KUNTHAVAI NAACIYAAR GOVT. ARTS COLLEGE FOR WOMEN, THANJAVUR

DEPARTMENT OF PHYSICS

I M.SC PHYSICS

CLASSICAL DYNAMICS AND RELATIVITY

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UNIT – I

FUNDAMENTAL PRINCIPLES AND LAGRANGIAN FORMULATION

Mechanics of a Particle

Let m be the mass of the given particle, o be the fixed origin and $\vec{r}$ be the radius vector of m with respect to 'O' at any instant of time 't'.

The ***velocity*** of m with respect to 'O' is the rate of change of displacement.

(ie) velocity $\vec{v}=\frac{d\vec{r}}{dt} = \dot{\vec{r}}$

Acceleration is the rate of change of velocity.

i.e acceleration is given by

$$a = \frac{d\boldsymbol{v}}{dt} = \frac{d\dot{\boldsymbol{r}}}{dt} = \ddot{\boldsymbol{r}}$$

Conservation of Linear Momentum

The linear momentum of a particle of mass 'm' with velocity 'v' is mv and it is denoted by $\vec{p} = m\vec{v} = m\dot{\vec{r}}$

By Newton's second law of motion ,

$$\vec{F} = ma = m\,\frac{dv}{dt} = \frac{dp}{dt}$$

$\vec{p} = m\vec{v}$ is the linear momentum

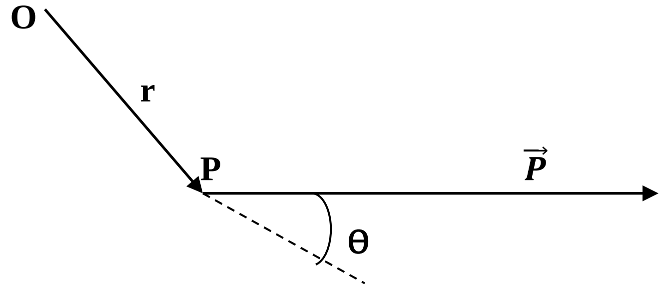
If the external force acting on the particle is zero, then

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

Or $\vec{p} = m\vec{v}$ = constant. Thus in the absence of external force, the linear momentum is conserved.

Conservation of Angular Momentum

The Angular momentum of a particle of mass 'm' with respect to a fixed point 'O' is defined to be $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p}$ is the linear momentum of the particle.



The moment of a force $\vec{F}$ with respect to a fixed origin is defined as $\vec{N} = \vec{r} \text{ x} \vec{F}$

Prove that moment of force is the rate of change of angular momentum.

(ie) To Prove $\vec{N} = \frac{d\vec{L}}{dt}$.

We have,

$$\vec{L} = \vec{r} \text{ x } \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \text{ x } \vec{p})$$

$$= \frac{d\vec{r}}{dt} \text{ x } \vec{p} + \vec{r} \text{ x } \frac{d\vec{p}}{dt}$$

$$= \vec{v} \text{ x } \vec{p} + \vec{r} \text{ x } \vec{F}$$

$$= \vec{v} \text{ x } m\vec{v} + \vec{r} \text{ x } \vec{F}$$

$$= m(\vec{v} \text{ x } \vec{v}) + \vec{r} \text{ x } \vec{F}$$

$$= 0 + \vec{r} \text{ x } \vec{F}$$

$$= \vec{r} \text{ x } \vec{F}$$

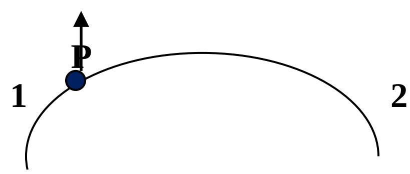
$$= \vec{N}$$

Hence $\vec{N} = \frac{d\vec{L}}{dt}$.

If the total torque, $\vec{N}$, is zero then $\dot{L} = 0$, and the angular momentum $\vec{L}$ is conserved.

Conservation of Energy

The work done by the external force $\vec{F}$ upon the particle in going from point 1 to point 2 is defined by $W_{12}=\int_1^2 \vec{F}\, d\vec{s}$ where $d\vec{s}$ corresponds to an infinitesimal displacement.



We Know that,

$$W_{12}=\int_1^2 \vec{F} d\vec{s}$$

Newton's second law

$$\vec{F} = ma$$

Multiply by ds on both sides

$$\vec{F}.ds = ma.ds$$

$$\vec{F}.ds = \left(m\frac{dv}{dt}.v\right)dt$$

[v=ds/dt, ds=vdt]

$$\vec{F}.ds = \left(mv.\frac{dv}{dt}\right)dt$$

$\left(mv.\frac{dv}{dt}\right)$ this can be written as $\frac{d}{dt}(\frac{1}{2}mv^2)$

Therefore,

$$W_{12} = \int_1^2 F.ds = \int_1^2 \frac{d}{dt}(\frac{1}{2}mv^2)\text{dt}$$

$$W_{12} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$W_{12} = T_2 - T_1$$

Also the forces are derivable from scalar potential energy function in the manner

$$F = -\nabla V$$

$$W_{12} = \int_1^2 F.ds = \int_1^2 -\nabla V.ds$$

$$W_{12} = \int_1^2 -\frac{dV}{ds}ds = \int_1^2 -dV = -(V_2 - V_1)$$

$$W_{12} = V_1 - V_2$$

Therefore

$$T_2 - T_1 = V_1 - V_2$$

$$T_1 + V_1 = T_2 + V_2 = constant.$$

In general T+V = constant. Thus the total energy is conserved

MECHANICS OF A SYSTEM OF PARTICLES

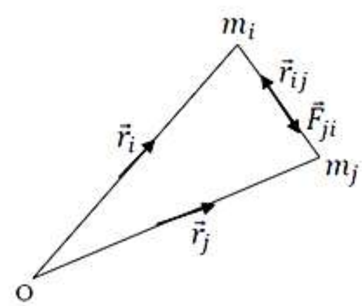
Consider a system consists of two or more particles. Force acting on i^{th} particle is given by

$$F_i = F_i^e + \sum_{j=1}^{N} F_{ij} - - - - - - - - 1$$

$F_i^e = external\ force\ from\ outside\ the\ system$

$F_{ij} = internal\ force\ on\ the\ i\ th\ particle\ due\ to\ j\ th\ particle.$

$\sum_{j=1}^{N} F_{ij} = total\ internal\ force\ due\ to\ all\ other\ particles$
$(j = 1\ to\ N) on\ i\ th\ particle.$



According to Newton's second law,

$$F_i = m_i a_i = \frac{dp_i}{dt} = \dot{P}_i$$

$$F_i = m_i \frac{dv_i}{dt} = m_i \frac{d^2 r_i}{dt^2}$$

For all particles in the system,

$$\sum \dot{P}_i = \frac{d^2}{dt^2} \sum_i m_i r_i ------2$$

From eqn 1

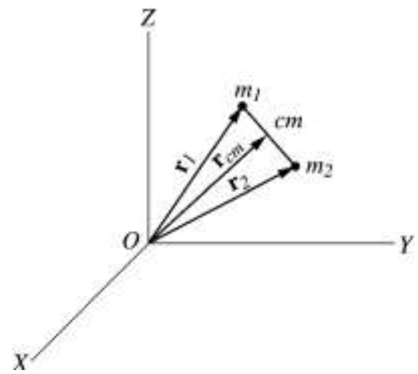
$$F_i = F_i^e + \sum_i \sum_j F_{ij} ------3$$

But$F_i^j = -F_j^i$. That is, $F_i^j + F_j^i = 0$. The second term in 3 becomes zero.

$[the\ sum\ of\ equal\ and\ opposite\ forces\ cancel\ each\ other\ and\ becomes\ zero]$

Now equating eqn 2 and 3

$$F_i^e = \frac{d^2}{dt^2}\sum_i m_i r_i ------4$$



Centre of mass R of a system is defined as,

$$R = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{\sum_i m_i r_i}{M}$$

$$\sum_i m_i r_i = MR ------5$$

Substitute eqn 5 in 4

$$F_i^e = M\frac{d^2 R}{dt^2} = Ma ------6$$

Thus the acceleration of centre of mass is due to only external force.

Conservation of linear momentum

From eqn 5

$$MR = \sum_i m_i r_i ------7$$

Differentiate the above eqn 7 with respect to 't'

$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + \cdots$$

$$MV = m_1 v_1 + m_2 v_2 + \cdots = \sum m_i v_i$$

$$\sum m_i v_i = P$$

$$is\ the\ linear\ momentum\ of\ all\ the\ particles\ in\ the\ system.\ Therefore$$

$$\boldsymbol{P = MV} \ ------\mathbf{8}$$

Thus total linear momentum is equal to product of total mass of system and velocity.

Differentiate eqn 8 with respect to ‘t’

$$\frac{dP}{dt} = \frac{d}{dt}(MV) = M\frac{dV}{dt} = M\frac{d^2R}{dt^2}$$

From eqn 6, total external force,

$$F_i^e = M\frac{d^2R}{dt^2} = \frac{dP}{dt} = \frac{d}{dt}(MV) -----9$$

When $F^e = 0$

$$P = MV = \sum m_i v_i = constant -----10$$

Thus if total external force on the system is zero, its total linear momentum is constant.

Conservation of Angular momentum

If L_1, L_2 ……….are the angular momenta of various particles of a system, the total angular momentum

$$L = L_1 + L_2 + \cdots = (r_1 \times p_1) + (r_2 \times p_2) + \ \dots \dots \dots$$

Therefore

$$L = \sum_{i=1}^{N} (r_i \times p_i) ------11$$

Differentiate eqn 11 with respect to ‘t’

$$\frac{dL}{dt} = \sum_i \left[\frac{d}{dt}(r_i \times p_i)\right] = \sum_i \left(\frac{dr_i}{dt} \times p_i\right) + \left(r_i \times \frac{dp_i}{dt}\right)$$

$$= \sum (\dot{r}_i \times p_i) + (r_i \times \dot{p}_i)$$

$[F = {}^{dp}/_{dt} = \dot{P}]$

$$= \sum (v_i \times mv_i) + (r_i \times F_i)$$

$$\frac{dL}{dt} = 0 + \sum r_i \times F_i - - - - - 12$$

Multiply by r_i on both sides for system of particles in eqn 1. Therefore,

$$\sum_i (r_i \times F_i) = \sum_i (r_i \times F_i^e) + \sum_i \sum_j (r_i \times F_{ij}) - - - - - 13$$

In the above term the last term becomes zero.

$$\sum_i (r_i \times F_i) = \sum_i (r_i \times F_i^e)$$

$$\sum_i (r_i \times F_i) = N$$

From eqn. 12 torque,

$$\boldsymbol{N} = \frac{\boldsymbol{dL}}{\boldsymbol{dt}}$$

Thus, the rate of change of angular momentum is equal to applied external torque on the system. If N = 0, $L = L_1 + L_2 + \ldots\ldots\ldots.. =$ constant. In the absence of external torque the angular momentum is conserved.

Conservation of Energy for a System of Particles

Consider a system of particles located at $r_1, r_2, \ldots\ldots.r_N$ and having masses $m_1, m_2 \ldots.m_N$. Forces acting on the particles can be derived from a potential function. Force on the i^{th} particle can be written as,

$$F_i = -\nabla_i V \text{ --------------- } 1$$

Newton's second law takes the form

$$\vec{F} = ma = m\,\frac{dv}{dt} = \frac{dp}{dt} = \frac{d}{dt}\,(mv)$$

Substitute for F in eqn. 1

$$\frac{d}{dt}\,(m_i v_i) = -\nabla_i\, V \text{ ---------------2}$$

Multiply by v_i on both sides. Summing over i particles we get

$$\sum_{i=1}^{N} m_i v_i\,.\frac{d}{dt}\,v_i = -\sum_{i=1}^{N} v_i\,.\nabla_i\, V \ldots\ldots\ldots\ldots\ldots 3$$

Consider LHS

$m_i v_i \frac{d}{dt}\, v_i$ can be written as, $\frac{d}{dt}\,\left({}^{1}\!/_{2}\; mv^2\right) = \frac{1}{2}\, m2\, v\,\frac{dv}{dt} = mv\frac{dv}{dt}$

LHS becomes

$$\frac{d}{dt}\sum_{i=1}^{N}\frac{1}{2}\, m_i v_i^2 \ldots\ldots\ldots\ldots\ldots\ldots..4$$

$$T = \sum_{i=1}^{N}\frac{1}{2}\, m_i v_i^2$$

$$LHS = \frac{dT}{dt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots.5$$

Consider RHS,

$$V_i = \frac{dr_i}{dt}$$

Therefore,

$$-\sum_{i=1}^{N}\;\nabla_i\, V.\frac{dr_i}{dt} = -\sum_{i=1}^{N}\frac{dV}{dr_i}.\frac{dr_i}{dt}$$

$$RHS = -\frac{dV}{dt} \ldots\ldots\ldots\ldots\ldots\ldots..6$$

5=6

$$\frac{dT}{dt} = -\frac{dV}{dt} \ldots\ldots\ldots\ldots\ldots\ldots..7$$

$$\frac{dT}{dt} + \frac{dV}{dt} = 0$$

$$\frac{dE}{dt} = 0 \ldots\ldots\ldots\ldots\ldots..8$$

Where E=T+V is the total energy. Thus if the force acting on the ith particle can be obtained as the gradient of a potential, then the total energy is given by E=T+V, is a constant of the motion.

Constraints

The limitations or the geometrical restrictions on the motion of a particles or system of particles are known as constraints.

Holonomic constraint

Let $\vec{r_1}, \vec{r_2} \ldots \vec{r_n}$ be the position coordinates of the system of particles. If the conditions of the constraints can be expressed as the equations connecting the coordinates of the particle having the form $f(\vec{r_1}, \vec{r_2} \ldots \vec{r_n}, t)=0$, then the constraints are said to be holonomic constraints.

"Constraints on the position of a system of particles are called holonomic constraints".

<u>Examples</u>

- The constraints involved in the rigid body in which the distance between any two particles is always fixed are holonomic.
- The constraints involved when a particle is restricted to move along a curve are holonomic.

- Simple pendulum with rigid support
- A bead moving on a circular ring or in an abacus.

Non holonomic constraints

"Constraints on the velocities of the particles in the system are called non holonomic"

- Rolling disc on a rough surface without slipping.
- Molecules in a gas-constraints involved in the motion of molecules in a gas container are non-holonomic.

Generalized Coordinates

The minimum possible number of independent coordinates required to specify the configurations of a system at any intent of time is known as the generalized coordinates.

It is denoted by the letters, $q_1, q_2, \ldots q_n$.

If $q_1, q_2, \ldots q_n$ are the generalized coordinates of the system then $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$ are the components of the velocities corresponding to the above coordinates. The generalized coordinates must satisfy the following two conditions.

1. The values of the coordinates determine the configuration of the system.

2. They may be varied arbitrarily and independently of each other, without violating the constraints of the system.

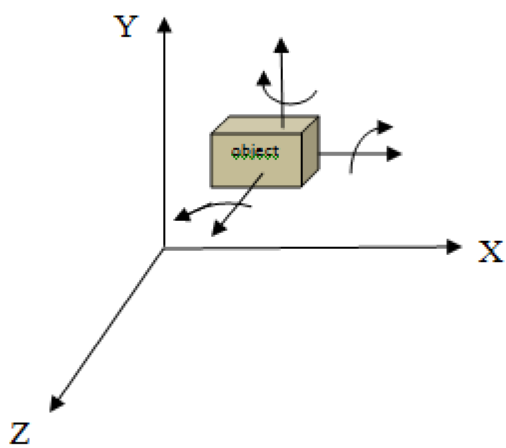
Examples

1. Consider a particle which moves in space, we can fixed in the position of the particle by using the coordinates x, y, z. Hence we require 3 generalized coordinates to fix the particles which moves in space.
2. When a particle moves in a plane it may be described by Cartesian coordinates x and y or the polar coordinate r, θ. So the generalized coordinates are two.
3. Consider a particle which is constraint to move only on a sphere of radius a. Then the generalized coordinates required are 2 namely θ *and* φ (longitude and latitude).
4. The beads of an abacus has the generalized coordinate x (the Cartesian coordinate along the horizontal wire)

Degrees of freedom

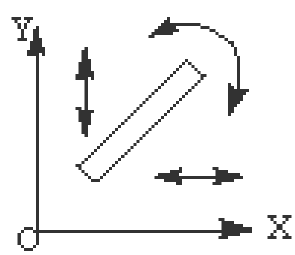
"The number of coordinates required to specify the position of a system is called the degrees of freedom of the system". The number of independent ways in which a mechanical system can move without violating any constraint is called the number of degrees of freedom of the system. It is indicated by the least possible number of coordinates to describe the system. The degree of freedom for a system containing n particles is 3N- k where k is the number of constraints on the system.

1. **Rigid body in a space -** Consider an rigid object moving in three dimensional space as in below diagram.



Here the object has three translational motion along X, Y and Z axis and three rotational motion about X, Y and Z axis. Therefore, the total degrees of freedom (DOF) are six.

2. **Rigid body in a plane**



The rigid bar can be translated along the *x* axis, translated along the *y* axis, and *rotated* about its centroid. Therefore the degrees of freedom are three.

D'Alemberts Principle

(It is a differential method to obtain Lagrange's equation)

This method is based on the principle of virtual work. For a system to be in equilibrium the resultant force acting on each particle must be zero. i.e., $F_i = 0$, where F_i is the force acting on i^{th} particle.

The virtual work is given by

$$F_i.\delta r_i = 0$$

δr_i *is the virtual displacement*

Summing all over the particles

$$\sum_i F_i.\delta r_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..1$$

From Newton's second law,

$$F_i = \frac{dP_i}{dt} = \dot{P}_i$$

P_i is the momentum of i^{th} particle due to F_i

$$F_i - \dot{P}_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.2$$

Replace F_i by $F_i - \dot{P}_i$ in equation 1

$$\sum_i (F_i - \dot{P}_i).\delta r_i = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..3$$

If constraints are present in the system then F_i can be written as

$$F_i = F_i^a + f_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots..4$$

F_i^a *is the applied or actual force and* f_i *is the constraint force*

Put eqn 4 in 3

$$\sum_i [(F_i^a + f_i) - \dot{P}_i).\delta r_i = 0$$

$$\sum_i (F_i^a - \dot{P}_i).\delta r_i + \sum_i f_i.\delta r_i = 0$$

If constraint force vanishes second term is 0

$$\sum_i (F_i^a - \dot{P}_i).\delta r_i = 0 \ldots\ldots\ldots\ldots.5$$

This is D'Alembert's Principle

Lagrange's equation from D'Alembert's Principle

Consider a system of particles. The position vectors of the particles in a system i.e., r_1, r_2, r_3 …….r_i are expressed as the functions of generalized co-ordinates q_1, q_2, q_3, …..q_n and time 't'.

$$r_i = r_i(q_1, q_2 \cdots\cdots q_n, t) - - - - - 1$$

Differentiating eqn. 1 with respect to time partially,

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots\cdots\cdots \frac{\partial r_i}{\partial t}\frac{dt}{dt}$$

$$v_i = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} - - - - - 2$$

Virtual displacement δr_i in terms of generalized co-ordinates from eqn. 1 is given by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j - - - - - - 3$$

From D'Alemberts principle (dropping superscripts)

$$\sum_i (F_i - \dot{P}_i).\delta r_i = 0 - - - - 4$$

Put eqn. 3 in 4

$$\sum_i (F_i - \dot{P}_i).\sum_j \frac{\partial r_i}{\partial q_j}\delta q_j = 0$$

$$\sum_j F_i \frac{\partial r_i}{\partial q_j}\delta q_j - \sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\delta q_j = 0 - - - - - 5$$

We define the term

$$\sum_{ij} F_i \frac{\partial r_i}{\partial q_j} = Q_j \; - - - - 6 \qquad \textit{as generalized force}$$

Put eqn. 6 in 5

$$\sum_j Q_j\,\delta q_j - \sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\delta q_j = 0 - - - - - 7$$

Consider second term in eqn. 7

$$\sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\ \delta q_j = \sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}\ \delta q_j - - - - - 8$$

Let,

$$\frac{d}{dt}\left(\dot{r}_i\ \cdot \frac{\partial r_i}{\partial q_j}\right) = \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

$$\ddot{r}_i \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(\dot{r}_i\ \cdot \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) - - - - - 9$$

Put eqn. 9 in 8

$$\sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\ \delta q_j = \sum_{ij} \left\{\frac{d}{dt}\left(m_i \dot{r}_i\ \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right\}\ \delta q_j$$

$$= \sum_{ij} \left\{\frac{d}{dt}\left(m_i v_i\ \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i v_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right\}\ \delta q_j - - - 10$$

But,

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial v_i}{\partial q_j} - - - - - 11$$

Differentiate eqn. 2 with respect to $\dot{q}_j$

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} - - - - - - 12$$

Put eqn. 11 and 12 in eqn. 10

$$\sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\ \delta q_j = \sum_{ij} \left\{\frac{d}{dt}\left(m_i v_i\ \cdot \frac{\partial v_i}{\partial \dot{q}_j}\right) - m_i v_i \cdot \left(\frac{\partial v_i}{\partial q_j}\right)\right\}\ \delta q_j$$

$$\sum_{ij} \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left[\sum \frac{1}{2} m_i v_i^2\right]\right\}\ \delta q_j$$

$$Therefore, \qquad \sum_{ij} \dot{P}_i \frac{\partial r_i}{\partial q_j}\ \delta q_j = \sum_{ij} \left\{\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j}\right\}\ \delta q_j - - - - 13$$

Put eqn. 13 in eqn. 7

$$\sum_j Q_j\ \delta q_j - \sum_{ij}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

$$\boldsymbol{Q_j} = \sum_{\boldsymbol{ij}}\left[\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\frac{\boldsymbol{\partial T}}{\boldsymbol{\partial \dot{q}_j}}\right) - \frac{\boldsymbol{\partial T}}{\boldsymbol{\partial q_j}}\right] - - - - - - 14$$

This is the general form of Lagrange's equation.

For conservative system the force is derived from potential function V.

$$F_i = -\nabla_i V = -\frac{\partial V}{\partial r_i}$$

Generalized force

$$Q_j = \sum_{ij} F_i \frac{\partial r_i}{\partial q_j}$$

$$= \sum_{ij} \nabla_i V \ \frac{\partial r_i}{\partial q_j} = -\sum_{ij}\frac{\partial V}{\partial r_i}\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

From eqn. 14

$$\sum_{ij}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = -\frac{\partial V}{\partial q_j}$$

$$\sum_{ij}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0$$

(Since V is not a function of $\dot{q}_j$ that is, $\frac{\partial V}{\partial \dot{q}_j}$=0)

$$\sum_{ij}\left[\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0$$

Define a new function called Lagrangian L of the system (L = T-V)

$$\sum_{ij}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0$$

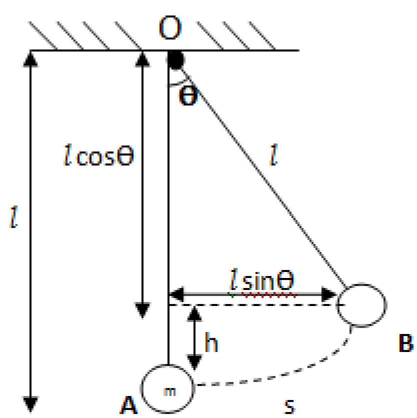
The above equation is known as Lagrange's equation of motion.

Application of Lagrangian formulation

Simple Pendulum

A simple pendulum consists of a mass *m* hanging from a string of length *l* and fixed at a point. When displaced to an initial angle and released, the pendulum will swing back and forth with periodic motion.

ϴ is the angular displacement of the simple pendulum from equilibrium position. ϴ is chosen as the generalized coordinate.



The kinetic energy is given by

$$T = \frac{1}{2} m v^2$$

[ϴ=arc/radius = s/*l*. *T*herefore, s=*l*ϴ . velocity v =ds/dt = d/dt(lϴ)= $l\dot{\theta}$]

$$T = \frac{1}{2} m(l\dot{\theta})^2 = \frac{1}{2} m\, l^2\dot{\theta}^2$$

Potential energy is V= mgh = mg (OA-OC) = mg (*l* – *l* cosϴ) = mg*l* (1-cosϴ)

Lagrangian L = T – V

$$L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} ml^2\, 2\,\dot{\theta} = ml^2\dot{\theta}$$

$$\frac{\partial L}{\partial \Theta} = -mgl\ sin\Theta$$

Substitute in Lagrangian equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}}\right) - \frac{\partial L}{\partial \Theta} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\Theta}\right) + mgl\ sin\Theta = 0$$

$$ml^2\ddot{\theta} + mgl\ sin\Theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\ sin\Theta = 0$$

For small amplitude $sin\Theta \approx \theta$

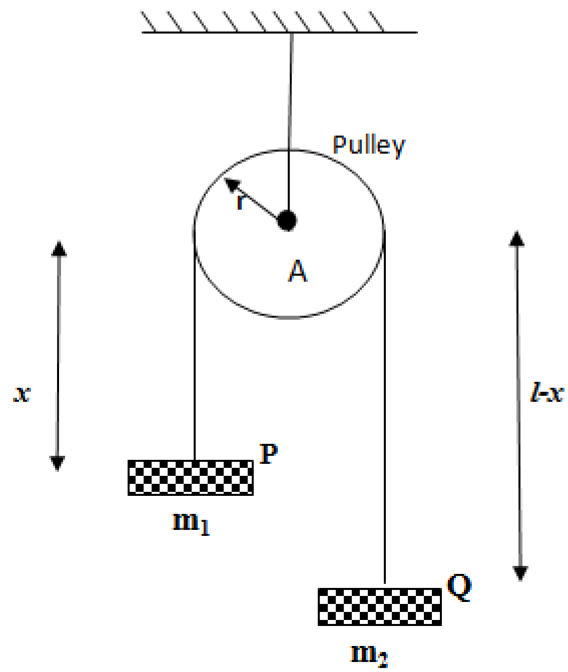
$$\ddot{\theta} + \frac{g}{l}\ \Theta = 0$$

This is the equation of motion for simple pendulum.

Atwood's Machine

Atwood's Machine is a system of two masses, connected by an inextensible string passing over a small smooth pulley. It is an example conservative system with holonomic constraint.

The schematic representation of Atwood's machine is shown below. It consists of two masses m_1 and m_2 suspended over a frictionless pulley of radius 'r' connected by a string of length '*l*'. Let '*x*' be the vertical distance from the pulley to mass m_1. Then mass m_2 will be at a distance *l*-*x* from pulley. PA = *x*, PQ = *l* and QA = *l*-*x*. Here x is the independent coordinate.



$$Kinetic\ energy\ of\ mass\ m_1 = \frac{1}{2}\ m_1 \dot{x}^2$$

$$Kinetic\ energy\ of\ mass\ m_2 = \frac{1}{2}\ m_2 \dot{x}^2$$

$$Total\ Kinetic\ energy\ of\ the\ system = T = \frac{1}{2}\ m_1 \dot{x}^2 + \frac{1}{2}\ m_2 \dot{x}^2$$

$$T = \frac{1}{2}(m_1 + m_2)\ \dot{x}^2$$

$$Potential\ energy\ of\ mass\ m_1 = -m_1 gx$$

$$Potential\ energy\ of\ mass\ m_2 = -m_2 g(l-x)$$

$$Total\ Potential\ energy\ of\ the\ system = V = -m_1 gx + (-m_2 g(l-x))$$

$$V = -m_1 gx - m_2 g(l-x)$$

$$Lagrangian\ L = T - V$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 gx + m_2 g(l-x)$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 gx + m_2 gl - m_2 gx$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + gx(m_1 - m_2) + m_2 gl$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

Lagrange's equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}[(m_1 + m_2)\dot{x}] - (m_1 - m_2)g = 0$$

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$\ddot{x} = \frac{(m_1 - m_2)g}{(m_1 + m_2)}$$

This is the required equation of motion of a system of two masses, connected by an inextensible string passing over a small smooth pulley.

UNIT – II MOTION UNDER CENTRAL FORCE

'Central force is that force which is always directed towards or away from a fixed centre and magnitude is a function of distance from the centre point'

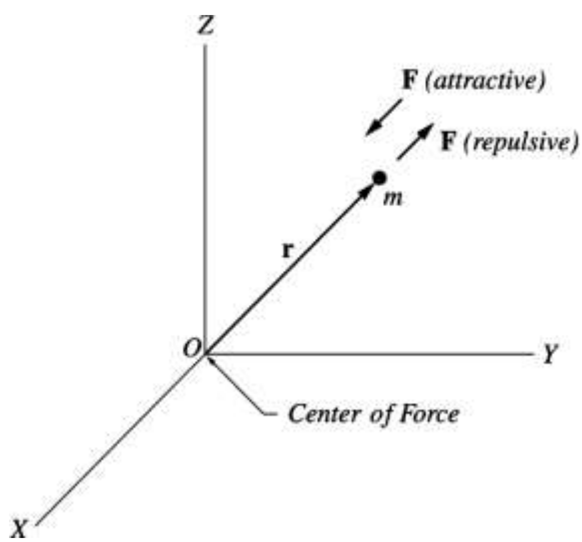
$$\vec{F} = f\,(r)\hat{r}$$

$f\,(r)\ is\ the\ function\ of\ r\ \ and\ \hat{r}$ is the unit vector.

The problem of finding the motion of a particle under a central force is one of the most important problems in physics, because it is closely related to mechanics of nature, that is, motion of planets, satellites etc.

The force between two interacting particle is primarily a central force. If one of the particle is heavier than other, although due to Newton's third law, force acting on both the particles will be the same. The acceleration of heavier particle will be too small than lighter one and it can be neglected and heavier particle can be regarded at rest. By locating origin at heavier particle, the problem of two bodies simply reduces to one body problem.

$$f\,(r)\ <\ 0\ \rightarrow attractive\ force$$
$$f\,(r)\ >0\ \rightarrow repulsive\ force$$



Thus central force is the force on a body or an object is always towards a fixed point (origin). OR "central force is the force that is directed along the line joining the object and the origin.

Some examples of central forces are

1. Gravitational force.
2. Coulomb force
3. Simple harmonic motion
4. Projectile motion
5. Uniform circular motion
6. Electrostatic forces and magneto static forces.

Conservation of energy

Motion of a particle with mass 'm' subject to a central force

$$\overrightarrow{F}(r) = f\ (r)\hat{r}$$

Unit vector $\hat{r} = \frac{\vec{r}}{r}$

If V(r) is potential energy then,

$$f\ (r) = -\frac{dV(r)}{dr}$$

$$\overrightarrow{F}(r) = -\nabla\ V\ (r)$$

$$V\ (r)\ is\ a\ scalar\ function.$$

The curl of gradient of scalar function is zero.

That is, $\nabla \times \nabla\ V(r) = 0$

$$\boldsymbol{\nabla} \times \vec{\boldsymbol{F}} = \mathbf{0}$$

Thus central force is always a conservative force.

Conservation of angular momentum

Torque

$$\overrightarrow{N} = \vec{r} \times \overrightarrow{F}(r)$$

$$= \vec{r} \times \hat{r}\ f\ (r)$$

$$= \vec{r} \times \frac{\vec{r}}{r}\ f\ (r) = 0$$

Thus $N = \frac{dL}{dt} = 0$

Angular momentum L through the central force is constant.

KEPLER PROBLEM

The inverse square law of force is most important of all the central force laws. It results in the deduction of Kepler's laws. The planets move around the sun under the influence of gravitational force which is an inverse square law of force. Hence we deduce the Keplers laws on the basis of inverse square law of force.

Inverse square law of force is given by,

$$f(r) = -\frac{k}{r^2}$$

Law of elliptical orbit - All the planets move in an elliptical orbit around the sun being at one of the foci.

Law of areas - The radius vector connecting the sun and the planet sweeps at equal areas is equal intervals of time. ie, areal velocity is constant.

Harmonic law - The square of the period of revolution of any planet about the sun is proportional to the cube of the semi major axis.

Deduction of first law:

The central force varies inversely as the square of the distance. That is

$$f(r) = -\frac{K}{r^2} ------1$$

K is the constant. The corresponding potential energy will be

$$V(r) = -\frac{K}{r}$$

For $u = 1/r$, the inverse square law force [$f(r) = -K/r^2$] is given by

$$f\left(\frac{1}{u}\right) = -Ku^2$$

Thus the differential equation of the orbit can be expressed as

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{J^2u^2}Ku^2 \quad [\because f(r) = -Ku^2] \quad \rightarrow 2$$

J is the angular momentum and u is a variable.

or $$\frac{d^2u}{d\theta^2} + u - \frac{mK}{J^2} = 0 \quad \rightarrow 3$$

Let $$x = u - \frac{mK}{J^2}.$$

Then $$\frac{d^2x}{d\theta^2} + x = 0 \quad \rightarrow 4$$

which has the solution

$$x = A \cos(\theta - \theta') \quad \rightarrow 5$$

where A and θ' are the constants of integration.

Since $x = u - \frac{mK}{J^2}$ and $u = \frac{1}{r}$, we can write eq. **5** as

$$\frac{1}{r} - \frac{mK}{J^2} = A \cos(\theta - \theta')$$

or $$\frac{1}{r} = \frac{mK}{J^2} + A \cos(\theta - \theta') \quad \rightarrow 6$$

Multiply by J^2 / mK

$$\frac{J^2/mK}{r} = 1 + \frac{J^2 A}{mK} \cos(\theta - \theta')$$

$$\frac{l}{r} = 1 + e \cos(\theta - \theta') \quad \rightarrow \mathbf{7}$$

Where

$$\frac{J^2}{mK} = l \text{ and } \frac{J^2 A}{mK} = e \quad \rightarrow 8$$

In the above equation 'e' is known as eccentricity.

Equation 7 represents a conic section and therefore coinciding with Kepler's first law of planetary motion.

$$r = \frac{l}{1 + e\,cos\theta} ------9$$

general equation of a conic section with one focus at the origin and eccentricity e, given by

$$e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$$ 10

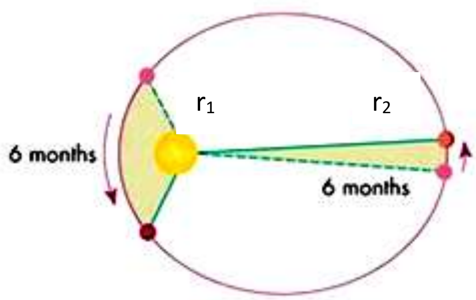
$$E = \frac{mK^2}{2J^2}(e^2 - 1) - - - - - - -11$$

The magnitude of e decides the nature of the orbit.

Value of e	*Value of total energy E*	*Conic*
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	Ellipse
$e = 0$	$E = -\frac{mK^2}{2J^2}$	Circle

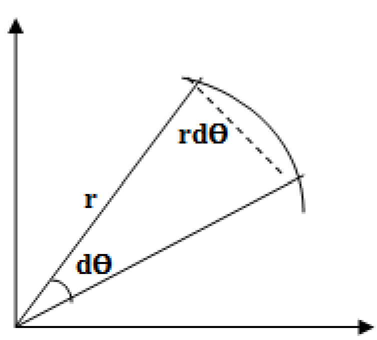
Deduction of Kepler's second law

When planet moves in orbit the radius vector sweeps equal area in equal interval of time.



Areal velocity $v_1r_1 = v_2r_2$

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m}, \text{ a constant}$$



If vector r rotates by an angle dΘ in time dt, the area swept out by r in time dt.

$$dA = \frac{1}{2}\, r\, (r d\theta)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m}$$

Thus areal velocity is constant. Thus second law is proved.

Deduction of Kepler's third law

For $e < 1$ or $E < 0$, the orbit is elliptical

From equation 7

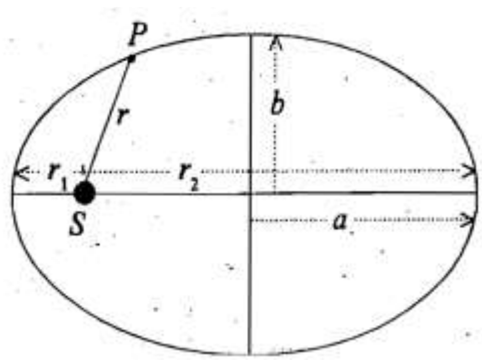
$$\frac{l}{r} = 1 + e \cos(\theta - \theta')$$

where $$l = \frac{J^2}{mK} \text{ and } e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$$

When $\theta - \theta' = 0$ or $\cos(\theta - \theta') = 1$, the value of $r = r_1$ is minimum and when $\theta - \theta' = \pi$ or $\cos(\theta - \theta') = -1$, the value of $r = r_2$ is maximum . The apsidal distances r_1 and r_2 are known as *perihelion* and *aphelion* and are given by

$$\frac{l}{r_1} = 1 + e \text{ or } r_1 = \frac{l}{1+e} \longrightarrow 12$$

and $$\frac{l}{r_2} = 1 - e \text{ or } r_2 = \frac{l}{1-e} \longrightarrow 13$$



The semimajor axis (a) of the ellipse is one-half the sum of these two apsidal (turning) distances, *i.e.*,

$$a = \frac{r_1 + r_2}{2} = \frac{1}{2}\left[\frac{l}{1+e} + \frac{l}{1-e}\right] = \frac{l}{1-e^2} \longrightarrow 14$$

From equation 11

$$E = \frac{mK^2}{2J^2}\,(e^2 - 1)$$

$$(e^2 - 1) = \frac{2EJ^2}{mK^2}$$

$$\left.\begin{aligned} (1 - e^2) &= -\frac{2EJ^2}{mK^2} \\ from\ eqn.8\ we\ have,\quad l &= \frac{J^2}{mK} \end{aligned}\right\} \quad 15$$

Substitute15 in eqn 14

$$a = -\frac{J^2}{mK}\cdot\frac{mK^2}{2EJ^2} = -\frac{K}{2E}$$

$$E = -\frac{K}{2a} \longrightarrow 16$$

Thus in case of an elliptical orbit, the total energy depends solely on the major axis.

If T be the periodic time in which the particle or radius vector completes one revolution then the area of the orbit is

$$A = \int_0^A dA = \int_0^T \left(\frac{1}{2}r^2\dot{\theta}\right) dt = \int_0^T \frac{J}{2m}\,dt = \frac{JT}{2m} \longrightarrow 17$$

But area of the ellipse $A = \pi ab$, $\longrightarrow$ 18

where a and b are the semi-major and semi-minor axes of the ellipse respectively.

Equating eqn. 17 and 18

$$T = \frac{2\pi\, abm}{J} \longrightarrow 19$$

From eqn. 14

$$a = \frac{l}{1-e^2} = \frac{J^2}{mK(1-e^2)} \quad \text{or} \quad 1-e^2 = \frac{J^2}{mKa}$$

But according to the property of the ellipse

$$b = a\sqrt{1-e^2} = a\sqrt{\frac{J^2}{mKa}} = a^{\frac{1}{2}} \frac{J}{\sqrt{mK}}$$

Substitute the above eqn. in T

$$T = \frac{2\pi am}{J} \cdot \frac{a^{1/2} J}{\sqrt{mK}} \quad \text{or} \quad T = 2\pi\, a^{3/2}\sqrt{\frac{m}{K}} \longrightarrow \mathbf{20}$$

This gives the time period in the elliptical orbit.

Squaring both sides of equation 20,

$$T^2 = 4\pi^2 a^3 \frac{m}{K} \quad \text{or} \quad T^2 \propto a^3$$

Thus the square of period of revolution of a planet around the sun is proportional to the cube of semi-major axis of the elliptical orbit. This is known as ***Kepler's third law of planetary motion.***

Virial Theorem (R.E. Clausius, 1870)

Let us consider a system of parcitcles with position vector $\mathbf{r}_i$ and applied force $\mathbf{F}_i$ According to Newton's second law, the equations of motion are

$$\mathbf{F}_i = \dot{\mathbf{p}}_i$$

We introduce a quantity λ ; defined by

$$\lambda = \sum_i \mathbf{p}_i \bullet \mathbf{r}_i \longrightarrow 1$$

where the sum is taken for all particles of the system.

The total time derivative of λ is

$$\frac{d\lambda}{dt} = \sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i + \sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i$$

First term on RHS can be written as

$$\sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T$$

Second term can be written as

$$\sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i = \sum_i \mathbf{F}_i \bullet \mathbf{r}_i$$

Hence,
$$\frac{d\lambda}{dt} = \frac{d}{dt}\left[\sum_i \mathbf{p}_i \bullet \mathbf{r}_i\right] = 2T + \sum_i \mathbf{F}_i \bullet \mathbf{r}_i$$

The time average of above equation over a time interval τ is obtained as

$$\frac{1}{\tau}\int_0^\tau \frac{d\lambda}{dt}\,dt = \overline{\frac{d\lambda}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i . \mathbf{r}_i} \text{ or } \frac{1}{\tau}[\lambda(\tau) - \lambda(0)] = \overline{2T} + \overline{\sum_i \mathbf{F}_i . \mathbf{r}_i} \longrightarrow 2$$

In case of periodic motion, τ is chosen as period and all coordinates repeat after this interval of time. In such a case, $\lambda(\tau) = \lambda(0)$ and then left hand side of eq. 2 vanishes, *i.e.*,

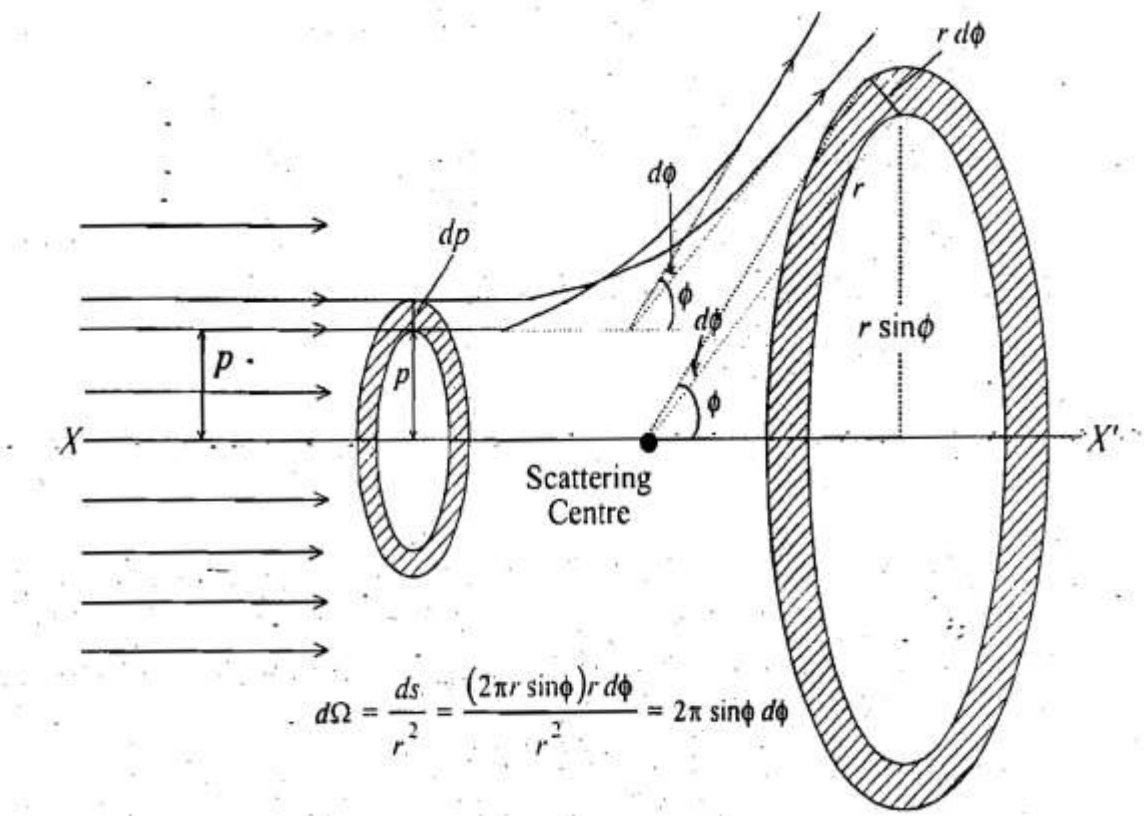
$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} = 0 \text{ or } \overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} \longrightarrow 3$$

Eq. 3 is called the ***Virial theorem*** and the quantity $-\frac{1}{2}\overline{\sum \mathbf{F}_i \bullet \mathbf{r}_i}$ is known as the ***Virial of Clausius.***

Scattering in a Central Force field

Let us consider a uniform beam of particles incident on a centre of force. All the particles of the beam have the same mass and energy. The *intensity* of the beam I_0 is the number of particles crossing unit area per unit time normal to the direction of the beam. This I_0 is also called *flux density*. We assume that the force between an incident particle and the particle at the centre of force falls off to zero at large distances.

When a particle approaches the centre of force, it will interact [for example, an α-particle (+ve charge) will experience replusion from the positively charged nucleus] so that its path will deviate from the



Scattering–impact parameter (p) and angle of scattering (ϕ)

incident straight line trajectory. After passing the centre of force, as the particle goes away, the force acting on it will decrease and finally at large distances, the force will become zero. This results again in the straight line motion but in general in a different direction and we say that the particle has been ***scattered.***

Scattering cross-section : Consider a uniform beam of particles, moving with a flux of I_0 particles per unit area towards a scattering centre (*e.g.*, an atom of a target). Imagine that the scattering centre presents an area $d\sigma$ perpendicular to the path of the beam such that whatever particles hit $d\sigma$ area are scattered into a solid angle $d\Omega$. Thus the number of particles, scattered into $d\Omega$ solid angle per second are $I_0\, d\sigma$. If $I(\Omega)$ (intensity of the scattered particles) is defined as the number of particles scattered in the direction Ω per unit solid angle per unit time, then the number of scattered particles in the small solid angle $d\Omega$ about Ω direction is given by

$$I_0 d\sigma = I(\Omega)\, d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0}$$

The quantity $\frac{d\sigma}{d\Omega} = \sigma(\Omega)$ is called ***differential scattering cross-section*** or simply scattering cross-section for scattering in Ω-direction *i.e.*,

$$\sigma(\Omega) = \frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0}$$

Thus the differential scattering cross-section is the ratio of the number of the scattered particles per second per unit solid angle and the flux density of the incident particles.

The total cross-section (σ) is given by

$$\sigma = \int \sigma(\Omega)\, d\Omega = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

Scattering angle (ϕ) : The angle between the incident and scattered directions of the particle is called scattering angle and is denoted by ϕ.

Impact Parameter (p) : If we draw a perpendicular on the direction of the incident particle from the scattering centre, then the length of this perpendicular is known as impact parameter p.

As the force is central, there must be complete symmetry about the axis (XX') of the incident beam. The solid angle $d\Omega$ is given by*

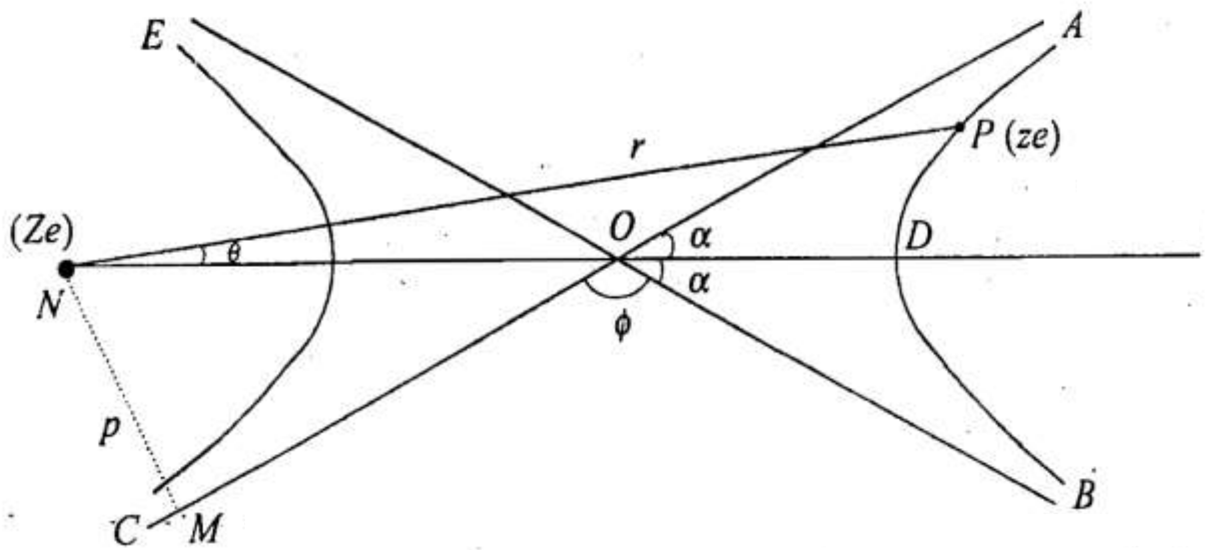
$$d\Omega = 2\pi \sin\phi\, d\phi$$

RUTHERFORD SCATTERING CROSS-SECTION

In the Rutherford scattering a positively charged particle of charge ze and mass m is scattered by a heavy nucleus N. The nucleus is assumed to be at rest during the collision. The charge on the nucleus is Ze, where Z is the atomic number. Suppose the positively charged particle is moving towards the heavy nucleus with initial velocity v_0. As the particle approaches the nucleus, the repulsive force $(Zze^2/4\pi\varepsilon_0 r^2)$ increases rapidily and the particle changes from a straight line path to a hyperbola ADB, having one focus at N as shown in Fig. The asymptotes AO and BO to the hyperbola give the direction of the incident and scattered particle. The angle COB is the scattering angle ϕ and NM is the impact parameter p.

The charged particle is moving in a central force field, hence the equation of its path is given by

$$\frac{l}{r} = 1 + e\cos\theta \quad \longrightarrow \quad 1$$



where $l = J^2/mK$, $e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$ and θ', the constant of integration, has been taken to be zero. Here the force F is given by

$$F = \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{r^2} = -\frac{K}{r^2}$$

Therefore, $$K = -\frac{Zze^2}{4\pi\varepsilon_0}$$

Hence $$l = -\frac{J^2 4\pi\varepsilon_0}{Zze^2 m} \text{ and } e = \sqrt{1 + \frac{2EJ^2(4\pi\varepsilon_0)^2}{Z^2 z^2 e^4 m}}$$

As the initial velocity of the particle is v_0, its total energy is given by

$$E = \frac{1}{2} m v_0^2 \text{, whence } m v_0 = \sqrt{2mE}$$

According to the law of conservation of angular momentum,

$$m v_0 p = m r^2 \dot{\theta} = J \text{, whence } m v_0 = \frac{J}{p}$$

Therefore, $$\frac{J}{p} = \sqrt{2mE} \quad \text{or} \quad J = p\sqrt{2mE}$$

This gives

$$e = \sqrt{1 + \frac{2E\left(p\sqrt{2mE}\right)^2 (4\pi\varepsilon_0)^2}{m z^2 Z^2 e^4}} \text{ or } e = \sqrt{1 + \left(\frac{2Ep\,4\pi\varepsilon_0}{zZe^2}\right)^2}$$

Obviously, $e > 1$, because $(2Ep/zZe^2)^2$ is a positive quantity.

Hence equation 1 represents the path of the charged particle as a hyperbola

Since the hyperbolic path must be symmetric about the direction of the periapsis, the scattering angle ϕ is given by

$$\phi = \pi - 2\alpha \text{ or } \alpha = \frac{\pi}{2} - \frac{\phi}{2}$$

where α is the angle between the direction of the incoming asymptote and the periapsis direction (OD).

Further the asymptotic direction is that for which r is infinite (∞) and then $\theta \to \alpha$.

From equation 1

$$1 + e\cos\alpha = 0 \text{ or } \cos\alpha = -\frac{1}{e} \text{ or } \cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = -\frac{1}{e} \text{ or } \sin\frac{\phi}{2} = -\frac{1}{e}$$

Thus $$\operatorname{cosec}\frac{\phi}{2} = -e$$

Squaring it, we get

$$\operatorname{cosec}^2\frac{\phi}{2} = e^2 \text{ or } 1 + \cot^2\frac{\phi}{2} = 1 + \left[\frac{2Ep\ 4\pi\varepsilon_0}{zZe^2}\right]^2$$

$$\cot\frac{\phi}{2} = \frac{2Ep\,(4\pi\varepsilon_0)}{zZe^2}$$

From which one can find the ***scattering angle*** ϕ.

expression for ***impact parameter***

$$p = \frac{zZe^2 \cot\phi/2}{2E(4\pi\varepsilon_0)}$$

Differentiating it, we get

$$\frac{dp}{d\phi} = -\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\operatorname{cosec}^2\frac{\phi}{2}$$

Substituting the value of p and dp/d∅ in the differential cross-section $\sigma(\Phi)$ eqn.

$$\sigma(\phi) = -\frac{p}{\sin\phi}\left[\frac{dp}{d\phi}\right]$$

$$\sigma(\phi) = -\frac{zZe^2 \cot\frac{\phi}{2}}{2E\,(4\pi\varepsilon_0)\, 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\left[-\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\right]\operatorname{cosec}^2\frac{\phi}{2}$$

$$\sigma(\phi) = \frac{1}{4}\left[\frac{zZe^2}{(4\pi\varepsilon_0)\,2E}\right]^2 \operatorname{cosec}^4\frac{\phi}{2}$$

This is the well known expression for the ***Rutherford scattering cross-section***. Thus the scattering cross-section or the number of particles scattered per second along the direction ϕ are proportional to

(1) $\operatorname{cosec}^4\frac{\phi}{2}$,

(2) the square of the charge on the nucleus (Ze),

(3) the square of the charge on the particle (ze), and

(4) inversely proportional to the square of the initial kinetic energy E.

Thus if N_ϕ is the number of particles scattered along the angle ϕ per second, then one can represent

$$N_\phi = C \operatorname{cosec}^4 \phi/2$$

where C is a constant

ARTIFICIAL SATELLITES

We have studied the motion of a planet and its orbit around the sun. In fact, a body which revolves constantly round a comparatively much larger body is said to be satellite. We know that the earth and other planets revolve round the sun in their specified orbits. The moon revolves round the earth and the planets Jupiter and Saturn have six and nine moons respectively revolving around them. All these are the examples of *natural satellites.* Each one of these satellites is attracted by its primary with a force, given by Newton's law of gravitation.

Scientists have also been able to placed man-made satellites, revolving round the earth or sun. They are called **artificial satellites**. The theory discussed above for the orbits and planetary motion is valid for the discussion of satellites.

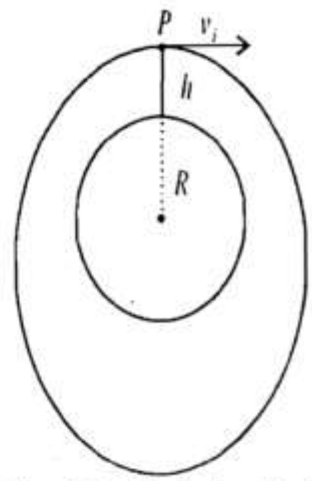
An artificial satellite of the earth is a body, place in a stable orbit around the earth with the help of multistage rocket. In order to launch a satellite in a stable orbit, first it is necessary to take the satellite to the altitude h, where at the point P by some mechanism, it is given the necessary orbitting velocity, called the **insertion velocity** v_i

The total energy of the satellite at P relative to the earth is given by

$$E = \frac{1}{2}mv_i^2 - \frac{GMm}{R+h}$$

where m is the mass of the satellite and M that of the earth, having radius R.

The orbit will be an ellipse, a parabola or hyperbola, depending on whether E is negative, zero or positive. In each case, the centre of the earth is at one focus of the path.



Elliptical path of a body projected horizontally from a height h above the earth's surface for $v_i^2 < 2GM/(R+h)$

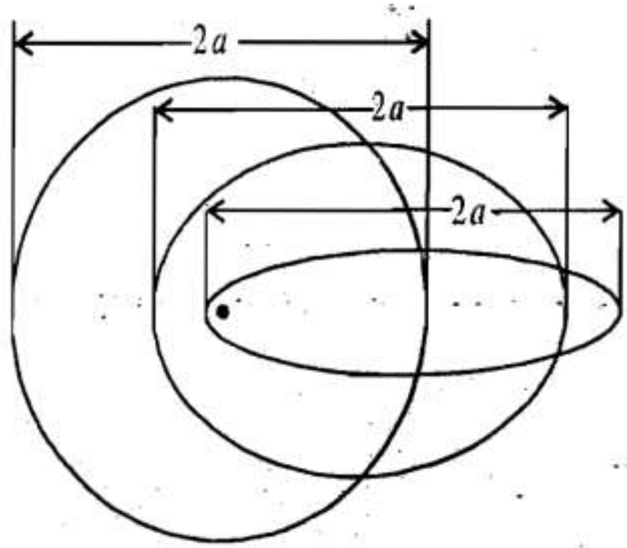
Therefore, the satellite will be moving in an elliptical orbit if

$$v_i^2 < \frac{2GM}{R+h}$$

The total energy E determines the size or semi-major axis of the orbit. However the shape or eccentricity e of the orbit is determined by both total energy E and angular momentum J by the relation :

$$e = \sqrt{1 + \frac{2EJ^2}{mK^2}} \qquad \ldots(64)$$

with $K = GMm$. For elliptical orbits, layer the angular momentum, the less elongated is the orbit



Elliptical orbits for different values of the angular momentum J with same enegry E; various orbits have the same focus and semi-major axis, but differing in eccentricity.

Geostationary Orbit : If the height of an artificial satellite at equator above the earth's surface is such that its period of revolution is exactly equal to the period of rotation of the earth, then the satellite would appear stationary over a point on earth's equator. Such a satellite is called **geostationary satellite** and its orbit is called **geostationary orbit**. Therefore for a geostationary satellite, we must have the orbit (i) *to be geosynchronous* (ii) *to be circular* and (iii) *to stay over the geographical equator of the earth.*

The height of geostationary satellite is

$$h = a_s - R = 35,786 \text{ km}.$$

The geostatinary orbit is often called **parking orbit**. Artificial satellites used for telecasting are put in parking orbits.

Uses of Artificial Satellites : Artificial satellites are used in the following :

(1) Distant transmission of radio and TV signals.

(2) To study upper regions of the atmosphere.

(3) High altitude satellites for astronomical observations (as the effects of atmosphere are not present).

(4) Weather forecasting.

(5) Eearth measurements (gravitation and magnetic fields).

UNIT – III RIGID BODY DYNAMICS AND SMALL OSCILLATIONS

A rigid body is defined as a system of particles in which the distance between any two particles remains fixed throughout the motion. Thus a system of N particles is said to be a rigid body if it is subjected to holonomic constraints of the form

$$r_{ij} = C_{ij}$$

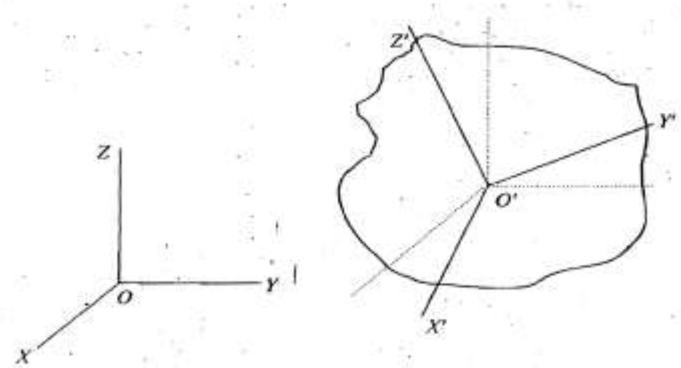
where r_{ij} is the distance between ith and jth particles and C_{ij} is the constant. In a rigid body motion, the deformations, occurring in actual bodies, are neglected and a rigid body maintains its shape during its motion.

We may describe the motion of a rigid body by using two coordinate systems –

(1) Body coordinate system : A coordinate system, fixed in the rigid body, is called a body coordinate system and its axes are called body set of axes.

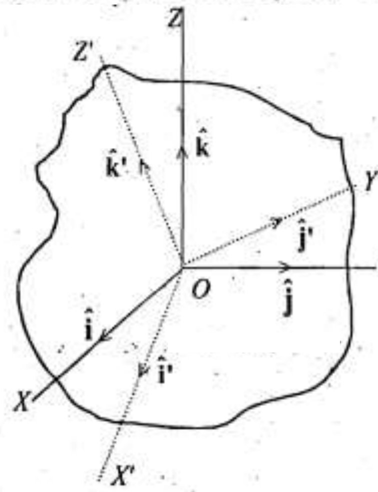
(2) Space coordinate system : The axes of such a coordinate system are fixed in the space are called space set of axes.

XYZ represents the space reference system with origin O and $X'Y'Z'$ the body coordinate system, fixed in the rigid body with origin O'. We choose the origin O' of the body set of axes to coincide with the centre of mass of the rigid body. Clearly three coordinates are required to specify the origin O' of this body set of axes relative to the origin O of the space reference system.



XYZ – Space set of axes ; $X'Y'Z'$ – Body set of axes

Let **R** (X, Y, Z) be the position vector of O' relative to O. Further, for the general motion of the rigid body, the orientation of the body set of axes $X'Y'Z'$ is described by three angles relative to a coordinate system with common origin O' and axes parallel to the space set of axes (XYZ). Thus three coordinates of the origin O' and these three angles constitute six independent coordinates which provide complete configuration of the rigid body in motion at any instant of time.



XY Z - Space set of axes ;
X'Y'Z' - Body set of axes

For convenience, first consider the origins of space set of axes and body set of axes to be the same (O). In order to specify the orientation of the body set of axes, we may use the direction cosines of body set of axes ($X'Y'Z'$) relative to the space set of axes (XYZ).

Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be the unit vectors along X, Y, Z axes and $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ along X', Y', Z' axes respectively. If C_{11}, C_{12}, C_{13} be the direction cosines of the X' axis (or $\hat{\mathbf{i}}'$ unit vector) with respect to X, Y, Z axes respectively, then

$$C_{11} = \cos(X', X) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{i}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}$$
$$C_{12} = \cos(X', Y) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{j}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} \qquad \text{--1}$$
$$C_{13} = \cos(X', Z) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{k}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}}$$

Thus $\hat{\mathbf{i}}' = C_{11}\hat{\mathbf{i}} + C_{12}\hat{\mathbf{j}} + C_{13}\hat{\mathbf{k}}$

or $\hat{\mathbf{i}}' = (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$ — 2

Similarly, $\hat{\mathbf{j}}' = C_{21}\hat{\mathbf{i}} + C_{22}\hat{\mathbf{j}} + C_{23}\hat{\mathbf{k}}$

and $\hat{\mathbf{k}}' = C_{31}\hat{\mathbf{i}} + C_{32}\hat{\mathbf{j}} + C_{33}\hat{\mathbf{k}}$

where C_{21}, C_{22}, C_{23} are the direction cosines of Y' -axis and C_{31}, C_{32}, C_{33} those of Z'-axis with respect to X, Y, Z axes respectively.

These sets of nine direction cosines then completely specify the orientation of the X', Y', Z' axes with respect to X, Y, Z axes. With the help of these direction cosines, we can also relate the coordinates of a given point from one system to another. If **r** be the position vector of a point with coordinates (x, y, z) and (x', y', z') in the two systems, then

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

and $$\mathbf{r} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}'$$

Now, $x' = (\mathbf{r} \cdot \hat{\mathbf{i}}') = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (C_{11}\hat{\mathbf{i}} + C_{12}\hat{\mathbf{j}} + C_{13}\hat{\mathbf{k}})$

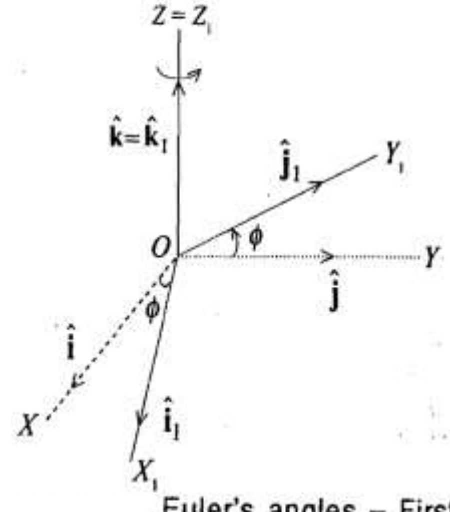
or $x' = C_{11}x + C_{12}y + C_{13}z$

Similarly, $y' = C_{21}x + C_{22}y + C_{23}z$

and $z' = C_{31}x + C_{32}y + C_{33}z$

EULER'S ANGLES

We are interested in knowing three independent parameters to specify the orientation of body set of axes relative to the space set of axes. For this purpose, we use three angles. These angles may be chosen in various ways, but the most commonly used set of three angles are the Euler's angles, represented by ϕ, θ and ψ.



Euler's angles – First rotation ϕ, defining precession angle.

We can reach an arbitrary orientation of the body set of axes X' Y' Z' from space set of axes $(X\ Y\ Z)$ by making three successive rotations performed in a specific order.

(1) First rotation (ϕ) **:** First the space set of axes is rotated through an angle ϕ counter-clockwise about the Z-axis so that Y-Z plane takes the new position Y_1- Z_1 and this new plane Y_1- Z_1 contains the Z'-axis of the body coordinate system. Now the new position of the coordinate system is X_1 Y_1 Z_1 (with $Z = Z_1$) If $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ are the unit vectors along X, Y, Z axes and $\hat{\mathbf{i}}_1, \hat{\mathbf{j}}_1, \hat{\mathbf{k}}_1$ along X_1, Y_1, Z_1 axes respectively, then the transformation to this new set of axes from space set of axes is represented by the equations

$$\hat{\mathbf{i}}_1 = \cos\phi\ \hat{\mathbf{i}} + \sin\phi\ \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}}_1 = -\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}}$$

$$\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}$$

or

$$\begin{pmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix}$$

Thus XYZ axes are transformed to $X_1\ Y_1\ Z_1$ by the matrix of transformation

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The angle ϕ is called the ***precession angle***.

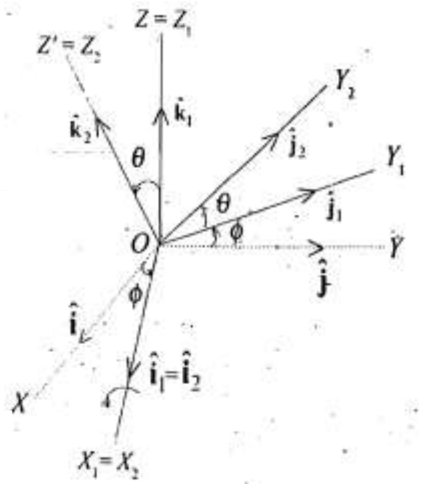
(2) Second rotation (θ) : Next intermediate axes $X_1\ Y_1\ Z_1$ are rotated about X_1 axis counter-clockwise through an angle θ to the position $X_2\ Y_2\ Z_2$ so that Y_1, Z_1 axes acquire the positions Y_2, Z_2 with $Z_2 = Z'$

This also results the plane $X_2.Y_2$ in plane $X'\ Y'$. If $\hat{\mathbf{i}}_2, \hat{\mathbf{j}}_2, \hat{\mathbf{k}}_2$ are unit vectors along X_2, Y_2, Z_2 axes respectively, then

$$\hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_1$$

$$\hat{\mathbf{j}}_2 = \cos\theta\, \hat{\mathbf{j}}_1 + \sin\theta\, \hat{\mathbf{k}}_1$$

$$\hat{\mathbf{k}}_2 = -\sin\theta\, \hat{\mathbf{j}}_1 + \cos\theta\, \hat{\mathbf{k}}_1$$



. Euler's angles - Second rotation θ, defining nutation angle

$$\begin{pmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{pmatrix}$$

In this case the matrix of transformation is

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

The angle θ is called the ***nutation angle***. The $X_2 = X_1$ axis is at the intersection of the X-Y and X_2-Y_2 planes and is called the ***line of nodes***.

(3) Third rotation (ψ) : Finally the third rotation is performed about $Z_2 = Z'$ axis through an angle ψ counter-clockwise so that X_2, Y_2 axes coincide $X_3 = X'$, $Y_3 = Y'$

Thus these three rotations ϕ, θ and ψ bring the space set of axes to coincide with body set of axes. The ϕ, θ and ψ are the Euler's angles and completely specify the orientation of the $X'\ Y'\ Z'$ system relative to the $X\ Y\ Z$ system. These ϕ, θ and ψ angles can be taken as three generalized coordinates. Now

$$\hat{\mathbf{i}}_3 = \hat{\mathbf{i}}' = \hat{\mathbf{i}}_2 \cos\psi + \hat{\mathbf{j}}_2 \sin\psi$$

$$\hat{\mathbf{j}}_3 = \hat{\mathbf{j}}' = -\hat{\mathbf{i}}_2 \sin\psi + \hat{\mathbf{j}}_2 \cos\psi$$

$$\hat{\mathbf{k}}_3 = \hat{\mathbf{k}}' = \hat{\mathbf{k}}_2$$

$$\begin{pmatrix} \hat{\mathbf{i}}' \\ \hat{\mathbf{j}}' \\ \hat{\mathbf{k}}' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{pmatrix}$$

So the transformation matrix is given by

$$\boldsymbol{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The angle Ψ is called the body angle.

The complete matrix of transformation A will be A = BCD.

$$\boldsymbol{A} = \boldsymbol{BCD} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

The inverse transformation matrix from body set of axes to space set of axes is given by $A^{-1} = A_T$ because $\boldsymbol{A}$ represents a proper orthogonal matrix. Thus

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix}$$

Moments and Products of Inertia

If a rigid body is taken as a rigid collection of particles then angular momentum is given as,

$$\vec{L} = \sum_i m\,(r_i \times v_i)$$

L_x, L_y, L_z are angular momentum component. ω_x, ω_y, ω_z are angular velocity.
$Let\ I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}, I_{yx}, I_{zy}, I_{xz}$ be nine coefficients. They are written as 3x3 matrix.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$L = I\,\vec{\omega}$$

When I operates on angular velocity vector $\vec{\omega}$ a physically different vector, the angular momentum $\vec{L}$ results.

Therefore I is termed as moment of inertia tensor.

$$I_{xx} = \sum m_i(r_i^2 - x_i^2) = \sum m_i(y_i^2 + z_i^2)$$

$$I_{yy} = \sum m_i(r_i^2 - y_i^2) = \sum m_i(z_i^2 + x_i^2)$$

$$I_{zz} = \sum m_i(r_i^2 - z_i^2) = \sum m_i(x_i^2 + y_i^2)$$

are called **moment of inertia** coefficients.

I_{xx} - moment of inertia of the body about x-axis.
I_{yy}- moment of inertia of the body about y-axis.
I_{zz} - moment of inertia of the body about z-axis.

$$I_{xy} = I_{yx} = -\sum m_i x_i y_i$$

$$I_{yz} = I_{zy} = -\sum m_i y_i z_i$$

$$I_{zx} = I_{xz} = -\sum m_i z_i x_i$$

are termed as **products of inertia**.

Euler's equation of motion for a rigid body

Lagrange's method

Euler's angles completely describe the orientation of the rigid body when it is rotating with one point fixed. The Euler's angle Φ, Θ, Ψ are taken as generalized coordinates and components of the applied torque as the generalized forces corresponding to these angles. The Lagrangian is given by,

$$L = T - V$$

$$L = T\left(\dot{\Phi}, \dot{\Theta}, \dot{\Psi}, \Phi, \Theta, \Psi\right) - V(\Phi, \Theta, \Psi)$$

$$L = \frac{1}{2}\left(I_1\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2\right) - V(\Phi, \Theta, \Psi) ----1$$

I_1, I_2, I_3 are the principal moment of inertia.

For a fixed point, the kinetic energy depends on Euler's angle Φ, Θ, Ψ via the angular velocity components along the principal axes x, y and z.

The angle Ψ happens to be the angle of rotation about the principal z-axis, so that angular velocity $\omega_\Psi = \dot{\Psi}$ and the generalized force or the z-component of torque is

$$N_Z = -\frac{\partial V}{\partial \Psi}$$

Therefore, the Lagrange's equation for Ψ- coordinate is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\Psi}}\right) - \frac{\partial T}{\partial \Psi} = -\frac{\partial V}{\partial \Psi} = N_Z ----2$$

For convenience, we reproduce here the angular velocity components expressed in terms of Euler's angles and kinetic energy expressed in terms of ω_x, ω_y , ω_z as referred to the principal axes. ω_x, ω_y , ω_z are angular velocity components.

$$\omega_x = \dot{\Phi}\sin\Theta\, sin\Psi + \dot{\Theta}\cos\Psi$$

$$\omega_y = \dot{\Phi}\sin\Theta\, cos\,\Psi - \dot{\Theta}\sin\Psi$$

$$\omega_z = \dot{\Phi} cos\,\Theta - \dot{\Psi}$$

Kinetic energy $T = \frac{1}{2}\left(I_1\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2\right)$

Therefore,

$$\frac{\partial\omega_x}{\partial\dot{\Psi}} = 0 \qquad \frac{\partial\omega_y}{\partial\dot{\Psi}} = 0 \qquad \frac{\partial\omega_z}{\partial\dot{\Psi}} = 1$$

$$\frac{\partial\omega_x}{\partial\Psi} = \omega_y \qquad \frac{\partial\omega_y}{\partial\Psi} = -\omega_x \qquad \frac{\partial\omega_z}{\partial\Psi} = 0$$

$$\frac{\partial \mathrm{T}}{\partial\dot{\Psi}} = \frac{\partial \mathrm{T}}{\partial\omega_x}\frac{\partial\omega_x}{\partial\dot{\Psi}} + \frac{\partial \mathrm{T}}{\partial\omega_y}\frac{\partial\omega_y}{\partial\dot{\Psi}} + \frac{\partial \mathrm{T}}{\partial\omega_z}\frac{\partial\omega_z}{\partial\dot{\Psi}}$$

$$\frac{\partial \mathrm{T}}{\partial\dot{\Psi}} = \mathrm{I}_1\omega_x.0 + \mathrm{I}_2\omega_y.0 + \mathrm{I}_3\omega_z.1$$

$$\frac{\partial \mathrm{T}}{\partial\dot{\Psi}} = \mathrm{I}_3\omega_z - - - -3$$

Similarly,

$$\frac{\partial \mathrm{T}}{\partial\Psi} = \frac{\partial \mathrm{T}}{\partial\omega_x}\frac{\partial\omega_x}{\partial\Psi} + \frac{\partial \mathrm{T}}{\partial\omega_y}\frac{\partial\omega_y}{\partial\Psi} + \frac{\partial \mathrm{T}}{\partial\omega_z}\frac{\partial\omega_z}{\partial\Psi}$$

$$\frac{\partial \mathrm{T}}{\partial\Psi} = \mathrm{I}_1\omega_x\omega_y + \mathrm{I}_2\omega_y(-\omega_x) + \mathrm{I}_3\omega_z.0$$

$$\frac{\partial \mathrm{T}}{\partial\Psi} = (\mathrm{I}_1 - \mathrm{I}_2)\omega_x\omega_y - - - - - - -4$$

Substitute eqn. 3 and 4 in 2

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{I}_3\omega_z) - (\mathrm{I}_1 - \mathrm{I}_2)\omega_x\omega_y = N_Z$$

$$\mathrm{I}_3\dot{\omega}_z - \omega_x\omega_y(\mathrm{I}_1 - \mathrm{I}_2) = N_Z - - - 5$$

Similarly by cyclic permutation,

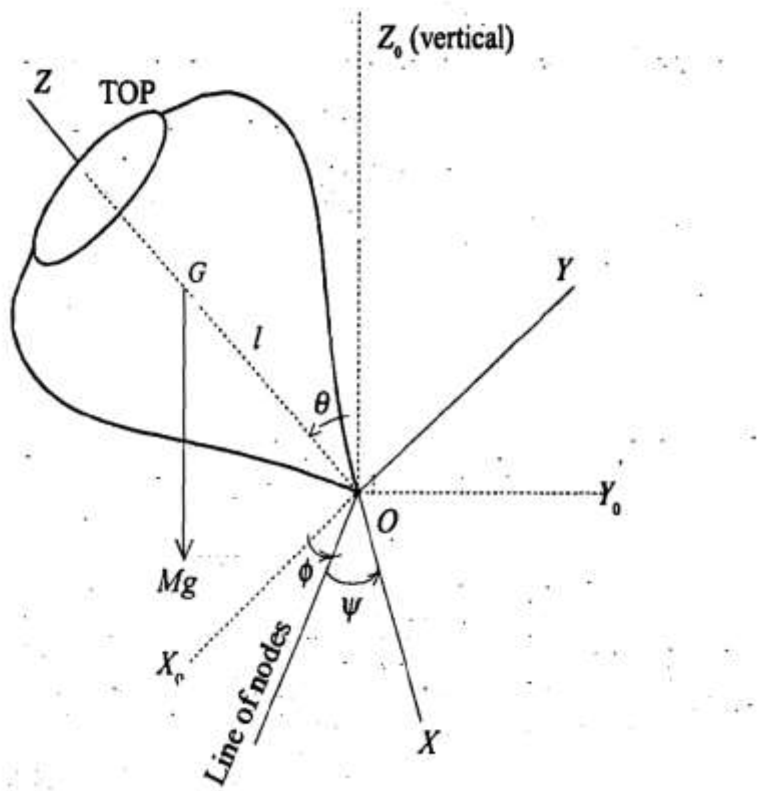
$$\mathrm{I}_1\dot{\omega}_x - \omega_y\omega_z(\mathrm{I}_2 - \mathrm{I}_3) = N_x - - - 6$$

$$\mathrm{I}_2\dot{\omega}_y - \omega_z\omega_x(\mathrm{I}_3 - \mathrm{I}_1) = N_y - - - 7$$

Equations 5,6 and 7 are known as Euler’s equation.

MOTION OF A HEAVY SYMMETRICAL TOP

Let us consider a spinning symmetrical top in a uniform gravitational field with one point O on the symmetry axis fixed in space. Such a top is called a heavy symmetrical top and its examples are child's top, gyroscope etc. Let G be the centre of gravity of the top and l be the distance from the fixed point O to $C.G.$ We take the symmetry axis as one of the principal axes and choose it Z-axis fixed in the body so that X, Y are the other two principal axes [**Fig.1**] and $I_1 = I_2$. The force acting on the top is Mg, the force due to gravity. Let $X_0 Y_0 Z_0$ be the fixed set of axes; X_0 and Y_0 are in the horizontal plane and Z_0 is vertical. As O is the fixed point of the top, the motion can be described in terms of the three Euler's angles ϕ, θ and ψ:



Euler's angles specifying the orientation of a heavy symmetrical top ($I_1 = I_2$)

(i) θ is the angle of inclination of Z-axis from the vertical (Z_0-axis).

(ii) ϕ is the azimuth of the top about vertical (Z_0-axis), *i.e.*, the angle in the horizontal plane between X_0 and line of nodes, and

(iii) ψ is the rotation angle of the top about its own Z-axis i.e., the angle between line of nodes and X-axis (body axis).

The Lagrangian for the top is

$$L = T - V = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 - Mgl\cos\theta$$

where I_3 is the principal moment of inertia about the symmetry axis.

From Euler's geometrical equation,

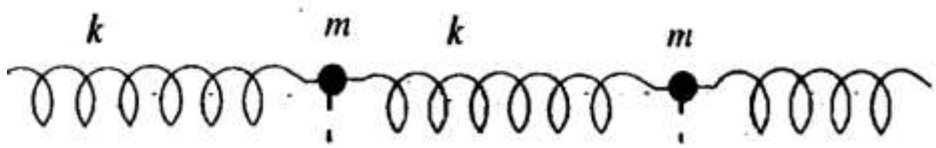
$$\omega_1^2 + \omega_2^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \text{ and } \omega_3^2 = (\dot{\psi} + \dot{\phi}\cos\theta)^2$$

So that $$L = \frac{1}{2} I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta$$

Theory of Small Oscillations

Normal modes and frequencies

Consider three springs arranged as shown below.



The two masses are equal. Let x_1 and x_2 be displacements of the left and right masses. The middle spring is stretched or compressed by $x_2 - x_1$.

Solving for x_1 and x_2 we get,

$$x_1 = A_1\cos(\omega_1 t + \Phi_1) + A_2\cos(\omega_2 t + \Phi_2)$$

$$x_2 = A_1\cos(\omega_1 t + \Phi_1) - A_2\cos(\omega_2 t + \Phi_2)$$

A_1 *and* A_2 are amplitudes of mode 1 and mode 2 respectively.

Φ_1 *and* Φ_2 are phase constants of mode 1 and mode 2.

If $A_2 = 0$

$$x_1 = A_1\cos(\omega_1 t + \Phi_1)$$

$$x_2 = A_1\cos(\omega_1 t + \Phi_1)$$

If $A_1 = 0$

$$x_1 = A_2\cos(\omega_2 t + \Phi_2)$$

$$x_2 = -A_2\cos(\omega_2 t + \Phi_2)$$

Thus if $A_2 = 0$, the two masses oscillate together in phase with frequency ω_1.

If $A_1 = 0$, the two masses oscillate with frequency ω_2 opposite to each other.. that is out of phase by π.

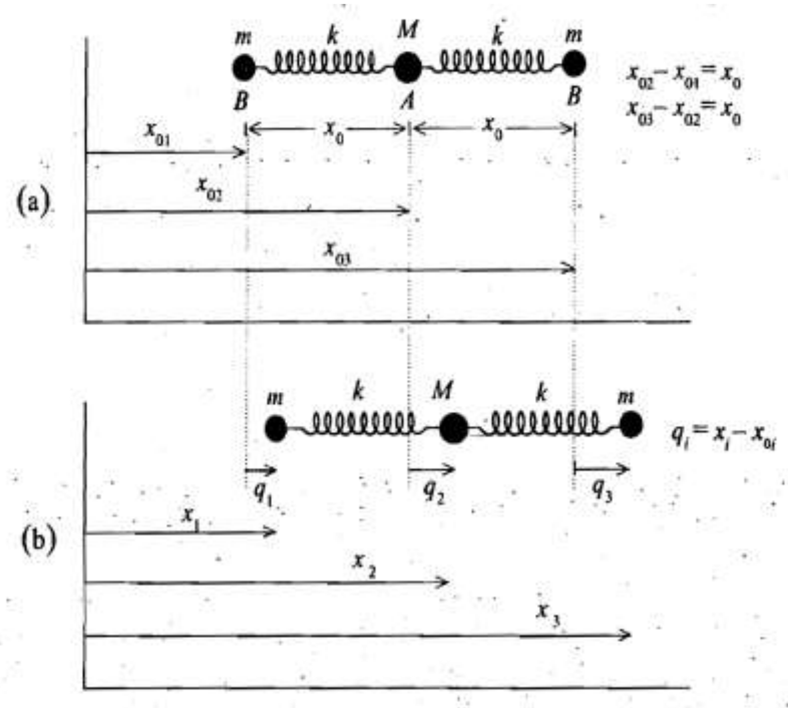
"the two such modes of oscillation involving a single frequency are called normal modes of vibration"

For a given normal mode, all the coordinates (x_1 and x_2) oscillate with same frequency.

ω_1 *and* ω_2 are known as normal frequency.

Linear triatomic molecule

Let us consider a linear triatomic molecule of the type AB_2 (*e.g.*, CO_2) in which A atom is in the middle and B atoms are at the ends [Fig. ...]. The mass of A atom is M and that of each of the B atom is m. The interatomic force between A and B atom is approximated by elastic force of spring force constant k. The motion of the three atoms is constrained along the line joining them. There are three coordinates marking the positions of three atoms on the line. If x_1, x_2 and x_3 are the positions of the three atoms at any instant from some arbitrary origin, then



Longitudinal oscillations of a linear symmetric triatomic molecule
(a) Equilibrium configuration, (b) Configuration at any instatnt *t*

$$T = \frac{1}{2} m \left[\dot{x}_1^2 + \dot{x}_3^2\right] + \frac{1}{2} M\dot{x}_2^2$$

and
$$V = \frac{1}{2} k (x_2 - x_1 - x_0)^2 + \frac{1}{2} k (x_3 - x_2 - x_0)^2$$

where x_0 is the distance between any A and B atoms in the equilibrium configuration.

Let us define the generalized coordinates as

$$q_1 = x_1 - x_{01}, q_2 = x_2 - x_{02}, q_3 = x_3 - x_{03},$$

where
$$x_{02} - x_{01} = x_{03} - x_{01} = x_0.$$

Then
$$T = \frac{1}{2} m [\dot{q}_1^2 + \dot{q}_3^2] + \frac{1}{2} M\dot{q}_2^2$$

and
$$V = \frac{1}{2} k(q_2 - q_1)^2 + \frac{1}{2} k(q_3 - q_2)^2$$

Thus the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices are

$$\boldsymbol{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \text{ and } \boldsymbol{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

The secular equation is

$$\left|\boldsymbol{V} - \omega^2 \boldsymbol{I}\right| = \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

whence
$$\omega^2 (k - m\omega^2)[k(M + 2m) - \omega^2 Mm] = 0$$

The solutions of this equation are

$$\omega_1 = 0,\ \omega_2 = \sqrt{\frac{k}{m}} \text{ and } \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$$

The first eigen value $\omega_1 = 0$ corresponds to non-oscillatory motion and refers to translatory motion of the molecule as a whole rigidly.

To determine the eigenvectors, we use the equation

$$(V-\omega_k^2 T)\,a_k=0 \text{ or } \begin{pmatrix} k-m\omega_k^2 & -k & 0 \\ -k & 2k-M\omega_k^2 & -k \\ 0 & -k & k-m\omega_k^2 \end{pmatrix}\begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix}=0$$

Let us now discuss the eigen vectors for the three modes of vibrations.

(1) For $\omega_1 = 0$,

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}=0$$

or $$a_{11}-a_{21}=0, -a_{11}+2a_{21}-a_{31}=0,\ -a_{21}+a_{31}=0$$

or $$a_{11}=a_{21}=a_{31}=\alpha \quad \text{(say)}.$$

Thus for $\omega_1 = 0$, the eigen vector is given by

$$a_1=\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$$

(2) For $\omega_2=\sqrt{k/m}$,

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k-\dfrac{Mk}{m} & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}=0$$

or $$a_{22}=0,\ -a_{12}-a_{32}=0$$

Therefore, $a_{22}=0,\ a_{12}=-a_{32}=\beta$ (say)

Thus, for $\omega_2=\sqrt{k/m},\ a_2=\begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix}$

(3) For $$\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)},$$

$$\begin{pmatrix} -\frac{2mk}{M} & -k & 0 \\ -k & -\frac{kM}{m} & -k \\ 0 & -k & -\frac{2mk}{M} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

which yields

$$\frac{2m}{M}a_{13} + a_{23} = 0\,,\ a_{13} + \frac{M}{m}a_{23} + a_{33} = 0,\ \ a_{23} + \frac{2m}{M}a_{33} = 0$$

Therefore, $a_{13} = a_{33} = \gamma$ (say) and $a_{23} = -(2m/M)\gamma$

Thus for $$\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)},\ \boldsymbol{a}_3 = \begin{pmatrix} \gamma \\ -\frac{2m}{M}\gamma \\ \gamma \end{pmatrix}$$

Now, the $\boldsymbol{A}$ matrix is

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

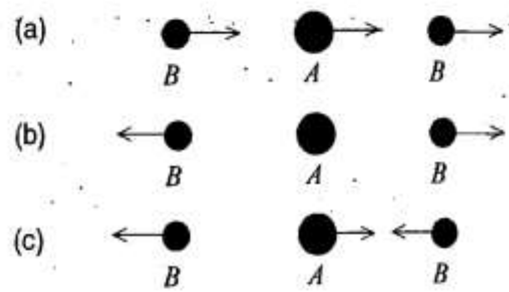
We impose the condition

$$\overline{\boldsymbol{A}}\ \boldsymbol{T}\boldsymbol{A} = \boldsymbol{I}$$

i.e., $$\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -\frac{2m}{M}\gamma & \gamma \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or $$\begin{pmatrix} \alpha^2(2m+M) & 0 & 0 \\ 0 & 2\beta^2 m & 0 \\ 0 & 0 & 2\gamma^2 m\left(1+\frac{2m}{M}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $$\alpha = \frac{1}{\sqrt{2m+M}},\ \beta = \frac{1}{\sqrt{2m}},\gamma = \frac{1}{\sqrt{2m(1+2m/M)}}.$$



Longitudinal normal modes of the triatomic molecule :

(a) Mode 1, all the three atoms are displaced equally in the same direction,

(b) Mode 2, A atom does not vibrate and B atoms oscillate with equal amplitudes but in opposite directions,

(c) B atoms vibrate in phase with equal amplitudes and the middle atom A vibrates in opposite phase with different amplitude.

UNIT – IV

HAMILTON'S FORMULATION

Hamilton's canonical equations of motion

The Hamiltonian, in general, is a function of generalized coordinates q_k, generalized momenta p_k and time t, *i.e.*,

$$H = H(q_1, q_2, \ldots, q_k, \ldots, q_n, p_1, p_2, \ldots, p_k, \ldots, p_n, t)$$

We may write the differential dH as

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \longrightarrow 1$$

$H = \sum_k p_k \dot{q}_k - L$ and hence

$$dH = \sum_k \dot{q}_k \, dp_k + \sum_k p_k \, d\dot{q}_k - dL \longrightarrow 2$$

Also, $L = L(q_1, q_2, \ldots, q_k, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k, \ldots, \dot{q}_n, t)$

Therefore, $$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

But $\dot{p}_k = \dfrac{\partial L}{\partial q_k}$ and $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$

Therefore, $$dL = \sum_k \dot{p}_k \, dq_k + \sum_k p_k \, d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

Substituting for dL

$$dH = \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k - \frac{\partial L}{\partial t} dt \longrightarrow 3$$

Comparing the coefficients of dp_k, dq_k and dt in eqs. **1 and 3**

$$\left.\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}\right\}$$

This equation is known as Hamilton's equations or Hamilton's canonical equations of motion.

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

DEDUCTION OF HAMILTON'S EQUATIONS FROM VARIATIONAL PRINCIPLE

According to Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

where $L = T - V = L(q_k, \dot{q}_k, t)$.

in terms of Hamiltonian H

$$H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Hence the Hamilton's principle in the new form is obtained as

$$\delta \int_{t_1}^{t_2} \left(\sum_k p_k \dot{q}_k - H \right) dt = 0$$

This is known as ***modified Hamilton's principle***.

The δ-variation of q_k and p_k coordinates at constant t can be expressed in terms of a parameter α common to all points of the path of integration in phase space [(similar to eq.(3)] as

$$\delta q_k = \frac{\partial q_k}{\partial \alpha}\delta\alpha = \eta_k\,\delta\alpha \text{ and } \delta p_k = \frac{\partial p_k}{\partial \alpha}\delta\alpha = \eta'_k\,\delta\alpha$$

where η_k and η'_k are arbitrary subject to the conditions

$$\eta_k(t_1) = \eta_k(t_2) = \eta'_k(t_1) = \eta'_k(t_2) = 0$$

Therefore, the δ-variation of the integral is

$$\delta\int_{t_1}^{t_2}\left[\sum_k p_k\dot{q}_k - H\right]dt = \int_{t_1}^{t_2}\sum_k\left[\left(\frac{\partial p_k}{\partial\alpha}\dot{q}_k + p_k\frac{\partial \dot{q}_k}{\partial \alpha}\right)\delta\alpha - \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial\alpha}\delta\alpha - \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \alpha}\delta\alpha\right]dt$$

$$= \delta\alpha\int_{t_1}^{t_2}\sum_k\left[\eta'_k\,\dot{q}_k + p_k\frac{\partial\dot{q}_k}{\partial\alpha} - \eta_k\frac{\partial H}{\partial q_k} - \eta'_k\frac{\partial H}{\partial p_k}\right]dt$$

But $$\int_{t_1}^{t_2} p_k\frac{\partial\dot{q}_k}{\partial\alpha}dt = \int_{t_1}^{t_2} p_k\frac{d}{dt}\left[\frac{\partial q_k}{\partial\alpha}\right]dt = \int_{t_1}^{t_2} p_k\frac{d\eta_k}{dt}dt$$

$$= \left[p_k\eta_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\dot{p}_k\eta_k dt = -\int_{t_1}^{t_2}\dot{p}_k\eta_k dt \quad [\because\ \eta_k(t_1) = \eta_k(t_2) = 0]$$

Also in view the modified Hamilton's principle [eq. (25)], the δ–variation of the integral must be zero. Therefore, we obtain from (28)

$$\delta\alpha\int_{t_1}^{t_2}\sum_k\left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\eta'_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\eta_k\right]dt = 0$$

or $$\int_{t_1}^{t_2}\sum_k\left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\delta q_k\right]dt = 0$$

Since $\dot{q}_k$ and p_k are independent variables, the integral will be zero, when

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0 \text{ and } \dot{p}_k + \frac{\partial H}{\partial q_k} = 0 \text{ or } \dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

These are the desired ***Hamilton's canonical equations.***

PRINCIPLE OF LEAST ACTION

According to the principle of least action for a conservative system

$$\Delta\int_{t_1}^{t_2}\sum_k p_k\dot{q}_k dt = 0 \longrightarrow 1$$

where the quantity $W = \int_{t_1}^{t_2}\sum_k p_k\dot{q}_k dt$ is sometimes called ***abbreviated action.***

Eq. 1 was established by Maupertuis (1668-1759) and therefore it is usually referred ***Maupertuis principle of least action.***

Proof : Let us consider Hamilton's principle function (or action integral) S, given by

$$S = \int_{t_1}^{t_2} L\, dt$$

The Δ-variation of S is

$$\Delta S = \Delta\int_{t_1}^{t_2} L\, dt = \left[\delta + \Delta t\frac{d}{dt}\right]\int_{t_1}^{t_2} L\, dt$$

$$= \delta\int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2}\Delta t\, d(L) = \delta\int_{t_1}^{t_2} L\, dt + [L\,\Delta t]_{t_1}^{t_2} = \int_{t_1}^{t_2}\delta L\, dt + [L\Delta t]_{t_1}^{t_2}\ [\because \delta(dt) = 0]$$

$$= \int_{t_1}^{t_2}\sum_k\left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k\right]dt + [L\,\Delta t]_{t_1}^{t_2} \longrightarrow 2$$

In the present case $\delta q_k \neq 0$ at the end points, hence $\delta\int_{t_1}^{t_2} L\, dt$ is not equal to zero. Now, according to Lagrange's equations, we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \text{ or } \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] \longrightarrow 3$$

Also

$$\delta\dot{q}_k = \frac{d}{dt}[\delta q_k] \longrightarrow 4$$

Using 2 and 3 the quantity in the first term of eqn 1 is

$$\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right]\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}[\delta q_k] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right] = \frac{d}{dt}[p_k\,\delta q_k] \longrightarrow 5$$

Δ-operation is

$$\Delta = \delta + \Delta t \frac{d}{dt}$$

$$\Delta q_k = \delta q_k + \Delta t \frac{dq_k}{dt} \text{ or } \delta q_k = \Delta q_k - \Delta t \dot{q}_k \text{ or } p_k \delta q_k = p_k \Delta q_k - p_k \dot{q}_k \Delta t$$

Hence $$\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \frac{d}{dt}\left[p_k \Delta q_k\right] - \frac{d}{dt}\left[p_k \dot{q}_k \Delta t\right]$$

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt}\left[p_k \Delta q_k\right] - \frac{d}{dt}\left[p_k \dot{q}_k \Delta t\right]\right] dt + \left[L \Delta t\right]_{t_1}^{t_2}$$

$$= \sum_k \int_{t_1}^{t_2} \left[d\left(p_k \Delta q_k\right) - d\left(p_k \dot{q}_k \Delta t\right)\right] + \left[L \Delta t\right]_{t_1}^{t_2}$$

$$= \sum_k \left[p_k \Delta q_k\right]_{t_1}^{t_2} - \sum_k \left[p_k \dot{q}_k \Delta t\right]_{t_1}^{t_2} + \left[L \Delta t\right]_{t_1}^{t_2}$$

As $\Delta q_k = 0$ at the end points, $\left[p_k \Delta q_k\right]_{t_1}^{t_2} = 0$.

$$\Delta \int_{t_1}^{t_2} L dt = \left[\left(L - \sum_k p_k \dot{q}_k\right)\Delta t\right]_{t_1}^{t_2}$$

or $$\Delta \int_{t_1}^{t_2} L\, dt = -\left[H \Delta t\right]_{t_1}^{t_2} \qquad \left[\because H = \sum_k p_k \dot{q}_k - L\right]$$

Now, if we restrict to systems for which $\partial H/\partial t = 0$ and to variations for which H remains constant (conservative systems), then

$$\Delta \int_{t_1}^{t_2} H\, dt = \int_{t_1}^{t_2} H d(\Delta t) = \left[H \Delta t\right]_{t_1}^{t_2}$$

Substituting for $\left[H \Delta t\right]_{t_1}^{t_2}$ i

$$\Delta \int_{t_1}^{t_2} L\, dt = -\Delta \int_{t_1}^{t_2} H\, dt \text{ or } \Delta \int_{t_1}^{t_2} [H + L]\, dt = 0$$

or $$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad \left[\because H = \sum_k p_k \dot{q}_k - L\right].$$

This is what is known as ***principle of least action.***

CANONICAL TRANSFORMATION

In several problems, we may need to change one set of position and momentum coordinates into another set of position and momentum coordinates. Suppose that q_k and p_k are the old position and momentum coordinates and Q_k and P_k are the new ones. Let these coordinates be related by the following transformations :

$$\left.\begin{aligned} P_k &= P_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \\ \text{and} \quad Q_k &= Q_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \end{aligned}\right\} \qquad \ldots(1)$$

Now, if there exists a Hamiltonian H' in the new coordinates such that

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad \text{and} \quad \dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad \ldots(2)$$

where $$H' = \sum_{k=1}^{n} P_k \dot{Q}_k - L' \qquad \ldots(3)$$

and L' substituted in the Hamilton's principle

$$\delta \int L' dt = 0 \qquad \ldots(4)$$

gives the correct equations of motion in terms of the new coordinates P_k and Q_k, then the transformations (1) are known as *canonical* (or *contact*) *transformations.*

Legendre Transformations

This is a mathematical technique used to change the basis from one set of coordinates to another. If $f(x, y)$ is a function of two variables x and y, then the differential of this function can be written as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \text{ or } df = u\,dx + v\,dy \qquad \ldots(5)$$

where $$u = \partial f / \partial x \quad \text{and} \quad v = \partial f / \partial y \qquad \ldots(6)$$

Now, we want to change the basis from (x, y) to (u, v) so that u is now an independent variable and x is a dependent one. Let f' be a function of u and y such that

$$f' = f - ux \qquad \ldots(7)$$

Then, $$df' = df - u\,dx - x\,du$$

Substituting for df from (5), we get

$$df' = u\,dx + v\,dy - u\,dx - x\,du$$

or $$df' = v\,dy - x\,du \qquad ...(8)$$

But f' is a function of u and y, therefore

$$\therefore\ df' = \frac{\partial f'}{\partial u}du + \frac{\partial f'}{\partial y}dy \qquad ...(9)$$

Comparing eqs. (8) and (9), we get

$$x = -\frac{\partial f'}{\partial u} \text{ and } v = \frac{\partial f'}{\partial y} \qquad ...(10)$$

$\therefore$ *These are the necessary relations for* ***Legendre transformations.***

Generating Functions

For canonical transformations, the Lagrangian L in p_k, q_k coordinates and L' in P_k, Q_k coordinates must satisfy the Hamilton's principle, *i.e.*,

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \quad \text{and} \quad \delta\int_{t_1}^{t_2} L'dt = 0 \qquad ...(11)$$

But $$L = \sum_{k=1}^{n} p_k\,\dot{q}_k - H \quad \text{and} \quad L' = \sum_{k=1}^{n} P_k\,\dot{Q}_k - H',$$

therefore, $$\delta\int_{t_1}^{t_2}\left[\sum_k p_k\,\dot{q}_k - H\right]dt = 0 \qquad ...(12)$$

and $$\delta\int_{t_1}^{t_2}\left[\sum_k P_k\,\dot{Q}_k - H'\right]dt = 0 \qquad ...(13)$$

Subtracting eq. (13) from eq. (12), we get

$$\delta\int_{t_1}^{t_2}\left[\left(\sum_k p_k\,\dot{q}_k - H\right) - \left(\sum_k P_k\,\dot{Q}_k - H'\right)\right]dt = 0 \qquad ...(14)$$

In δ-variation process, the condition $\delta\int f\,dt = 0$ is to be satisfied, in general, by $f = dF/dt$, where F is an arbitrary function. Therefore,

$$\delta\int_{t_1}^{t_2}\frac{dF}{dt}dt = 0 \qquad ...(15)$$

where $$dF/dt = L - L' \qquad ...(16\text{ a})$$

or $$\frac{dF}{dt} = \left(\sum_k p_k\,\dot{q}_k - H\right) - \left(P_k\,\dot{Q}_k - H'\right) \qquad ...(16\text{b})$$

The function F is known as the generating function. The meaning of the name will be clear later on. The first bracket in (16) is a function of p_k, q_k and t and the second as a function of P_k, Q_k and t. F is therefore, in general, a function of $(4n+1)$ variables p_k, q_k, P_k, Q_k and t. It is to be remembered that the variables are subjected to the transformation equations (1) and therefore F may be regarded as the function of $(2n+1)$ variables, comprising t and any $2n$ of the p_k, q_k, P_k, Q_k. Thus we see that F can be written as a function of $(2n+1)$ independent variables in the following four forms :

(*i*) $F_1(q_k, Q_k, t)$, (*ii*) $F_2(q_k, P_k, t)$, ...(17)

(*iii*) $F_3(p_k, Q_k, t)$, and (*iv*) $F_4(p_k, P_k, t)$

The choice of the functional form of the generating function F depends on the problem under consideration.

Case I : If we choose the form (*i*), *i.e.*,

$$F_1 = F_1(q_1, q_2, ..., q_k, ..., q_n, Q_1, Q_2, ..., Q_k, ..., Q_n, t) \quad ...(18)$$

then
$$\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k}\dot{Q}_k + \frac{\partial F_1}{\partial t} \quad ...(19)$$

Subtracting (19) from (16 b), we can write

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)\dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)\dot{Q}_k + H' - H - \frac{\partial F_1}{\partial t} = 0$$

or
$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)dq_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)dQ_k + \left[H' - H - \frac{\partial F_1}{\partial t}\right]dt = 0 \quad ...(20)$$

As q_k, Q_k and t may be regarded as independent variables,

$$p_k = \frac{\partial}{\partial q_k}F_1(q_k, Q_k, t),\ P_k = -\frac{\partial}{\partial Q_k}F_1(q_k, Q_k, t)$$

and
$$H' - H = \frac{\partial}{\partial t}F_1(q_k, Q_k, t) \quad ...(21)$$

In principle, first equation of (21) may be solved to give

$$Q_k = Q_k(q_k, p_k, t) \quad ...(22)$$

Substituting this in the second equation of (21), one gets

$$P_k = P_k(q_k, p_k, t) \quad ...(23)$$

In fact, these are the transformation equations (1). Thus we find that transformation equations can be derived from a knowledge of the function F. This is why F is known as the ***generating function of the transformation***.

Case II : If the generating function is of the type $F_2(q_k, P_k, t)$, then it can be dealt with by affecting a Legendre transformation of $F_1(q_k, Q_k, t)$.

In case of Legendre transformation (7) :

$$f' = f - ux, \text{ where } u = \partial f/\partial x$$

Here, since $P_k = -\partial F_1/\partial Q_k$, we have $u = -P_k$, $x = Q_k$, $f' = F_2$ and $f = F_1$.

Therefore ,
$$F_2(q_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k \quad ...(24)$$

Evidently, F_2 is independent of Q_k variables, because

$$\frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0 \text{ as } \frac{\partial F_1}{\partial Q_k} = -P_k \text{ in (21).}$$

Using eq.(16)

$$\left(\sum_k p_k\dot{q}_k - H\right) - \left(\sum_k P_k\dot{Q}_k - H'\right) = \frac{dF_1}{dt} = \frac{d}{dt}\left[F_2 - \sum_k P_k Q_k\right]$$

or
$$\frac{dF_2}{dt} = \sum_k p_k\dot{q}_k + \sum_k Q_k\dot{P}_k + H' - H \quad ...(25)$$

Total time derivative of $F_2(q_k, P_k, t)$ is

$$\frac{dF_2}{dt} = \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k + \frac{\partial F_2}{\partial t} \quad ...(26)$$

From (25) and (26), we get

$$p_k = \frac{\partial F_2}{\partial q_k},\ Q_k = \frac{\partial F_2}{\partial P_k} \text{ and } H' - H = \frac{\partial F_2}{\partial t} \quad ...(27)$$

If we look (21) and (27), we find $\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t}$. Further as $\frac{\partial F_1}{\partial q_k} = \frac{\partial F_2}{\partial q_k}$, first equation of (21) and that of (27) are identical. Second equation of (27) appears to be different from the second equation of (21), but in fact it is a rearrangement of it.

Case III : We can again relate the third type of generating function $F_3(p_k, Q_k, t)$ to F_1 by a Legendre transformation in view of the relation $p_k = \partial F_1/\partial q_k$. Here $u = p_k$, $x = q_k$, $f' = F_3$ and $f = F_1$. Therefore,

$$F_3(p_k, Q_k, t) = F_1(q_k, Q_k, t) - \sum_k p_k q_k \quad ...(28)$$

or
$$F_1(q_k, Q_k, t) = F_3(p_k, Q_k, t) + \sum_k p_k q_k$$

Using eq. (16), we have

$$\left(\sum_k p_k\dot{q}_k - H\right) - \left(\sum_k P_k\dot{Q}_k - H'\right) = \frac{dF_1}{dt} = \frac{d}{dt}\left(F_3 + \sum p_k q_k\right)$$

or
$$\frac{dF_3}{dt} = -\sum_k \dot{p}_k q_k - \sum_k P_k\dot{Q}_k + H' - H$$

Also,
$$\frac{dF_3}{dt} = \sum_k \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \frac{\partial F_3}{\partial t}$$

Therefore, the new transformation equations are

$$q_k = -\frac{\partial F_3}{\partial p_k},\ P_k = -\frac{\partial F_3}{\partial Q_k} \text{ and } H' - H = \frac{\partial F_3}{\partial t} \quad ...(29)$$

Case IV : Using Legendre transformations, the generating function $F_4(p_k, P_k, t)$ can be connected to $F_1(q_k, Q_k, t)$ as

$$F_4(p_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k - \sum_k p_k q_k \qquad ...(30)$$

Using eq. (16), we have

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{d}{dt}\left(F_4 - \sum_k P_k Q_k + \sum_k p_k q_k\right)$$

or
$$\frac{dF_4}{dt} = -\sum_k q_k \dot{p}_k + \sum_k Q_k \dot{P}_k + H' - H$$

But
$$\frac{dF_4}{dt} = \sum_k \frac{\partial F_4}{\partial p_k} \dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k} \dot{P}_k + \frac{\partial F_4}{\partial t}$$

A comparison of the above two equations gives the fourth set of transformation equations:

$$q_k = -\frac{\partial F_4}{\partial p_k},\ Q_k = \frac{\partial F_4}{\partial P_k},\ H' - H = \frac{\partial F_4}{\partial t} \qquad ...(31)$$

POISSON'S BRACKET

we have shown that in the case of infinitesimal contact transformations, the changes in the conjugate variables p_k and q_k are given by

$$\delta q_k = \in \frac{\partial G}{\partial p_k} \text{ and } \delta p_k = -\in \frac{\partial G}{\partial q_k} \qquad ...(1)$$

where $\in$ is an infinitesimal parameter and the generating function $G(q_k, p_k)$ is arbitrary. Now let us consider some function $F(q_k, p_k)$. The change in the value of $F(q_k, p_k)$ with the changes δq_k and δp_k in the coordinates q_k and p_k respectively can be expressed as

$$\delta F = \sum_k \left(\frac{\partial F}{\partial q_k}\delta q_k + \frac{\partial F}{\partial p_k}\delta p_k\right) \qquad ...(2)$$

If the transformation (1), generated by the function G, is applied, we get

$$\delta F = \sum_k \left[\frac{\partial F}{\partial q_k}\left(\in \frac{\partial G}{\partial p_k}\right) + \frac{\partial F}{\partial p_k}\left(-\in \frac{\partial G}{\partial q_k}\right)\right]$$

Since the parameter $\in$ is independent of q_k and p_k, we have

$$\delta F = \in \left[\sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right)\right] \qquad ...(3)$$

The quantity in the big bracket in (3) is called the ***Poisson bracket*** of two functions or dynamical variables $F(q_k, p_k)$ and $G(q_k, p_k)$ and is denoted by $[F, G]$. This definition of Poisson bracket is true for F and G, being functions of time. Thus

$$\delta F = \in [F, G] \qquad ...(4)$$

If the functions F and G depend upon the position coordinates q_k, momentum coordinates p_k and time t, the Poisson bracket of F and G is defined as

$$[F,G]_{q,p} = \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right) \qquad ...(5)$$

For brevity, we may drop the subscripts q, p and write the Poisson bracket as $[F, G]$.

The total time derivative of the function F can be written as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial p_k}\dot{p}_k\right) \qquad ...(6)$$

Using, Hamilton's equations $\dot{q}_k = \frac{\partial H}{\partial p_k}$ and $-\dot{p}_k = \frac{\partial H}{\partial q_k}$, eq. (6) is obtained to be

$$\frac{dF}{dt} = \dot{F} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \qquad ...(7)$$

In view of the definition of Poisson's bracket given by eq. (5), we obtain

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F,H] \qquad ...(8)$$

From this equation we see that the function F is a constant of motion, if

$$\frac{dF}{dt} = 0 \quad \text{or} \quad \frac{\partial F}{\partial t} + [F,H] = 0 \qquad ...(9)$$

Now, if the function F does not depend on time explicitly, $\frac{\partial F}{\partial t} = 0$ and then the condition for F to be constant of motion is obtained to be

$$[F,H] = 0 \qquad ...(10)$$

Thus *if a function F does not depend on time explicitly and is a constant of motion, its Poisson bracket with the Hamiltonian vanishes*. In other words, *a function whose Poisson bracket with Hamiltonian vanishes is a constant of motion*. This result does not depend whether H itself is constant of motion.

Equations of motion in Poisson bracket form : Special cases of (8) are

(1) $F = q_k, \quad \dot{q}_k = [q_k, H]$ ...(11*a*)

(2) $F = p_k, \quad \dot{p}_k = [p_k, H]$ ...(11*b*)

(3) $F = H, \quad \dot{H} = \frac{\partial H}{\partial t}$ ...(11*c*)

Properties of Poisson brackets and Fundamental Poission brackets : The Poisson bracket has the property of antisymmetry, given by

$$[F, G] = -[G, F] \quad ...(12)$$

because $$[F,G] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right] = -\sum_k \left[\frac{\partial G}{\partial q_k}\frac{\partial F}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k}\right] = -[G,F]$$

Thus Poisson bracket does not obey the commutative law of algebra. As an application of the Poisson brackets, we are giving below some of the special cases :

(1) When $G = q_l$,

$$[F,q_l] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial q_l}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial q_l}{\partial q_k}\right] = -\sum_k \frac{\partial F}{\partial p_k}\delta_{lk}$$

or $$[F,q_l] = -\frac{\partial F}{\partial p_l} \quad ...(13)$$

Also if $F = q_k$, $[q_k, q_l] = -\frac{\partial q_k}{\partial p_l} = 0$ ...(14)

and if $F = p_k$, $[p_k, q_l] = -\frac{\partial p_k}{\partial p_l} = -\delta_{kl}$ ...(15)

(2) When $G = p_l$, $[F, p_l] = \sum_k \frac{\partial F}{\partial q_k}\delta_{kl}$

or $$[F,p_l] = \frac{\partial F}{\partial q_l} \quad ...(16)$$

For $F = p_k$, $[p_k, p_l] = \frac{\partial p_k}{\partial q_l} = 0$ ...(17)

and for $F = q_k$, $[q_k, p_l] = \frac{\partial q_k}{\partial q_l} = \delta_{kl}$ ...(18)

The above results can be summarized as follows :

$$[q_k, q_l] = [p_k, p_l] = 0 \quad ...(19)$$

and $$[q_k, p_l] = \delta_{kl} \quad ...(20)$$

where δ_{kl} is the kronecker delta symbol with the property

$$\delta_{kl} = 0 \text{ for } k \neq 1 \text{ and } \delta_{kl} = 1 \text{ for } k = l$$

Equations (19) and (20) are called the ***fundamental Poisson's brackets.***

Further from the definition of Poisson bracket of any two dynamical variables or functions, one can obtain the following identities :

(i) $[F, F] = 0$ ...(21)

(ii) $[F, C] = 0$, C = constant ...(22)

(iii) $[CF, G] = C[F, G]$ ...(23)

(iv) $[F_1 + F_2, G] = [F_1, G] + [F_2, G]$ ...(24)

(v) $[F, G_1 G_2] = G_1[F, G_2] + [F, G_1]G_2$ ...(25)

(vi) $\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$ ...(26)

(vii) $[F, [G, K]] + [G, [K, F]] + [K, [F, G]] = 0$ (Jacobi's identity) ...(27)

INVARIANCE OF POISSON BRACKET WITH RESPECT TO CANONICAL TRANSFORMATION

Poisson brackets are invariant under canonical transformations. First we shall prove this statement for fundamental Poisson brackets and then in general.

Fundamental Poisson brackets under canonical transformation : The fundamental Poisson brackets are invariant under canonical transformation means that if

$$[q_k, q_l] = [p_k, p_l] = 0, \ [q_k, p_l] = \delta_{kl} \qquad ...(37)$$

and the transformation $(q_k, p_k) \to (Q_k, P_k)$ is canonical, then

$$[Q_k, Q_l] = [P_k, P_l] = 0, \ [Q_k, P_l] = \delta_{kl} \qquad ...(38)$$

According to the definition of Poisson bracket [eq. (5)], we have

$$[F, G]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right) \qquad ...(39)$$

Therefore,

$$[Q_k, Q_l]_{q,p} = \sum_i\left[\frac{\partial Q_k}{\partial q_i}\frac{\partial Q_l}{\partial p_i} - \frac{\partial Q_k}{\partial p_i}\frac{\partial Q_l}{\partial q_i}\right] \qquad ...(40)$$

$$\frac{\partial p_k}{\partial Q_l} = \frac{\partial}{\partial Q_l}\frac{\partial F_1}{\partial q_k} = \frac{\partial}{\partial q_k}\frac{\partial F_1}{\partial Q_l} = -\frac{\partial P_l}{\partial q_k} \qquad ...(41)$$

$$\frac{\partial p_k}{\partial P_l} = \frac{\partial}{\partial P_l}\frac{\partial F_2}{\partial q_k} = \frac{\partial}{\partial q_k}\frac{\partial F_2}{\partial P_l} = \frac{\partial Q_l}{\partial q_k} \quad ...(42)$$

$$\frac{\partial q_k}{\partial Q_l} = -\frac{\partial}{\partial Q_l}\frac{\partial F_3}{\partial p_k} = -\frac{\partial}{\partial p_k}\frac{\partial F_3}{\partial Q_l} = \frac{\partial P_l}{\partial p_k} \quad ...(43)$$

$$\frac{\partial q_k}{\partial P_l} = -\frac{\partial}{\partial P_l}\frac{\partial F_4}{\partial p_k} = -\frac{\partial}{\partial p_k}\frac{\partial F_4}{\partial P_l} = -\frac{\partial Q_l}{\partial p_k} \quad ...(44)$$

Hence eq. (40) is [using (41) and (43)]

$$[Q_k, Q_l]_{q,p} = \sum_i \left(-\frac{\partial Q_k}{\partial q_i}\frac{\partial q_i}{\partial P_l} - \frac{\partial Q_k}{\partial p_i}\frac{\partial p_i}{\partial P_l} \right) = -\frac{\partial Q_k}{\partial P_l} = 0 \quad ...(45)$$

because Q_k and P_k are independent variables. Also we note that

$$[Q_k, Q_l]_{Q,P} = \sum_i \left(-\frac{\partial Q_k}{\partial Q_i}\frac{\partial Q_l}{\partial P_i} - \frac{\partial Q_k}{\partial P_i}\frac{\partial Q_l}{\partial Q_i} \right) = 0$$

Therefore, $\left[Q_k, Q_l\right]_{q,p} = \left[Q_k, Q_l\right]_{Q,P} = 0$ ...(46)

Similarly we can prove

$$[P_k, P_l]_{q,p} = [P_k, P_l]_{Q,P} = 0 \quad ...(47)$$

Now, $$\left[Q_k, P_l\right]_{q,p} = \sum_i \left(\frac{\partial Q_k}{\partial q_i}\frac{\partial P_l}{\partial p_i} - \frac{\partial Q_k}{\partial p_i}\frac{\partial P_l}{\partial q_i} \right)$$

Using eqs. (41) and (43), we obtain

$$\left[Q_k, P_l\right]_{q,p} = \sum_i \left(\frac{\partial Q_k}{\partial q_i}\frac{\partial q_i}{\partial Q_l} + \frac{\partial Q_k}{\partial p_i}\frac{\partial p_i}{\partial Q_l} \right) = \frac{\partial Q_k}{\partial Q_l} = \delta_{kl} \quad ...(48)$$

By definition $[Q_k, P_l]_{Q,P} = \delta_{kl}$ ...(49)

Thus $$\left[Q_k, P_l\right]_{q,p} = \left[Q_k, P_l\right]_{Q,P} = \delta_{kl} \quad ...(50)$$

Eqs. (46) , (47) and (50) show the invariance of fundamental Poisson brackets with respect to canonical transformation.

Show that transformation defined by

$$q = \sqrt{2P}\sin Q, \quad p = \sqrt{2P}\cos Q$$

is canonical by using Poisson bracket.

Solution : The transformation is

$$q = \sqrt{2P}\sin Q, \quad p = \sqrt{2P}\cos Q$$

From these equations, we can write the transformation as

$$\tan Q = \frac{q}{p} \text{ and } P = \tfrac{1}{2}\left(q^2 + p^2\right) \quad ...(i)$$

In order to show that the given transformation is canonical, the Poisson bracket conditions are

$$[Q, Q] = [P, P] = 0 \text{ and } [Q, P] = 1 \quad ...(ii)$$

Here, $$[Q, Q] = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = 0 \quad ...(iii)$$

Similarly, $$[P, P] = 0 \quad ...(iv)$$

Also $$[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \quad ...(v)$$

But from (i),

$$\sec^2 Q\frac{\partial Q}{\partial q} = \frac{1}{p}, \ \frac{\partial P}{\partial p} = p, \ \sec^2 Q\frac{\partial Q}{\partial p} = -\frac{q}{p^2}, \frac{\partial P}{\partial q} = q$$

Substituting these values in (v), we get

$$[Q, P] = \frac{\cos^2 Q}{p}p + \frac{q\cos^2 Q}{p^2}q = \cos^2 Q + \frac{q^2}{p^2}\cos^2 Q$$

$$= \cos^2 Q\left[1 + \frac{q^2}{p^2}\right] = \cos^2 Q\left[1 + \tan^2 Q\right]$$

$$= \cos^2 Q\sec^2 Q = 1 \quad ...(vi)$$

Thus we prove the conditions (ii) which means that the given transformation is canonical.

HAMILTON – JACOBI METHOD

Solution to harmonic oscillator problem

Let us consider a one-dimensional harmonic oscillator. The force acting on the oscillator at a displacement q is

$$F = -kq$$

where k is force constant.

Potential energy, $V = \int_0^q kq\,dq = \frac{1}{2}kq^2$

Kinetic energy, $T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Hamiltonian, $H = T + V$ (conservative system)

or $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

But $p = \frac{\partial S}{\partial q}$, therefore

$$H = \frac{1}{2}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2$$

Hence the Hamilton-Jacobi equation corresponding to this Hamiltonian is

$$\frac{1}{2m}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2 + \frac{\partial S}{\partial t} = 0$$

As the explicit dependence of S on t is involved only in the last term of left hand side of eq , a solution to this equation can be assumed in the form

$$S = S_1(q) + S_2(t)$$

Thus $$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \frac{1}{2}kq^2 = -\frac{\partial S_2}{\partial t}$$

Setting each side of eq. equal to a constant, say α, we get

$$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \frac{1}{2}kq^2 = \alpha \quad \text{and} \quad -\frac{\partial S_2}{\partial t} = \alpha$$

So that $\dfrac{\partial S_1}{\partial q} = \sqrt{2m\left(\alpha - \frac{1}{2}kq^2\right)}$ and $-\dfrac{\partial S_2}{\partial t} = \alpha$

Integrating, we get

$$S_1 = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq + C_1 \text{ and } S_2 = -\alpha t + C_2$$

Therefore, $$S = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq - \alpha t + C$$

where $C = (C_1 + C_2)$ the constant of integration. It is to be noted that C is an additive constant and will not affect the transformation, because to obtain the new **position coordinate** ($Q = \partial S/\partial P$ or $\beta = \partial S/\partial \alpha$) only partial derivative of S with respect to α (= P, new momentum) is required. This is why this additive constant C has no effect on transformation and is dropped. Thus

$$S = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq - \alpha t$$

We designate the constant α as the new momentum P. The new constant coordinate ($Q = \beta$) is obtained by the transformation

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{\alpha - \frac{1}{2}kq^2}} - t = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \frac{kq^2}{2\alpha}}} - t$$

or $$\beta = \sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} - t$$

Therefore, $$\sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} = t + \beta \text{ or } \sin^{-1} q\sqrt{\frac{k}{2\alpha}} = \sqrt{\frac{k}{m}}\,(t + \beta)$$

Writing $\omega = \sqrt{k/m}$, we obtain

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin \omega\,(t + \beta)$$

which is the *familliar solution of the harmonic oscillator*,

UNIT – V

RELATIVITY

In Newtonian mechanics; space and time are completely separable and the transformations connecting the space-time coordinates of a particle are the Galilean transformations. These transformations are valid as far as Newton's laws are concerned, but fail in the field of electrodynamics. Principle of relativity, when applied to the electromagnetic phenomena, asserts that the speed of light in vacuum is a constant of nature. This starement has been confirmed by several experiments and led Einstein to formulate the special theory of relativity. In view of this theory, space and time are not independent of each other and the correct transformation equations are Lorentz transformations.

Principle of Relativity

Absolute velocity of a body has no meaning. The velocity has a meaning only when it is measured relative to some other body or frame of reference. If two bodies are moving with uniform relative velocity, it is impossible to decide which of them is at rest or which of them is moving. This is known as ***principle of relativity***. However, acceleration has an absolute meaning. For example, if we are sitting in a windowless accelerated aircraft, we can perform an experiment and measure its acceleration. But if the aircraft is moving with uniform velocity, we cannot measure its velocity. Of course, we measure its velocity relative to a body outside. Thus the principle of relativity can be alternatively stated as follows :

It is impossible to perform an experiment which will measure the state of uniform velocity of a system by observations, confined to that system.

The motion of a body itself has no meaning unless, we do not know with respect to which this motion has been measured. This led Newton to think about the absolute space and it represents an absolute frame with respect to which every motion should be measured. However, in view of this principle of relativity, we can not perform an experiment which will measure the uniform velocity of a reference system relative to the absolute frame by observations confined to that system.

In the unaccelerated windowless ship all experiments performed inside it will appear the same whether this ship is stationary or in uniform motion. Newton stated the principle of relativity as follows :

The motions of bodies included in a given space are the same among themselves whether that space is at rest or moving uniformly forward in a straight line.

Study of the physical laws involves the measurements of accelerations, forces etc among bodies. The principle of relativity can be stated in an elegant form as follows :

The basic laws of physics are identical in all inertial systems which move with uniform velocity with respect to one another.

This principle is called ***Galilean*** or ***Newtonian principle of relativity*** and sometimes it is named as ***hypothesis of Galilean invariance***. In fact, the principle of relativity is a fundamental postulate and is entirely consistent with the theory of special relativity. If any two inertial systems, moving with constant relative velocity, are connected by Galilean transformations, the principle of relativity is modified as :

The basic laws of physics are invariant in form in two reference systems connected by Galilean transformations.

This statement is somewhat special than the principle of relativity in the sense that it means the assumptions that the time and the space intervals are independent of the frame of reference. We shall see later in the theory of special relativity that the Galilean transformations are not correct, but the appropriate exact transformation equations are the Lorentz transformation equations for connecting any two frames in uniform relative motion. Thus, the principle of relativity may be stated as :

The basic laws of physics are invariant in form in two inertial frames connected by Lorentz transformations.

Postulates of Special theory of Relativity

The two fundamental postulates of the special theory of relativity are the following :

(1) *All the laws of physics have the same form in all inertial systems, moving with constant velocity relative to one another*. This postulate is just the principle of relativity.

(2) *The speed of light is constant in vacuum in every inertial system*. This postulate is an experimental fact and asserts that the speed of light does not depend on the direction of propagation in vacuum and the relative velocity of the source and the observer. In fact, *the second postulate is contained in the first because it predicts the speed of light c to be constant of nature.*

The name special theory of relativity comes from the fact that this theory permits the independence of the physical laws of those coordinate systems which are moving with constant velocity relative to one another. Later, Einstein propounded his *general theory of relativity* which allows for the independence of the physical laws of all coordinate systems, having any general relative motion.

These two postulates of special theory of relativity look to be very simple, but they have revolutionised the physics with far reaching consequences. First we deduce transformation equations, connecting any two inertial systems moving with constant relative velocity. The transformation should be such that they are applicable to both Newtonian mechanics and electromagnetism. Such transformations were deduced by Einstein in 1905 and are known as Lorentz transformations because Lorentz deduced them first in his theory of electromagnetism.

Four Dimensional formaulation – Minskowski' space

In accordance with the two postulates of the special theory of relativity, namely the constancy of the speed of light in vacuum and the invariance of the basic laws of physics in inertial frames, we deduced earlier the Lorentz transformations. These transformations connect the space-time coordinates of an event in two inertial frames S and S' and are given by

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z \text{ and } t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, the frame S' is moving with constant velocity v along X-axis relative to the frame S.

We find that in relativistic mechanics, the space and time coordinates depend on each other. The time coordinate of one inertial system depends on both the space and time coordinates of another system [$t' = g(t - vx/c^2)$]. Therefore, instead of treating the space and time coordinates separately, it is natural to seek the way so that both the coordinates are dealt together similarly. In fact, H. Minkowski was the first to develop a procedure in which the time coordinate is treated similar to the three space coordinates.

Minkowski considered a four dimensional cartesian space in which the position is specified by three coordinates x, y, z and the time is referred by a fourth coordinate ict. If we write $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$, then an event is represented by the position vector (x_1, x_2, x_3, x_4) in this four dimensional space. Of course the fourth dimension, referring to time, is imaginary. This four dimensional space is called ***Minkowski*** or ***world space***. It is also referred as ***space-time continuum*** and sometimes briefly as ***four-space***. The square of the magnitude of the position vector in such a four-space has the form

$$s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad ...(1)$$

Lorentz transformations are designed so that the speed of light remains constant in S and S' inertial frames (S' is moving with constant velocity v relative to S) and this condition is equivalent to require that the position vector in the four-space is held invariant under the transformations, *i.e.*,

$$s'^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

or

$$s^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

or

$$s^2 = \sum_{\mu=1}^{4} x'^2_\mu = \sum_{\mu=1}^{4} x^2_\mu \qquad ...(2)$$

This equation is analogous to the distance-preserving orthogonal transformation for rotation from one frame of reference to another in three dimensional space. Thus the coordinates, x_1, x_2, x_3, x_4, chosen above, form an orthogonal coordinate system in four dimensions and eq. (2) implies that the transformations which we are seeking, correspond to a rotation in a fourdimensional space. In fact, these orthogonal transformations in the four-dimensional Minkowski space are the Lorentz transformations.

Deduction of Lorentz Tansformations : In order to prove the statement that the Lorentz transformations can be regarded as orthogonal transformations due to rotation of axes in the Minkowski space, we deduce these transformations in the four-space.

The frame S' is moving with constant velocity v along X-axis relative to the inertial frame S and hence we may have

$$y' = y \quad \text{and} \quad z' = z \quad \text{or} \quad x'_2 = x_2 \quad \text{and} \quad x'_3 = x_3 \qquad ...(3)$$

Thus from (2), the transformations should be such that

$$x'^2_1 + x'^2_4 = x^2_1 + x^2_4 \qquad ...(4)$$

In order to keep this requirement, we consider two orthogonal coordinate systems X_1X_4 and $X'_1 X'_4$ in the same plane (plane of the paper) with the same origin O. The axes of $X'_1 X'_4$ system correspond to rotation θ with respect to those of X_1X_4 system, *i.e.*, the axes of the former coordinate system are inclined with the later through an angle θ. We observe that

$$OP^2 = x^2_1 + x^2_4 = x'^2_1 + x'^2_4$$

where the coordinates in two coordinate systems are related as

$$\begin{aligned} x'_1 &= x_1 \cos\theta + x_4 \sin\theta \\ x'_4 &= -x_1 \sin\theta + x_4 \cos\theta \end{aligned} \qquad ...(5)$$

In matrix notation

$$\begin{pmatrix} x'_1 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad ...(6)$$

Also, $x_1 = x'_1 \cos\theta - x'_4 \sin\theta$

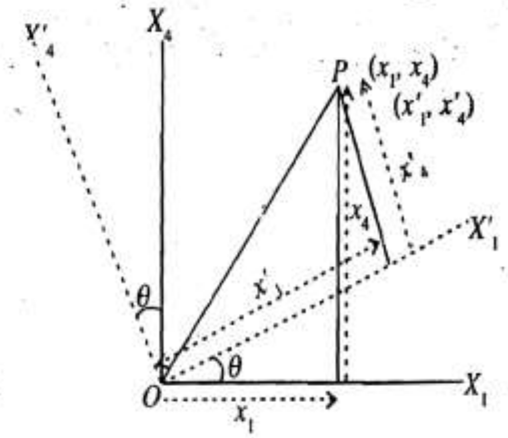
$x_4 = x'_1 \sin\theta + x'_4 \cos\theta$ ...(7)

When $x'_1 = 0$, $x_1 = -x'_4 \sin\theta$

and $x_4 = x'_4 \cos\theta$

So that $\tan\theta = -\dfrac{x_1}{x_4} = -\dfrac{x}{ict} = \dfrac{iv}{c}$ ...(8)

where $x'_1 = x' = 0$ corresponds to the coordinate of the point O' (S'-frame) relative to O (S-frame); *i.e.*, $x = vt$ or $\dfrac{x}{t} = v$.



Rotation of orthogonal coordinates axes and invariance of $OP_2 = x_1^2 + x_4^2 - x'^2_1 + x'^2_4$

Therefore from (8),

$$\sin\theta = \frac{iv/c}{\sqrt{1-v^2/c^2}} = \frac{i\gamma v}{c} \text{ and } \cos\theta = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma \qquad \text{(say)}$$

Hence eqs. (5) can be expressed as

$$x'_1 = \gamma x_1 + i\gamma \frac{v}{c} x_4 = \gamma\left(-x_1 + i\frac{v}{c}x_4\right) \text{ and } x'_4 = -i\gamma\frac{v}{c}x_1 + \gamma x_4 = \gamma\left(-\frac{iv}{c}x_1 + x_4\right).$$

If we add $x'_2 = x_2$, and $x'_3 = x_3$, the transformation equations are

$$x'_1 = \gamma\left(x_1 + i\frac{v}{c}x_4\right), x'_2 = x_2, x'_3 = x_3, \text{ and } x'_4 = \gamma\left(-i\frac{v}{c}x_1 + x_4\right) \qquad ...(9)$$

In fact, these are the Lorentz transformations. This may be seen by putting $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = ict$ in eq. (9), *i.e.*,

$$x' = \gamma(x - vt),\ y' = y,\ z' = z \text{ and } t' = \gamma\left(t - vx/c^2\right) \qquad ...(10\,a)$$

In matrix notation, the Lorentz transformations from S-frame to S'-frame can be represented as

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad ...(10b)$$

or $$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \quad ...(10c)$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. and $a_{\mu\nu}$ are the elements of the above square matrix .

The inverse Lorentz transformations are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} \quad ...(11a)$$

or $$x_\mu = \sum_{\nu=1}^{4} a_{\nu\mu} x'_\nu \quad ...(11b)$$

because $\sum_\nu a_{\nu\mu} x'_\nu = \sum_\nu a_{\nu\mu} \sum_\lambda a_{\nu\lambda} x_\lambda = \sum_\lambda \sum_\nu a_{\nu\mu} a_{\nu\lambda} x_\lambda = \sum_\lambda \delta_{\mu\lambda} x_\lambda = x_\mu$.

Remember that for orthogonal transformations

$$\sum_\nu a_{\mu\nu} a_{\lambda\nu} = \sum_\nu a_{\nu\mu} a_{\nu\lambda} = \delta_{\mu\lambda}$$

Here x_μ and x'_μ satisfy the condition (2), *i.e.*,

$$\sum_{\mu=1}^{4} x'^2_\mu = \sum_{\mu=1}^{4} x^2_\mu$$

The four coordinates x_1, x_2, x_3 and x_4 or x, y, z and ict, define the position vector in the four-space and may be termed as ***four-position vector***. We shall discuss more about four-vectors later.

WORLD POINT AND WORLD LINE

A physical event in Minkowski space is described by a point with four coordinates (x_1, x_2, x_3, x_4) $(x_4 = ict)$. This point in the four-space is called ***world point***. In this space, the motion of a particle (*i.e.*, a particle at various instants) corresponds to a line, known as ***world line***. A particle in uniform rectilinear motion corresponds to a straight world line. The relative position (in space-time) of one event with respect to another would be represented by ***line element***, joining the two events.

In order to show the interdependence of space and time more clearly and to represent them geometrically, we consider only one space axis, X-axis and ignore Y and Z axes. The time axis is represented perpendicular to X-axis by $T = ct$, so that the dimensions of the coordinates are the same.

The Lorentz transformations for x and t are

$$x' = \gamma(x - vt) \quad \text{or} \quad x' = \gamma(x - \beta T) \qquad ...(12)$$

and

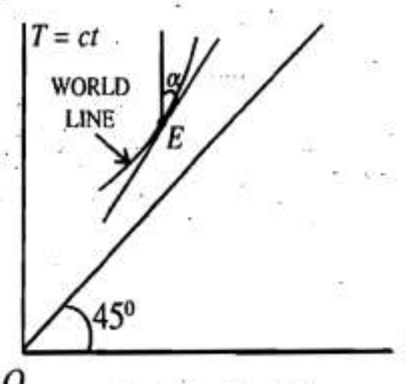
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{or} \quad T' = \gamma(T - \beta x) \qquad ...(13)$$

where $\beta = v/c$.

We observe that *the Lorentz transformations for space and time in this form possess symmetry*. In this X-T coordinate system in the Minkowski space, we have represented the motion of a particle by a world line [Fig.14.2]. The inclination α of a tangent at any point E of a world line is given by

$$\tan\alpha = \frac{dx}{dT} = \frac{dx}{cdt} = \frac{u}{c} \qquad ...(14)$$

where we must have $u < c$ for a material particle. This means that $\alpha < 45^\circ$ for a material particle. If the particle velocity (u) is



World line

constant, $\tan\alpha$ is also constant. Hence the world line for a particle moving with constant velocity is a straight line. For light signal, $u = c$ and therefore $\alpha = 45^\circ$. Thus the world line for light signal is a straight line making an angle 45° with the X-axis.

FOUR-VECTORS

A vector in four dimensional Minkowski space is called a ***four-vector***. Its components transform from one frame to another similar to Lorentz transformations.

An event in four domensional space is represented by a world point (x_1, x_2, x_3, x_4). The Lorentz transformations from S-frame to S'-frame correspond to orthogonal transformations in the four-space and are represented as

$$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \text{ or } \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

with the condition

$$\sum_{\mu=1}^{4} x'^2_\mu = \sum_{\mu=1}^{4} x^2_\mu$$

We may represent the position vector of a world point by

$$x_\mu = (x_1, x_2, x_3, x_4) = (\mathbf{r}, ict)$$

where (x_1, x_2, x_3) or (x, y, z) represent the position vector $\mathbf{r}$ of a point in three dimensional space and $x_4 = ict$ or $x_4 = iT$. $\mathbf{r}\,(= x, y, z)$ is the space part and ict is the time part of the four dimensional position vector x_μ.

A four-vector A_μ is a vector in four dimensional space with components A_1, A_2, A_3 and A_4 and is represented as

$$A_\mu = (A_1, A_2, A_3, A_4) = (\mathbf{A}, iA_t)$$

where $\mathbf{A}(= A_1, A_2, A_3)$ is the space component and $A_4 (= iA_t)$ is the time component. These components transform from S-frame to S'-frame similar to Lorentz transformations, *i.e.*,

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

or $$A'_{\mu} = \sum_{\nu=1}^{4} a_{\mu\nu} A'_{\nu}$$

i.e., $A'_1 = \gamma(A_1 + i\beta A_4), A'_2 = A_2, A'_3 = A_3, A'_4 = \gamma(-i\beta A_1 + A_4)$
These transformations are governed by the condition

$$\sum_{\mu=1}^{4} A'^2_{\mu} = \sum_{\mu=1}^{4} A^2_{\mu} \quad \text{or} \quad A'_{\mu} A'_{\mu} = A_{\mu} A_{\mu}$$

The square of the magnitude of the four vector is given by

$$A_{\mu} A_{\mu} = A_1^2 + A_2^2 + A_3^2 + A_4^2$$

or $$\sum_{\mu=1}^{4} A^2_{\mu} = A_1^2 + A_2^2 + A_3^2 - A_t^2$$

Let two vectors A_{μ} and B_{μ} be $A_{\mu} = (A_1, A_2, A_3, A_4)$ with $A_4 = iA_t$ and $B_{\mu} = (B_1, B_2, B_3, B_4)$ with $B_4 = iB_t$.

The scalar product of the four-vectors A_{μ} and B_{μ} is defined as

$$A_{\mu} B_{\mu} = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4$$

or $$A_{\mu} B_{\mu} = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_t B_t$$

This scalar product is invariant under Lorentz transformations *i.e.*,

$$A'_\mu B'_\mu = A_\mu B_\mu$$

because

$$A'_1 B'_1 + A'_2 B'_2 + A'_3 B'_3 + A'_4 B'_4$$

$$= \gamma^2 (A_1 + i\beta A_4)(B_1 + i\beta B_4) + A_2 B_2 + A_3 B_3 + \gamma^2(-i\beta A_1 + A_4)(-i\beta B_1 + B_4)$$

$$= A_1 B_1 \gamma^2 (1 - \beta^2) + A_2 B_2 + A_3 B_3 + A_4 B_4 \gamma^2 (-\beta^2 + 1)$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4$$

Momentum four vector p_μ : The components of four-momentum p_μ are defined by

$$p_1 = m_0 u_1 = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}} = m u_x = p_x$$

$$p_2 = m_0 u_2 = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}} = m u_y = p_y$$

$$p_3 = m_0 u_3 = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}} = m u_z = p_z$$

$$p_4 = m_0 u_4 = \frac{m_0 ic}{\sqrt{1 - u^2/c^2}} = imc = i\frac{E}{c}$$

Hence,

$$p_\mu = (p_1, p_2, p_3, p_4) = (p_x, p_y, p_z, imc) = (\mathbf{p}, iE/c) \text{ with } \mathbf{p} = m\mathbf{u}$$

The square of the magnitude of the four-momentum is given by

$$p_\mu p_\mu = p^2 - \frac{E^2}{c^2} = -(E^2 - p^2 c^2)/c^2 \quad \text{or} \quad p_\mu p_\mu = -m_0^2 c^2$$

This p_μ is also called ***energy- momentum four-vector***.

THOMAS PRECESSION

“Thomas precession is a kinematic effect in the flat space time of special relativity. In the curved space time of general relativity,”

For a given inertial frame, if a second frame is Lorentz-boosted relative to it, and a third boosted relative to the second, but non-collinear with the first boost, then the Lorentz transformation between the first and third frames involves a combined boost and rotation, known as the "Wigner rotation" or "Thomas rotation". For accelerated motion, the accelerated frame has an inertial frame at every instant. Two boosts a small time interval (as measured in the lab frame) apart leads to a Wigner rotation after the second boost. In the limit the time interval tends to zero, the accelerated frame will rotate at every instant, so the accelerated frame rotates with an angular velocity.

The precession can be understood geometrically as a consequence of the fact that the space of velocities in relativity is hyperbolic, and so parallel transport of a vector (the gyroscope's angular velocity) around a circle (its linear velocity) leaves it pointing in a different direction, or understood algebraically as being a result of the non-commutativity of Lorentz transformations.

Elements of general theory of relativity

General relativity, also known as the general theory of relativity, is the geometric theory of gravitation published by Albert Einstein in 1915 and is the current description of gravitation in modern physics. General relativity generalizes special relativity and refines Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time or four-dimensional spacetime. In particular, the *curvature of spacetime* is directly related to the energy and momentum of whatever matter and radiation are present.

Some predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include gravitational time dilation, gravitational lensing, the gravitational redshift of light, the gravitational time delay and singularities/black holes. The predictions of general relativity in relation to classical physics have been confirmed in all observations and experiments to date. Although general relativity is not the only relativistic theory of gravity, it is the simplest theory that is consistent with experimental data.

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