KUNTHAVAI NAACCHIYAAR GOVT. ARTS COLLEGE FOR WOMEN (AUTONOMOUS),

THANJAVUR – 7.

I M. Sc. PHYSICS

STUDY MATERIAL

SUBJECT : ELECTROMAGNETIC THEORY

SUB.CODE : 18KP2P05

UNIT - I

Electrostatics

Electrostatics, as the name implies, is the study of stationary electric charges. A rod of plastic rubbed with fur or a rod of glass rubbed with silk will attract small pieces of paper and is said to be **electrically charged**. The charge on plastic rubbed with fur is defined as **negative**, and the charge on glass rubbed with silk is defined as **positive**.

Electrically charged objects have several important characteristics:

- Like charges repel one another; that is, positive repels positive and negative repels negative.
- Unlike charges attract each another; that is, positive attracts negative.
- Charge is conserved. A neutral object has no net charge. If the plastic rod and fur are initially neutral, when the rod becomes charged by the fur, a negative charge is transferred from the fur to the rod. The net negative charge on the rod is equal to the net positive charge on the fur.

Coulomb's law

Coulomb's law gives the magnitude of the electrostatic force (*F*) between two charges:

$$F = \frac{kq_1q_2}{r^2}$$

where q_1 and q_2 are the charges, *r* is the distance between them, and *k* is the proportionality constant. The SI unit for charge is the coulomb.

If the charge is in coulombs and the separation in meters, the following approximate value for *k* will give the force in newtons: $k = 9.0 \times 10^9\ N \cdot m^2/C^2$. The direction of the electrostatic force depends upon the signs of the charges. Like charges repel, and unlike charges attract.

Coulomb's law can also be expressed in terms of another constant (ε_0), known as the **permittivity of free space:**

$$\varepsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$$

When the permittivity constant is used, Coulomb's law is

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

The most fundamental electric charge is the charge of one proton or one electron. This value *(e)* is *e* = 1.602 × 10^{-19} coulombs. It takes about 6.24 × 10^{18} excess electrons to equal the charge of one coulomb; thus, it is a very large static charge.

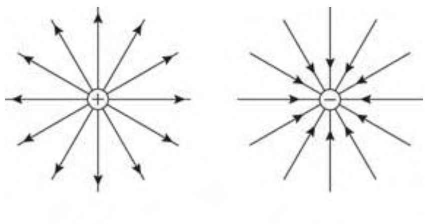
Electric fields and lines of force

When a small positive test charge is brought near a large positive charge, it experiences a force directed away from the large charge. If the test charge is far from the large charge, the electrostatic force given by Coulomb's law is smaller than when it is near. This data of direction and magnitude of an electrostatic force, due to a fixed charge or set of fixed charges, constitutes an electrostatic field. The **electric field** is defined as the force per unit charge exerted on a small positive test charge (q_0) placed at that point. Mathematically,

$$E = \frac{F}{q_0}$$

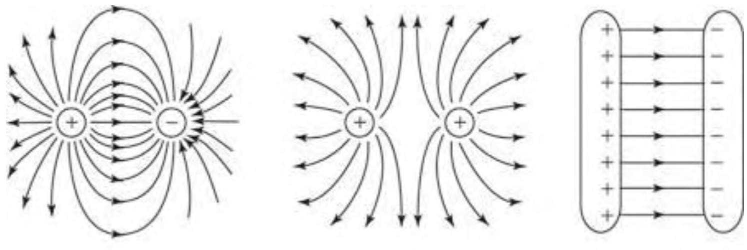
Both the force and electric field are vector quantities. The test charge is required to be small so that the field of the test charge does not affect the field of the set charges being examined. The SI unit for electric field is newtons per coulomb (N/C).

The following figure is a pictorial representation of the electric fields surrounding a positive charge (a) and a negative charge (b). These lines are called **field lines** or **lines of force**.



(a) (b)

The electric fields for opposite charges, similar charges, and oppositely charged plates is shown below.



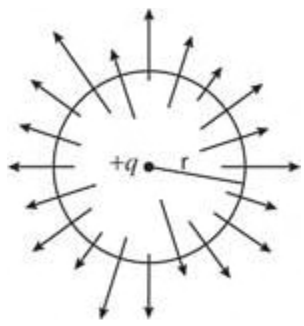
(a) (b) (c)

The rules for drawing electric field lines for any static configuration of charges are

- The lines begin on positive charges and terminate on negative charges.
- The number of lines drawn emerging from or terminating on a charge is proportional to the magnitude of the charge.
- No two field lines ever cross in a charge-free region. (Because the tangent to the field line represents the direction of the resultant force, only one line can be at every point.)
- The line approaches the conducting surface perpendicularly.

Electric flux

Electric flux is defined as the number of field lines that pass through a given surface. In Figure, lines of electric flux emerging from a point charge pass through an imaginary spherical surface with the charge at its centre.



This definition can be expressed as follows:

$\Phi = \sum E \cdot A$, where Φ (the Greek letter phi) is the electric flux, **E** is the electric field, and **A** is area perpendicular to the field lines.

Electric flux is measured in $N \cdot m^2 / C^2$ and is a scalar quantity. If the surface under consideration is not perpendicular to the field lines, then the expression is

$\Phi = \sum EA \cos \theta$.

In general terms, flux is the closed integral of the dot product of the electric field vector and the vector ΔA. The direction of ΔA is the outward drawn normal to the imaginary surface.

Mathematically, $\Phi = \Phi E \cdot dA$. The accepted convention is that flux lines are positive if leaving a surface and negative if entering a surface.

Gauss's law

Gauss's law provides a method to calculate any electric field; however, its only practical use is for fields of highly symmetric distributions of fixed charges. The law states that the net electric flux through any real or imaginary closed surface is equal to the net electric charge enclosed within that surface divided by ε. As a result, if no charge exists with a given closed surface, then there are as many flux lines entering the surface as there are leaving it. The imaginary surface necessary to apply Gauss's law is called the **gaussian surface**. Algebraically,

$$\sum EA \cos \theta = \frac{Q}{\varepsilon_o}$$

or in integral form,

$$\int \mathrm{E} \cdot d\mathrm{A} = \frac{Q}{\varepsilon_o}$$

where θ is the angle between the direction of *E* and the outward direction of normal to the surface and ε is the permittivity constant.

In the calculation of the electric field due to a point charge, as the electric field is perpendicular to the gaussian surface and directed outward, θ is 90 degrees, and cos θ = 1. Gauss's law is

$$EA = \frac{q}{\varepsilon_o}$$

Substitute in the area of a sphere, and the left side reduces to

$$E4\pi r^2 = \frac{q}{\varepsilon_o}$$

or

$$E = \frac{q}{4\pi r^2 \varepsilon_o}$$

which is the same expression obtained from Coulomb's law and the definition of electric field in terms of force.

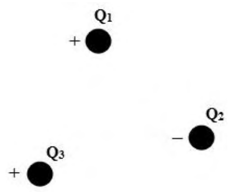
Types of Electric Charge Distributions

In the above the charge distribution is explained by considering the point charges. In addition, there is possibility to distribute the charge continuously along line, in a volume or on surface. Therefore, there are four types of charge distributions namely

1. Point charge
2. Line charge
3. Surface charge
4. Volume charge

Point Charge

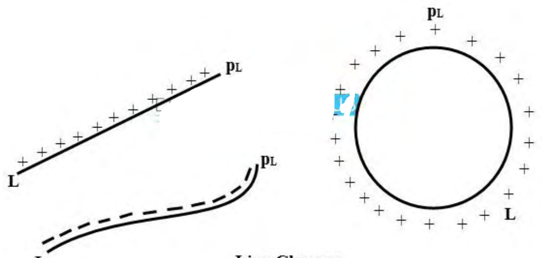
Compared with region surrounded by a surface carrying a charge, the dimensions of that surface are very small, then that charge is treated as point charge. The point charge can be negative or positive. And it has a position but do not have dimensions.



Line Charge

A charge possibly spread all along the line whether infinitely or finitely. A charge that uniformly distributed all along the line is called as line charge as shown in figure. The charge density of a line charge is defined as the charge per unit length and denoted as pL.

It is measured in coulomb per meter and is constant all along the length of the line charge.



Line Charges

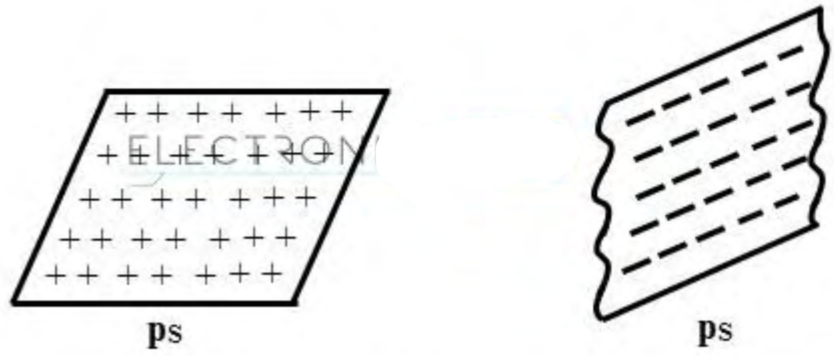
The total charge for the entire length L is obtained by applying line integral to the charge dQ (is equal to the pL) on dl as

$$Q = \oint pL\, dl$$

Surface Charge

A surface charge is also called as sheet of charge in which the charge is uniformly distributed over a two dimensional surface. The area of the two dimensional surface is in square meters. The surface charge density is defined as the charge per unit surface area and is denoted as pS.

It is measured in coulomb per meter square and it is constant over the surface carrying the charge. The example of the surface charge distribution is the plate of a charged parallel plate capacitor.



Surface Charge Distributions

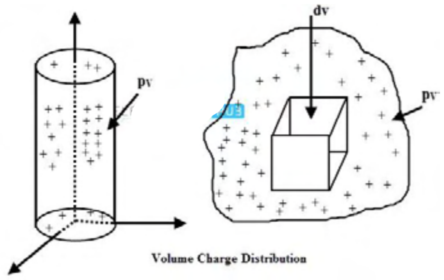
In surface charge distribution, the total charge distribution is determined by considering the charge dQ on elementary surface area ds over that surface. Thus, it considers the surface integral rather than normal integral. Mathematically

$$Q = \int_{S} dQ = \int_{S} \mathrm{p_S}\, dS$$

The distribution of the charge is considered as an infinite sheet of charge if the dimensions of the surface is charge is very large compared to the distance at which the effects of charge to be considered.

Volume Charge

A volume charge is the charge which is distributed uniformly in a given volume. The volume charge density is defined as the charge per unit volume and is denoted as pV. It is measured in coulomb per meter cube. The example of this volume charge is the charged cloud.



The total charge within a given volume is obtained by integrating dQ by a differential volume dv. This integral is called as volume integral and is given as

$$Q = \int_{vol} pv \ \ dv$$

Electrostatic Field

Once the charge distribution of a particle is known, we can determine the electric field of that particle. As we know that substances constitute the point charges (electrons and protons) which results field from each of these point charges.

Suppose if a particle bearing a unit positive charge is placed at a specified point, then it experience a force which is called electric force F. the region in which this force exist is called as the region of electric field.

This is defined as the force per unit charge exerted on a tiny positive charge placed at that point and it is a vector quantity with a unit of Newton/coulomb.

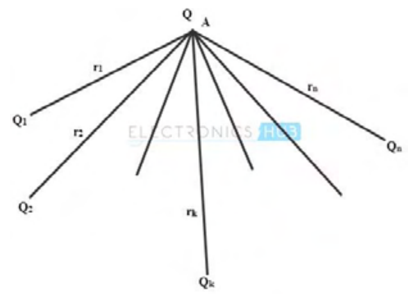
The electrostatic field E is given as

E = F/q or F = E q where F is the electrostatic force and q is the net electric test charge.

The above equation states that regardless of a second charge on the space surrounded by the net charge, it produces an electric field around that space. Consider that if the two point charges are exist in the region as shown in the above figure. Then the electric field intensity by considering Q2 as one coulomb is given as

$E = (Q1/ (4 \pi \epsilon o r^2_{12})) \times r1$

Where r12 is the distance between Q1 and Q2 and r1 is the unit vector in the direction of line joining Q1 with Q2. And also F12 = Q2 E



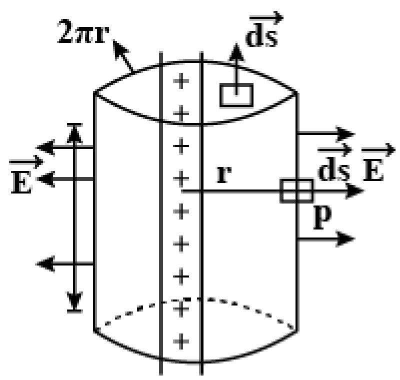
Consider that the above figure in which the region consists of n two point charges, then the electric field intensity at point A is given as

EA = Σk =1n (Qk / (4 π ϵor^2k)) × rk1

Electrostatic potential

Imagine moving a small test charge *q′* from point A to point B in the uniform field between parallel plates. The work done in transferring the charge equals the product of the force on the test charge and the parallel component of displacement, using the same definition of work given in the section on mechanics. This work can also be expressed in terms of **E** from the definition of electric field as the ratio of force to charge: *W* · d, E = F/ *q* and *W* = *q′*.

Electric field due to an infinitely long straight uniformly charged wire



Consider an uniformly charged wire of infinite length having a constant linear charge density λ (Charge per unit length). Let P be a point at a distance r from the wire and E be the electric field at the point P. A cylinder of length l, radius r, closed at each end by plane caps normal to the axis is chosen as Gaussian surface. Consider a very small area ds on the Gaussian surface. By symmetry, the magnitude of the electric field will be the same at all points on the curved surface of the cylinder and directed radially outward. E and ds are along the same direction.

The electric flux (φ) through curved surface $=\oint E.ds \cos\theta$

$\varphi=\oint E.ds$ [$\because \theta=0; \cos\theta=1$]

$=E(2\pi rl)$ [The surface area of the curved part]

since E and ds are right angles 2πrl to each other, the electric flux through the plane caps =0. ∴ Total flux through the Gaussian surface, $\varphi= E(2\pi rl)$.

The net charge enclosed by Gaussian surface is, $q=\lambda l$

$= E\ (2\pi rl)\ \varepsilon_0\ \lambda\ l$

or $E=(1/2\pi\varepsilon_0 r)\ \lambda$

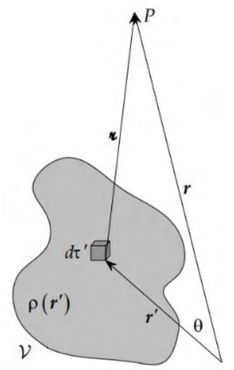
The direction of electric field E is radially outward, if line charge is positive and inward, if the line charge is negative.

The Multipole Expansion

A multipole expansion is a mathematical series representing a function that depends on angles (usually the two angles on a sphere). These series are useful because they can often be truncated, meaning that only the first few terms need to be retained for a good approximation to the original function. Multipole expansions are very frequently used in the study of electromagnetic and gravitational fields, where the fields at distant points are given in terms of sources in a small region. The multipole expansion with angles is often combined with an expansion in radius. Such a combination gives an expansion describing a function throughout three-dimensional space.

The multipole expansion is expressed as a sum of terms with progressively finer angular features. For example, the initial term called the zeroth, or monopole, moment is a constant, independent of angle. The following term the first, or dipole, moment varies once from positive to negative around the sphere. Higher-order terms (like the quadrupole and octupole) vary more quickly with angles. A multipole moment usually involves powers (or inverse powers) of the distance to the origin, as well as some angular dependence.

Consider an arbitrary charge distribution $\rho(r')$. The electrostatic potential due to this charge distribution at a given point r is to be determined. It is assumed that this point is at a large distance from the charge distribution, that is if r' varies over the charge distribution, then $r \gg r'$.



Now, the Coulomb potential for an arbitrary charge distribution is given by

$$V(r)=1/4\pi\epsilon_0\int\rho(r')|r-r'|dV' \qquad (1)$$

Here,

$$|r-r'|=\sqrt{|r^2-2r\cdot r'+r'_2|} \qquad (2)$$

$$=r|1-2\hat{r}\cdot r'/r+(r'/r)^2| \qquad (3)$$

where

$$\hat{r}=r/r \qquad (4)$$

Thus, using the fact that r is much larger than r', we can write

$1/|r-r'|=(1/r)1/\sqrt{|1-(2\hat{r}\cdot r'/r)+(r'/r)^2|}$ (5)

and using the binomial expansion,

$'|=(1/r)1/\sqrt{|1-(2\hat{r}\cdot r'/r)+(r'/r)2|} = 1-\hat{r}\cdot r'/r + 1/2r^2(r'^2-3(\hat{r}\cdot r')^2)+O(r'/r)^3$ (6)

(we neglect the third and higher order terms).

Binomial Theorem

The binomial theory can be used to expand specific functions into an infinite series:

$(1+x)^s = \sum s!/(n!(s-n)!)x^n = 1+(s/1!)x + (s(s-1)/2!)x^2 + (s(s-1)(s-2)/3!)x^3+\ldots$ (8)

Equation 5 can be rewritten as

$1/|r-r'|=(1/r)1/\sqrt{1+\epsilon}$ (9)

where

$\epsilon=-2\hat{r}\cdot r'/r+(r'/r)^2$ (10)

Applying the Binomial Theorem to Equation 9 (with $s=-1/2$) results in

$1/|r-r'|=1/r(1-(1/2)\epsilon+(3/8)\epsilon^2-(5/16)\epsilon^3+\ldots)$ (11)

Equation 6 originates from substituting Equation 10 into Equation 11.

The Expansion

Inserting Equation 6 into Equation 1 shows that the potential can be written as

$V(r)=(1/4\pi\epsilon_0 r)\int\rho(r')(1-\hat{r}\cdot r'/r+1/2r^2(3(\hat{r}\cdot r')^2-r'^2)+O(r'/r)^3)\, dV'$ (12)

This can be written as

$V(r)=V_{mon}(r)+V_{dip}(r)+V_{quad}(r)+\ldots$ (13)

The first (the zeroth-order) term in the expansion is called the monopole moment, the second (the first-order) term is called the dipole moment, the third (the second-order) is called the quadrupole moment, the fourth (third-order) term is called the octupole moment, and the fifth (fourth-order) term is called the hexadecapole moment. Given the limitation of Greek numeral prefixes, terms of higher order are conventionally named by adding "-pole" to the number of poles—e.g., 32-pole (i.e., dotriacontapole) and 64-pole (hexacontatetrapole).

These moments can be expanded thusly

$$V_{mon}(r) = 1/4\pi\epsilon_0 r \int \rho(r')dV' \qquad (14)$$

$$V_{dip}(r) = -1/4\pi\epsilon_0 r^2 \int \rho(r')(\hat{r}\cdot r')dV' \qquad (15)$$

$$V_{quad}(r) = 1/8\pi\epsilon_0 r^3 \int \rho(r')(3(\hat{r}\cdot r')^2 - r'^2)dV' \qquad (16)$$

and so on.

In principle, a multipole expansion provides an exact description of the potential and generally converges under two conditions:

1. if the sources (e.g. charges) are localized close to the origin and the point at which the potential is observed is far from the origin; or
2. the reverse, i.e., if the sources are located far from the origin and the potential is observed close to the origin.

In the first (more common) case, the coefficients of the series expansion are called exterior multipole moments or simply multipole moments whereas, in the second case, they are called interior multipole moments.

The Monopole Scalar

Observe that

$$V_{mon}(r) = 1/4\pi\epsilon 0 r \int \rho(r')dV' = q/4\pi\epsilon_0 r \quad (17)$$

is a scalar, (actually the total charge in the distribution) and is called the *electric monopole*. This term indicates point charge electrical potential with charge qq.

The Dipole Vector

If a charge distribution has a net total charge, it will tend to look like a monopole (point charge) from large distances. We can write

$V_{dip}(r) = -\hat{r}/4\pi\epsilon_0 r^2 \cdot \int \rho(r')r'dV'$ (18)

The vector

$p = \int \rho(r')r'dV'$ (19)

is called the electric dipole. And its magnitude is called the dipole moment of the charge distribution. This terms indicates the linear charge distribution geometry of a dipole electrical potential.

The Quadrupole Tensor

Let r^and r′ be expressed in Cartesian coordinates as (r1,r2,r3) and (x1,x2,x3). Then, $(\hat{r} \cdot r')^2 = (r_i x_i)^2 = r_i r_j x_i x_j$

We define a dyad to be the tensor r^r^ given by

$(\hat{r}\hat{r})_{ij} = r_i r_j$ (20)

Quadrupole tensor

$T = \int \rho(r')(3(r'r') - Ir'^2)dV'$ (21)

Then, we can write V_{qua} as the tensor contraction

$V_{qua}(r) = -\hat{r}\hat{r}/4\pi\epsilon_0 r^3 :: T$ (22)

this term indicates the three dimensional distribution of a quadruple electrical potential.

Model Questions

Part – A

1. Define linear charge density.
2. Define surface charge density.
3. Define volume charge density.
4. What is electric charge density?
5. State Coulomb's law.
6. Define electric intensity.
7. What is electric potential?
8. State Gauss's law.
9. Give the integral form of Gauss's law.
10. Give the differential form of Gauss's law.
11. Write the Poisson's equation and explain
12. Write the Laplace's equation and explain
13. What do you mean by multipole expansion?
14. What do you mean by electrostatic energy?

Part – B

1. Explain the various types of charge density.
2. State and explain Coulomb's law.
3. Get an expression for electric field intensity for continuous charge distribution.
4. Get an expression for electric potential for continuous charge distribution.
5. Establish Gauss's theorem for electrostatic field.
6. Write a short note on electrostatic energy.

Part – C

1. Establish Gauss's theorem for electrostatic field and deduce Poisson's equation and Laplace's equation from it.
2. Using Gauss law find the electric potential at a point due to a straight uniformly charged wire.
3. Show that the potential at any external point due to a charge distribution can be expressed as the contribution of the moments of monopole, dipole and quadrupole etc.
4. Show that the energy density in electrostatic field is U = 1/2 (D.E)

UNIT – II

Boundary conditions on Electric field intensity (E)

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. These discontinuities can be described mathematically as boundary conditions and used to constrain solutions for the associated electromagnetic quantities. In this section, we derive boundary conditions on the electric field intensity E.

Consider a region consisting of only two media that meet at an interface defined by the mathematical surface S.

At the surface of a perfectly-conducting region, E may be perpendicular to the surface (two leftmost possibilities), but may not exhibit a component that is tangent to the surface (two rightmost possibilities).

If either one of the materials is a perfect electrical conductor (PEC), then S is an equipotential surface; i.e., the electric potential V is constant everywhere on S. Since E is proportional to the spatial rate of change of potential ($E=-\nabla V$). The component of E that is tangent to a perfectly-conducting surface is zero.

This is sometimes expressed informally as follows:

$E_{tan}=0$ on PEC surface (1)

where "Etan" is understood to be the component of E that is tangent to S. Since the tangential component of E on the surface of a perfect conductor is zero, the electric field at the surface must be oriented entirely in the direction perpendicular to the surface.

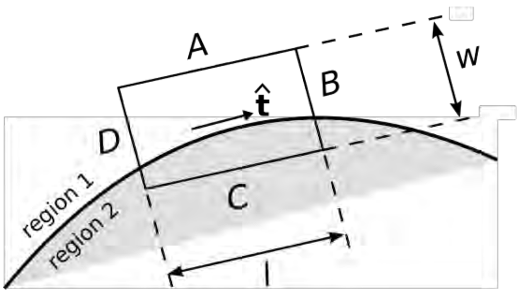
$E\times\hat{n}=0$ (on PEC surface) (2)

where n^ is either normal (i.e., unit vector perpendicular to the surface) to each point on S. This expression works because the cross product of any two vectors is perpendicular to either vector, and any vector which is perpendicular to n^ is tangent to S.

The desired boundary condition can be obtained directly from Kirchoff's Voltage Law.

$\oint_C E \cdot dl = 0$ (3)

Let the closed path of integration take the form of a rectangle centred on S.



Let the sides A, B, C, and D be perpendicular or parallel to the surface, respectively. Let the length of the perpendicular sides be w, and let the length of the parallel sides be l. From KVL we have

$\oint_C E \cdot dl = \int_A E \cdot dl + \int_B E \cdot dl + \int_C E \cdot dl + \int_D E \cdot dl = 0$

Now, let us reduce w and l together while (1) maintaining a constant ratio $w/l \ll 1$ (2) keeping C centred on S. In this process, the contributions from the B and D segments become equal in magnitude but opposite in sign; i.e.,

$\int_B E \cdot dl + \int_D E \cdot dl \rightarrow 0$ (4)

This leaves

$\oint_C E \cdot dl \rightarrow \int_A E \cdot dl + \int_C E \cdot dl \rightarrow 0$ (5)

Let us define the unit vector $\hat{t}$ ("tangent") as shown in Fig. When the lengths of sides A and C become sufficiently small,

$E_1 \cdot \hat{t} \Delta l - E_2 \cdot \hat{t} \Delta l \rightarrow 0$ (6)

where E_1 and E_2 are the fields evaluated on the two sides of the boundary and $\Delta l \rightarrow 0$ is the length of sides A and C while this is happening.

The tangential component of E must be continuous across an interface between dissimilar media.

This is a generalization of the result we obtained earlier for the case in which one of the media was a PEC – in that case, the tangent component of EE on the other side of the interface must be zero because it is zero in the PEC medium.

As before, $E1 \times \hat{n} = E2 \times \hat{n}$ on S (7)

or, as it is more commonly written:

$\hat{n} \times (E1 - E2) = 0$ on S (8)

Equation 8 is the boundary condition that applies to E for both the electrostatic and the general (time-varying) case.

Separation of variables : Cartesian coordinates

Consider the Laplace equation in Cartesian coordinates

$\nabla^2\phi(x,y,z) = (\partial^2/\partial x^2 + (\partial 2/\partial y^2 + (\partial 2/\partial z^2)\ \phi(x,y,z) = 0$

Under certain circumstances it is possible to write the solution in the product form

$\phi(x,y,z) = \phi_x(x)\phi_y(y)\phi_z(z)$

In this case the Laplace equation becomes

$(1/\phi_x(x))d^2\phi_x(x)/dx^2 + (1/\phi_y(y))d^2\phi_y(y)/dy^2 + (1/\phi_z(z))d^2\phi_z(z)/dz^2 = 0$

This implies that,

$d^2\phi_x(x)/dx^2 = \alpha\ \phi_x(x)$

$d^2\phi_y(y)/dy^2 = \beta\ \phi_y(y)$

$d^2\phi_z(z)/dz^2 = \gamma\ \phi_z(z)$

with $\alpha + \beta + \gamma = 0$

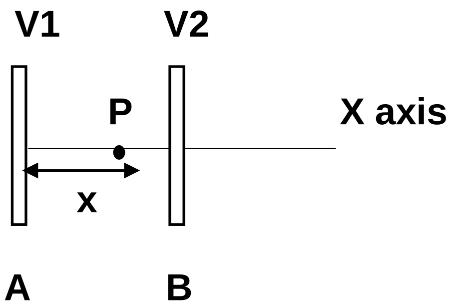
That is, the three dimensional partial differential equation separates into three one dimensional ordinary differential equation with a constraint. The constraint on α, β and γ implies that at least one of them is negative or one of them is positive or two of the functions are sinusoidal and the third is hyperbolic or two are hyperbolic and one sinusoidal.

The values of α, β and γ are to be determined by boundary conditions. It should be apparent that this method is useful if the boundary conditions of the problem are imposed on rectangular box.

This is the method of separation of variables. The method of separation of variables is useful when the problem has a symmetry and there is a corresponding orthogonal coordinate system in which the Laplacian operator (∇^2) is separable.

Application:

Potential at a point between the plates of a parallel plate capacitor



Let V1 and V2 to be the potential source of plates A and B respectively. Let the plates be perpendicular to x axis and O be the origin. The potential at any point between the plates will depend upon x only and so the corresponding form of Laplace Equation is

$$\partial^2V/\partial x^2 = 0$$

$\partial V/\partial x = C1$

$V = C1\, x + C2$

Where C1 and C2 are arbitrary constants to be evaluated by boundary conditions.

Now as at $x = 0$, $V = V1$ and $x = d$, $V = V2$

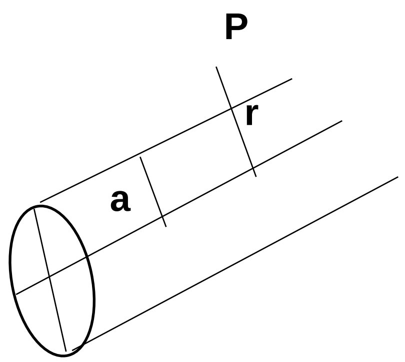
Therefore $V1 = C1.0 + C2$; $C2 = V1$

and $V2 = C1d + C2$; $C1 = (V2-V1)/d$

therefore $V = (V2-V1/d)\, x + V1 = V1 - ((V1-V2)/d)\, x$

This is the required result and represents the potential at any point between the plates.

Potential at a point due to a cylindrical shell



In this case due to symmetry V does not depend on and z so Laplace equation reduces to

$(1/r)\, \partial/\partial r(r\partial V/\partial r) = 0$

$\partial/\partial r(r\partial V/\partial r) = 0$

$r\partial V/\partial r = A$ or $\partial V/\partial r = A/r$ (1)

so $V = A \log r + B$ (2)

To determine the arbitrary constants A and B

$E = -(\partial V/\partial r) = \lambda^*/2\pi\epsilon_0 a$

And $V = V0$ for $r = R$

In the light of the above boundary conditions (1) and (2) give respectively

$A/a = -\lambda/2\pi\epsilon 0a$ and $V0 = A \log R + B$

$A = -\lambda/2\pi\epsilon 0$ and $B = V0 + \lambda/2\pi\epsilon 0 \log R0$

Therefore $V = V0 + (\lambda/2\pi\epsilon 0) \log R0/r$

The method of images

1. The method of images is a nice way to solve problems. The difficulty is that it really only works well in a very few cases with special symmetry.

2. The method is usually applied to situations where there is a known charge near a perfectly conducting surface.

3. Three examples are as follows: (1) a point charge above a conducting sheet, (2) a line charge parallel to a conducting cylinder, and (3) a point charge outside a conducting sphere.

4. The method of images involves some luck. The shape of the surface must have the right symmetry so that it can be replaced by a simple, finite collection of charges known as the image charges.

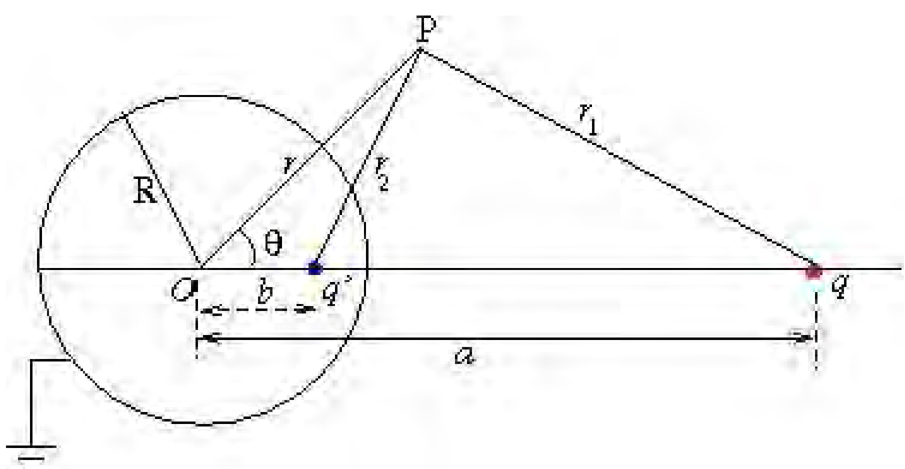
5. Luck is required because the potential due to the given charge plus the image charges must be such that the potential on the original conducting surface just turns out to be a constant.

6. One can often replace a general shaped conductor with an infinite set of charges but this is not so useful. Only when a small, finite number of charges is required is the replacement useful and corresponds to the method of images.

Method of Images for a spherical conductor

The method of images can be applied to the case of a charge in front of a grounded spherical conductor. The method is not as straightforward as the case of plane conductors but works equally well. Consider a charge q kept at a distance *a* from the centre of the grounded sphere. We wish to obtain an expression for the potential at a point P which is at the position (r, θ), the potential obviously will not depend on the azimuthal angle and hence that coordinate has been suppressed. Let the point P be at a distance r_1 from the location of the charge q.

The image charge is located at a distance b from the centre along the line joining the centre to the charge q. The line joining the charges and the centre is taken as the reference line with respect to which the angle θ is measured. Let P be at a distance *b* from the image charge. Let q' be the image charge.



The potential at P is given by $\varphi(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right)$. Using the property of triangle, we can express the potential at P as,

$$\varphi(r, \theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br\cos\theta}}\right]$$

Since the potential vanishes at $r = R$ for all values of θ, the signs of q and q' must be opposite, and we must have,

$$q^2(R^2 + b^2 - 2bR\cos\theta) = q'^2(\,R^2 + a^2 - 2aR\cos\theta\,)$$

In order that this relation may be true for all values of θ, the coefficient of $\cos\theta$ from both sides of this equation must cancel,

$$2bRq^2 = 2aRq'^2$$

Since q andq' have opposite sign, this gives,

$$\boxed{q' = -\sqrt{\frac{b}{a}}\; q}$$

Substituting this in the θ independent terms above, we get,

$$R^2 + b^2 = \frac{b}{a}(R^2 + a^2)$$

which gives $ab = R^2$. Thus $b = \frac{R^2}{a}, q' = -\frac{R}{a}q$.

It follows that if the object charge is outside the sphere, the image charge is inside the sphere. Using these, the potential at P is given by

$$\begin{aligned}\varphi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2br\cos\theta}}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{\frac{R}{a}}{\sqrt{r^2 + \left(\frac{R^4}{a^2}\right) - 2\left(\frac{R^2}{a}\right)r\cos\theta}}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{R}{\sqrt{r^2a^2 + R^4 - 2R^2ar\cos\theta}}\right]\end{aligned}$$

The electric field can be obtained by computing gradient of the potential. It can be easily verified that the tangential component of the electric field $E_t = E_\theta = 0$. The normal component is given by,

$$\begin{aligned}E_n = E_r &= -\frac{\partial\varphi}{\partial r} \\ &= -\frac{q}{4\pi\epsilon_0}\left[\frac{-r + a\cos\theta}{(r^2 + a^2 - 2ar\cos\theta\,)^{\frac{3}{2}}} + \frac{R(a^2r - R^2a\cos\theta)}{(r^2a^2 + R^4 - 2R^2ar\cos\theta\,)^{\frac{3}{2}}}\right]\end{aligned}$$

The charge density on the surface of the sphere is $\epsilon_0 E_n(r = R)$and is given by

$$\sigma(R,\theta) = -\frac{q}{4\pi}\left[\frac{-R + a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta\,)^{\frac{3}{2}}} + \frac{R(a^2R - R^2a\cos\theta)}{(R^2a^2 + R^4 - 2R^3a\cos\theta\,)^{\frac{3}{2}}}\right]$$

$$= \frac{q}{4\pi R}\frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta\ )^{\frac{3}{2}}}$$

The charge density is opposite to the sign of q since $R^2 - a^2 < 0$.

It is interesting to note that unlike in the case of a conducting plane, the magnitude of the image charge is not equal to that of the object charge but has a reduced value,

$$Q_{ind} = 2\pi\int_0^{\pi}\sigma(R,\theta)R^2\sin\theta d\theta = \frac{q}{2}\int_{-1}^{+1}\frac{R(R^2-a^2)}{(R^2+a^2-2aR\mu)^{3/2}}\frac{d\mu}{(-2aR)}$$

$$= \frac{qR(a^2-R^2)}{4aR}\left(R^2+a^2-2aR\mu\right)^{1/2}\Big|_{-1}^{1} = -q\frac{R}{a}$$

$$Q_{ind} = 2\pi\int_0^{2\pi}\sigma(R,\theta)R^2\sin\theta d\theta = \frac{q}{2}\int_{-1}^{+1}\frac{R(R^2-a^2)}{(R^2+a^2-2aR\mu)}\frac{d\mu}{(-2aR)}$$

$$= \frac{qR(a^2-R^2)}{4aR}(R^2+a^2-2aR\mu)|_{-1}^{+1} = -q\frac{R}{a}$$

Thus the potential can be written as

$$\boxed{\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r}-\vec{a}|} - \frac{\left(\frac{R}{a}\right)q}{\left|\vec{r}-\frac{R^2}{a^2}\vec{a}\right|}\right]}$$

The following figure shows the variation of charge density on the surface as a function of the angle θ. As expected, when the charge comes closer to the sphere, the charge density peaks around $\theta=0$.

Since the distance between the charge and its image is $a - b$, the force exerted on the charge by the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{|a-b|^2} = \frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{|a^2-R^2|^2}$$

For $a \gg R$, the force is proportional to the inverse cube of the distance of the charge from the centre. For $a \approx R$, let α be the distance of q from the surface of the sphere, $a = R + \alpha$; $\alpha \ll R$,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2R(R+\alpha)}{|2R\alpha+\alpha^2|^2} \sim \frac{1}{\alpha^2}$$

Model questions

Unit – II

Part – A

1. What do you mean by boundary conditions?
2. Write the boundary conditions on the surface between two dielectric media of different permittivity.
3. Express Laplace equation in Cartesian coordinates.
4. Express Laplace equation in Cylindrical coordinates.
5. Define method of images.

Part – B

1. Derive the boundary conditions on the surface between two dielectric media of different permittivity.
2. Explain the method of images for the solution of electrostatic problems.
3. Obtain the solution of Laplace equation in Cartesian coordinates.
4. Deduce the solution of Laplace equation for a problem in Cylindrical coordinates.

Part – C

1. A point charge q is placed at a distance d from an infinite conductor held at zero potential. Using method of images, find the surface charge density of induced charge and force between the charge and the plane.
2. Calculate the potential at an external point due to a point charge which is kept near an infinite conductor held at zero potential.
3. Write down the Laplace equation in Cartesian coordinate and find its solution. What will the solution for potential at a point between the plates of a parallel plate capacitor?
4. Write down the Laplace equation in Cylindrical coordinate and find its solution. What will the solution for potential at a point due to a Cylindrical shell?

Books for study and reference

1. Electromagnetic theory by K. K. CHOPRA and G. C. AGARWAL
   K. Nath & Co., Meerut

2. Electromagnetic theory and Electrodynamics by SATYAPRAKASH
   Kedarnath & Ramanan Co., Meerut