbf{T}_{LM}$
$\mathbf{T}_0$: $\mathbf{T}_{00}$ $\mathbf{T}_0$: $\mathbf{T}_{00}$	x ⟶		$\mathbf{T}_0$: $\mathbf{T}_{00}$
$\mathbf{T}_0$: $\mathbf{T}_{00}$ $\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$	x ⟶		$\mathbf{T}_0$: None $\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$
$\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$ $\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$	x ⟶	$\mathbf{T}_2$	$\mathbf{T}_0$: $\mathbf{T}_{00}$ $\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$ $\mathbf{T}_2$: $\mathbf{T}_{22}$ $\mathbf{T}_{21}$ $\mathbf{T}_{20}$ $\mathbf{T}_{2-1}$ $\mathbf{T}_{2-2}$
$\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$ $\mathbf{T}_2$: $\mathbf{T}_{22}$ $\mathbf{T}_{21}$ $\mathbf{T}_{20}$ $\mathbf{T}_{2-1}$ $\mathbf{T}_{2-2}$	x ⟶	$\mathbf{T}_3$	$\mathbf{T}_0$: None $\mathbf{T}_1$: $\mathbf{T}_{11}$ $\mathbf{T}_{10}$ $\mathbf{T}_{1-1}$ $\mathbf{T}_2$: $\mathbf{T}_{22}$ $\mathbf{T}_{21}$ $\mathbf{T}_{20}$ $\mathbf{T}_{2-1}$ $\mathbf{T}_{2-2}$ $\mathbf{T}_3$: $\mathbf{T}_{33}$ $\mathbf{T}_{32}$ $\mathbf{T}_{31}$ $\mathbf{T}_{30}$ $\mathbf{T}_{3-1}$ $\mathbf{T}_{3-2}$ $\mathbf{T}_{3-3}$

Problem 1 (Kronecker delta) Recall that the *Kronecker delta* is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1}$$

where $i, j \in \mathbb{Z}$ are integers.

(a) (5 points) Let $i, j \in \mathbb{Z}$ be integers, and let $\{f_i\}$ be a set of numbers. Show that the Kronecker delta has the following sifting property:

$$\sum_{i=-\infty}^{\infty} f_i \delta_{ij} = f_j \tag{2}$$

(b) (5 points) Recall that the *Dirac delta* "function" has the following sifting property:

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0) \tag{3}$$

Compare (2) and (3). Can you see the similarity between Kronecker's delta δ_{ij} and Dirac's delta $\delta(x)$? In what sense are they analogous?

(c) (5 points) How are the components of the rank-two identity tensor related to the Kronecker delta?

- The Number of **Components (N)** required for the description of a TENSOR of the $\mathbf{n^{th}}$ **Rank** in a **k**-dimensional space is:

$$\mathbf{N = k^n} \qquad (4\text{-}1)$$

<u>EXAMPLES</u>

(a) For a 2-D space, only four components are required to describe a second rank tensor.

(b) For a 3-D space, the number of components $N = 3^n$

Scalar quantities $\Rightarrow$ $3^0 \Rightarrow$ Rank Zero

Vector quantities $\Rightarrow$ $3^1 \Rightarrow$ Rank One

Stress, Strain $\Rightarrow$ $3^2 \Rightarrow$ Rank Two

Elastic Moduli $\Rightarrow$ $3^4 \Rightarrow$ Rank Four

$$\textit{Third Invariant}: I_3(A) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \det(A)$$

$$\begin{aligned}
= \; & \varepsilon_{111} A_{11} A_{21} A_{31} + \varepsilon_{112} A_{11} A_{21} A_{32} + \varepsilon_{113} A_{11} A_{21} A_{33} + \\
& \varepsilon_{121} A_{11} A_{22} A_{31} + \varepsilon_{112} A_{12} A_{21} A_{32} + \varepsilon_{123} A_{11} A_{22} A_{33} + \\
& \varepsilon_{131} A_{11} A_{23} A_{31} + \varepsilon_{132} A_{11} A_{23} A_{32} + \varepsilon_{133} A_{11} A_{23} A_{33} + \\
& \varepsilon_{211} A_{12} A_{21} A_{31} + \varepsilon_{212} A_{12} A_{21} A_{32} + \varepsilon_{213} A_{12} A_{21} A_{33} + \\
& \varepsilon_{221} A_{12} A_{22} A_{31} + \varepsilon_{232} A_{12} A_{23} A_{32} + \varepsilon_{223} A_{12} A_{22} A_{33} + \\
& \varepsilon_{231} A_{12} A_{23} A_{31} + \varepsilon_{232} A_{12} A_{23} A_{32} + \varepsilon_{233} A_{12} A_{23} A_{33} + \\
& \varepsilon_{311} A_{13} A_{21} A_{31} + \varepsilon_{312} A_{13} A_{21} A_{32} + \varepsilon_{313} A_{13} A_{21} A_{33} + \\
& \varepsilon_{321} A_{13} A_{22} A_{31} + \varepsilon_{322} A_{13} A_{22} A_{32} + \varepsilon_{323} A_{13} A_{22} A_{33} + \\
& \varepsilon_{331} A_{13} A_{23} A_{31} + \varepsilon_{332} A_{13} A_{23} A_{32} + \varepsilon_{333} A_{13} A_{23} A_{33}
\end{aligned}$$

eliminating all terms equal to zero gives:

$$\begin{aligned}
& \varepsilon_{123} A_{11} A_{22} A_{33} + \varepsilon_{132} A_{11} A_{23} A_{32} + \\
& \varepsilon_{213} A_{12} A_{21} A_{33} + \varepsilon_{231} A_{12} A_{23} A_{31} + \\
& \varepsilon_{312} A_{13} A_{21} A_{32} + \varepsilon_{321} A_{13} A_{22} A_{31}
\end{aligned}$$

applying the ± 1 rule for the permutation tensor ε gives:

$$\begin{aligned}
& A_{11} A_{22} A_{33} - A_{11} A_{23} A_{32} - \\
& A_{12} A_{21} A_{33} + A_{12} A_{23} A_{31} + \\
& A_{13} A_{21} A_{32} - A_{13} A_{22} A_{31}
\end{aligned}$$

The relationship between tensors and differential forms is most clearly seen from the following definition:

A tensor **T** is a multilinear mapping at a point of a manifold, of r one-forms ω_i and s tangent vectors ξ_j (where r, s = 0, 1, 2,...) to the real line $\Re$, given by

$$\mathbf{T}: (\omega_1, \omega_2, \ldots, \omega_r; \xi_1, \xi_2, \ldots, \xi_s) \rightarrow \mathbf{T}(\omega_1, \omega_2, \ldots, \omega_r; \xi_1, \xi_2, \ldots, \xi_s) \in \Re$$

$\mathbf{T}(\omega_1, \omega_2, \ldots, \omega_r; \xi_1, \xi_2, \ldots, \xi_s)$ is the contraction of r one-forms ω_i and s tangent vectors ξ_j, producing a scalar. The rank of **T** is given by $r + s$, as symbolized by $\binom{r}{s}$; **T** is said to be contravariant of index r (raised index) and covariant of index s (lowered index).

To define contravariant and covariant note the following change of coordinates:

$$(x^0, x^1, x^2, x^3) \rightarrow (x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})$$

A tensor which transforms like $T^{i\,\prime} = \left(\frac{\partial x^{i\,\prime}}{\partial x^j}\right) T^j$ is called a **contravariant tensor**.

A tensor which transforms like $T_{i\prime} = \left(\frac{\partial x^j}{\partial x^{i\prime}}\right) T_j$ is called a **covariant tensor**.

In some areas, tensor fields are so ubiquitous that they are often simply called "tensors". So a third-order tensor has three indices. You can imagine it like a many matrices that is not a rectangle on the paper, but a cuboid in the room. The matrix is a special case of a second-order ("two-dimensional") tensor. The common interpretation of tensors is as multilinear functional.

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Eigen value and Eigen Vectors

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot (-3) \\ 1 \cdot 3 + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

On the other hand the vector

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is *not* an eigenvector, since

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and this vector is not a multiple of the original vector *x*.

The identity matrix *I* (whose general element I_{ij} is 1 if $i=j$, and 0 otherwise) maps every vector to itself. Therefore, every vector is an eigenvector of *I*, with eigenvalue 1.

PROBLEMS

Calculate the Eigen Value and Eigen Vectors for the given below value

$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -0.4472 \\ 0.8944 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

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$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3 - \lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

$$\left(\mathbf{A} + \omega_2^2 \mathbf{I}\right) \mathbf{v}_2 = 0$$

$$\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v}_2 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0$$

$$v_{2,1} = v_{2,2}$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Unit – III

Complex Analysis

Functions of Complex Variables

Functions of (x, y) that depend only on the combination (x + iy) are called functions of a complex variable and functions of this kind that can be expanded in power series in this variable are of particular interest. A complex function is a function from complex numbers to complex numbers. In other words, it is a function that has a subset of the complex numbers as a domain and the complex numbers as a codomain. Complex functions are generally supposed to have a domain that contains a nonempty open subset of the complex plane.

Complex variable, In mathematics, a variable that can take on the value of a complex number. In basic algebra, the variables x and y generally stand for values of real numbers. The algebra of complex numbers (complex analysis) uses the complex variable z to represent a number of the form a + bi

The fact that the variables are complex isn't very difficult, as they are still variables. The complex numbers have a richer structure than their real components combined. You can see this even for some basic functions as roots, logarithms or trig functions when applied to complex numbers

Cauchy- Riemann conditions

Cauchy-Riemann conditions and Alembert-Euler conditions, they are the partial differential equations that must be satisfied by the real and imaginary parts of a complex-valued function f of one (or several) complex variable so that f is holomorphic.

(a) State, but do not prove, the Cauchy-Riemann equations.

(b) Use the Cauchy Riemann equations to prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(c) Follow steps (i)-(ii) below to show that, if

$$|f(z)| = c \qquad \text{for all } z \in D,$$

where $c \geq 0$ is a constant, then the function f is constant.

(i) Notice that $|f(z)| = c$ is equivalent to

$$u^2(x,y) + v^2(x,y) = c^2,$$

and differentiate the above relation twice with respect to x and twice with respect to y to deduce that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^2 v}{\partial y^2} = 0.$$

(ii) Add the last two relations above and then use (b) to deduce that u, v are constant.

(d) Let $f\colon D \to \mathbb{C}$ be a complex differentiable function. Show that if $\overline{f}\colon D \to \mathbb{C}$ is given by $\overline{f}(z) = \overline{f(z)}$ for $z \in D$ and $\overline{f}$ is also complex differentiable on D, then f is constant.

Functions of (x, y) that depend only on the combination (x + iy) are called functions of a complex variable and functions of this kind that can be expanded in power series in this variable are of particular interest. A complex function is a function from complex numbers to complex numbers.

$$a)\, f(z) = z^2 + 5iz + 3 - i$$
$$= (x+iy)^2 + 5i(x+iy) + 3 - i$$
$$= (x^2 - y^2 - 5y + 3) + i(5x + 2xy - 1)$$
$$= u + iv.$$

$$\frac{\partial u}{\partial x} = 2x; \frac{\partial v}{\partial y} = 2x; \frac{\partial u}{\partial y} = -2y - 5; \frac{\partial v}{\partial x} = 5 + 2y$$

$So, Cauchy - Riemann\ verified \Rightarrow f(z)\ analytic$

$b)$

$$f(z) = ze^{-z}$$
$$Let\, z = x + iy$$
$$f(z) = (x+iy)e^{-(x+iy)}$$
$$= e^{-x}(x+iy)e^{-iy}$$
$$= e^{-x}(x+iy)(\cos y - i\sin y) \qquad \left[e^{-i\theta} = \cos\theta - i\sin\theta\right]$$
$$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y)$$
$$= u + iv$$
$$u = e^{-x}(x\cos y + y\sin y)$$
$$\frac{du}{dx} = e^{-x}(\cos y) - e^{-x}(x\cos y + y\sin y)$$
$$\frac{du}{dv} = e^{-x}(-x\sin y + \sin y + y\cos y)$$
$$v = e^{-x}(y\cos y - x\sin y)$$
$$\frac{dv}{dy} = e^{-x}(\cos y) - e^{-x}(x\cos y + y\sin y)$$
$$\frac{dv}{dx} = -e^{x}(-x\sin y + \sin y + y\cos y)$$
$$\text{Since}\, \frac{du}{dx} = \frac{dv}{dy}\ and\ \frac{du}{dy} = -\frac{dv}{dx}$$

Hence Cauchy Riemann equations satisfy.

Yes function is analytic

Cauchy’s integral theoremand integral formula

- If a function F(x) is analytic in region R then its derivative at any point z=a is also analytic in R and it is given by
- $F'(a) = \frac{1}{2\pi i} \oint \frac{F(x)}{(z-a)^2} . dz$
- In general,
- $F^n(a) = \frac{n!}{2\pi i} \oint \frac{F(x)}{(z-a)^{n+1}} . dz$

the plane and $f: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then for a in $G - \{\gamma\}$

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Proof. Define $\varphi: G \times G \to \mathbb{C}$ by $\varphi(z,w) = [f(z) - f(w)]/(z-w)$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. It follows that φ is continuous; and for each w in G, $z \to \varphi(z,w)$ is analytic (Exercise 1). Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is a continuous integer-valued function of w, H is open. Moreover $H \cup G = \mathbb{C}$ by the hypothesis.

Define $g: \mathbb{C} \to \mathbb{C}$ by $g(z) = \int_\gamma \varphi(z,w) dw$ if $z \in G$ and $g(z) = \int_\gamma (w-z)^{-1} f(w) dw$ if $z \in H$. If $z \in G \cap H$ then

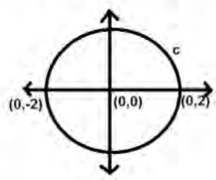
$$\begin{aligned}\int_\gamma \varphi(z,w)\, dw &= \int_\gamma \frac{f(w) - f(z)}{w-z} dw \\ &= \int_\gamma \frac{f(w)}{w-z} dw - f(z) n(\gamma; z) 2\pi i \\ &= \int_\gamma \frac{f(w)}{w-z} dw.\end{aligned}$$

Hence g is a well-defined function.

Problems

1. Evaluate $\oint_c \frac{dz}{z+4}$ where c is circle |z| = 2.

$$\mathbf{f(z)} = \frac{\mathbf{1}}{\mathbf{z+4}}$$



f (z) is not analytic at z = - 4. Here c is |z| = 2 which is a circle with center (0,0) and radios 2.

2. Calculate the value of I using Cauchy's integral theoremand integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a}\, dz$$

$$\begin{aligned} I &= \int_{|z=2|} \frac{1}{z^2+1}\, dz \\ &= \int_{|z=2|} \frac{1}{(z-i)(z+i)}\, dz \\ &= \int_{C_1} \frac{\frac{1}{z+i}}{z-i}\, dz + \int_{C_2} \frac{\frac{1}{z-i}}{z+i}\, dz \\ &= 2\pi i \cdot \left[\frac{1}{z+i}\right]_{z=i} + 2\pi i \cdot \left[\frac{1}{z-i}\right]_{z=-i} \\ &= 2\pi i \cdot \left(\frac{1}{2i} - \frac{1}{2i}\right) = 0. \end{aligned}$$

Taylor's series

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. A Taylor series is an idea used in computer science, calculus, chemistry, physics and other kinds of higher-level mathematics. It is a series that is used to create an estimate (guess) of what a function looks like. There is also a special kind of Taylor series called a Maclaurin series. To find the Taylor Series for a function we will need to determine a general formula for f(n)(a) f (n) (a) . This is one of the few functions where this is easy to do right from the start. To get a formula for f(n)(0) f (n) (0) all we need to do is recognize that, f(n)(x) =exn=0,1,2,3, …

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad for\,|x| < 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad for\,|x| < 1$$

Problems

$$\begin{aligned}
\cos(x) &= \frac{d}{dx}\sin(x) \\
&= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\
&= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}
\end{aligned}$$

Determine the Taylor series about point $x_0 = 0$ for the function $\dfrac{1}{1-x}$.

Also determine the radius of convergence of the series.

- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} n!x^n$. The radius of convergence of the series is $\rho = 1$
- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n$. The radius of convergence of the series is $\rho = 1$
- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} (n-1)!x^n$. The radius of convergence of the series is $\rho = 1$
- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. The radius of convergence of the series is $\rho = 1$
- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. The radius of convergence of the series is $\rho = 2$

Laurent's series

Laurent series with complex coefficients are an important tool in complex analysis, especially to investigate the behaviour of functions near singularities. for N = 1, 2, 3, 4, 5, 6, 7 and 50. As N → ∞, the approximation becomes exact for all (complex) numbers x except at the singularity x = 0. In mathematics, the **Laurent series** of a complex function f(z) is a representation of that function as a power **series** which includes terms of negative degree. It may be **used to** express complex functions in cases where a Taylor **series expansion** cannot be applied

Problems

Find the value of cos(x) using Laurent series

$$\begin{aligned}\cos(x) &= \frac{d}{dx}\sin(x) \\ &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\ &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\end{aligned}$$

$$\frac{d(\sin x)}{dx} = \cos x.$$

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Then:

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}$$

Cauchy’s residue theorems

In complex analysis, a discipline within mathematics, the residue theorem, sometimes called Cauchy's residue theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals and infinite series as well. Cauchy's residue theorem is a consequence of Cauchy's integral formula. f(z0) = 1. 2π i. ∮

In mathematics, more specifically complex analysis, the residue is a complex number proportional to the contour integral of a monomorphic function along a path enclosing one of its singularities. (More generally, residues can be calculated for any function.

The different types of singularity of a complex function f(z) are discussed and the definition of a residue at a pole is given. The residue theorem is used to evaluate contour integrals where the only singularities of f(z) inside the contour are poles.

Theorem

If f(z) is analytic at a point z, then the derivative f (z) is continuous at z. If f(z) is analytic at a point z, then f(z) has continuous derivatives of all order at the point z. Equations (2, 3) are known as the Cauchy-Riemann equations. They are a necessary condition for f = u + iv to be analytic.

A zero of a monomorphic function f is a complex number z such that f(z) = 0. A pole of f is a zero of 1/f.

There are basically three types of singularities (points where f(z) is not analytic) in the complex plane. An isolated singularity of a function f(z) is a point z0 such that f(z) is analytic on the punctured disc $0 < |z - z0| < r$ but is undefined at z = z0. We usually call isolated singularities poles.

The concept and the term "singularity" were popularized by Vernor Vinge in his 1993 essay The Coming Technological Singularity, in which he wrote that it would signal the end of the human era, as the new superintelligence would continue to upgrade itself and would advance technologically at an incomprehensible rate.

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \oint \frac{f(w)dw}{(w-a)^{n+1}},$$

the integral being taken around a circular contour in Ω *with center* a*.*

In particular, $\mathrm{Res}(f; a) = c_{-1}$.

PROOF: Assume without loss of generality that $a = 0$. Use Cauchy's integral formula to write

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)dw}{w-z},$$

where the contour γ is the union of two circles, one of radius larger than $|z|$ and traversed counterclockwise, one of radius smaller than $|z|$ and traversed clockwise. Now on the larger circle expand

$$\frac{1}{w-z} = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$$

with uniform convergence in w; on the smaller circle expand

$$\frac{1}{w-z} = -\sum_{k=0}^{\infty} \frac{w^k}{z^{k+1}}$$

again with uniform convergence in w. Substitute these expansions into the Cauchy integral formula and interchange summation and integration (justified by uniform convergence); the result follows. ■

Unit – IV

Special Functions

Gamma Functions

To extend the factorial to any real number x > 0 (whether or not x is a whole number), the gamma function is defined as Γ(x) = Integral on the interval [0, ∞] of ∫ 0∞t $^{x-1}$ e^{-t} dt. Using techniques of integration, it can be shown that Γ (1) = 1.

The gamma function is applied in exact sciences almost as often as the well-known factorial symbol. It was introduced by the famous mathematician L. Euler (1729) as a natural extension of the factorial operation from positive integers to real and even complex values of this argument.

Gamma is the rate of change in an option's delta per 1-point move in the underlying asset's price. Gamma is an important measure of the convexity of a derivative's value, in relation to the underlying. A delta hedge strategy seeks to reduce gamma in order to maintain a hedge over a wider price range Beta Functions

In mathematics, the beta function, also called the Euler integral of the first kind, is a special function that is closely related to the gamma function and to binomial coefficients. It is defined by the integral for complex number inputs x, y such that Re x > 0, Re y > 0

In Physics and string approach, the beta function is used to compute and represent the scattering amplitude for Reggae trajectories. Apart from these, you will find many applications in calculus using its related gamma function also.

<u>Gamma Function</u>

$$\Gamma[a] = \int_0^{\infty} e^{-x} x^{a-1} dx$$

$$\Gamma[a] = (a-1)\Gamma[a\text{-}1]$$

If a is an integer the gamma function can be written as

$$\Gamma[a] = (a-1)!$$

Special cases

$$\Gamma[0] = 0$$

$$\Gamma[1] = 1$$

$$\Gamma[1/2] = \sqrt{\pi}$$

Gamma is the rate of change in an option's delta per 1-point move in the underlying asset's price. Gamma is an important measure of the convexity of a derivative's value, in relation to the underlying. A delta hedge strategy seeks to reduce gamma in order to maintain a hedge over a wider price range Beta Functions

To extend the factorial to any real number x > 0 (whether or not x is a whole number), the gamma function is defined as Γ(x) = Integral on the interval [0, ∞] of ∫ 0∞t $^{x-1}$ e^{-t} dt. Using techniques of integration.

Properties of Gamma function

1) $\Gamma(m+1) = m\Gamma m$
2) $\Gamma(m+1) = m!$ When m is a positive integer.
3) $\Gamma(m+a) = (m+a-1)(m+a-2)\ldots\ldots\ldots a\Gamma a$, when n is a positive integer.
4) $\Gamma m = 2\int_0^{\infty} e^{-x^2} x^{2m-1}\,dx \quad (m > 0)$
5) $\frac{\Gamma m}{t^m} = \int_0^{\infty} e^{-tx} x^{m-1}\,dx \quad (m > 0)$
6) $\Gamma\frac{1}{2} = \sqrt{\pi}$
7) $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$
8) $\int_0^{1} x^n (\log x)^m\,dx = \frac{(-1)^m}{(n+1)^{m+1}}\Gamma(m+1)$

Gamma functions is also defined as

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx.$$

The gamma function has the following properties:

$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Use this information to show that the expected value of X is equal to $\sigma\sqrt{\frac{\pi}{2}}$.

To extend the factorial to any real number x > 0 (whether or not x is a whole number), the gamma function is defined as Γ(x) = Integral on the interval [0, ∞] of ∫ 0∞$t^{x-1} e^{-t}$ dt. Using techniques of integration.

Beta Function $\mathbf{B}(a,b)$

Another definite integral which is related to the $\mathbf{\Gamma}$-function is the Beta function $\mathbf{B}(a,b)$ which is defined as

$$\mathrm{B}(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1}\,dt, \qquad a>0,\;\; b>0 \tag{1.72}$$

The relationship between the $\mathbf{B}$-function and the $\mathbf{\Gamma}$-function can be demonstrated easily. By means of the new variable

$$u = \frac{t}{(1-t)}$$

Therefore Eq. 1.72 becomes

$$\mathrm{B}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\,du \qquad a>0,\;\; b>0 \tag{1.73}$$

Find the value of the above function using beta and gammafunction

$$\begin{aligned}
\Gamma(u)\Gamma(v) &= \int_0^1\int_0^\infty e^{-x}x^{u-1}y^{u-1}x^{v-1}(1-y)^{v-1}\,xdxdy \\
&= \int_0^\infty e^{-x}x^{u+v-1}dx\int_0^1 y^{u-1}(1-y)^{u-1}\,dy \\
&= \Gamma(u+v)B(u,v)
\end{aligned}$$

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

Find the value of the above function using beta and gammafunction

$$
\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{1/2-1}\exp(-x)dx \\
&= \int_0^\infty x^{-1/2}\exp(-x)dx \\
&= 2\int_0^\infty \exp\left(-t^2\right)dt \qquad \text{(change of variable: } t = x^{1/2}\text{)} \\
&= 2\left(\int_0^\infty \exp\left(-t^2\right)dt\int_0^\infty \exp\left(-t^2\right)dt\right)^{1/2} \\
&= 2\left(\int_0^\infty \exp\left(-t^2\right)dt\int_0^\infty \exp\left(-s^2\right)ds\right)^{1/2} \\
&= 2\left(\int_0^\infty\int_0^\infty \exp\left(-t^2-s^2\right)dtds\right)^{1/2} \\
&= 2\left(\int_0^\infty\int_0^\infty \exp\left(-s^2u^2-s^2\right)sduds\right)^{1/2} \qquad \text{(change of variable: } t = su\text{)} \\
&= 2\left(\int_0^\infty\int_0^\infty \exp\left(-\left(1+u^2\right)s^2\right)sdsdu\right)^{1/2} \\
&= 2\left(\int_0^\infty\left[-\frac{1}{2\left(1+u^2\right)}\exp\left(-\left(1+u^2\right)s^2\right)\right]_0^\infty du\right)^{1/2} \\
&= 2\left(\int_0^\infty\left[0+\frac{1}{2\left(1+u^2\right)}\right]du\right)^{1/2} \\
&= 2^{1/2}\left(\int_0^\infty \frac{1}{1+u^2}du\right)^{1/2} \\
&= 2^{1/2}\left([\arctan(u)]_0^\infty\right)^{1/2} \\
&= 2^{1/2}(\arctan(\infty)-\arctan(0))^{1/2} \\
&= 2^{1/2}\left(\frac{\pi}{2}-0\right)^{1/2} \\
&= \pi^{1/2}
\end{aligned}
$$

Find the value of the above function using beta and gammafunction

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\,dt \\ &= \lim_{b\to\infty} -e^{-b} - (-e^0) \\ &= 1\end{aligned}$$

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty t^\alpha e^{-t}\,dt \quad \left\{\begin{array}{l}\text{use integration by parts:}\\ u = t^\alpha, dv = e^{-t}dt\end{array}\right. \\ &= \lim_{b\to\infty}\left(-b^\alpha e^{-b} - 0\right) - \int_0^\infty \alpha t^{\alpha-1}(-e^{-t})\,dt \\ &= \alpha\Gamma(\alpha), \text{ if } \alpha > 0\end{aligned}$$

Find the value of the above function using beta and gammafunctio

$$\int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}dx$$

$$y = \sqrt[3]{x} \Rightarrow x = y^3, dy = \frac{1}{3}x^{-\frac{2}{3}}dx \Rightarrow dx = 3y^2dy$$

$$\therefore 3\int_0^\infty y^2 y^{\frac{3}{2}} e^{-y}dy = 3\int_0^\infty y^{\frac{7}{2}}e^{-y}dy = 3\Gamma\left(\frac{9}{2}\right) = 3\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$Therefore, \int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}dx = \frac{315}{16}\sqrt{\pi}$$

Find the value of the above function using beta and gammafunction

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\,dt \\ &= \lim_{b\to\infty} -e^{-b} - (-e^0) \\ &= 1\end{aligned}$$

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty t^\alpha e^{-t}\,dt \quad \left\{\begin{array}{l}\text{use integration by parts:}\\ u = t^\alpha, dv = e^{-t}dt\end{array}\right. \\ &= \lim_{b\to\infty}\left(-b^\alpha e^{-b} - 0\right) - \int_0^\infty \alpha t^{\alpha-1}(-e^{-t})\,dt \\ &= \alpha\Gamma(\alpha), \text{ if } \alpha > 0\end{aligned}$$

Find the value of the above function using beta and gammafunction

$$\Gamma(x)\Gamma(y) = \int_0^\infty u^{x-1}e^{-u}\,du \int_0^\infty v^{y-1}e^{-v}\,dv = \int_0^\infty\int_0^\infty u^{x-1}v^{y-1}e^{-u-v}\,du\,dv$$

$$= \left| \begin{array}{l} u = \varrho\cos^2(\varphi) \qquad v = \varrho\sin^2(\varphi) \\ J = \det\begin{pmatrix} \cos^2(\varphi) & -2\varrho\cos(\varphi)\sin(\varphi) \\ \sin^2(\varphi) & 2\varrho\sin(\varphi)\cos(\varphi) \end{pmatrix} = 2\varrho\sin(\varphi)\cos(\varphi) \end{array} \right|$$

$$= 2\int_0^\infty\int_0^{\pi/2} e^{-\varrho}\varrho^{x+y-1}\cos^{2x-1}(\varphi)\sin^{2y-1}(\varphi)\,d\varrho\,d\varphi$$

$$= 2\int_0^\infty e^{-\varrho}\varrho^{x+y-1}\,d\varrho \cdot \int_0^{\pi/2}\cos^{2x-1}\sin^{2y-1}(\varphi)(\varphi)\,d\varphi$$

$$= \left| \begin{array}{l} \cos^2(\varphi) = t \\ \qquad dt = 2\sin(\varphi)\cos(\varphi)\,d\varphi \end{array} \right|$$

$$= \Gamma(x+y)\cdot\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \Gamma(x+y)B(x,y).$$

Cases

- $B(m,n) = B(n,m)$
- $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$
- $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$
- $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Find the value of the above function using beta and gammafunction

$$
\begin{aligned}
B(x,y) &= \int_0^\infty t^{x-1}(1+t)^{-x-y}dt \\
&= \int_0^1 \left(\frac{s}{1-s}\right)^{x-1}\left(1+\frac{s}{1-s}\right)^{-x-y}\left(\frac{1}{1-s}\right)^2 ds \\
&= \int_0^1 \left(\frac{s}{1-s}\right)^{x-1}\left(\frac{1}{1-s}\right)^{-x-y}\left(\frac{1}{1-s}\right)^2 ds \\
&= \int_0^1 s^{x-1}\left(\frac{1}{1-s}\right)^{x-1-x-y+2} ds \\
&= \int_0^1 s^{x-1}\left(\frac{1}{1-s}\right)^{1-y} ds \\
&= \int_0^1 s^{x-1}(1-s)^{y-1}ds
\end{aligned}
$$

Find the value of the above function using beta and gammafunction

$$
\begin{aligned}
&\int_0^\infty x^{3/2}(1+2x)^{-5}dx \\
&= \int_0^\infty \left(\frac{1}{2}t\right)^{3/2}(1+t)^{-5}\frac{1}{2}dt \qquad \text{(by a change of variables: } t = 2x\text{)} \\
&= \left(\frac{1}{2}\right)^{5/2}\int_0^\infty t^{3/2}(1+t)^{-5}dt \\
&= \left(\frac{1}{2}\right)^{5/2}\int_0^\infty t^{5/2-1}(1+t)^{-5/2-5/2}dt \\
&= \left(\frac{1}{2}\right)^{5/2}B\left(\frac{5}{2},\frac{5}{2}\right) \qquad \text{(using the integral representation of the Beta function)}
\end{aligned}
$$

Find the value of the above function using beta and gammafunction

$$
\begin{aligned}
\Gamma(u)\Gamma(v) &= \int_0^1\int_0^\infty e^{-x}x^{u-1}y^{u-1}x^{v-1}(1-y)^{v-1}\,xdxdy \\
&= \int_0^\infty e^{-x}x^{u+v-1}dx\int_0^1 y^{u-1}(1-y)^{u-1}\,dy \\
&= \Gamma(u+v)B(u,v) \\
B(u,v) &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}
\end{aligned}
$$

Find the value of the above function using beta and gammafunction

$$\int_0^\infty x^{3/2}(1+2x)^{-5}dx$$

$$= \int_0^\infty \left(\frac{1}{2}t\right)^{3/2}(1+t)^{-5}\frac{1}{2}dt \qquad \text{(by a change of variables: } t = 2x\text{)}$$

$$= \left(\frac{1}{2}\right)^{5/2}\int_0^\infty t^{3/2}(1+t)^{-5}dt$$

$$= \left(\frac{1}{2}\right)^{5/2}\int_0^\infty t^{5/2-1}(1+t)^{-5/2-5/2}dt$$

$$= \left(\frac{1}{2}\right)^{5/2}B\left(\frac{5}{2},\frac{5}{2}\right) \qquad \text{(using the integral representation of the Beta function)}$$

Gamma functions and Beta functions

Definition 1

The gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx.$$

Theorem 2

(1) $\Gamma(n+1) = n\Gamma(n)$ for each $n \in (0,\infty)$.

(2) $\Gamma(n+1) = n!$ for each $n \in \mathbb{N}$.

(3) $\displaystyle\int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$ for each $n \in (0,\infty)$ and for each $a \in (0,\infty)$.

Unit – v

Group Theory

Mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms.

A group is a monoid each of whose elements is invertible. The study of groups is known as group theory. If there are a finite number of elements, the group is called a finite group and the number of elements is called the group order of the group.

Group

Group is a collection of individuals who have relations to one another that make them interdependent to some significant degree. As so defined, the term group refers to a class of social entities having in common the property of interdependence among their constituent members.

A group is a finite or infinite set of elements together with a binary operation (called the group operation) that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property.

A Number group is a set of numbers that results from dividing all the numbers in the number field into groups of numbers to be studied. ... More often than not, one or more number group is not represented in a drawing. If you are using groups of 10, for example, a drawing might be missing a number from the 20s.

Subgroup

A subgroup of a group G is a subset of G that forms a group with the same law of composition. For example, the even numbers form a subgroup of the group of integers with group law of addition. Any group G has at least two subgroups: the trivial subgroup {1} and G itself.

A subgroup H of the group G is a normal subgroup if $g^{-1} H g = H$ for all $g \in G$. If $H < K$ and $K < G$, then $H < G$ (subgroup transitivity). if H and K are subgroups of a group G Then $H \cap K$ is also a subgroup.

Observe that every group G with at least two elements will always have at least two subgroups, the subgroup consisting of the identity element alone and the entire group itself. The subgroup H={e} of a group G is called the trivial subgroup. A subgroup that is a proper subset of G is called a proper subgroup.

If the group is seen multiplicatively, the order of an element a of a group, sometimes also called the period length or period of a, is the smallest positive integer m such that $a^m = e$, where e denotes the identity element of the group, and a^m denotes the product of m copies of a.

Coset

A subset of a mathematical group that consists of all the products obtained by multiplying either on the right or the left a fixed element of the group by each of the elements of a given subgroup.

Although derived from a subgroup, cosets are not usually themselves subgroups of G, only subsets. If the left cosets and right cosets are the same then H is a normal subgroup and the cosets form a group called the quotient or factor group.

In mathematics, specifically group theory, a subgroup H of a group G may be used to decompose the underlying set of G into disjoint equal-size pieces called cosets. There are two types of cosets: left cosets and right cosets. Furthermore, H itself is a coset, which is both a left coset and a right coset.

Class

A class of groups is a set theoretical collection of groups satisfying the property that if G is in the collection then every group isomorphic to G is also in the collection.

The class group is a measure of the extent to which unique factorization fails in the ring of integers of K. The order of the group, which is finite, is called the class number of K. ... For example, the class group of a Dedekind domain is trivial if and only if the ring is a unique factorization domain.

A class group is an instance of the Data-Admin-DB-Class Group class. A class group instance causes the system to store the instances corresponding to two or more concrete classes that share a common key format in a single database table. The name of the class group is a prefix of the names of the member classes.

Multiplication Table

There is a very important rule about group multiplication tables called rearrangement theorem, which is that every element will only appear once in each row or column.[1]In group theory, when the column element is A and row element is B, then the corresponding multiplication is AB, which means B operation is performed.

In group theory, the symmetry group of a geometric object is the group of all transformations under which the object is invariant, endowed with the group operation of composition. ... A frequent notation for the symmetry group of an object X is G = Sym(X).

Multiplication Table for four groups

(a)

∘	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(c)

∘	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

(b)

∘	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(d)

∘	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

Multiplication Table for six groups

$z = x^{-1}y^{-1}$

x \ y	a	b	c	d	e	f
a	a	b	c	e	d	f
b	b	a	d	f	c	e
c	c	e	a	b	f	d
d	e	c	f	d	a	b
e	d	f	b	a	e	c
f	f	d	e	c	b	a

Character table

In group theory, a branch of abstract algebra, a character table is a two-dimensional table whose rows correspond to irreducible representations, and whose columns correspond to conjugacy classes of group elements. Nonaxial groups, C1 · Cs · Ci, -, -, -, -. Cn groups, C2 · C3 · C4 · C5 · C6 · C7 · C8. Dn groups, D2 · D3 · D4 · D5 · D6 · D7 · D8. Cnv groups, C2v · C3v · C4v · C5v.

n group theory, a branch of abstract algebra, a character table is a two-dimensional table whose rows correspond to irreducible representations, A character table summarizes the behaviour of all of the possible irreducible representations of a group under each of the symmetry operations of the group.

Character table for c_{2v}

The group order of C2v is 4. Each point group is characterized by each own multiplication table. C2V files contain protected information about the license terms and data stored in deployed Sentinel protection keys.

When you check in a C2V file, you can view the identifying information for the Sentinel protection key associated with the file, including the Batch Code, ID The character table is a square matrix where the columns are indexed by conjugacy classes and the rows are indexed by representations.

If the structure of the group is clearly understood, then the columns can easily be specified, which means that we know the number of rows as well as the number of column

C_{2v}	E	C_2	σ_v (xz)	$\sigma_{v'}$ (yz)		
A_1	1	1	1	1	z	x^2, y^2, z^2
A_2	1	1	-1	-1	R_z	xy
B_1	1	-1	1	-1	x, R_y	xz
B_2	1	-1	-1	1	y, R_x	yz

C_{2v} *Character Table to be used for water*

Point group

Symmetry operations

C_{2v}	E	C_2	$\sigma_v(xz)$	$\sigma_v'(yz)$		
A_1	1	1	1	1	z	x^2, y^2, z^2
A_2	1	1	−1	−1	R_z	xy
B_1	1	−1	1	−1	x, R_y	xz
B_2	1	−1	−1	1	y, R_x	yz

Mülliken symbols

Characters
+1 symmetric behavior
-1 antisymmetric

Each row is an irreducible representation

Character table for c_{3v}

The number of elements h is called the order of the group. Thus C3v is a group of order 6. Cyclic point groups are typically Abelian; others are usually not. Def.: The number of elements in a (sub)group is called it order. - The order of the C3v point group is 6. This point group contains only two symmetry operations: E the identity operation. σ a mirror plane.

Number of symmetry elements, h = 6. Number of irreducible representations, n = 3. Abelian group, no. Number of subgroups, 2. Subgroups, Cs, C3. e.g. NH3, point group C3v: Which C3v symmetry operations are the inverse of which… and which are together in one class? E-1 = E.

In C3v there are three classes and hence three irreducible representations. 2) The characters of all operations in the same class are equal in each given irreducible (or reducible) representation. ...

The character of a matrix is the sum of all its diagonal elements (also called the trace of a matrix). Number of symmetry elements, h = 6. Number of irreducible representations, n = 3. Abelian group, no. Number of subgroups, 2. Subgroups, Cs, C3.

Ammonia belongs to the symmetry group design- nated C3v, characterized by a three-fold axis with three vertical planes of symmetry. The order of the C_{3v} point group is 6, and the order of the principal axis (C_3) is 3. The group has three irreducible representations. The C_{3v} point group is isomorphic to D_3. It is also isomorphic to the Symmetric Group Sym (3), the group of all permutations of order 3

C_{3v}	E	C_3	C_3^2	σ_1	σ_2	σ_3
E	E	C_3	C_3^2	σ_1	σ_2	σ_3
C_3	C_3	C_3^2	E	σ_3	σ_1	σ_2
C_3^2	C_3^2	E	C_3	σ_2	σ_3	σ_1
σ_1	σ_1	σ_2	σ_3	E	C_3	C_3^2
σ_2	σ_2	σ_3	σ_1	C_3^2	E	C_3
σ_3	σ_3	σ_1	σ_2	C_3	C_3^2	E

Group multiplication table for C_{3v}

The number of elements h is called the order of the group. Thus C3v is a group of order 6. Cyclic point groups are typically Abelian; others are usually not. Def.: The number of elements in a (sub)group is called it order. - The order of the C3v point group is 6. This point group contains only two symmetry operations: E the identity operation. σ a mirror plane.

The group multiplication table for the $\mathbf{C}_{3v}$ point group.

	First operation					
Second operation	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^2$	$\hat{\sigma}_v$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$
$\hat{E}$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^2$	$\hat{\sigma}_v$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$
$\hat{C}_3$	$\hat{C}_3$	$\hat{C}_3^2$	$\hat{E}$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$	$\hat{\sigma}_v$
$\hat{C}_3^2$	$\hat{C}_3^2$	$\hat{E}$	$\hat{C}_3$	$\hat{\sigma}_v''$	$\hat{\sigma}_v$	$\hat{\sigma}_v'$
$\hat{\sigma}_v$	$\hat{\sigma}_v$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'$	$\hat{E}$	$\hat{C}_3^2$	$\hat{C}_3$
$\hat{\sigma}_v'$	$\hat{\sigma}_v'$	$\hat{\sigma}_v$	$\hat{\sigma}_v''$	$\hat{C}_3$	$\hat{E}$	$\hat{C}_3^2$
$\hat{\sigma}_v''$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'$	$\hat{\sigma}_v$	$\hat{C}_3^2$	$\hat{C}_3$	$\hat{E}$

Group multiplication table for C_{3v} for Ammonia

C_{3v}	$\hat{E}$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'''$	$\hat{C}_3$	$\hat{C}_3^{-1}$
$\hat{E}$	$\hat{E}$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'''$	$\hat{C}_3$	$\hat{C}_3^{-1}$
$\hat{\sigma}_v'$	$\hat{\sigma}_v'$	$\hat{E}$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'''$
$\hat{\sigma}_v''$	$\hat{\sigma}_v''$	$\hat{C}_3^{-1}$	$\hat{E}$	$\hat{C}_3$	$\hat{\sigma}_v'''$	$\hat{\sigma}_v'$
$\hat{\sigma}_v'''$	$\hat{\sigma}_v'''$	$\hat{C}_3$	$\hat{C}_3^{-1}$	$\hat{E}$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$
$\hat{C}_3$	$\hat{C}_3$	$\hat{\sigma}_v'''$	$\hat{\sigma}_v'$	$\hat{\sigma}_v''$	$\hat{C}_3^{-1}$	$\hat{E}$
$\hat{C}_3^{-1}$	$\hat{C}_3^{-1}$	$\hat{\sigma}_v''$	$\hat{\sigma}_v'''$	$\hat{\sigma}_v'$	$\hat{E}$	$\hat{C}_3$

Homomorphism

An isomorphism is a special type of homomorphism. The Greek roots "homo" and "morph" together mean "same shape." There are two situations where homomorphism's arise: when one group is a subgroup of another; when one group is a quotient of another. The corresponding homomorphism's are called embedding and quotient maps.

A group homomorphism is a map between two groups such that the group operation is preserved: for all, where the product on the left-hand side is in and on the right-hand side in.

1. A group homomorphism f: G→G′ is a map from G to G′ satisfying. f(xy)=f(x)f(y) for any x, y∈G.
2. A map f: G→G′ is called surjective if for any a∈G′, there exists x∈G such that. f(x)=a.
3. A surjective group homomorphism is a group homomorphism which is surjective.

- If (G, •) and (H, *) are groups, then a function f:G→ H is a homomorphism if

  f(x•y) = f(x) * f(y) for all x, y in G

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Source

1. https://scihub.wikicn.top/