SEMESTER : I

ALLIED COURSE : I - Mathematics

Inst Hour	: 5
Credit	: 3
Code	: 18K1CH/PAM1

CALCULUS AND VECTOR CALCULUS

(For B.Sc., Physics & Chemistry Major)

UNIT 1:

Successive Differentiation – n^{th} derivative of standard functions – Fractional Expressions of the form $\frac{f(x)}{\varphi(x)}$ – Trigonometrical Transformation - Leibnitz's Theorem (proof not needed) for the n^{th} derivative of a product of functions – applicable to suitable problems

(Chapter 3-Sec 1.1-1.6, 2.1, 2.2 of Text Book 1)

UNIT 2:

Curvature – Circle and Centre of Curvature - Radius of Curvature in Cartesian only.

(Chapter 10- Sec 2.1-2.3 of Text Book 1)

UNIT 3:

General properties of definite integrals -Reduction formula for (when n is a positive integer)

1] $\int e^{ax}x^n dx$ 2] $\int \sin^n x dx$ 3] $\int \cos^n x dx$ 4] $\int_0^b e^{ax}x^n dx$ 5] $\int_0^{\frac{\pi}{2}} \sin^n x dx$ 6] $\int_0^{\frac{\pi}{2}} \cos^n x dx$

7] Without proof $\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x dx$ - and illustrations (problems only)

Evaluation of double & triple integrals (omitting -changing the order of integration)

(Chapter 1- Sec11, Sec13-13.1, 13.3, 13.4, 13.5 and Chapter 5 Sec 2.2 & 4 of Text Book 2)

UNIT 4:

Vector Differentiation – Gradient of a vector - Directional Derivative – Unit Normal Vector - Divergence & Curl of a vector, Solenoidal & Irrotational vectors – Vector Identities.

(Chapter 2 – Sec 2.1, 2.2, 2.2.1-2.2.4, 2.3, 2.3.1, 2.3.2, 2.4, 2.4.1-2.4.3, 2.5, 2.5.1 of Text Book 3)

UNIT 5:

Vector integration –Line Integral- surface integral - Volume integral – simple problems. Gauss Divergence Theorem – Stoke's Theorem – problems only (Verification of the theorems).

(Chapter 3- Sec: 3.2-3.7 Chapter 4 Sec: 4.2, 4.2.3, 4.4, 4.4.3 of Text book 3)

Text Books :

[1] S.Narayanan, T.K.Manickavasagam Pillai, Calculus Volume I, S.V Publishers 2004

[2] S.Narayanan, T.K.Manickavasagam Pillai, Calculus Volume II, S.V Publishers 2003

[3] K.Viswanatham ,S.Selvaraj ,Vector Analysis , Emerald Publishers 1984

Reference Books:

[1] A.Singaravelu,Calculus .

[2] M.L.Khanna, Vector Calculus

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.

Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.

Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

3.3.18 HOD

ALLIED MATHEMATICS FOR PHYSICS & CHEMISTRY

TITLE OF THE PAPER: CALCULUS AND VECTOR CALCULUS

PAPER CODE: 18KICH/PAM1

UNIT - I

SUCCESSIVE DIFFERENTIATION

Let y be a function of x, then it can be differentiable with respect to x and it may be denoted as y' or y_1 or $\frac{d}{dx}(y) = \frac{dy}{dx}$.

Here y' or y_1 or $\frac{dy}{dx}$ is the first derivative

y'' or y_2 or $\frac{d^2y}{dx^2}$ is the second derivative.

$\vdots$

y^n or y_n or $\frac{d^ny}{dx^n}$ is the n^{th} derivative.

The Symbols of the Successive Derivatives are:

$$\frac{d}{dx}(y) = \frac{dy}{dx} = Dy$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y$$

$$\vdots$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny$$

If $y = f(x)$ the Successive Derivatives are

$f'(x), f''(x), \dots, f^n(x)$ (or)

$y', y'', y''', \dots, y^n$ (or)

$y_1, y_2, y_3, \dots, y_n$

The n^{th} derivative:

If $y = e^{ax}$, then $\frac{dy}{dx} = ae^{ax}$; $\frac{d^2y}{dx^2} = a^2e^{ax}$; . . . .

$$\frac{d^ny}{dx^n} = a^ne^{ax}.$$

STANDARD RESULTS:

1. If $y = (ax+b)^m$ then

$$y_1 = \frac{dy}{dx} = m(ax+b)^{m-1}\cdot(a) = (a)(m)(ax+b)^{m-1}$$

$$y_2 = \frac{d^2y}{dx^2} = (a)(m)(m-1)(ax+b)^{m-2}(a) = (a^2)(m)(m-1)(ax+b)^{m-2}$$

$$y_3 = \frac{d^3y}{dx^3} = (a^2)(m)(m-1)(m-2)(ax+b)^{m-3}\cdot(a)$$

$$y_3 = (a^3)(m)(m-1)(m-2)(ax+b)^{m-3}.$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_n = (a^n)(m)(m-1)(m-2)\cdots\cdots(m-n+1)(ax+b)^{m-n}.$$

2. Find the n^{th} derivative of $\log(ax+b)$.

Soln:

Let $y = \log(ax+b)$

$$y_1 = \frac{d}{dx}(y) = \frac{d}{dx}\{\log(ax+b)\} = \frac{1}{ax+b}(a) = a(ax+b)^{-1}.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\{a(ax+b)^{-1}\} = a\frac{d}{dx}\{(ax+b)^{-1}\} = a(ax+b)^{-2}(-a)$$

$$y_2 = (-a^2)(ax+b)^{-2}.$$

$$\vdots$$

$$y_n = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}\{\log(ax+b)\} = D^n\{\log(ax+b)\} = (D^{n-1})(D^n\{\log(ax+b)\})$$

$$= D^{n-1}\left\{\frac{d}{dx}[\log(ax+b)]\right\} = D^{n-1}\left\{\frac{1}{ax+b}(a)\right\} = D^{n-1}\{(a)(ax+b)^{-1}\}$$

$$= (a)D^{n-1}\{(ax+b)^{-1}\} = (a)\cdot\frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}$$

$$\therefore\ y_n = D^n\{\log(ax+b)\} = \frac{(-1)^{n-1}(n-1)!\,(a^n)}{(ax+b)^n}.$$

Find the n^{th} derivative of $\frac{1}{ax+b}$.

Soln:

Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$.

$y_1 = \frac{dy}{dx} = \frac{d}{dx}\{(ax+b)^{-1}\} = (-1)(ax+b)^{-2}(a)$

$y_1 = (-a)(ax+b)^{-2}(a)$.

$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\{(-1)(ax+b)^{-2}(a)\}$

$y_2 = (-1)(-2)(ax+b)^{-3}(a^2)$

$y_3 = \frac{d}{dx}(y_2) = \frac{d}{dx}\{(-1)(-2)(ax+b)^{-3}(a^2)\}$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4}(a^3)$.

$\vdots$

$y_n = \frac{d}{dx}(y_{n-1}) = \frac{d^ny}{dx^n} = (-1)(-2)(-3)\cdots(-n)(ax+b)^{-(n+1)}(a^n)$

$y_n = (-1)(-2)(-3)\cdots(-n)(ax+b)^{-(n+1)}(a^n)$

$y_n = (-1)^n(1)(2)(3)\cdots(n)(ax+b)^{-(n+1)}(a^n)$

$y_n = D^n(y) = D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n(n!)(a^n)}{(ax+b)^{n+1}}$.

If $y = \sin(ax+b)$ find y_n.

Soln: $y = \sin(ax+b)$

$y_1 = \frac{d}{dx}(\sin(ax+b)) = \cos(ax+b)(a)$

$y_1 = (a)\cos(ax+b) = (a)\sin(ax+b+\pi/2)$

$y_1 = (a)\sin(ax+b+\pi/2)$.

$y_2 = \frac{d}{dx}\{(a)\sin(ax+b+\pi/2)\} = (a)\frac{d}{dx}\{\sin(ax+b+\pi/2)\}$

$= (a)\cos(ax+b+\pi/2)(a) = (a^2)\cos(ax+b+\pi/2)$.

$y_2 = (a^2)\sin(ax+b+\pi/2+\pi/2)$

$\therefore y_2 = (a^2)\sin(ax+b+2\pi/2)$.

Likewise $y_3 = a^3\sin(ax+b+3\pi/2)$

$\vdots$

$y_n = a^n\sin(ax+b+n\pi/2)$.

Find the n^{th} derivative of $e^{ax}\sin(bx+c)$.

Soln:

$$y = e^{ax}\sin(bx+c).$$

$$y_1 = \frac{d}{dx}(y) = \frac{d}{dx}\{e^{ax}\sin(bx+c)\}$$

$$= (a)(e^{ax})\sin(bx+c) + (e^{ax})(b)(\cos(bx+c))$$

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c) \longrightarrow ①.$$

Take $a = r\cos\alpha$ and $b = r\sin\alpha$

$$a^2+b^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = r^2(\cos^2\alpha+\sin^2\alpha) = r^2(1) = r^2.$$

$$a^2+b^2 = r^2 \Rightarrow r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$$

$$\frac{b}{a} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha;\ \frac{b}{a} = \tan\alpha. \Rightarrow \alpha = \tan^{-1}(b/a).$$

Substitute $a = r\cos\alpha$ and $b = r\sin\alpha$ in ①.

$$① \rightarrow y_1 = e^{ax}(a\sin(bx+c) + b\cos(bx+c))$$

$$= e^{ax}\{(r\cos\alpha)\sin(bx+c) + (r\sin\alpha)\cos(bx+c)\}$$

$$= (e^{ax})(r)\{(\cos\alpha)\sin(bx+c) + (\sin\alpha)\cos(bx+c)\}$$

$$\therefore y_1 = r.e^{ax}.\sin(bx+c+\alpha) \longrightarrow ②.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\{r(e^{ax})\sin(bx+c+\alpha)\}$$

$$= (r)\frac{d}{dx}\{(e^{ax})\sin(bx+c+\alpha)\}.$$

$$= r\{(a.e^{ax})\sin(bx+c+\alpha) + (e^{ax})(b)\cos(bx+c+\alpha)\}$$

$$= (r.e^{ax})\{a\sin(bx+c+\alpha) + b\cos(bx+c+\alpha)\}.$$

$$= (r.e^{ax})\{(r\cos\alpha)\sin(bx+c+\alpha) + (r\sin\alpha)\cos(bx+c+\alpha)\}$$

$$= r^2.e^{ax}\{(\cos\alpha)\sin(bx+c+\alpha) + (\sin\alpha)\cos(bx+c+\alpha)\}$$

$$y_2 = r^2.e^{ax}.\sin(bx+c+2\alpha) \longrightarrow ③.$$

$$\vdots$$

$$y_n = \frac{d^ny}{dx^n} = (r^n)(e^{ax})\sin(bx+c+n\alpha) \longrightarrow ④.$$

Substitute $r = (a^2+b^2)^{1/2}$ and $\alpha = \tan^{-1}(b/a)$ in ④.

$$y_n = \{(a^2+b^2)^{1/2}\}^n.(e^{ax})\sin(bx+c+n.\tan^{-1}(b/a)).$$

$$= (a^2+b^2)^{n/2}(e^{ax})\sin(bx+c+n.\tan^{-1}(b/a)).$$

$$y_n = (e^{ax})(a^2+b^2)^{n/2}\sin(bx+c+n.\phi)$$

Similarly, $D^n\{e^{ax}\cos(bx+c)\} = r^n e^{ax}\cos(bx+c+n.\phi)$

where $r = (a^2+b^2)^{1/2}$ & $\phi = \tan^{-1}(b/a)$.

Find y_n for $y = \dfrac{x^2}{(x-1)^2(x+2)}$

<u>Soln:</u> $y = \dfrac{x^2}{(x-1)^2(x+2)}$

Resolving into Partial Fractions.

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$A = 5/9$; $B = 1/3$; $C = 4/9$.

$$\therefore y = \frac{5/9}{(x-1)} + \frac{1/3}{(x-1)^2} + \frac{4/9}{(x+2)}$$

$$y = \frac{5}{9}(x-1)^{-1} + \frac{1}{3}(x-1)^{-2} + \frac{4}{9}(x+2)^{-1}$$

$$y_n = D^n\left\{\frac{5}{9}(x-1)^{-1}\right\} + D^n\left\{\frac{1}{3}(x-1)^{-2}\right\} + D^n\left\{\frac{4}{9}(x+2)^{-1}\right\}$$

$$= \left(\frac{5}{9}\right)\frac{n!(-1)^n}{(x-1)^{n+1}} + \left(\frac{1}{3}\right)\frac{(n+1)!(-1)^n}{(x-1)^{n+2}} + \left(\frac{4}{9}\right)\frac{n!(-1)^n}{(x+2)^{n+1}}$$

$$\therefore y_n = (-1)^n(n!)\left\{\frac{5/9}{(x-1)^{n+1}} + \frac{1/3(n+1)}{(x-1)^{n+2}} + \frac{4/9}{(x+2)^{n+1}}\right\}.$$

Find y_n where $y = \dfrac{1}{x^2+a^2}$.

<u>Soln:</u> $y = \dfrac{1}{x^2+a^2} \Rightarrow y = \dfrac{1}{2ai}\left\{\dfrac{1}{x-ai} - \dfrac{1}{x+ai}\right\}$

$$y = \frac{1}{2ai}\left\{(x-ai)^{-1} - (x+ai)^{-1}\right\}$$

using the formula for $D^n(ax+b)^{-1} = \dfrac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}$

$$y_n = D^n\left\{\left(\frac{1}{2ai}\right)\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)\right\} = D^n\left\{\frac{1}{2ai}\left((x-ai)^{-1} - (x+ai)^{-1}\right)\right\}$$

$$= \frac{1}{2ai}\left\{\frac{(-1)^n(n!)}{(x-ai)^{n+1}} - \frac{(-1)^n(n!)}{(x+ai)^{n+1}}\right\}$$

$$\therefore y_n = \frac{(-1)^n(n!)}{2ai}\left\{\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right\}.$$

If $y = a\cos(\log x) + b\sin(\log x)$

show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

<u>Soln:</u> $y = a\cos(\log x) + b\sin(\log x)$.

$$\frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x} \longrightarrow (1).$$

Multiply ① by x on bothsides,

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x) \longrightarrow ②$$

Differentiate ② w.r.t x on bothsides,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

Multiply by x on bothsides,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -a\cos(\log x) - b\sin(\log x)$$

(ie) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(a\cos(\log x) + b\sin(\log x))$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \quad \therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$

Find the n^{th} differential Co-efficient of $\cos^5\theta\sin^7\theta$.

Soln: let $x = \cos\theta + i\sin\theta \Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x - \frac{1}{x} = 2i\sin\theta$

By De Moivre's Theorem,

$x^n = \cos n\theta + i\sin n\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$2^5\cos^5\theta = (x+\frac{1}{x})^5$ and $2^7(i)^7\sin^7\theta = (x-\frac{1}{x})^7$

Hence,

$$(2^5\cos^5\theta)(2^7i^7\sin^7\theta) = (x+\tfrac{1}{x})^5(x-\tfrac{1}{x})^7 = (x+\tfrac{1}{x})^5(x-\tfrac{1}{x})^5(x-\tfrac{1}{x})^2$$

$$= (x^2-\tfrac{1}{x^2})^5(x-\tfrac{1}{x})^2 = (x^{10}-5x^6+10x^2-10/x^2+5/x^6-1/x^{10})(x^2+1/x^2-2)$$

$$= (x^{12}-1/x^{12}) - 2(x^{10}-1/x^{10}) - 4(x^8-1/x^8) + 10(x^6-1/x^6) + 5(x^4-1/x^4) - 20(x^2-1/x^2)$$

$\therefore 2^{12}(i)^7\cos^5\theta\sin^7\theta$

$$= 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta) + 5(2i\sin 4\theta) - 20(2i\sin 2\theta).$$

$2^{12}(-i)\cos^5\theta\sin^7\theta$

$$= 2i(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5.\sin 4\theta - 20\sin 2\theta)$$

$$\cos^5\theta\sin^7\theta = \frac{2i}{2^{12}(-i)}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$$

$$= \frac{-1}{2^{11}}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$$

$D^n(\cos^5\theta\sin^7\theta)$

$= D^n\left\{\frac{-1}{2^{11}}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)\right\}$

$= \left(\frac{-1}{2^{11}}\right)D^n\{\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta\}$

$\therefore D^n(\cos^5\theta\sin^7\theta) = \frac{-1}{2^{11}}\{(12)^n\sin(12\theta + n\pi/2) - (2)(10)^n\sin(10\theta + n\pi/2)$
$- (4)(8)^n\sin(8\theta + n\pi/2) + (10)(6)^n\sin(6\theta + n\pi/2)$
$+ (5)(4)^n\sin(4\theta + n\pi/2) - (20)(2)^n\sin(2\theta + n\pi/2)\}$

If $y = ae^{mx} + be^{-mx}$ ST $\frac{d^2y}{dx^2} - m^2y = 0$.

<u>Soln:</u> $y = ae^{mx} + be^{-mx} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(ae^{mx} + be^{-mx})$

$\frac{dy}{dx} = a\frac{d}{dx}(e^{mx}) + b\frac{d}{dx}(e^{-mx}) = a.m.e^{mx} - b.m.e^{-mx}$

$\frac{dy}{dx} = m(ae^{mx} - be^{-mx}) \Rightarrow \frac{d^2y}{dx^2} = m\{a.me^{mx} - b(-m)e^{-mx}\}$

$\frac{d^2y}{dx^2} = m\{a.m.e^{mx} + bme^{-mx}\} = m^2(ae^{mx} + be^{-mx}) = m^2y.$

$\frac{d^2y}{dx^2} = m^2y \Rightarrow \frac{d^2y}{dx^2} - m^2y = 0.$

If $y = \sin(m\sin^{-1}x)$ ST $(1-x^2)y_2 - xy_1 + m^2y = 0$.

<u>Soln:</u> $y = \sin(m\sin^{-1}x) \Rightarrow \sin^{-1}(y) = m\sin^{-1}(x) \longrightarrow (1)$.

Differentiate (1) w.r.t x on bothsides.

$\frac{d}{dx}(\sin^{-1}(y)) = \frac{d}{dx}(m\sin^{-1}(x)) = m\frac{d}{dx}(\sin^{-1}x)$.

$\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = m\cdot\frac{1}{\sqrt{1-x^2}} \longrightarrow (2)$.

(ie) $\sqrt{1-x^2}\cdot\frac{dy}{dx} = m\sqrt{1-y^2} \longrightarrow (3)$.

Squaring the bothsides (3) $\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2) \to (4)$

Differentiate (4) w.r.t on bothsides.

$(1-x^2)\left(2\cdot\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - (2x)\left(\frac{dy}{dx}\right)^2 = (m^2)(-2y)\frac{dy}{dx}$.

Divide by $2\frac{dy}{dx}$ on bothsides.

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$.

(ie) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

(ie) $(1-x^2)y_2 - xy_1 + m^2y = 0$. (7)

LEIBNITZ FORMULA FOR THE n^{th} DERIVATIVE OF A PRODUCT:

If u and v are functions of x.

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \; ; \; D(uv) = vD(u) + u \cdot D(v).$$

$$D^2(uv) = vD^2(u) + uD^2(v) + 2D(u)D(v)$$

$$D^3(uv) = vD^3(u) + 3D^2(u)D(v) + 3D(u)D^2(v) + uD^3(v).$$

Using Binomial Theorem,

$$D^n(uv) = v \cdot D^n(u) + nC_1 D^{n-1}(u)D(v) + nC_2 D^{n-2}(u)D^2(v)$$
$$+ nC_3 D^{n-3}(u)D^3(v) + \cdots + nC_r D^{n-r}(u)D^r(v)$$
$$+ u \cdot D^n(v).$$

Example:

Find the n^{th} differential Co-efficient of x^3e^{5x}.

Soln: Let $u = e^{5x}$; $v = x^3$.

$$D^n(e^{5x}x^3) = D^n(e^{5x})(x^3) + nC_1 D^{n-1}(e^{5x})D(x^3) + nC_2 D^{n-2}(e^{5x})D^2(x^3)$$
$$+ nC_3 D^{n-3}(e^{5x})D^3(x^3).$$

$$D^n(e^{5x}x^3) = (5^n)(e^{5x})(x^3) + nC_1 D^{n-1}(e^{5x})(3x^2)$$
$$+ (5^{n-2})(nC_2)(e^{5x})(6x) + (nC_3)(5^{n-3})(6).$$

If $y = a\cos(\log x) + b\sin(\log x)$ show that $x^2y_2 + xy_1 + y = 0$ and also prove that $x^2y_{n+2} + xy_{n+1}(2n+1) + (n^2+1)y_n = 0$.

Soln:

$$y = a\cos(\log x) + b\sin(\log x) \longrightarrow (1)$$

Differentiating (1) w.r.t x on bothsides.

$$y_1 = -a\sin(\log x)(1/x) + b\cos(\log x)(1/x)$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x) \longrightarrow (2).$$

Differentiate (2) w.r.t x on bothsides,

$$xy_2 + y_1 = -a\cos(\log x)(1/x) - b\sin(\log x)(1/x)$$

$$x(xy_2 + y_1) = -x\{a\cos(\log x)(1/x) + b\sin(\log x)(1/x)\}$$

$$x^2y_2 + xy_1 = -\{a\cos(\log x) + b\sin(\log x)\}$$

$$x^2y_2 + xy_1 = -y \Rightarrow x^2y_2 + xy_1 + y = 0 \longrightarrow (3).$$

Differentiate (3) w.r.t x for n times and using Leibnitz Formula.

$$D^n\{x^2y_2 + xy_1 + y\} = 0$$

$D^n(x^2y_2)+D^n(xy_1)+D^n(y)=0.$

$D^n(y_2)(x^2)+nc_1D^{n-1}(y_2)D(x^2)+nc_2D^{n-2}(y_2)D^2(x^2)$
$+D^n(y_1)(x)+nc_1D^{n-1}(y_1)D(x)+D^n(y)=0.$

$(y_{n+2})(x^2)+(n)(y_{n+1})(2x)+\frac{n(n-1)}{2}(y_n)(2)$
$+(y_{n+1})(x)+(n)(y_n)(1)+y_n=0.$

$x^2y_{n+2}+(2xn)y_{n+1}+\frac{2(n^2-n)}{2}y_n+xy_{n+1}+ny_n+y_n=0.$

$x^2y_{n+2}+(2xn)y_{n+1}+(n^2-n)y_n+xy_{n+1}+ny_n+y_n=0.$

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-n+n+1)y_n=0.$

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$

If $y=\sinh(m\sinh^{-1}x)$ prove that
$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$

Soln:

$y=\sinh(m\sinh^{-1}x) \Rightarrow \sinh^{-1}(y)=m\sinh^{-1}(x) \longrightarrow ①.$

Differentiate ① w.r.t x,

$\frac{1}{\sqrt{1+y^2}}\cdot\frac{dy}{dx}=m\cdot\frac{1}{\sqrt{1+x^2}} \Rightarrow \sqrt{1+x^2}\cdot\frac{dy}{dx}=m\sqrt{1+y^2}.$

squaring on bothsides, $(1+x^2)\left(\frac{dy}{dx}\right)^2=m^2(1+y^2)\rightarrow ②$

Differentiate ② w.r.t x,

$(1+x^2)\left(2\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2(2x)=m^2(2y)(y_1).$

$(1+x^2)(2y_1)(y_2)+(y_1)^2(2x)=m^2(2yy_1)\longrightarrow ③$

Divide ③ by $2y_1$ on bothsides.

$(1+x^2)y_2+xy_1=m^2y \Rightarrow (1+x^2)y_2+xy_1-m^2y=0\longrightarrow ④$

Differentiate ④ for n^{th} derivative and applying Leibnitz Theorem.

$D^n\{(1+x^2)y_2+xy_1-m^2y\}=0.$

$D^n(y_2)(1+x^2)+nc_1D^{n-1}(y_2)D(1+x^2)+nc_2D^{n-2}(y_2)D^2(1+x^2)$
$D^n(y_1)(x)+nc_1D^{n-1}(y_1)D(x)-D^n(m^2y)=0.$

$y_{n+2}(1+x^2)+(n)(y_{n+1})(2x)+\frac{n(n-1)}{2}(y_n)(2)+y_{n+1}(x)$
$+n(y_n)(1)-m^2(y_n)=0.$

$(1+x^2)y_{n+2}+(2xn)y_{n+1}+\frac{n^2-n}{2}(2)y_n+(x)y_{n+1}$
$+ny_n-m^2y_n=0.$

$\therefore (1+x^2)y_{n+2}+(2n+1)(x)y_{n+1}+(n^2-n+n-m^2)y_n=0.$

$(1+x^2)y_{n+2}+(2n+1)(x)y_{n+1}+(n^2-m^2)y_n=0.$

If $y = (x+\sqrt{1+x^2})^m$ show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Soln: $y = (x+\sqrt{1+x^2})^m \longrightarrow ①$.

Differentiate ① w.r.t x.

$$y_1 = m(x+\sqrt{1+x^2})^{m-1}\left(1+\frac{1}{2\sqrt{1+x^2}}(2x)\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\cdot y_1 = m(x+\sqrt{1+x^2})^{m-1}\cdot(x+\sqrt{1+x^2})$$

$$\sqrt{1+x^2}(y_1) = m(x+\sqrt{1+x^2})^m \Rightarrow \sqrt{1+x^2}(y_1) = my \longrightarrow ②.$$

squaring ② on bothsides.

$$(1+x^2)(y_1)^2 = m^2y^2 \Rightarrow (1+x^2)(y_1^2) - m^2y^2 = 0 \longrightarrow ③.$$

Differentiate ③ w.r.t x

$$(1+x^2)(2y_1)(y_2) + (y_1)^2(2x) - m^2(2y)(y_1) = 0.$$

Divide by $2y_1$ on bothsides.

$$(1+x^2)(y_2) + x(y_1) - m^2y = 0 \longrightarrow ④.$$

Applying Leibnitz Theorem and Differentiate ④ for n times.

$$D^n\{(1+x^2)y_2 + xy_1 - m^2y\} = 0.$$

$$D^n(y_2)(1+x^2) + nC_1 D^{n-1}(y_2)D(1+x^2) + nC_2 D^{n-2}(y_2)D^2(1+x^2)$$
$$+ D^n(y_1)(x) + nC_1 D^{n-1}(y_1)D(x) - m^2D^n(y) = 0.$$

$$(y_{n+2})(1+x^2) + (n)(y_{n+1})(2x) + \frac{n(n-1)}{2}(y_n)(2)$$
$$+ (y_{n+1})(x) + (n)(y_n)(1) - m^2(y_n) = 0.$$

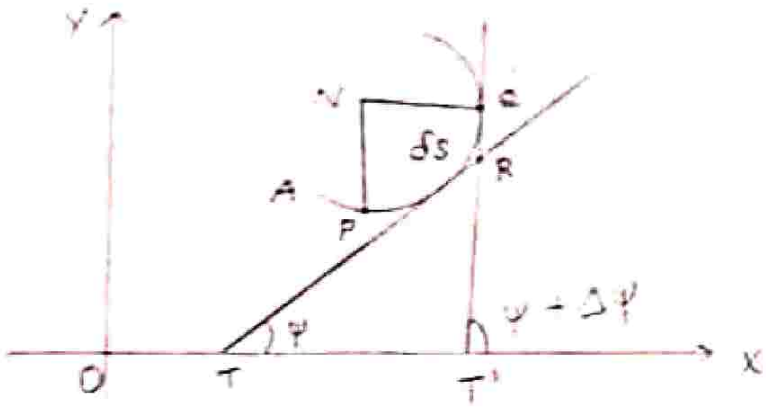
$$(1+x^2)y_{n+2} + (2xn)y_{n+1} + \frac{(n^2-n)}{2}(2)y_n + (x)y_{n+1}$$
$$+ n(y_n) - m^2(y_n) = 0.$$

$$(1+x^2)y_{n+2} + (2xn+x)y_{n+1} + (n^2-n+n-m^2)y_n = 0$$

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

UNIT II

CURVATURE



CENTRE OF CURVATURE:

Let P be a given point on a given curve, and Q be any other point on it. Let the normals at P and Q intersect in N. If N tends to a definite position C as Q tends to P, then C is called the centre of curvature of the curve at P.

CURVATURE:

The reciprocal of the distance CP is called the curvature of the curve at P.

CIRCLE OF CURVATURE:

The circle with its centre at C and radius CP is called the circle of curvature of the curve at P.

RADIUS OF CURVATURE:

The distance CP is called the radius of curvature of the curve at P. The radius of curvature is usually denoted by ρ.

Example 1:

Find the radius of curvature (ρ) for the catenary whose intrinsic equation is $s = a\tan\psi$.

Soln:

$$s = a\tan\psi$$

$$\rho = \frac{ds}{d\psi} = \frac{d}{d\psi}(a\tan\psi)$$

$$\rho = a\sec^2\psi.$$

Example 2:

Find the radius of curvature (ρ) for the cycloid $s = 4a\sin\psi$.

Soln:

$$s = 4a\sin\psi$$

$$\rho = \frac{ds}{d\psi} = \frac{d}{d\psi}(4a\sin\psi).$$

$$\rho = 4a\cos\psi.$$

CARTESIAN FORMULA FOR THE RADIUS OF CURVATURE:

$$\frac{dy}{dx} = \tan\psi.$$

$$\frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx} = \sec^2\psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$\frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} \text{ as } \frac{dx}{ds} = \cos\psi$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{ds}{d\psi} = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

Example 1:

What is the radius of curvature of the curve $x^4+y^4=2$ at the point $(1,1)$?

Soln:

$$x^4+y^4=2 \longrightarrow (1).$$

Diff. (1) w.r.t x on bothsides,

$$4x^3+4y^3\frac{dy}{dx}=0.$$

$$x^3+y^3\frac{dy}{dx}=0.$$

$$y^3\frac{dy}{dx}=-x^3 \Rightarrow \frac{dy}{dx}=\frac{-x^3}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{(-x^3)(3y^2)\frac{dy}{dx}+y^3(3x^2)}{y^6}$$

$$= \frac{-3x^3y^2\frac{dy}{dx}+3x^2y^3}{y^6} = \frac{3x^2y^2\left(y-x\frac{dy}{dx}\right)}{y^6}$$

$$\frac{d^2y}{dx^2} = \frac{3x^2y-3x^3\,dy/dx}{y^4}$$

At the point $(1,1)$ $\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-x^3}{y^3} = \frac{-1}{1} = -1.$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,1)} = \frac{3(1)(1)-3(1)(-1)}{(1)} = \frac{3+3}{1} = 6.$$

$$\rho = \frac{\left\{1+(dy/dx)^2\right\}^{3/2}}{d^2y/dx^2} \Rightarrow \rho_{(1,1)} = \frac{(1+(-1)^2)^{3/2}}{6} = \frac{2^{3/2}}{6} = \frac{2\sqrt{2}}{6}$$

$$\boxed{\rho = \sqrt{2}/3}$$

4. Find ρ for $y = c\cosh x/c$ at $(0, c)$. ④

Soln:

Given $y = c\cosh(x/c)$

$$\frac{dy}{dx} = c \cdot \sinh(x/c) \cdot 1/c = \sinh(x/c).$$

$$\left(\frac{dy}{dx}\right)_{(0,c)} = \sinh(0/c) = \sinh(0) = 0.$$

$$\frac{d^2y}{dx^2} = \cosh(x/c)\, 1/c$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,c)} = \cosh(0/c)(1/c) = 1/c \cosh(0)$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,c)} = 1/c.$$

$$\rho = \frac{\left\{1 + (dy/dx)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho_{(0,c)} = \frac{(1+0)^{3/2}}{1/c} = \frac{1}{1/c} = c.$$

$$\boxed{\rho_{(0,c)} = c}$$

5. Show that the radius of curvature at any point of the Catenary $y = c\cosh x/c$ is equal to the length of the portion of two normal intercepted between the curve and the axis of x.

Soln:

$$y = c\cosh(x/c) \Rightarrow \frac{dy}{dx} = (c)\sinh(x/c)(1/c)$$

$$\frac{dy}{dx} = \sinh(x/c) \Rightarrow \frac{d^2y}{dx^2} = \cosh(x/c)(1/c)$$

$$\rho = \frac{\{1+(\frac{dy}{dx})^2\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\{1+(\sinh x/c)^2\}^{3/2}}{(1/c)\cosh(x/c)}$$

$$= \frac{\{1+\sinh^2(x/c)\}^{3/2}}{(1/c)\cosh(x/c)} = \frac{(\cosh^2(x/c))^{3/2}}{1/c \cosh x/c}$$

$$= \frac{\cosh^3 x/c}{\cosh x/c}(c) = c\,co$$

$$= c\cosh^2(x/c)$$

$$\rho = y^2/c.$$

Again at any point (x,y) the normal

$$= y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2} = y\cosh x/c = y^2/c$$

6. Find the radius of curvature for the curve $\sqrt{x}+\sqrt{y}=1$ at $(1/4, 1/4)$.

Soln: $\sqrt{x}+\sqrt{y} = 1 \longrightarrow (1)$.

Diff. (1) w.r.t x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}} \Rightarrow \left(\frac{dy}{dx}\right)_{(1/4,1/4)} = -\sqrt{\frac{1/4}{1/4}} = -1$$

⑥

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{x}\,\frac{1}{2\sqrt{y}}\,\frac{dy}{dx} + \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{\sqrt{y}}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{y}}(dy/dx)}{x}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1/4,1/4)} = \frac{\frac{\sqrt{1/4}}{2\sqrt{1/4}} - \frac{\sqrt{1/4}}{2\sqrt{1/4}}(-1)}{1/4} = \frac{\frac{1}{2}+\frac{1}{2}}{1/4}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1/4,1/4)} = \frac{1}{1/4} = 4.$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho_{(1/4,1/4)} = \frac{\{1+(-1)^2\}^{3/2}}{4} = \frac{2^{3/2}}{4}$$

$$= \frac{2^1 \cdot 2^{1/2}}{2^2} = \frac{2^{1/2}}{2} = \frac{\sqrt{2}}{2}$$

$$\rho_{(1/4,1/4)} = \frac{1}{\sqrt{2}}.$$

⑦

1. Find the radius of curvature of the curve $xy^2 = a^3 - x^3$ at $(a, 0)$.

Soln:

$$xy^2 = a^3 - x^3 \longrightarrow (1)$$

Diff (1) w.r.t x,

$$(x)(2y)\frac{dy}{dx} + y^2(1) = -3x^2.$$

$$2xy\frac{dy}{dx} + y^2 = -3x^2.$$

$$2xy\frac{dy}{dx} = -3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{-(3x^2+y^2)}{2xy}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \frac{-(3a^2+0)}{0} = \infty$$

$$\left(\frac{d^2y}{dx^2}\right)_{(a,0)} = \infty.$$

∴ The formula for ρ is $\dfrac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

$$\frac{dx}{dy} = \frac{-2xy}{3x^2+y^2} \Rightarrow \left(\frac{dx}{dy}\right)_{(a,0)} = \frac{-2(a)(0)}{3a^2+0}$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0.$$

$$\frac{d^2x}{dy^2} = \frac{(3x^2+y^2)\left(-2\left(x+y\frac{dx}{dy}\right)\right) + 2xy\left(6x\frac{dx}{dy}+2y\right)}{(3x^2+y^2)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-(3a^2+0)(2)(a+0) + 2(a)(0)}{(3a^2)^2} = \frac{-(3a^2)(a)(2)}{9a^4}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\rho = \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$$

$$\rho_{(a,0)} = \frac{(1+0)^{3/2}}{-2/3a} = \frac{-1}{2/3a}$$

$$\therefore \boxed{\rho_{(a,0)} = \frac{-3a}{2}} \Rightarrow \rho = \frac{3a}{2}.$$

8. Find ρ for the curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln:

$$x^3+y^3=3axy \longrightarrow (1).$$

Diff (1) w.r.t x,

$$3x^2+3y^2\frac{dy}{dx} = 3ay+3ax\frac{dy}{dx}.$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay-3x^2.$$

$$3\frac{dy}{dx}(y^2-ax) = 3(ay-x^2)$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}.$$

$$\left(\frac{dy}{dx}\right)_{(3a/2,\,3a/2)} = \frac{a(3a/2)-(3a/2)^2}{(3a/2)^2-a(3a/2)}$$

$$= \frac{3a^2/2 - 9a^2/4}{9a^2/4 - 3a^2/2}$$

$$= \frac{(12a^2-9a^2)/4}{(9a^2-12a^2)/4} = \frac{3a^2}{-3a^2}$$

$$\left(\frac{dy}{dx}\right)_{(3a/2,\,3a/2)} = -1.$$

$$\frac{d^2y}{dx^2} = \frac{(y^2-ax)\left(a\frac{dy}{dx}-2x\right)-(ay-x^2)\left(2y\frac{dy}{dx}-a\right)}{(y^2-ax)^2}.$$

$$\left(\frac{d^2y}{dx^2}\right)_{(3a/2,3a/2)} = \frac{\left((3a/2)^2-a(3a/2)\right)\left(a(-1)-2(3a/2)\right)-\left(a(3a/2)-(3a/2)^2\right)\cdot\left(2(3a/2)(-1)-a\right)}{\left((3a/2)^2-a(3a/2)\right)^2}$$

$$= \frac{(9a^2/4-3a^2/2)(-a-3a)-(3a^2/2-9a^2/4)(-3a-a)}{(9a^2/4-3a^2/2)^2}$$

$$= \frac{(9a^2/4-6a^2/4)(-4a)-(6a^2/4-9a^2/4)(-4a)}{\left(\frac{9a^2}{4}-\frac{6a^2}{4}\right)^2}$$

$$= \frac{(3a^2/4)(-4a)-(-3a^2/4)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$= \frac{-3a^3-3a^3}{\frac{9a^4}{16}} = \frac{-6a^3}{\frac{9a^4}{16}} = -6a^3 \times \frac{16}{9a^4}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(3a/2,3a/2)} = \frac{-32a^3}{3a^4} = \frac{-32}{3a}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\left(\frac{d^2y}{dx^2}\right)} \Rightarrow \rho_{(3a/2,3a/2)} = \frac{(1+(-1)^2)^{3/2}}{-32/3a}$$

$$= \frac{2^{3/2}}{-32/3a} = \frac{2\cdot 2^{1/2}}{-32/3a} = 2\cdot 2^{1/2} \times \frac{-3a}{32}$$

$$= \frac{-(2^{1/2})(3a)}{16} = \frac{-(3a)\sqrt{2}}{16}.$$

$$\therefore \rho = \frac{(3a)\sqrt{2}}{16}.$$

RADIUS OF CURVATURE IN PARAMETRIC CO-ORDINATE:

Let $x = f(t)$ and $y = \phi(t)$.

Then $\frac{dx}{dt} = f'(t)$, $\frac{dy}{dt} = \phi'(t)$.

$$\therefore \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)} \Rightarrow \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\phi'(t)}{f'(t)}\right)$$

$$= \frac{d}{dt}\left(\frac{\phi'(t)}{f'(t)}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{f'(t)\cdot\phi''(t) - \phi'(t)\cdot f''(t)}{(f'(t))^2}\cdot\frac{1}{f'(t)}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1+\left(\frac{\phi'(t)}{f'(t)}\right)^2\right\}^{3/2}}{\frac{f'(t)\cdot\phi''(t)-\phi'(t)\cdot f''(t)}{(f'(t))^3}} = \frac{\left\{1+\frac{(\phi'(t))^2}{(f'(t))^2}\right\}^{3/2}}{\frac{f'(t)\phi''(t)-\phi'(t)\cdot f''(t)}{(f'(t))^3}}$$

$$= \frac{\left\{\frac{(f'(t))^2+(\phi'(t))^2}{(f'(t))^2}\right\}^{3/2}}{\frac{f'(t)\phi''(t)-\phi'(t)f''(t)}{(f'(t))^3}} = \frac{\frac{\{(f'(t))^2+(\phi'(t))^2\}^{3/2}}{(f'(t)^2)^{3/2}}}{\frac{f'(t)\phi''(t)-\phi'(t)f''(t)}{(f'(t))^3}}$$

$$= \frac{\frac{((f'(t))^2+(\phi'(t))^2)^{3/2}}{(f'(t))^3}}{\frac{f'(t)\phi''(t)-\phi'(t)f''(t)}{(f'(t))^3}}$$

$$\rho = \frac{(f'(t)^2+\phi'(t)^2)^{3/2}}{f'(t)\phi''(t)-\phi'(t)f''(t)} \Rightarrow \rho = \frac{(f'^2+\phi'^2)^{3/2}}{f'\cdot\phi''-\phi'\cdot f''}$$

$$\frac{1}{\rho} = \frac{f'\cdot\phi''-\phi'\cdot f''}{(f'^2+\phi'^2)^{3/2}} = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$$

10. Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos\theta/2$.

Soln :

$x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$.

$$\frac{dx}{d\theta} = a(1+\cos\theta) \;;\; \frac{dy}{d\theta} = a(\sin\theta)\cdot$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{dy}{dx} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

$$\frac{dy}{dx} = \tan\theta/2\cdot$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta/2)$$

$$= \frac{d}{d\theta}(\tan\theta/2)\frac{d\theta}{dx} = \frac{1}{2}\sec^2\theta/2\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\theta/2\cdot\frac{1}{a\,2\cos^2\theta/2}$$

$$= \frac{1}{2}\left(\frac{1}{\cos^2\theta/2}\right)\cdot\frac{1}{2a\cos^2\theta/2} = \frac{1}{4a\cos^4\theta/2}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\theta/2)^{3/2}}{\frac{1}{4a\cos^4\theta/2}}$$

$$= \frac{(\sec^2\theta/2)^{3/2}}{1/4a\cos^4\theta/2} = \frac{\sec^3\theta/2}{1/4a\cos^4\theta/2} = \sec^3\theta/2\times$$

$$= \sec^3\theta/2\times 4a\cos^4\theta/2 = \frac{4a\cos^4\theta/2}{\cos^3\theta/2}$$

$$\rho = 4a\cos\theta/2\cdot$$

11. Find ρ at the point 't' of the curve $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$.

Soln:

$x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t.$$

$$\therefore \frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$= (\sec^2 t)\frac{1}{at\cos t} = \frac{1}{\cos^2 t}\cdot\frac{1}{at\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$$

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2 t)^{3/2}}{\frac{1}{at\cos^3 t}}$$

$$= \frac{(\sec^2 t)^{3/2}}{\frac{1}{at\cos^3 t}} = \frac{\sec^3 t}{\frac{1}{at\cos^3 t}}$$

$$= \sec^3 t \times at\cos^3 t = \frac{1}{\cos^3 t} \times at\cos^3 t$$

$$\therefore \rho = at.$$

ALLIED MATHEMATICS FOR PHYSICS & CHEMISTRY

TITLE OF THE PAPER: CALCULUS AND VECTOR CALCULUS

PAPER CODE : 18KICH/PAM1

UNIT III

PROPERTIES OF DEFINITE INTEGRALS

Property 1: $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Proof: $LHS = \int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a) \rightarrow ①$.

$RHS = -\int_b^a f(x)dx = -[F(x)]_b^a$

$= -[F(a)-F(b)] = F(b)-F(a) \longrightarrow ②$

① & ② $\Rightarrow \int_a^b f(x)dx = -\int_b^a f(x)dx.$

Example: $\int_1^2 x^2 dx = -\int_2^1 x^2 dx = 7/3.$

Property 2: $\int_a^b f(x)dx = \int_a^b f(y)dy.$

Proof: $LHS = \int_a^b f(x)dx = F(b)-F(a) \longrightarrow ①$

$RHS = \int_a^b f(y)dy = F(b)-F(a) \longrightarrow ②$

① & ② $\rightarrow \int_a^b f(x)dx = \int_a^b f(y)dy.$

Example: $\int_1^3 x^3 dx = \int_1^3 y^3 dy = \int_1^3 u^3 du$

$\int_1^3 x^3 dx = [x^4/4]_1^3 = \frac{(3)^4-(1)^4}{4} = \frac{81-1}{4} = \frac{80}{4} = 20.$

$\therefore \int_1^3 x^3 dx = \int_1^3 y^3 dy = \int_1^3 u^3 du = 20.$

Property 3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$ $a<c<b$

Proof: $LHS = \int_a^b f(x)dx = F(b)-F(a) \longrightarrow ①$

$$RHS = \int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a) \longrightarrow ②$$

$$① \& ② \Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

Proof: $RHS = \int_0^a f(a-x)dx \longrightarrow ①$

Limits: $a-x = y$; $x=0 \Rightarrow y=a$ & $x=a \Rightarrow y=0.$

$$a-x=y \Rightarrow -dx = dy.$$

$$① \Rightarrow \int_0^a f(a-x)dx = \int_a^0 f(y)(-dy) = -\int_a^0 f(y)dy$$

$$= \int_0^a f(y)dy = \int_0^a f(x)dx$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

Property 5: $\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx.$

Proof: $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx \longrightarrow ①.$

In the first integral of RHS put $x=-y$ and $dx=-dy.$

$$\therefore \int_{-a}^0 f(x)dx = \int_a^0 f(-y)(-dy) = -\int_a^0 f(-y)dy$$

$$= \int_0^a f(-y)dy$$

$$= \int_0^a f(-x)dx. \longrightarrow ②.$$

$$② \text{ in } ① \Rightarrow \int_{-a}^a f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx.$$

Corollary 1:

If $f(x)$ is an odd function, $f(-x) = -f(x)$

$$\int_{-a}^a f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx \Rightarrow \int_{-a}^a f(x)dx = 0$$

Example: $\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx = 0.$

Corollary 2:

If $f(x)$ is an even function, $f(-x) = f(x)$

$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$

Example: $\int_{-\pi/2}^{\pi/2} \cos^4 x\, dx = 2\int_0^{\pi/2} \cos^4 x\, dx =$

Property 6: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

Proof: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \longrightarrow (1).$

In the second integral of RHS.

$x = 2a - y \Rightarrow dx = -dy.$

Limits: $x = a \Rightarrow y = a$ & $x = 2a \Rightarrow y = 0.$

$\therefore \int_a^{2a} f(x)dx = \int_a^0 f(2a-y)(-dy) = -\int_a^0 f(2a-y)dy$

$= \int_0^a f(2a-y)dy = \int_0^a f(2a-x)$

$\int_a^{2a} f(x)dx = \int_0^a f(2a-x)dx \longrightarrow (2).$

(2) in (1) $\int_0^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

Corollary 1:

If $f(x) = f(2a-x)$, $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx.$

Corollary 2: $\int_0^{\pi} \sin^n x\, dx = 2\int_0^{\pi/2} \sin^n x\, dx.$

proof:

$\int_0^{\pi} \sin^n x\, dx = \int_0^{2\pi/2} \sin^n x\, dx = 2\int_0^{\pi/2} \sin^n x\, dx.$

Corollary 3: If $f(x) = -f(2a-x)$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$$

$$\int_0^{2a} f(x)dx = -\int_0^a f(2a-x)dx + \int_0^a f(2a-x)dx$$

$$\int_0^{2a} f(x)dx = 0.$$

Example:

Evaluate $\int_0^{\pi/2} \log(\sin x)dx$.

<u>Soln</u>: $I = \int_0^{\pi/2} \log(\sin x)dx \longrightarrow (1).$

$a = \pi/2 \Rightarrow f(a-x) = I = \int_0^{\pi/2} \log(\sin(\pi/2 - x))dx$

$$I = \int_0^{\pi/2} \log(\cos x)dx \longrightarrow (2).$$

$(1)+(2) \Longrightarrow 2I = \int_0^{\pi/2} \log(\sin x)dx + \int_0^{\pi/2} \log(\cos x)dx.$

$= \int_0^{\pi/2} \{\log(\sin x) + \log(\cos x)\}dx$

$= \int_0^{\pi/2} \log(\sin x \cos x)dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$

$= \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} \log(2)dx.$

$= \int_0^{\pi/2} \log(\sin 2x)dx - \log(2)\int_0^{\pi/2} dx.$

$= \int_0^{\pi/2} \log(\sin 2x)dx - \log 2[x]_0^{\pi/2}$

$= \int_0^{\pi/2} \log(\sin 2x)dx - [\log 2(\pi/2) - \log 2(0)]$

$2I = \int_0^{\pi/2} \log(\sin 2x)dx - \pi/2 \log(2) \longrightarrow (3).$

$2I = I_1 + I_2 = I_1 - \pi/2 \log 2.$

$I_1 = \int_0^{\pi/2} \log(\sin 2x)dx$

$2x = y \Rightarrow 2dx = dy\ ;\ x = 0 \Rightarrow y = 0\ \&\ x = \pi/2 \Rightarrow y = \pi$

$I_1 = \int_0^{\pi/2} \log(\sin 2x)dx = \int_0^{\pi} \log(\sin y)dy/2 = \frac{1}{2}\int_0^{\pi} \log(\sin y)dy$

$= 1/2 \int_0^{\pi} \log(\sin x)dx \quad \left(\int_0^a f(x)dx = \int_0^a f(y)dy\right).$

$\therefore \int_0^{\pi/2} \log(\sin 2x) = I \longrightarrow (4).$

(4) in (3) $\Rightarrow 2I = I - \pi/2 \log 2 \Rightarrow 2I - I = -\pi/2 \log 2.$

$I = -\pi/2 \log 2 = (-1)\pi/2 \log 2.$

$I = \pi/2 \log(2)^{-1} \Rightarrow \boxed{I = \pi/2 \log(1/2).}$

13.3 REDUCTION FORMULAE

i) $I_n = \int \sin^n x\,dx$

Proof:

$I_n = \int \sin^n x\,dx$

$= \int \sin^{n-1} x \sin x\,dx = \int \sin^{n-1} x\, d(-\cos x)$

$u = \sin^{n-1} x \Rightarrow du = (n-1)\sin^{n-2} x \cos x\,dx.$

$dv = d(-\cos x) \Rightarrow v = -\cos x.$ $\left(\int u\,dv = uv - \int v\,du\right)$

$I_n = \sin^{n-1} x(-\cos x) - \int(-\cos x)(n-1)\sin^{n-2} x \cos x\,dx.$

$= -\sin^{n-1} x \cos x + \int \cos x (n-1)\sin^{n-2} x \cos x\,dx.$

$= -\sin^{n-1} x \cos x + (n-1)\int \cos^2 x \sin^{n-2} x\,dx.$

$= -\sin^{n-1} x \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \{\sin^{n-2} x - \sin^n x\}\,dx.$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$

$I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

$\therefore (n-1+1)I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$

(ii) $\int \cos^n x \, dx$.

proof:

$$I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

$$= \int \cos^{n-1} x \, d(\sin x) \qquad \left(\int u\,dv = uv - \int v\,du\right)$$

$$u = \cos^{n-1} x \Rightarrow du = (n-1)\cos^{n-2} x(-\sin x)\,dx$$

$$dv = d(\sin x) \Rightarrow v = \int dv \Rightarrow v = \int d(\sin x) = \sin x.$$

$$I_n = \cos^{n-1} x \sin x - \int \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx.$$

$$= \cos^{n-1} x \sin x + \int (n-1)\sin^2 x \cos^{n-2} x \, dx.$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x \, dx.$$

$$= \cos^{n-1} x \sin x + (n-1)\left\{\int \cos^{n-2} x \, dx - \int \cos^n x \, dx\right\}$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx.$$

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\cos^{n-1} x \sin x + (n-1) I_{n-2} = I_n + (n-1) I_n.$$

$$(n-1+1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}.$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}, \ n \text{ is even}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots\cdots\cdots 2/3, \ n \text{ is odd}$$

Examples:

① $$\int_0^{\pi/2} \cos^8 x \, dx = \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-6} \cdot \pi/2.$$

$$\int_0^{\pi/2} \cos^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi/2 = \frac{35\pi}{256}.$$

② $\int_0^{\pi/2} \cos^5 x\,dx$

<u>Soln:</u> $\int_0^{\pi/2} \cos^5 x\,dx = \frac{5-1}{5}\cdot\frac{5-3}{5-2} = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$ //

$$\boxed{\begin{array}{l} \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2} \quad n \text{ is even.} \\ \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\cdots\cdots\frac{2}{3} \quad n \text{ is odd} \end{array}}$$

<u>Examples:</u>

① $\int_0^{\pi/2} \sin^6 x\,dx = \frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\cdot\frac{\pi}{2} = \frac{5}{6_2}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$.

② $\int_0^{\pi/2} \sin^7 x\,dx = \frac{7-1}{7}\cdot\frac{7-3}{7-2}\cdot\frac{7-5}{7-4} = \frac{6^2}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}$.

Evaluate $\int_0^{\pi/2} x(1-x^2)^{1/2}\,dx$.

<u>Soln:</u> $x = \sin\theta$; $dx = \cos\theta\,d\theta$

Limits: $x = 0 \Rightarrow \theta = 0$ & $x = \pi/2 \Rightarrow \theta = 1$.

$\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta = \int_0^{\pi/2} \cos^2\theta\,d(-\cos\theta) = \left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} = 1/3$ //

$I_{m,n} = \int \sin^m x \cos^n x\,dx$.

Proof: I

$I_{m,n} = \int \sin^m x \cos^n x\,dx = \int \sin^m x \cos^{n-1} x \cos x\,dx$

$= \int \sin^m x \cos^{n-1} x\,d(\sin x) = \int \cos^{n-1} x\,d\left(\frac{\sin^{m+1} x}{m+1}\right)$

$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1}\,d(\cos^{n-1} x)$.

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \frac{n-1}{m+1}\int \sin^{m+1} x \cos^{n-2} x(-\sin x)\,dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\,dx$.

$\left(I_1 = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1}\right)$

$= I_1 + \frac{n-1}{m+1}\int \sin^m x \sin^2 x \cos^{n-2} x\,dx$

$= I_1 + \frac{n-1}{m+1}\int \sin^m x(1-\cos^2 x)\cos^{n-2} x\,dx$.

$= I_1 + \frac{n-1}{m+1}\int \{\sin^m x \cos^{n-2} x - \sin^m x \cos^n x\}\,dx$

$$I_{m,n} = I_1 + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx - \frac{n-1}{m+1}\int \sin^m x \cos^n x\,dx$$

$$= I_1 + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+1+n-1}{m+1}\right) = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n}\left(\frac{m+n}{m+1}\right) = \frac{\cos^{n-1} x \sin^m x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\therefore (m+n)\cdot I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$$

Examples:

(1) $\int_0^{\pi/2} \sin^6 x \cos^5 x\,dx = \frac{4}{11}\cdot\frac{2}{9}\cdot\frac{1}{7} = \frac{8}{693}$

(2) $\int_0^{\pi/2} \sin^6 x \cos^4 x\,dx = \frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{512}$.

MULTIPLE INTEGRALS.

DOUBLE INTEGRAL

Let $f(x,y)$ be a function of x alone and y as a constant and integrating it between $x = f_1(y)$ and $x = f_2(y)$ and then integrating the resulting function of y between c and $y = d$. The region R is called the region of integration corresponding to the interval of integration (a,b), $\int_R f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$.

Example

Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.

Soln: $x^2+y^2=a^2 \Rightarrow y^2 = a^2-x^2$ & $y = \sqrt{a^2-x^2}$

Limits:

$y=0$ in $x^2+y^2=a^2 \Rightarrow x^2=a^2 \Rightarrow x=a$.

$x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$

$$\iint xy\,dx\,dy = \int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a\left\{\int_0^{\sqrt{a^2-x^2}} xy\,dy\right\}dx.$$

$$= \int_0^a x\,dx\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} = \frac{1}{2}\int_0^a x\,dx\left[y^2\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x\left[(\sqrt{a^2-x^2})^2-0\right]dx = \frac{1}{2}\int_0^a x(a^2-x^2)\,dx$$

$$= \frac{1}{2}\int_0^a (a^2x-x^3)\,dx = \frac{1}{2}\left\{\frac{a^2x^2}{2}-\frac{x^4}{4}\right\}_0^a.$$

$$= \frac{1}{2}\left\{\frac{a^2}{2}\left[a^2-0\right]-\frac{1}{4}\left[a^4-0\right]\right\}.$$

$$= \frac{1}{2}\left\{\frac{a^2}{2}(a^2)-\frac{1}{4}(a^4)\right\} = \frac{1}{2}\left\{\frac{a^4}{2}-\frac{a^4}{4}\right\}.$$

$$= \frac{1}{2}\left\{\frac{2a^4-a^4}{4}\right\} = \frac{1}{2}\cdot\frac{a^4}{4}$$

$$\iint xy\,dx\,dy = \frac{a^4}{8}.$$

Example:

Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x, y are each ≥ 0 and $x+y \leq 1$.

Soln:

$x+y=1$

Limits: $x+y=1 \rightarrow y=1-x \rightarrow y=0$ to $1-x$.

$y=0 \rightarrow x=1 \rightarrow x=0$ to 1.

$$\iint (x^2+y^2)\,dx\,dy = \int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx$$

$$= \int_0^1\left[x^2y+y^3/3\right]_0^{1-x}dx = \int_0^1\left\{x^2(1-x)+\frac{(1-x)^3}{3}\right\}dx$$

$$= \int_0^1\left[(x^2-x^3)+\frac{1}{3}(1^3-x^3-3x+3x^2)\right]dx.$$

$$= \frac{1}{3}\int_0^1\left[3(x^2-x^3)+(1-x^3-3x+3x^2)\right]dx$$

$$= \frac{1}{3}\int_0^1(3x^2-3x^3+1-x^3-3x+3x^2)\,dx.$$

$= \frac{1}{3}\int_0^1 (1-3x+6x^2-4x^3)\,dx.$

$= \frac{1}{3}\left[x - \frac{3x^2}{2} + \frac{6x^3}{3} - \frac{4x^4}{4}\right]_0^1$

$= \frac{1}{3}\left[1 - 3/2 + 2 - 1\right] = \frac{1}{3}\left(\frac{4-3}{2}\right) = \frac{1}{3}\cdot\frac{1}{2}$

$\therefore \iint (x^2+y^2)\,dx\,dy = \frac{1}{6}.$

TRIPLE INTEGRALS

If $f(x,y,z)$ is continuous and a single valued function of x, y and z over the region of space R enclosed by the surface S. Let R be subdivided into subregions ΔR_{rst} the Triple Integral of $f(x,y,z)$ over R is defined by $r=n,\ s=m,\ t=p.$

$$\int f(x,y,z)\,dV = \lim_{\substack{n\to\infty \\ m\to\infty \\ p\to\infty}} \sum_{\substack{r=1 \\ s=1 \\ t=1}} f(\xi_{rst}, \eta_{rst}, \zeta_{rst})\,\Delta V_{rst}$$

The Triple Integral R is considered to be subdivided into planes parallel to the three Co-ordinate planes. Then $\Delta V_{rst} = \Delta x_r \Delta y_s \Delta z_t$

$$\therefore \int_R f(x,y,z)\,dV = \int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,dx\,dy\,dz.$$

Example!

Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Soln:
$$x^2+y^2+z^2=a^2 \longrightarrow (1).$$

Limits:

$(1) \Rightarrow z^2 = a^2-y^2-x^2 \Rightarrow z=0$ to $\sqrt{a^2-y^2-x^2}$

$z=0$ in $(1) \Rightarrow x^2+y^2=a^2 \Rightarrow y^2=a^2-x^2$ & $y=0$ to $\sqrt{a^2-x^2}$

$z=y=0$ in $(1) \Rightarrow x^2=a^2 \Rightarrow x^2=a^2$ & $x=0$ to a.

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dx\,dy\,dz = \int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx.$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx = \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\left[z^2\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx.$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\left(\sqrt{a^2-y^2-x^2}\right)^2 dy\,dx = \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} (xy)(a^2-y^2-x^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} (a^2xy - xy^3 - x^3y)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{a^2xy^2}{2} - \frac{xy^4}{4} - \frac{x^3y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a\left[\frac{a^2x(\sqrt{a^2-x^2})^2}{2} - \frac{x(\sqrt{a^2-x^2})^4}{4} - \frac{x^3(\sqrt{a^2-x^2})^2}{2}\right]dx$$

$$= \frac{1}{2}\int_0^a\left(\frac{a^2x(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4} - \frac{x^3(a^2-x^2)}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a\left(\frac{2a^2x(a^2-x^2)}{4} - \frac{x(a^4+x^4-2a^2x^2)}{4} - \frac{2x^3(a^2-x^2)}{4}\right)dx$$

$$= \frac{1}{2}\cdot\frac{1}{4}\int_0^a (2a^4x - 2a^2x^3 - a^4x - x^5 + 2a^2x^3 - 2a^2x^3 + 2x^5)\,dx$$

$$= \frac{1}{8}\int_0^a (2a^4x - 2a^2x^3 - a^4x + x^5)\,dx$$

$$= \frac{1}{8}\left[\frac{2a^4x^2}{2} - \frac{2a^2x^4}{4} - \frac{a^4x^2}{2} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[a^4(a)^2 - \frac{a^2(a)^4}{2} - \frac{a^4(a)^2}{2} + \frac{(a)^6}{6}\right]$$

$$= \frac{1}{8}\left[a^6 - \frac{a^6}{2} - \frac{a^6}{2} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left(\frac{a^6}{6}\right) = \frac{a^6}{48}.$$

$$\therefore \int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{a^6}{48}.$$

ALLIED MATHEMATICS FOR PHYSICS & CHEMISTRY

TITLE OF THE PAPER : CALCULUS AND VECTOR CALCULUS

PAPER CODE : 18KICH/PAMI

UNIT – IV

DIFFERENTIAL OPERATORS

2.1 VECTOR DIFFERENTIAL OPERATOR ∇:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}, \ f \text{ is a scalar.}$$

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f$$

$$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}.$$

2.2 GRADIENT:

$\phi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of ϕ is defined by

$$\text{grad}\,\phi = \nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}.$$

2.2.1 $\text{Grad}(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi$.

proof:

$$\text{Grad}(\phi \pm \psi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\phi \pm \psi)$$

$$\text{Grad}(\phi \pm \psi) = \vec{i}\frac{\partial}{\partial x}(\phi \pm \psi) + \vec{j}\frac{\partial}{\partial y}(\phi \pm \psi) + \vec{k}\frac{\partial}{\partial z}(\phi \pm \psi)$$

$$= \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right) \pm \left(\vec{i}\frac{\partial \psi}{\partial x} + \vec{j}\frac{\partial \psi}{\partial y} + \vec{k}\frac{\partial \psi}{\partial z}\right)$$

$$\text{Grad}(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi.$$

2.2.2 $\text{Grad}(\phi\psi) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi$.

proof:

$$\text{grad}(\phi\psi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\phi\psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\phi\psi) + \vec{j}\frac{\partial}{\partial y}(\phi\psi) + \vec{k}\frac{\partial}{\partial z}(\phi\psi).$$

$$= \vec{i}\left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right) + \vec{j}\left(\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right) + \vec{k}\left(\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)\phi + \psi\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi.$$

$$\therefore \nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi.$$

2.3 DIVERGENCE:

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F}$$

$$= \vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}.$$

$$\vec{F}(x,y,z) = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\operatorname{div}\vec{F} = \nabla\cdot(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

2.3.1 $\operatorname{div}(\vec{A}+\vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}$

Proof:

$$\operatorname{div}(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B})$$

$$\operatorname{div}(\vec{A}+\vec{B}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{A}+\vec{B})$$

$$= \vec{i}\cdot\frac{\partial(\vec{A}+\vec{B})}{\partial x} + \vec{j}\cdot\frac{\partial(\vec{A}+\vec{B})}{\partial y} + \vec{k}\cdot\frac{\partial(\vec{A}+\vec{B})}{\partial z}$$

$$= \vec{i}\cdot\left(\frac{\partial\vec{A}}{\partial x} + \frac{\partial\vec{B}}{\partial x}\right) + \vec{j}\cdot\left(\frac{\partial\vec{A}}{\partial y} + \frac{\partial\vec{B}}{\partial y}\right) + \vec{k}\cdot\left(\frac{\partial\vec{A}}{\partial z} + \frac{\partial\vec{B}}{\partial z}\right)$$

$$= \left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{A}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{A}}{\partial z}\right) + \left(\vec{i}\cdot\frac{\partial\vec{B}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{B}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$$

$$\operatorname{div}(\vec{A}+\vec{B}) = \operatorname{div}\vec{A} + \operatorname{div}\vec{B}.$$

2.3.2 SOLENOIDAL VECTOR:

A vector $\vec{F}$ is called Solenoidal, if $\operatorname{div}\vec{F} = 0$ (ie) $\nabla\cdot\vec{F} = 0$.

Example: $\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx = 0.$

Corollary 2:

If $f(x)$ is an even function, $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

Example: $\int_{-\pi/2}^{\pi/2} \cos^4 x\, dx = 2\int_0^{\pi/2} \cos^4 x\, dx =$

Property 6: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

Proof: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \longrightarrow (1).$

In the second integral of RHS.

$x = 2a - y \Rightarrow dx = -dy.$

Limits: $x = a \Rightarrow y = a$ & $x = 2a \Rightarrow y = 0.$

$$\therefore \int_a^{2a} f(x)dx = \int_a^0 f(2a-y)(-dy) = -\int_a^0 f(2a-y)dy$$
$$= \int_0^a f(2a-y)dy = \int_0^a f(2a-x)$$

$$\int_a^{2a} f(x)dx = \int_0^a f(2a-x)dx \longrightarrow (2).$$

(2) in (1) $\int_0^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

Corollary 1:

If $f(x) = f(2a-x)$, $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx.$

Corollary 2: $\int_0^{\pi} \sin^n x\, dx = 2\int_0^{\pi/2} \sin^n x\, dx.$

proof:

$$\int_0^{\pi} \sin^n x\, dx = \int_0^{2\pi/2} \sin^n x\, dx = 2\int_0^{\pi/2} \sin^n x\, dx.$$

2.4.2 IRROTATIONAL VECTOR:

$\vec{F}$ is irrotational, if curl $\vec{F} = 0$, $\nabla \times \vec{F} = 0$.

2.4.3. Examples:

① Find grad r^n, $r^2 = x^2 + y^2 + z^2$.

Soln:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\text{grad } r^n = \nabla r^n.$$

$$= \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}.$$

$$\text{grad } r^n = \vec{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \vec{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \vec{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \vec{i}\left(nr^{n-1}\cdot\frac{x}{r}\right) + \vec{j}\left(nr^{n-1}\frac{y}{r}\right) + \vec{k}\left(nr^{n-1}\frac{z}{r}\right)$$

$$= \vec{i}(x\cdot n\cdot r^{n-2}) + \vec{j}(y\cdot n\cdot r^{n-2}) + \vec{k}(z\cdot n\cdot r^{n-2})$$

$$= (nr^{n-2})(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\text{grad } r^n = (nr^{n-2})\,\vec{r}.$$

② Show that

(i) $\vec{A} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal

(ii) $\vec{B} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

proof:

(i) $\text{div } \vec{A} = \nabla\cdot\vec{A}$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) - \frac{\partial}{\partial z}(3x^2y^2)$$

$$= (0) + 0 + 0$$

$$\text{div}\vec{A} = \nabla\cdot\vec{A} = 0$$

$\vec{A}$ is solenoidal.

(ii) $Curl\,\vec{B} = \nabla \times \vec{B}$; $\vec{B} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

$$\nabla \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}\left\{\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right\}$$
$$-\vec{j}\left\{\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right\}$$
$$+\vec{k}\left\{\frac{\partial}{\partial x}(3x^2z) - \frac{\partial}{\partial y}(6xy+z^3)\right\}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x)$$

$$\nabla \times \vec{B} = 0 \Rightarrow Curl\,\vec{B} = 0.$$

$\therefore \vec{B}$ is irrotational.

2.5.1 Examples:

① $\vec{V} = x^2y\vec{i} - 2zx\vec{j} + 2yz\vec{k}$, $curl\,curl\,\vec{V} = ?$

Soln:

$$Curl\,\vec{V} = \nabla \times \vec{V}$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & & \frac{\partial}{\partial z} \\ x^2y & -2zx & 2yz \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2zx)\right)$$
$$-\vec{j}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(-2zx) - \frac{\partial}{\partial y}(x^2y)\right)$$

$$Curl\,\vec{V} = \vec{i}(2z+2x) - \vec{j}(0) + \vec{k}(-2z-x^2)$$

$$Curl\,Curl\,\vec{V} = \nabla \times Curl\,\vec{V}$$
$$= \nabla \times \left[(2x+2z)\vec{i} - (x^2+2z)\vec{k}\right]$$

$$\text{curl curl}\,\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2z & 0 & -(x^2+2z) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-(x^2+2z))\right] - \vec{j}\left[\frac{\partial}{\partial x}(-(x^2+2z)) - \frac{\partial}{\partial z}(2x+2z)\right] + \vec{k}\left[-\frac{\partial}{\partial y}(2x+2z)\right]$$

$$= \vec{i}(0) - \vec{j}(-2x-2) + \vec{k}(0)$$

$$\text{curl curl}\,\vec{V} = \vec{j}(2x+2).$$

(2) $\phi = x^2y^3z^4$, find div grad ϕ and curl grad ϕ.

Soln:

$$\text{grad}\,\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial(x^2y^3z^4)}{\partial x} + \vec{j}\frac{\partial(x^2y^3z^4)}{\partial y} + \vec{k}\frac{\partial(x^2y^3z^4)}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(3x^2y^2z^4) + \vec{k}(4x^2y^3z^3).$$

(i) $\text{div grad}\,\phi = \nabla\cdot\nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x}\right)\vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\Big)\cdot\Big(\vec{i}(2xy^3z^4) + \vec{j}(3x^2y^2z^4) + \vec{k}(4x^2y^3z^3)\Big)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(3x^2y^2z^4) + \frac{\partial}{\partial z}(4x^2y^3z^3)$$

$$\text{div grad}\,\phi = 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2.$$

(ii) $\text{curl grad}\,\phi = \nabla\times\nabla\phi$

$$\nabla\times\nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x^2y^3z^3) - \frac{\partial}{\partial z}(3x^2y^2z^4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x^2y^3z^3) - \frac{\partial}{\partial z}(2xy^3z^4)\right] + \vec{k}\left[\frac{\partial}{\partial x}(3x^2y^2z^4) - \frac{\partial}{\partial y}(2xy^3z^4)\right]$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) \Rightarrow \text{curl grad}\,\phi = \nabla\times\nabla\phi = 0.$$

VECTOR IDENTITIES

1. $div(\phi\vec{F}) = \phi \, div\vec{F} + \vec{F}\cdot grad\,\phi$

$$\nabla\cdot \vec{F} = \phi(\nabla\cdot\vec{F}) + \vec{F}\cdot(\nabla\phi)$$

2. $curl(\phi\vec{F}) = \phi\, curl\vec{F} + (grad\,\phi)\times\vec{F}$

$$\nabla\times(\phi\vec{F}) = \phi(\nabla\times\vec{F}) + (\nabla\phi)\times\vec{F}.$$

3. $div(\vec{A}\times\vec{B}) = \vec{B}\cdot curl\vec{A} - \vec{A}\cdot curl\vec{B}$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

4. $Curl(\vec{A}\times\vec{B}) = \vec{A}\, div\vec{B} - \vec{B}\, div\vec{A} + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

5. $Curl(grad\,\phi) = 0$ (ie) $\nabla\times(\nabla\phi) = 0$

6. $div(curl\,\vec{A}) = 0$ (ie) $\nabla\cdot(\nabla\times\vec{A}) = 0$

7. $div(grad\,\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$$\nabla\cdot(\nabla\phi) = \nabla^2\phi.$$

$$\nabla^2 = \frac{\partial^2 x}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \text{Laplacian Operator}$$

Examples:

(1) If $u = x^2 - y^2$ P.T $\nabla^2 u = 0$.

Soln: $\nabla^2 u = \nabla\cdot(\nabla u) = \nabla\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)u$

$$= \nabla\cdot\left(\vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}\right)$$

$$= \nabla\cdot\left(\vec{i}\frac{\partial(x^2-y^2)}{\partial x} + \vec{j}\frac{\partial(x^2-y^2)}{\partial y} + \vec{k}\frac{\partial(x^2-y^2)}{\partial z}\right)$$

$$= \nabla\cdot\left(\vec{i}(2x) + \vec{j}(-2y) + \vec{k}(0)\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}2x + \vec{j}(-2y)\right)$$

$$= \frac{\partial}{\partial x}(2x) - \vec{j}\frac{\partial}{\partial y}(2y) + \vec{k}(0)$$

$$= 2 - 2$$

$$\Rightarrow \nabla^2 u = 0.$$

② $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, PT $\text{div}\,\vec{r} = 3$ & $\text{curl}\,\vec{r} = 0$

Soln:

$$\text{div}\,\vec{r} = \nabla\cdot\vec{r}$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$
$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$
$$= (1) + (1) + (1)$$
$$\text{div}\,\vec{r} = 3.$$

$$\text{curl}\,\vec{r} = \nabla\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\nabla\times\vec{r} = \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$\nabla\times\vec{r} = \vec{i}(0) + \vec{j}(0) + \vec{k}(0).$$

$$\nabla\times\vec{r} = 0.$$

③ Find the value of a if $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

Soln

$\vec{F}$ is solenoidal $\Rightarrow \nabla\cdot\vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left((x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right) = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0.$$

(ie) $1 + 1 + a = 0 \Rightarrow 2 + a = 0 \Rightarrow \boxed{a = -2}$.

④ $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$, ST $\nabla^2\vec{F} = 0$.

Soln $\nabla^2\vec{F} = \dfrac{\partial^2\vec{F}}{\partial x^2} + \dfrac{\partial^2\vec{F}}{\partial y^2} + \dfrac{\partial^2\vec{F}}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(y\vec{i} + z\vec{k}\right) + \frac{\partial}{\partial y}\left(x\vec{i} + z\vec{j}\right) + \frac{\partial}{\partial z}\left(y\vec{j} + x\vec{k}\right)$$

$$= 0 + 0 + 0$$

$$\Rightarrow \nabla^2\vec{F} = 0.$$

(5) Find gradϕ if $\phi = xyz$ at $(1,1,1)$

Soln: $\phi = xyz$, grad $\phi = \nabla\phi$.

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial(xyz)}{\partial x} + \vec{j}\frac{\partial(xyz)}{\partial y} + \vec{k}\frac{\partial(xyz)}{\partial z}$$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$(\nabla\phi)_{(1,1,1)} = \vec{i}(1) + \vec{j}(1) + \vec{k}(1) = \vec{i}+\vec{j}+\vec{k}.$$

(6) Find Directional Derivative of $f = x.y.z$ at $(1,1,1)$ in the direction of $\vec{i}+\vec{j}+\vec{k}$.

Soln: $$\hat{n} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{1^2+1^2+1^2}} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}.$$

$$\text{grad } f = \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= \vec{i}\frac{\partial(xyz)}{\partial x} + \vec{j}\frac{\partial(xyz)}{\partial y} + \vec{k}\frac{\partial(xyz)}{\partial z}$$

$$= \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

$$(\text{grad } f)_{(1,1,1)} = \vec{i}+\vec{j}+\vec{k}.$$

$$\text{Directional Derivative} = \text{grad} f \cdot \hat{n}$$

$$= (\vec{i}+\vec{j}+\vec{k})\cdot\left(\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}\right)$$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Directional Derivative $= \sqrt{3}$

Example:

$\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$.

Soln $$\nabla^2(r^n\vec{r}) = \left(\vec{i}\frac{\partial^2}{\partial x^2} + \vec{j}\frac{\partial^2}{\partial y^2} + \vec{k}\frac{\partial^2}{\partial y^2}\right)(r^n\vec{r})$$

$$\frac{\partial}{\partial x}(r^n\vec{r}) = nr^{n-1}\frac{\partial r}{\partial x}\cdot\vec{r} + r^n\cdot\frac{\partial \vec{r}}{\partial x}$$

$$\frac{\partial}{\partial x}(r^n \vec{r}) = n r^{n-1} \vec{r}\left(\frac{x}{r}\right) + r^n \vec{i} \qquad \left(\frac{\partial \vec{r}}{\partial x} = \vec{i}\right)$$

$$\frac{\partial}{\partial x}(r^n \vec{r}) = n r^{n-2} \vec{r} x + r^n \vec{i}$$

$$\frac{\partial^2}{\partial x^2}(r^n \vec{r}) = n\left[r^{n-2}\vec{r} + (n-2) r^{n-2} \frac{\partial r}{\partial x} x \vec{r} + r^{n-2} \frac{\partial \vec{r}}{\partial x} x\right]$$

$$\frac{\partial^2}{\partial x^2}(r^n \vec{r}) = n\left[r^{n-2}\vec{r} + (n-2) x^2 r^{n-4} \vec{r} + r^{n-2} x \vec{i}\right] + n r^{n-2} x \vec{i}$$

$$= \left[n(n-2) r^{n-4} x^2 + n r^{n-2}\right]\vec{r} + 2n r^{n-2} x \vec{i}$$

$$\therefore \sum \frac{\partial^2}{\partial x^2}(r^n \vec{r}) = \left[n(n-2) r^{n-4}(x^2+y^2+z^2) + 3n r^{n-2}\right]\vec{r} + 2n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \left[n(n-2) r^{n-2} + 3n r^{n-2}\right]\vec{r} + 2n r^{n-2}\vec{r}$$

$$= (n^2+3n) r^{n-2}\vec{r}$$

$$\nabla^2 r^n \vec{r} = n(n+3) r^{n-2} \vec{r}.$$

Example:

$\vec{A}$ & $\vec{B}$ are irrotational, PT $\vec{A} \times \vec{B}$ is solenoidal.

Soln:

$\vec{A}$ & $\vec{B}$ → irrotational

$curl\,\vec{A} = 0, \; curl\,\vec{B} = 0.$

$$div(\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A} \times \vec{B})$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$= \vec{B} \cdot curl\,A - \vec{A} \cdot curl\,\vec{B}$$

$$div(\vec{A} \times \vec{B}) = 0$$

$\therefore \vec{A} \times \vec{B}$ is irrotational.

ALLIED MATHEMATICS FOR PHYSICS & CHEMISTRY

TITLE OF THE PAPER: CALCULUS AND VECTOR CALCULUS

PAPER CODE: 18KICH/PAM1.

UNIT V

INTEGRATION OF VECTORS & INTEGRAL THEOREMS

LINE INTEGRAL:

An integral which is evaluated along a curve is called a line integral.

CIRCULATION:

The tangential line integral of a vector function $\vec{F}$ around a simple closed curve C, is called the circulation of $\vec{F}$ about C, $\oint_C \vec{F}\cdot d\vec{r}$

WORK DONE BY A FORCE:

If $\vec{F}(x,y,z)$ is a force acting on a particle which moves along a given curve C, then $\int_C \vec{F}\cdot d\vec{r}$ is the total work done by $\vec{F}$ and $\vec{F} = \int_C \vec{F}\cdot d\vec{r}$

SURFACE INTEGRAL:

An integral evaluated along a surface is called as a surface integral, $\int_S \phi\, dS$.

FLUX:

This is a surface integral in which the integral of the normal component of a vector point function over the surface is considered. The Flux of $\vec{F}$ over S is denoted by $\iint_S \vec{F}\cdot\hat{n}\, dS$.

Example 1

Evaluate $\int_V \vec{F}\, dv$, where $\vec{F} = xy\vec{i} - zx\vec{j} + \vec{k}$ and V is the octant of the sphere $x^2+y^2+z^2=4$; $x, y, z \geq 0$.

Soln

$$\int_V \vec{F}\, dv = \int_V (xy\vec{i} - zx\vec{j} + \vec{k})\, dv$$

$$= \vec{i}\int_V xy\, dv - \vec{j}\int_V zx\, dv + \vec{k}\int_V dv \longrightarrow (1).$$

$$\int_V xy\, dv = \iiint_V xy\, dx\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{\sqrt{4-z^2}}\int_{x=0}^{\sqrt{4-y^2-z^2}} xy\, dx\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{\sqrt{4-z^2}}\int_{x=0}^{\sqrt{4-y^2-z^2}} x\, dx\; y\, dy\, dz = \int_{z=0}^{2}\int_{y=0}^{\sqrt{4-z^2}} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y^2-z^2}} y\, dy\, dz$$

$$= \frac{1}{2}\int_0^2\int_0^{\sqrt{4-z^2}} y(4-y^2-z^2)\, dy\, dz$$

$$= \frac{1}{2}\int_0^2\int_0^{\sqrt{4-z^2}} (4y - y^3 - z^2 y)\, dy\, dz$$

$$= \frac{1}{2}\int_0^2 \left[\frac{4y^2}{2} - \frac{y^4}{4} - \frac{z^2y^2}{2}\right]_0^{\sqrt{4-z^2}} dz$$

$$= \frac{1}{2}\int_0^2 \left[\frac{4(4-z^2)}{2} - \frac{(4-z^2)^2}{4} - \frac{z^2(4-z^2)}{2}\right] dz$$

$$= \frac{1}{2}\int_0^2 \left[\frac{(16-4z^2)}{2} - \frac{(16+z^4-8z^2)}{4} - \frac{(4z^2-z^4)}{2}\right] dz$$

$$= \frac{1}{8}\int_0^2 \left[32 - 8z^2 - 16 - z^4 + 8z^2 - 8z^2 + 2z^4\right] dz$$

$$= \frac{1}{8}\int_0^2 (16 - 8z^2 + z^4)\, dz = \frac{1}{8}\left[16z - \frac{8z^3}{3} + \frac{z^5}{5}\right]_0^2$$

$$\int_V xy\,dV = \frac{1}{8}\left(32 - \frac{64}{3} + \frac{32}{5}\right)$$

$$= \frac{32}{8} - \frac{64}{8 \cdot 3} + \frac{32}{8 \cdot 5}$$

$$= 4 - \frac{8}{3} + \frac{4}{5}$$

$$= \frac{60 - 40 + 12}{15}$$

$$\int_V xy\,dV = \frac{32}{15} \longrightarrow (2)$$

Similarly $\int_V zx\,dV = \frac{32}{15} \longrightarrow (3)$

$$\int_V dV = V$$

$$= \frac{1}{8} \times \text{volume of the sphere}$$

$$= \frac{1}{8} \cdot \frac{4}{3}\pi \cdot 2^3$$

$$\int_V dV = \frac{4}{3}\pi \longrightarrow (4)$$

$$(1) \Rightarrow \int_V \vec{F}\,dV = \frac{32}{15}\vec{i} - \frac{32}{15}\vec{j} + \frac{4\pi}{3}\vec{k}$$

Example 2:

If $\vec{A} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along C given by $x = t$, $y = t^2$, $z = t^3$.

Soln:

$$\vec{A} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}.$$

$$\int_C \vec{A} \cdot d\vec{r} = \int_C \left[(3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right] \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (3x^2+6y)\,dx - (14yz)\,dy + 20xz^2\,dz$$

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left[(3x^2+6y)\frac{dx}{dt} - (14yz)\frac{dy}{dt} + (20xz^2)\frac{dz}{dt}\right]dt$$

$x = t \Rightarrow dx = dt$ & $\frac{dx}{dt} = 1$

$y = t^2 \Rightarrow dy = 2t\,dt$ & $\frac{dy}{dt} = 2t$

$z = t^3 \Rightarrow dz = 3t^2 dt$ & $\frac{dz}{dt} = 3t^2$

$$\int_C \vec{A}\cdot d\vec{r} = \int_{x,y,z=0}^{1} \left[(3x^2+6y)(1) - (14yz)(2t) + (20xz^2)(3t^2)\right]dt$$

$$= \int_0^1 \left[(3t^2+6t^2) - (14\cdot t^2\cdot t^3)(2t) + (20\cdot t\cdot t^6)(3t^2)\right]dt$$

$$= \int_0^1 \left[3t^2 + 6t^2 - 28t^6 + 60t^9\right]dt$$

$$= \left[\frac{3t^3}{3} + \frac{6t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= \left[\frac{3(1)}{3} + \frac{6(1)}{3} - \frac{28(1)}{7} + \frac{60(1)}{10}\right]$$

$$= 1 + 2 - 4 + 6$$

$$\int_C \vec{A}\cdot d\vec{r} = 5.$$

Example 3:

Find velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$.

Soln:

$$\text{Velocity } \vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$= \frac{d}{dt}(2\cos 3t)\vec{i} + \frac{d}{dt}(2\cos 3t)\vec{j} + \frac{d}{dt}(8t)\vec{k}$$

$$= 6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k}$$

$$|\vec{V}| = \sqrt{(6\cos 3t)^2 + (6\sin 3t)^2 + 8^2}$$

$$= \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$$

$|\vec{V}| = \sqrt{36(\cos^2 3t + \sin^2 3t) + 64} = \sqrt{36+64} = \sqrt{100}$

$|\vec{V}| = 10.$

Acceleration $= \dfrac{d\vec{V}}{dt} = \dfrac{d}{dt}(6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k})$

$= -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j}$

$\left|\dfrac{dV}{dt}\right| = \sqrt{18^2\sin^2 3t + 18^2\cos^2 3t}$

$= \sqrt{18^2(\sin^2 3t + \cos^2 3t)} = \sqrt{18^2}$

$\left|\dfrac{dV}{dt}\right| = 18.$

Example 4:

Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is the part of the sphere $x^2+y^2+z^2=1$ in the first octant.

Soln:

$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k},$

$\phi = x^2+y^2+z^2-1 \Rightarrow \nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4(x^2+y^2+z^2)}}$$

$$= \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{2\sqrt{x^2+y^2+z^2}} = x\vec{i}+y\vec{j}+z\vec{k}.$$

$\vec{F}\cdot\hat{n} = (yz\vec{i} + zx\vec{j} + xy\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k})$

$= xyz + xyz + xyz$

$\vec{F}\cdot\hat{n} = 3xyz$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$|\hat{n}\cdot\vec{k}| = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k} = z.$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R 3xyz\,\frac{dx\,dy}{z}.$$

$$= 3\iint_R xy\,dy\,dz = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{3}{2}\int_0^1 x(1-x^2)\,dx.$$

$$= \frac{3}{2}\int_0^1 (x-x^3)\,dx = \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{3}{2}\left(\frac{2-1}{4}\right) = \frac{3}{2}\left(\frac{1}{4}\right)$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{8}.$$

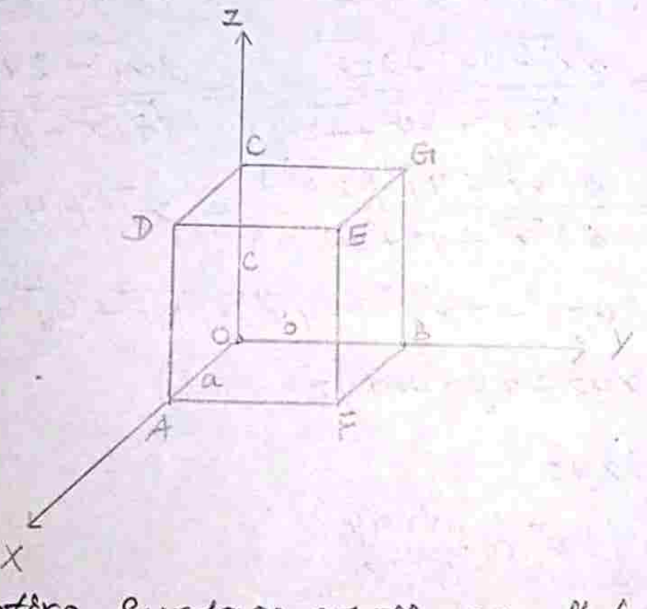
Example 5:

Verify Gauss Divergence Theorem for $\vec{F}=(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ and S is the surface of the rectangular parallelopiped $0\le x\le a, 0\le x\le b, 0\le x\le c$.

Soln:

Gauss divergence theorem is,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$



$S \Rightarrow$ entire surface of the parallelopiped.

$V \Rightarrow$ volume of the parallelopiped enclosed by S.

$\vec{i}, \vec{j}, \vec{k} \Rightarrow$ unit vectors along X, Y & Z axis

$S_1, S_2, S_3, S_4, S_5, S_6$ be the faces of DEFA, CGBO, EGBF, DCOA, DEGC, OAFB and the unit normal vectors along these faces are $\vec{i}, -\vec{i}, \vec{j}, -\vec{j}, \vec{k}$ and $-\vec{k}$.

To verify Gauss's Divergence Theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

(i) $\iiint_V \nabla\cdot\vec{F}\,dv$

$$\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$\nabla\cdot\vec{F} = 2x+2y+2z \quad \& \quad dv = dx\,dy\,dz$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a\int_0^b\int_0^c (x+y+z)\,dz\,dy\,dx$$

$$= 2\int_0^a\int_0^b \left(xz+yz+\frac{z^2}{2}\right)_0^c dy\,dx$$

$$= 2\int_0^a\int_0^b (xc+yc+c^2/2)\,dy\,dx$$

$$= 2\int_0^a \left(xcy+\frac{y^2c}{2}+\frac{c^2y}{2}\right)_0^b dx$$

$$= 2\int_0^a \left(xbc+\frac{b^2c}{2}+\frac{bc^2}{2}\right)dx$$

$$= 2\left(\frac{x^2bc}{2}+\frac{xb^2c}{2}+\frac{xbc^2}{2}\right)_0^a$$

$$= 2\left(\frac{a^2bc+ab^2c+abc^2}{2}\right)$$

$$= \frac{2(abc)(a+b+c)}{2}$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = abc(a+b+c) \longrightarrow (A).$$

(ii) $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

(a) $\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{DEFA} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]\cdot\vec{i}\,dy\,dz$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (x^2-yz)\,dy\,dz = \int_0^c\int_0^b (a^2-yz)\,dy\,dz \quad (x=a).$$

$$= \int_0^c \left[a^2y - \frac{y^2z}{2}\right]_0^b dz = \int_0^c \left(a^2b - \frac{b^2z}{2}\right) dz$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \left[a^2bz - \frac{b^2z^2}{4}\right]_0^c = \left[a^2bc - \frac{b^2c^2}{4}\right] \longrightarrow (1).$$

(b) $$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{CGBO} \left[(x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(-\vec{i})\,dy\,dz$$

$$= -\int_0^c\int_0^b (x^2-yz)\,dy\,dz = -\int_0^c\int_0^b (-yz)\,dy\,dz \quad (x=0)$$

$$= \int_0^c \left[\frac{zy^2}{2}\right]_0^b dz = \frac{1}{2}\int_0^c b^2z\,dz = \frac{1}{2}\left[\frac{b^2z^2}{2}\right]_0^c$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4} \longrightarrow (2).$$

(c) $$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{EGBF} \left[(x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(\vec{j})\,dx\,dz$$

$$= \int_0^a\int_0^c (y^2-xz)\,dz\,dx = \int_0^a\int_0^c (b^2-xz)\,dz\,dx \quad (y=b)$$

$$= \int_0^a \left[b^2z - \frac{xz^2}{2}\right]_0^c dx = \int_0^a \left(b^2c - \frac{xc^2}{2}\right) dx$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \left[xb^2c - \frac{x^2c^2}{4}\right]_0^a = ab^2c - \frac{a^2c^2}{4} \longrightarrow (3).$$

(d) $$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{DCOA} \left[(x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(-\vec{j})\,dx\,dz$$

$$= \int_0^a\int_0^c -(y^2-xz)\,dz\,dx = \int_0^a\int_0^c -(-xz)\,dz\,dx \quad (y=0)$$

$$= \int_0^a\int_0^c xz\,dz\,dx = \int_0^a \left[\frac{xz^2}{2}\right]_0^c dx = \int_0^a \frac{xc^2}{2}\,dx$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \frac{1}{2}\left[\frac{x^2c^2}{2}\right]_0^a = \frac{a^2c^2}{4} \longrightarrow (4).$$

(e) $$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DEGC} \left[(x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(\vec{k})\,dx\,dy$$

$$= \int_0^a\int_0^b (z^2-xy)\,dy\,dx = \int_0^a\int_0^b (c^2-xy)\,dy\,dx \quad (z=c)$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^a \left[c^2y - \frac{xy^2}{2}\right]_0^b dx = \int_0^a \left(bc^2 - \frac{xb^2}{2}\right) dx$$

$$= \left[xbc^2 - \frac{x^2b^2}{4}\right]_0^a = abc^2 - \frac{a^2b^2}{4} \longrightarrow (5).$$

(f) $$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OAFB} \left[(x^2-yz)\vec{i} + (y^2-xz)\vec{j} + (z^2-xy)\vec{k}\right]\cdot(-\vec{k})\,dy\,dx$$

$$= \int_0^a\int_0^b -(z^2-xy)\,dy\,dx = \int_0^a\int_0^b (xy)\,dy\,dx \quad (z=0)$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_0^b dx = \int_0^a \frac{xb^2}{2} dx = \left[\frac{x^2b^2}{4}\right]_0^a$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \frac{a^2b^2}{4} \longrightarrow (6).$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} \longrightarrow (7).$$

(1), (2), (3), (4), (5), (6) in (7)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \left(a^2bc - \frac{b^2c^2}{4}\right) + \left(\frac{b^2c^2}{4}\right) + \left(ab^2c - \frac{a^2c^2}{4}\right)$$

$$+ \left(\frac{a^2c^2}{4}\right) + \left(abc^2 - \frac{a^2b^2}{4}\right) + \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c) \longrightarrow (B).$$

$$(A) \,\&\, (B) \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

Hence Gauss's Divergence Theorem is verified.

Example 6

Evaluate by Stoke's Theorem $\int_C e^x dx + 2y\,dy - dz$

C is the curve $x^2+y^2=4$, $z=2$.

Soln: Stoke's Theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

$$\vec{F}\cdot d\vec{r} = (e^x dx + 2y\,dy - dz)$$

$\therefore \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(2y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}(e^x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(e^x)\right]$$

$\text{curl}\,\vec{F} = \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$

$\text{curl}\,\vec{F} = 0 \Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds = 0$

$\therefore \int_C \vec{F}\cdot d\vec{r} = 0$

(ie) $\int_C (e^x dx + 2y\,dy - dz) = 0$

Example: 7

If $\vec{F} = x^2\vec{i} + y^2\vec{j}$, evaluate $\int \vec{F}\cdot d\vec{r}$ along the line $y = x$ from $(0,0)$ to $(1,1)$.

Soln: $\vec{F} = x^2\vec{i} + y^2\vec{j}$; $y = x \Rightarrow dy = dx$.

$\therefore \vec{F} = x^2\vec{i} + x^2\vec{j}$ $(y = x)$; $d\vec{r} = dx\vec{i} + dy\vec{j}$

$d\vec{r} = dx\vec{i} + d\ \vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2\vec{i} + x^2\vec{j})\cdot(dx\vec{i} + dx\vec{j})$

$= x^2dx + x^2dx \Rightarrow \vec{F}\cdot d\vec{r} = 2x^2dx.$

$\int_{y=x} \vec{F}\cdot d\vec{r} = \int_0^1 2x^2dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2(1/3).$

$\therefore \int_{y=x} \vec{F}\cdot d\vec{r} = 2/3.$