Inst Hour	: 5
Credit	: 5
Code	: 18K1M01

SEMESTER : I
CORE COURSE : I

BASICS, DIFFERENTIAL CALCULUS AND TRIGONOMETRY

UNIT 1:
Sets – Mappings – Equivalence Relations – The Integers – Binary Operations – Partially Ordered Sets.
(Chapter 1 : Sections 1.1 to 1.6 of Text Book 1).
Methods of Successive Differentiation – Leibnitz's Theorem for the n^{th} derivative -Increasing & Decreasing functions, Maxima and Minima for one variable.
(Chapter 3: Sections 1.1-1.6, 2.1, 2.2, Chapter IV Sections 2.1, 2.2 &
Chapter 5: Sections 1.1-1.5 of Text Book 2)

UNIT 2:
Curvature – Radius of curvature in Cartesian & in Polar Coordinates – Centre of curvature – Evolutes & Involutes
(Chapter 10: Sections 2.1 –2.6 of Text Book 2)

UNIT 3:
Expansions of sin(nx), cos(nx), tan(nx) – Expansions of $\sin^n x$, $\cos^n x$ – Expansions of sin(x), cos(x), tan(x) in powers of x and related problems.
(Chapter 1: Sections 1.2 to 1.4 of Text Book 3)

UNIT 4:
Hyperbolic functions – Relation between hyperbolic & Circular functions- Inverse hyperbolic functions.
(Chapter 2: Sections 2.1 & 2.2 of Text Book 3)

UNIT 5:
Logarithm of a complex number -Summation of Trigonometric series-Difference method- Angles in arithmetic progression method –Gregory's Series
(Chapter 3 & Chapter 4 : Sections 4.1,4.2 & 4.4 of Text Book 3)

Text Book(s)

[1] M. L. Santiago, Modern Algebra, Tata McGraw – Hill Publishing Company Limited, 2001.
[2] S.Narayanan,T.K.Manickavasagam Pillai, Calculus Volume I, S.V Publications - 2004
[3] S. Arumugam, Issac & Somasundaram, Trigonometry and Fourier Series New Gamma Publications– 1999 Edition

Books for Reference

[1] S.Arumugam & others, Calculus Volume I.
[2] S.Narayanan, Trigonometry
[3] Ajit Kumar, S.Kumaresan, Bhaba Kumar Sarma, A Foundation Course in Mathematics, Narosa Publishing House.

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

UNIT - I

Standard Result on n^{th} Derivative :-

① If $y = e^{ax}$, Find y_n

Soln :-

$$y_1 = a e^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$\vdots$$

$$y_n = a^n e^{ax}$$

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② If $y = (ax+b)^m$, Find y_n

Soln :-

$$y_1 = m(ax+b)^{m-1} \cdot a$$

$$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$$

$$\vdots$$

$$y_n = m(m-1)(m-2) \ldots (m-n+1)(ax+b)^{m-n} a^n$$

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③ If $y = (ax+b)^{-1}$, Find y_n

Soln :-

$$y_1 = (-1)\, a\, (ax+b)^{-2}$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2$$

$$\vdots$$

$$y_n = (-1)^n n!\, a^n / (ax+b)^{n+1}.$$

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④ Find y_n, when $y = \log(ax+b)$

Soln :-

$$y_1 = \frac{1}{(ax+b)} \cdot a = a \cdot (ax+b)^{-1}$$

$$y_2 = a \cdot (-1)(ax+b)^{-2} \cdot a = a^2 (-1)(ax+b)^{-2}$$

$$\vdots$$

$$y_n = a^n (-1)^{n-1} (n-1)!\, (ax+b)^{-n}$$

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⑤ If $y = \sin(ax+b)$, Find y_n.

Soln :-

$$y_1 = a\cos(ax+b) = a \sin(\pi/2 + ax + b)$$

$$y_2 = a^2 \cos(\pi/2 + ax + b) = a^2 \sin(2\pi/2 + ax + b)$$

$$\vdots$$

$$y_n = a^n \sin(n\pi/2 + ax + b).$$

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⑥ If $y = \cos(ax+b)$, Find y_n

Soln :-

$$y_1 = -\sin(ax+b) \cdot a = a \cdot \cos(\pi/2 + ax + b)$$

$$y_2 = a^2 \cos(\pi/2 + ax + b)$$

$$y_n = a^n \cos(\pi/2 + ax + b)$$

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(7) If $y = e^{ax}\sin(bx+c)$, Find y_n

Soln :-

$y_1 = e^{ax}\cos(bx+c) + a.e^{ax}\sin(bx+c)$

$a = r\cos\theta \quad b = r\sin\theta$

$r = \sqrt{a^2+b^2} \quad \theta = \tan^{-1}(b/a)$

$a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$y_1 = e^{ax}(r\sin\theta\cos(bx+c) + r\cos\theta\sin(bx+c))$

$= r.e^{ax}(\sin(\theta+bx+c))$

$y_2 = r^2 e^{ax}(\sin(2\theta+bx+c))$

$\vdots$

$y_n = r^n e^{ax}(\sin(n\theta+bx+c))$

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(8) When $y = \cos 3x$, Find y_n

Soln :-

$y = \cos 3x$

$y_n = 3^n\cos\left(\frac{n\pi}{2} + 3x\right)$ $\left[\text{W.K.T } y_n = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)\right]$

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(9) If $y = \sin^2 x$, Find y_n

Soln :- $y = \sin^2 x$ $\left[\text{W.K.T } \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$y_n = -1/2\,[2^n\cos(n\pi/2 + 2x)]$

$= -2^{n-1}\cos(n\pi/2 + 2x)$

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(10) If $y = \sin 4x\cos x$, Find y_n

Soln :- $y = \sin 4x\cos x$ [W.K.T $2\sin A\cos B = \sin(A+B) + \sin(A-B)$]

$= 1/2\,(\sin 5x + \sin 3x)$

$y_n = 1/2\,[5^n\sin(n\pi/2 + 5x) + 3^n\sin(n\pi/2 + 3x)]$

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(11) Find y_n, Where $y = \frac{3}{(2x-1)(x+1)}$

Soln :- $y = 3/(2x-1)(x+1)$

$$\frac{3}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$

$3 = A(x+1) + B(2x-1)$

$x = -1 \Rightarrow B = -1$; $x = 1/2 \Rightarrow A = 2$

$\therefore y = 2/_{2x-1} - 1/_{x+1}$

$$y_n = \frac{2(-1)^n 2^n . n!}{(2x-1)^{n+2}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

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12) Find the n^{th} derivative of $x^2/(x+2)(x-1)^2$

Soln :-

$$y = x^2/(x+2)(x-1)^2$$

$$\frac{x^2}{(x+2)(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$x = 1 \Rightarrow B = 1/3$; $x = -2 \Rightarrow C = 4/9$; $x = 0 \Rightarrow A = 5/9$

$$y = 5/9 \frac{1}{(x-1)} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{(x+2)}$$

$$y_n = \frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n+1!}{3(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+2}}$$

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13) Find y_n, when $y = 1/x^2+a^2$.

Soln :-

$$\frac{1}{x^2+a^2} = \frac{A \quad 1}{(x+ia)(x-ia)}$$

$$\frac{1}{x^2+a^2} = \frac{A}{(x+ia)} + \frac{B}{(x-ia)}$$

$$1 = A(x-ia) + B(x+ia)$$

$x = ia \Rightarrow B = 1/2ia$; $x = -ia \Rightarrow A = -1/2ia$.

$$y = \frac{-1/2ia}{(x+ia)} + \frac{1/2ia}{(x-ia)}$$

$$y_n = \frac{-1 \, (-1)^n n! \, i^n}{2ia \, (x+ia)^{n+1}} + \frac{(-1)^n n! \, i^n}{2ia \, (x-ia)^{n+1}}$$

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(14) Find the n^{th} derivative of $\log\left(\frac{x+a}{x-a}\right)$.

Soln :-

(i) $$\log\left(\frac{x+a}{x-a}\right) = \log(x+a) - \log(x-a)$$

$$D^n\left[\log\left(\frac{x+a}{x-a}\right)\right] = D^n[\log(x+a)] - D^n[\log(x-a)]$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} - \frac{(-1)^{n-1}(n-1)!}{(x-a)^n}.$$

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(15) $\cos^3 2x$

Soln :-

$$D^n[\cos^3 2x] = D^n\left[\frac{3\cos x + \cos 3x}{4}\right]$$

$$= 3/4 \, D^n(\cos x) + 1/4 \, D^n(\cos 3x)$$

$$= 3/4 \cos(x + n\pi/2) + 1/4 \, 3^n \cos(3x + n\pi/2)$$

Trigonometrical Transformation :-

(15) Find the n^{th} diff/- co-eff of $\cos x \cos 2x \cos 3x$

Soln :-

$y = \cos x \cos 2x \cos 3x$ [$\because$ W.K.T $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$]

$= \frac{1}{2} \cos 2x [\cos 4x + \cos 2x]$

$= \frac{1}{2}[\cos 4x + \cos 2x] + \frac{1}{2} \cos^2 2x$

$= \frac{1}{4}(\cos 2x + \cos 6x) + \frac{1}{4}(1 + \cos 4x)$

$= \frac{1}{4} + \frac{1}{4}(\cos 2x + \cos 4x + \cos 6x)$

$D^n(\cos x \cos 2x \cos 3x) = \frac{1}{4}\{2^n \cos(n\pi/2 + 2x) +$
$4^n \cos(n\pi/2 + 4x) + 6^n \cos(n\pi/2 + 6x)\}$

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(16) Find the n^{th} diff/- co-eff of $\cos^5 \theta \sin^7 \theta$.

Soln :-

Let $x = \cos\theta + i\sin\theta$; $1/x = \cos\theta - i\sin\theta$

$x + 1/x = 2\cos\theta$; $x - 1/x = 2i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$; $1/x^n = \cos n\theta - i\sin n\theta$

$x^n + 1/x^n = 2\cos n\theta$; $x^n - 1/x^n = 2i\sin n\theta$

$2^5 \cos^5\theta = (x + 1/x)^5$; $2^7 i^7 \sin^7\theta = (x - 1/x)^7$

$\therefore 2^{12} i^7 \cos^5\theta \sin^7\theta = (x + 1/x)^5 (x - 1/x)^7$

$= (x + 1/x)^5 (x - 1/x)^5 (x - 1/x)^2$

$= (x^2 - 1/x^2)^5 (x - 1/x)^2$

$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)(x^2 - 2 + 1/x^2)$

$= [x^{12} - 2x^{10} + 2x^6 - x^4 - 4x^8 + 8x^6 - 8x^2 + 4 + 6x^4 -$
$12x^2 + 12/x^2 - 6/x^4 - 4 + 8/x^2 - 8/x^6 + 4/x^8 +$
$1/x^4 - 2/x^6 + 2/x^{10} - 1/x^{12}]$

$= (x^{12} - 1/x^{12}) - 2(x^{10} - 1/x^{10}) - 4(x^8 - 1/x^8) + 10(x^6 - 1/x^6)$
$+ 5(x^4 - 1/x^4) - 20(x^2 - 1/x^2)$

$= 2i \sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta)$
$+ 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$

$\div$ by $2i$ on both sides.

$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta +$
$10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta$.

$D^n(\cos^5\theta\sin^7\theta) = -\frac{1}{2^{11}}\{12^n \sin(n\pi/2 + 12\theta) - 10^n 2\sin(n\pi/2 + 10\theta) - 8^n 4\sin(n\pi/2 + 8\theta) + 6^n 10\sin(n\pi/2 + 6\theta) + 4^n 5\sin(n\pi/2 + 4\theta) - 2^n 20\sin(n\pi/2 + 2\theta)\}$

Formations of Equations involving Derivatives :-

① If $xy = ae^x + be^{-x}$. Prove that $x\, d^2y/dx^2 + 2\, dy/dx - xy = 0$

Soln :-

$xy = ae^x + be^{-x}$

Diff/- both sides with respect to x

$x\, dy/dx + y = ae^x - be^{-x}$

Again differentiating both sides

$x\, d^2y/dx^2 + dy/dx + dy/dx = ae^x + be^{-x}$

$x\, d^2y/dx^2 + 2\, dy/dx = xy$

$\therefore\ x\, d^2y/dx^2 + 2\, dy/dx - xy = 0$

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② Prove that if $y = \sin(m\sin^{-1}x)$, $(1-x^2)y_2 - xy_1 + M^2y = 0$.

Soln :-

$y = \sin(m\sin^{-1}x)$

$\sin^{-1}y = M\sin^{-1}x$

Diff/- w.r. to x on both sides

$\frac{1}{\sqrt{1-y^2}}\, dy/dx = M \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, dy/dx = M \cdot \sqrt{1-y^2}$

Squaring on both sides

$(1-x^2)(dy/dx)^2 = M^2(1-y^2)$

Again diff/- w.r. to x on both sides

$(1-x^2)\, 2(dy/dx)(d^2y/dx^2) + (dy/dx)^2(-2x) = -M^2 2y\frac{dy}{dx}$

Dividing by $2\, dy/dx$

$(1-x^2)\, d^2y/dx^2 + x\, dy/dx = -M^2y$.

$\therefore\ (1-x^2)y_2 - xy_1 + M^2y = 0$.

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③ If $x = \sin\theta$, $y = \cos p\theta$, P.T $(1-x^2)y_2 - xy_1 + p^2y = 0$

Soln :-

$dx/d\theta = \cos\theta$; $dy/d\theta = -p\sin\theta$; $dy/dx = \frac{-p\sin\theta}{\cos\theta}$

$x = \sin\theta \Rightarrow x^2 = \sin^2\theta \Rightarrow 1-x^2 = 1-\sin^2\theta$

$\therefore 1-x^2 = \cos^2\theta$

$\sqrt{1-x^2} = \cos\theta$

$y = \cos p\theta \Rightarrow y^2 = \cos p^2\theta \Rightarrow 1-y^2 = 1-\cos p^2\theta$

$\therefore 1-y^2 = \sin p^2\theta \Rightarrow \sqrt{1-y^2} = \sin p\theta$

$dy/dx = \dfrac{-p\sqrt{1-y^2}}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx} = -p\sqrt{1-y^2}$

Squaring on both sides

$(1-x^2)(dy/dx)^2 = p^2(1-y^2)$

Diff/-

$(1-x^2)\,2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + \left(\dfrac{dy}{dx}\right)^2(-2x) = -p^2\, 2y\dfrac{dy}{dx}$

$\div$ by $2(dy/dx)$

$(1-x^2)\,d^2y/dx^2 - x(dy/dx) = -p^2y$

$(1-x^2)\,y_2 - xy_1 + p^2y = 0$

Leibnitz Formula for the n^{th} derivative

If u and v are the functions of x, and assume the theorem to be true for some one value of n

$D^n(uv) = u_n v + nc_1 u_{n-1}v_1 + \cdots + nc_r u_{n-r}v_r + \cdots + uv_n$

1. If $y = \sin(m\sin^{-1}x)$. Prove that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.

Soln :-

$y = \sin(m\sin^{-1}x) \rightarrow 1 \Rightarrow \sin^{-1}y = M\sin^{-1}x \rightarrow 2$

Diff/- 2 w. r. to x.

$\dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx} = M\cdot\dfrac{1}{\sqrt{1-x^2}}$.

Squaring on both sides

$(1-x^2)(dy/dx)^2 = M^2(1-y^2)$.

Again diff/- w. r to x on both sides

$(1-x^2)\,d^2y/dx^2 - x\,dy/dx = -M^2y$

$(1-x^2)y_2 - xy_1 + M^2y = 0$

$(1-x^2)y_2 - (xy_1) + M^2y = 0 \rightarrow 3$

Apply Leibnitz theorem in eqn/- 3

$(1-x^2)y_{n+2} + nc_1 y_{n+1}(-2x) + nc_2(-2)y_n - (xy_{n+1} + nc_1 y_n) + M^2y_n = 0$

$(1-x^2)y_{n+2} - 2nxy_{n+1} + \frac{n(n-1)}{2}(-2)y_n - (xy_{n+1}) -$

$(ny_n) + m^2 y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} + (m^2 - n^2)y_n = 0$

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(2) If $y = a\cos(\log x) + b\sin(\log x)$

P.T $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Soln:-

$y = a\cos(\log x) + b\sin(\log x)$

Diff/- w.r to x on both sides.

$y_1 = -a\sin(\log x)\cdot 1/x + b\cos(\log x)\cdot 1/x$

$xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again diff/- w.r to x on both sides

$xy_2 + y_1 = -a\cos(\log x)\cdot 1/x - b\sin(\log x)\cdot 1/x$

$xy_2 + y_1 = -1/x\,(a\cos(\log x) + b\sin(\log x))$

$x^2y_2 + xy_1 = -y$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Apply Leibnitz theorem,

$x^2 y_{n+2} + nc_1 y_{n+1} 2x + nc_2 y_n (2) + xy_{n+1} + nc_1 y_n + y_n = 0$

$x^2y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}(2)y_n + xy_{n+1} + ny_n + y_n = 0$

$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

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(3) If $y = (x+\sqrt{1+x^2})^m$, Prove that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

Soln:-

$y = (x+\sqrt{1+x^2})^m \rightarrow (1)$

Diff/- w.r.to x on both sides

$y_1 = m(x+\sqrt{1+x^2})^{m-1} + \left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right)$

$\sqrt{1+x^2}\, y_1 = m(x+\sqrt{1+x^2})^{m-1}$

$\sqrt{1+x^2}\, y_1 = my$ (by (1))

Squaring on both sides

$(1+x^2)y_1^2 = m^2y^2$

Diff/- w.r.to x on both sides

$(1+x^2)2y_1y_2 + y_1^2(2x) = 2m^2yy_1$

Divide by $2y_1$ on both sides

$(1+x^2)y_2 + xy_1 - m^2y = 0$

$((1+x^2)y_2) + (xy_1) - m^2 y = 0$

Apply Lebintz theorem

$(1+x^2)y_{n+2} + nc_1 y_{n+1}(2x) + nc_2 y_n(2) + xy_{n+1} + {}^{n}c_1 y_n - m^2 y_n = 0$

$(1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2} 2y_n + (xy_{n+1}) + ny_n - m^2 y_n = 0$

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

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(4) If $y^{1/m} + y^{-1/m} = 2x$, then p.t

$(x^2-1)y_{n+2} + (n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Soln :-

$y^{1/m} + y^{-1/m} = 2x \rightarrow (1)$

Diff/- w.r. to x on both sides

let $y^{1/m} = z$; $y^{-1/m} = 1/z$

$(1) \Rightarrow z + 1/z = 2x \Rightarrow \frac{z^2+1}{z} = 2x$

$z^2+1 = 2zx \Rightarrow z^2 - 2zx + 1 = 0$

$z = \frac{2x \pm \sqrt{4x^2-4}}{2} = \frac{2(x \pm \sqrt{x^2-1})}{2} = (x \pm \sqrt{x^2-1})$

$y^{1/m} = (x \pm \sqrt{x^2-1}) \Rightarrow y = (x \pm \sqrt{x^2-1})^{1/m} \rightarrow (2)$

Diff/- (2) w.r. to x on both sides

$y_1 = M(x \pm \sqrt{x^2-1})^{M-1}\left(1 + \frac{2x}{2\sqrt{x^2-1}}\right)$

$= M(x \pm \sqrt{x^2-1})^{M-1}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$

$= M(x \pm \sqrt{x^2-1})^{m-1}\left(\frac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$

$y_1 = \frac{M(x \pm \sqrt{x^2-1})^M}{\sqrt{x^2-1}}$.

$(\sqrt{x^2-1})y_1 = M(x \pm \sqrt{x^2-1})^M$

$(\sqrt{x^2-1})y_1 = My$

Squaring on both sides.

$(x^2-1)y_1^2 = M^2y^2$

Diff/- w.r. to x on both sides

$(x^2-1)2y_1y_2 + y_1^2(2x) = M^2 2yy_1$

Divide by $2y_1$ &

Apply Lebintz theorem.

$(x^2-1)y_{n+2} + nc_1 y_{n+1}(2x) + nc_2 y_n(2) + xy_{n+1} + {}^{n}c_1 y_n - M^2 y_n = 0$

$(x^2-1)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2}(2)y_n + (xy_{n+1}) + ny_n - m^2y_n = 0$

$(x^2-1)y_{n+2} + (n+1)xy_{n+1} + (n^2-M^2)y_n = 0$.

Meaning of the Derivative :-

① prove that the function $f(x) = x^3 - 3x^2 + 6$ is positive for all values of $x > 2$.

Soln :-

$f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

If $x > 2$, then $f'(x)$ is positive

$\therefore f(x)$ is an increasing function for $x > 2$

But, $f(2) = 8 - 12 + 6 = 2$

$f'(2) = 8 - 12 + 6 = 2$

$\therefore x > 2$, $f(x)$ is positive.

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② For what values of x is $2x^3 - 9x^2 + 12x + 4$, a decreasing function.

Soln :-

$f(x) = 2x^3 - 9x^2 + 12x + 6$

$f'(x) = 6x^2 - 18x + 12$

$\div$ by 3.

$\Rightarrow$ $2x^2 - 6x + 4 = 0$

$(x-1)(x-2) = 0$

$x - 1 = 0$ (or) $x - 2 = 0$

$\therefore \boxed{x = 1}$ (or) $\boxed{x = 2}$

If the value of x lies between 1 and 2. $f'(x)$ is Negative and so in that range $f(x)$ is a decreasing function.

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③ Show that for $x > 0$, $x - \frac{1}{2}x^2 < \log(1+x) < x$.

Proof :-

Let $f(x) = x - \frac{1}{2}x^2 < \log(1+x) < x$

$f(x) = x - \frac{1}{2}x^2 - \log(1+x)$

$$f'(x) = 1 - x - \frac{1}{1+x}$$

$$= \frac{1(1+x) - x(1+x) - 1}{(1+x)}$$

$$= \frac{1+x - x - x^2 + 1}{1+x} = \frac{-x^2}{1+x}$$

If x is positive, $f'(x)$ is Negative.

$\therefore$ $f(x)$ is a decreasing function.

The value of $f(x)$ at $x=0$ is 0.

For all positive values of x, $f(x)$ is Negative

$$x - \tfrac{1}{2}x^2 < \log(1+x)$$

$$\therefore \quad x - \tfrac{1}{2}x^2 < \log(1+x).$$

Let, $F(x) = \log(1+x) < x$

$$F(x) = \log(1+x) - x$$

$$F'(x) = \frac{1}{1+x} - 1$$

$$= \frac{1 - 1 - x}{1+x} = \frac{-x}{1+x}.$$

$\because$ x is positive, $f'(x)$ is Negative.

$\therefore$ $f(x)$ is a decreasing function.

$$F(0) = 0.$$

So, if $x > 0$, $F(x)$ is Negative

$$\therefore \log(1+x) - x < 0$$

$$\therefore \log(1+x) < x.$$

Hence, $x - \tfrac{1}{2}x^2 < \log(1+x) < x$.

UNIT - II

Curvature and Radius of Curvature :-

A curve has a definite direction at every point on it. At any particular point say P, the direction of the curve is the same as that of the tangent to the curve at that point. The direction usually changes from point to point and the tangent line rotates as the point moves along the curve.

Curvature of Circle at any point :-

The circle is called the circle of curvature at P. So, it can be defined as that circle which touches the given curve at the point has a radius equal to the radius of curvature at the point and lies on the same side of the tangent as the curve. The radius of curvature is often denoted by ρ and so the curvature is $1/\rho$.

Formulas For Radius of Curvature :-

Cartesian Formula :-

$$\rho = \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2} \quad \text{(OR)}$$

$$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{y_2}$$

---

1. What is the Radius of Curvature of the curve $x^4+y^4=2$ at the point $(1,1)$.

Soln :-

$$x^4+y^4=2$$

Differentiating above equation, we get

$$4x^3 + 4y^3 \cdot \frac{dy}{dx} = 0$$

$$4y^3\, dy/dx = -4x^3$$

$$dy/dx = -4x^3/4y^3$$

$$dy/dx = -x^3/y^3$$

$$(dy/dx)_{(1,1)} = -1$$

$$d^2y/dx^2 = \frac{-[3y^3x^2 - 3x^3y^2 \cdot dy/dx]}{y^6}$$

$$= \frac{-3y^2[yx^2 + x^3]}{y^6} = -\frac{3}{y^4}(yx^2 + x^3)$$

$$(d^2y/dx^2)_{(1,1)} = -\frac{3}{1}(1+1) = -6.$$

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1 + ((-1)^2)]^{3/2}}{-6} = \frac{(1+1)^{3/2}}{-6} = \frac{2^{3/2}}{-6}$$

$$= \frac{2^1 \cdot 2^{1/2}}{-6} = 2^{1/2}/\cancel{-3}$$ (ommiting Negative sign)

$$\rho = \sqrt{2}/3$$

---

② Show that the Radius of Curvature at any point of the catenary $y = c \cosh x/c$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Soln :-

$$y = c \cosh x/c$$

$$y_1 = c \sinh x/c \cdot 1/c$$

$$y_1 = \sinh x/c$$

$$y_2 = \cancel{1/c} \cdot \cosh x/c \cdot 1/c$$

$$y_2 = 1/c \cosh x/c$$

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1+(\sinh x/c)^2]^{3/2}}{1/c \cdot \cosh x/c}$$

$$= \frac{c\,[\cosh^2 x/c]^{3/2}}{\cosh x/c} = \frac{c\,[\cosh x/c]^3}{\cosh x/c}$$

$$= c\,[\cosh x/c]^2$$

$$= c\,[y/c]^2$$

$$= c\left[\frac{y^2}{c^2}\right]$$

$$= y^2/c \rightarrow (1)$$

Length of the Normal

$$= y\left[1+(dy/dx)^2\right]^{1/2}$$

$$= c\,[\cosh x/c]\,[1+(\sinh x/c)^2]^{1/2}$$

$$= c\,[\cosh x/c]\,[\cosh^2 x/c]^{1/2}$$

$$= c\,[\cosh x/c]\sqrt{\cosh^2 x/c}$$

$$= c\,[\cosh x/c]\,[\cosh x/c]$$

$$= y \times y/c$$

$$= y^2/c \rightarrow (2)$$

- From (1) and (2),

$\therefore$ Radius of Curvature = Length of the Normal.

---

(3) If a curve is defined by the parametric equation $x = f(\theta)$, $y = \phi(\theta)$. Prove that the curvature is

$$1/\rho = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

Soln :-

$$dy/dx = \frac{dy/d\theta}{dx/d\theta} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\cdot {}^{1}/_{dx/d\theta}$$

$$= {}^{d}/_{d\theta}\left(\frac{y'}{x'}\right)\cdot\frac{1}{x'}$$

$$= \frac{x'y''-y'x''}{(x')^2}\cdot\frac{1}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{x'y''-y'x''}{(x')^3}$$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}$$

$$= \frac{\dfrac{x'y''-y'x''}{(x')^3}}{[1+(y'/x')^2]^{3/2}}$$

$$= \frac{\dfrac{x'y''-y'x''}{(x')^3}}{\dfrac{(x'^2+y'^2)^{3/2}}{(x'^2)^{3/2}}}$$

$$= \frac{x'y''-y'x''}{(x')^3}\cdot\frac{(x')^3}{(x'^2+y'^2)^{3/2}}$$

$$\frac{1}{\rho} = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$$

---

(4) Prove that the Radius of curvature at any point of the Cycloid $x = a(\theta+\sin\theta)$ and $y = a(1-\cos\theta)$ is $4a\cos\theta/2$.

Soln :-

$x = a(\theta + \sin\theta)$ $\quad\quad$ $y = a(1 - \cos\theta)$

$x' = a(1 + \cos\theta)$ $\quad\quad$ $y' = a(0 + \sin\theta)$

$x'' = a(0 - \sin\theta)$ $\quad\quad$ $y'' = a\cos\theta$

$x'' = -a\sin\theta$

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

$$= \frac{a(1+\cos\theta)(a\cos\theta) - a(0+\sin\theta)(-a\sin\theta)}{[(a(1+\cos\theta))^2 + (a\sin\theta)^2]}$$

$$= \frac{a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}{[a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)]}$$

$$= \frac{a^2\cos\theta + a^2}{a^3(2 + 2\cos\theta)^{3/2}} = \frac{a^2(1+\cos\theta)}{a^3[2(1+\cos\theta)]^{3/2}}$$

$$= \frac{a^2(1+\cos\theta)}{a^3[2(1+\cos\theta)]^{3/2}} = \frac{2\cdot\cos^2\theta/2}{a(2\cdot 2\cos^2\theta/2)^{3/2}}$$

$$= \frac{2\cos^2\theta/2}{a(4\cos^2\theta/2)^{3/2}}$$

$$= \frac{1}{4a\cos\theta/2}$$

$\rho = 4a\cos\theta/2$

Note :-

The Definition of the Radius of Curvature shows that its value depends only on the curve and not on the axes. Therefore by interchanging the axes of x and y, we obtain

$$\rho = \frac{[1+(dx/dy)^2]^{3/2}}{d^2x/dy^2}$$

The Formula is useful when the tangent is parallel to the y axis. (i.e) when $dy/dx = \infty$

---

(1) Find the radius of curvature of the curve $y^2 = x^3 + 8$ at $(-2, 0)$.

Soln :-

$$y^2 = x^3 + 8 \Rightarrow 2y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(-2,0)} = \frac{3(-2)^2}{2(0)} = \frac{12}{0} = \infty$$

Here, we get the value of dy/dx at $(-2,0)$ is ∞.

$\therefore$ To Find ρ, we use the formula.

$$\rho = \frac{[1+(dx/dy)^2]^{3/2}}{d^2x/dy^2} \quad ; \quad \frac{dx}{dy} = \frac{2y}{3x^2} ; \left(\frac{dx}{dy}\right)_{(-2,0)} = \frac{0}{12} = 0$$

$$\frac{d^2y}{dx^2} = \frac{3x^2 . 2 - 2y . 6x . dx/dy}{9x^4} = \frac{3(-2)^2 . 2 - 2(0)6(-2) . 0}{9(-2)^4}$$

$$= 12/72 = 1/6.$$

$$\rho = \frac{[1+(dx/dy)^2]^{3/2}}{d^2x/dy^2} = \frac{[1+0]^{3/2}}{1/6} = 6$$

$$\rho = 6.$$

Co.ordinate of Centre of Curvature :-

$$\bar{x} = x - y_1/y_2\,(1+y_1^2)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

Equation of the circle of curvature :-

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2.$$

---

(1) Find the co.ordinates of the centre of curvature of the curve $xy = 2$ at the point $(2,1)$

Soln :- $xy = 2$

Diff/- w.r. to x

$xy_1 + y(1) = 0$

$y_1 = -y/x \Rightarrow y_1(2,1) = -1/2$

$$y_2 = -\left[\frac{xy_1 - y(1)}{x^2}\right] \Rightarrow y_{2(2,1)} = -\left[\frac{2(-1/2)-1}{4}\right]$$

$y_2 = 2/4 = 1/2$

$$X = \frac{2 - (-1/2)[1+(-1/2)^2]}{1/2} = \frac{12+1}{4} = \frac{13}{4}$$

$$Y = \frac{1+(1+1/4)}{1/2} = \frac{1+5/4}{1/2} = 1+\frac{5}{2} = \frac{7}{2}.$$

$\therefore$ The centre of curvature is $(X, Y) = (13/4, 7/2)$.

---

(2) Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$, $X = 2a + 3at^2$, $Y = -2at^3$. Deduce the equation of the Evolute.

Soln:- $x = at^2 \Rightarrow dx/dt = a2t = 2at$

$y = 2at \Rightarrow dy/dt = 2a$

$$\therefore y_1 = dy/dx = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = 1/t.$$

$$y_2 = d^2y/dx^2 = \frac{d}{dt}(dy/dx)\left(\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\left(\frac{1}{2at}\right)$$

$$= (-1)t^{-1-1}\cdot\frac{1}{2at} = -\frac{1}{t^2}\cdot\frac{1}{2at}$$

$d^2y/dx^2 = -1/2at^3$.

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1+(1/t^2)]^{3/2}}{-1/2at^3} = -2at^3\left[1+\frac{1}{t^2}\right]^{3/2}$$

$$= \frac{-2at^3(1+t^2)^{3/2}}{t^3} = -2a(1+t^2)^{3/2}.$$

$$X = \frac{x - y_1(1+y_1^2)}{y_2} = \frac{at^2 + 1/t\,(1+(1/t)^2]}{1/2at^3}$$

$X = at^2 + (t^2+1)2a = at^2 + 2at^2 + 2a.$

$$Y = \frac{y + (1+y_1^2)}{y_2} = 2at - \frac{(1+1/t^2)}{1/2at^3}$$

$= 2at - (t^2+1)2at = 2at - 2at^3 + 2at$

$y = -2at^3$

Therefore the centre of curvature is $(2a+3at^2, -2at^3)$

$x = 2a + 3at^2$ $\qquad$ $y = -2at^3$

$3at^2 = x - 2a$ $\qquad$ $y = -2a(t^2)^{3/2}$

$t^2 = \frac{x-2a}{3a}$ $\qquad$ $y = -2a\left(\frac{x-2a}{3a}\right)^{3/2}$

$y = -2a\left(\frac{x-2a}{3a}\right)^{3/2} \Rightarrow y^2 = \frac{4}{27a}(x-2a)^3$.

$\Rightarrow 27ay^2 = 4(x-2a)^3$

which is the Evolute.

(3) Show that the Evolute of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ is another equation cycloid.

Soln :-

$x = a(\theta - \sin\theta)$ $\qquad$ $y = a(1-\cos\theta)$

$dx/d\theta = a(1-\cos\theta)$ $\qquad$ $dy/d\theta = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{a\cdot 2\sin\theta/2\cos\theta/2}{a\cdot 2\cdot\sin^2\theta/2} = \frac{\cos\theta/2}{\sin\theta/2}$$

$y_1 = dy/dx = \cot\theta/2$

$$y_2 = d^2y/dx^2 = \frac{d}{d\theta}(\cot\theta/2)\frac{1}{a(1-\cos\theta)} = -\operatorname{cosec}^2\theta/2\cdot\tfrac{1}{2}\frac{1}{a(2\sin^2\theta/2)}$$

$$y_2 = -\frac{1}{\sin^2\theta/2}\cdot\frac{1}{2}\cdot\frac{1}{a(2\sin^2\theta/2)} = -\frac{1}{4a\sin^4\theta/2}.$$

$$X = x - \frac{y_1(1+y_1^2)}{y_2} = \frac{a(\theta-\sin\theta) - (\cot\theta/2)(1+\cot\theta/2)^2}{-1/4a\sin^4\theta/2}$$

$$= a(\theta-\sin\theta) + \left(\frac{\cos\theta/2}{\sin\theta/2}\right)\left(1+\frac{\cos^2\theta/2}{\sin^2\theta/2}\right)(4a\sin^4\theta/2)$$

$$= a(\theta-\sin\theta) + 4a\sin\theta/2\cdot\cos\theta/2 = a(\theta-\sin\theta) + 2a\sin\theta$$

$X = a[\theta + \sin\theta]$

$$Y = y + \frac{(1+y_1^2)}{y_2} = a(1-\cos\theta) + \frac{1+\cot^2\theta/2}{-1/4a\sin^4\theta/2}$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2\cdot\frac{1}{\sin^2\theta/2}$$

$= a(1-\cos\theta) - 4a\sin^2\theta/2$

$= a(1-\cos\theta) - 2a(1-\cos\theta)$

$y = a(1-\cos\theta)$

The locus of (x, y) is

$x = a[\theta + \sin\theta]$; $y = -[a(1-\cos\theta)]$

This is also a cycloid.

Evolute and Involute :-

Evolute of a curve as the locus of the centre of curvature and deduced the equations of the Evolute of the parabola and ellipse.

If the Evolute itself be regarded as the original curve, a curve of which it is the called the Involute.

① Show that the Radius of curvature at any point of the curve $r = a(1-\cos\theta)$ varies as $\sqrt{r}$.

soln :-

$$r = a(1-\cos\theta) \rightarrow ①$$

Diff/- ① w.r. to θ

$$\frac{dr}{d\theta} = a\sin\theta \quad ; \quad \frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)\cdot a\cos\theta}$$

$$= \frac{a^3[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]^{3/2}}{a^2[1 - 2\cos\theta + \cos^2\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$$

$$= \frac{a(2-2\cos\theta)^{3/2}}{(3-3\cos\theta)} = \frac{a2\sqrt{2}(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3} a\sqrt{\frac{r}{a}}$$

$$= \frac{2\sqrt{2}\sqrt{a}}{3} \cdot \sqrt{r}$$

Hence the Radius of curvature varies as $\sqrt{r}$

(2) Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ is $\frac{a^n r^{-n+1}}{n+1}$

Soln :-

$$r^n = a^n \cos n\theta$$

Taking log on both sides, we get

$$\log r^n = \log a^n \cos n\theta$$

$$n \log r = \log a^n + \log \cos n\theta$$

By diff/. we get

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}$$

$$\frac{dr}{d\theta} = -r \tan n\theta .$$

$$\frac{d^2 r}{d\theta^2} = -\frac{dr}{d\theta} \tan n\theta - nr \sec^2 n\theta$$

$$= r \tan^2 n\theta - nr \sec^2 n\theta .$$

$$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta) - r\, d^2r/d\theta^2} .$$

$$= \frac{|r^2 + r^2 \tan^2 n\theta|^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2 + r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r \sec^3 n\theta}{\sec^2 n\theta + n \sec^2 n\theta}$$

$$= \frac{r a^n}{(1+n) r^n}$$

$$= \frac{a^n r^{-n+1}}{(n+1)}$$

UNIT - III

Expression for $\sin n\theta$, $\cos n\theta$, $\tan n\theta$.

Theorem :-

For any positive integer n

(i) $\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + \cdots$

(ii) $\sin n\theta = n\cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

(iii) $\tan n\theta = \dfrac{nC_1 \tan\theta - nC_3 \tan^3\theta + \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta + \cdots}$

Problem :-

(1) Expand $\sin 7\theta$ in powers of $\cos\theta$ and $\sin\theta$.

Hence prove that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

Soln :-

$$(\cos 7\theta + i\sin 7\theta) = (\cos\theta + i\sin\theta)^7$$

$$= \cos^7\theta + 7C_1 \cos^6\theta(i\sin\theta) + 7C_2 \cos^5\theta (i\sin\theta)^2 + 7C_3\cos^4\theta (i\sin\theta)^3 + 7C_4 \cos^3\theta (i\sin\theta)^4 + 7C_5 \cos^2\theta (i\sin\theta)^5 + 7C_6 \cos\theta (i\sin\theta)^6 + 7C_7 \cancel{\cos} (i\sin\theta)^7.$$

$$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta \sin^6\theta - i\sin^7\theta.$$

Equating ~~Real and~~ I.P.

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$$

$$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$= 7(1-\sin^2\theta)^3 - 35\cos^4\theta\,(1-\sin^2\theta)^2\sin^2\theta +$
$21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$

$= 7(1-\sin^6\theta - 3\sin^2\theta + 3\sin^4\theta) -$
$35(1-2\sin^2\theta + \sin^4\theta)\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta$
$- \sin^6\theta.$

$= 7 - 7\sin^6\theta - 21\sin^2\theta + 21\sin^4\theta - 35\sin^2\theta$
$+ 70\sin^4\theta - 35\sin^6\theta + 21\sin^4\theta -$
$21\sin^6\theta - \sin^6\theta.$

$= 7 - 64\sin^6\theta - 56\sin^2\theta + 112\sin^4\theta.$

---

(2) Prove that $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 16\cos^4\theta - 32\cos^2\theta + 1.$

Soln :-

$\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8$
$= \cos^8\theta + 8C_1\cos^7\theta(i\sin\theta) + 8C_2\cos^6\theta(i\sin\theta)^2$
$+ 8C_3\cos^5\theta(i\sin\theta)^3 + 8C_4\cos^4\theta(i\sin\theta)^4 +$
$8C_5\cos^3\theta(i\sin\theta)^5 + 8C_6\cos^2\theta(i\sin\theta)^6 +$
$8C_7\cos\theta(i\sin\theta)^7 + 8C_8(i\sin\theta)^8.$

Equating the Real part, we get.

$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta$
$- 28\cos^2\theta\sin^6\theta + \sin^8\theta.$

$= \cos^8\theta - 28\cos^6\theta(1-\cos^2\theta) + 70\cos^4\theta(1-\cos^2\theta)^2$
$- 28\cos^2\theta(1-\cos^2\theta)^2 + (1-\cos^2\theta)^4.$

$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta$
$- 140\cos^6\theta + 70\cos^8\theta - 28\cos^2\theta +$
$84\cos^4\theta - 84\cos^6\theta + 28\cos^8\theta +$
$1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta$

$= 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$

(8) Prove that

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

Soln :-

$$\tan 7\theta = \frac{7C_1 \tan\theta - 7C_3 \tan^3\theta + 7C_5 \tan^5\theta - 7C_7 \tan^7\theta}{1 - 7C_2 \tan^2\theta + 7C_4 \tan^4\theta - 7C_6 \tan^6\theta}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

---

(11) Find the equation whose roots are $\sin^2(2\pi/7)$, $\sin^2(4\pi/7)$, $\sin^2(8\pi/7)$. Hence prove that $\cos(4\pi/7) + \cos(8\pi/7) + \cos(16\pi/7) = 1/2$.

Soln :-

Let θ have one of the following values

$2\pi/7$ (or) $4\pi/7$ (or) $8\pi/7$.

$\theta = 2\pi/7$ (or) $4\pi/7$ (or) $8\pi/7$

$7\theta = 2\pi$ (or) 4π (or) 8π.

$\sin 7\theta = 0$

$\Rightarrow 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta = 0$

Dividing by $\sin\theta$

$7\cos^6\theta - 35\cos^4\theta \sin^2\theta + 21\cos^2\theta \sin^4\theta - \sin^6\theta = 0$

$7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2 \sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta = 0$

put $\sin\theta = t$

$7(1-t^2)^3 - 35(1-t^2)t^2 + 21(1-t^2)t^4 - t^6 = 0$

Now, subt. $t^2 = y$.

$7(1-y)^3 - 35(1-y)^2 y + 21(1-y)y^2 - y^3 = 0$

$7(1 - 3y + 3y^2 - y^3) - 35(1 - 2y + y^2)y + 21(1-y)y^2 - y^3 = 0.$

$$7-21y+21y^2-7y^3-35y+70y^2-35y^3+$$
$$21y^2-21y^3-y^3=0$$

$$64y^3-112y^2+56y-7=0.$$

Sum of the roots of this equation is

$$\sin^2\left(\frac{2\pi}{7}\right)+\sin^2\left(\frac{4\pi}{7}\right)+\sin^2(8\pi/7)=112/64.$$

$$1/2\,(1-\cos 4\pi/7)+1/2\,(1-\cos 8\pi/7)+1/2\,(1-\cos 16\pi/7)=112/64$$

$$3/2-1/2\,(\cos 4\pi/7+\cos 8\pi/7+\cos 16\pi/7)=7/4$$

$$3-(\cos 4\pi/7+\cos 8\pi/7+\cos 16\pi/7)=7/2$$

$$3-7/2=\cos\{4\pi/7+\cos 8\pi/7+\cos 16\pi/7\}$$

$$\cos 4\pi/7+\cos 8\pi/7+\cos 16\pi/7=-1/2.$$

---

(5) Prove that $\tan 2\pi/7\,\tan 4\pi/7\,\tan 6\pi/7=\sqrt{7}.$

Soln :-

Let θ be any one of the values of

$2\pi/7$ (or) $4\pi/7$ (or) $6\pi/7$.

$$7\theta=2\pi\ (\text{or})\ 4\pi\ (\text{or})\ 6\pi$$

$$\tan 7\theta=0.$$

$$\frac{7c_1\tan\theta-7c_3\tan^3\theta+7c_5\tan^5\theta-7c_7\tan^7\theta}{1-7c_2\tan^2\theta+7c_4\tan^4\theta-7c_6\tan^6\theta}=0$$

$$7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta=0$$

put $\tan\theta=x$.

$$x^7-21x^5+35x^3-x=0.$$

$$\therefore x(x^6-21x^4+35x^2-1)=0$$

$$\therefore x=0\quad(\text{or})\quad x^6-21x^4+35x^2-1=0$$

Suppose $x=0$, then $\tan\theta=0$

$$\theta=\pi$$

Which is contradiction,

$$x^6-21x^4+35x^2-7=0.$$

Further than

$\tan(2\pi/7) = \tan(\pi - 5\pi/7) = \tan 5\pi/7$.

$\tan(4\pi/7) = \tan(\pi - 3\pi/7) = -\tan 3\pi/7$

$\tan(6\pi/7) = \tan(\pi - \pi/7) = -\tan \pi/7$.

$\therefore$ The equation $x^6 - 21x^4 + 35x^2 - 7 = 0$.

has the six roots $\pm \tan 2\pi/7, \pm \tan 4\pi/7, \pm \tan 6\pi/7$.

put $x^2 = y$ in the above equation, it reduces to

$y^3 - 21y^2 + 35y - 7 = 0$ and this equation has roots.

$\tan^2 2\pi/7, \tan^2 4\pi/7, \tan^2 6\pi/7$.

$\therefore$ product of the roots

$\tan^2 2\pi/7 \ \tan^2 4\pi/7 \ \tan^2 6\pi/7 = 7$

$\therefore \tan 2\pi/7 . \tan 4\pi/7 . \tan 6\pi/7 = \sqrt{7}$.

P.2 <u>Expression for $\sin^n\theta$ and $\cos^n\theta$.</u>

<u>Theorem</u>

$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nc_1 \cos(n-2)\theta + nc_2\cos(n-4)\theta + \cdots\right].$$

problem.

① prove that $2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

Soln :- Let $x = \cos\theta + i\sin\theta$.

$x + 1/x = 2\cos\theta$; $x^n + 1/x^n = 2\cos n\theta$.

$(2\cos\theta)^6 = (x + 1/x)^6$

$= x^6 + 6x^4 + 15x^2 + 20 + (15/x^2) + (6/x^4) + (1/x^6)$

$= (x^6 + 1/x^6) + 6(x^4 + 1/x^4) + 15(x^2 + 1/x^2) + 20$.

$\div$ by 2.

$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

---

② prove that $\sin^5\theta = (1/2^4)[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$

Soln :- Let $x = \cos\theta + i\sin\theta$

$(2i\sin\theta)^5 = (x - 1/x)^5$

$= x^5 - 5C_1 x^4 \cdot 1/x + 5C_2 x^3 \cdot 1/x^2 - 5C_3 x^2 \cdot 1/x^3 +$

$5C_4 x \cdot 1/x^4 - 5C_5 1/x^5$.

$i^5 2^5 \sin^5\theta = x^5 - 5x^3 + 10x - 10/x + 5/x^3 - 1/x^5$

$i\, 2^5 \sin^5\theta = (x^5 - 1/x^5) - 5(x^3 - 1/x^3) + 10(x - 1/x)$

$\div$ by $2i$

$2^4 \sin^5\theta = 2i \sin 5\theta - 5 \underset{\uparrow}{\overset{\times 2i}{}} \sin 3\theta + 10 \times 2i \sin\theta$

$\sin^5\theta = 1/2^4 [\sin 5\theta - 5 \sin 3\theta + 10 \sin\theta]$.

---

(3) Expand $\cos^5\theta \sin^3\theta$ in a series of sines of multiples of θ.

Soln :-

Let $x = \cos\theta + i\sin\theta$

$x + 1/x = 2\cos\theta$; $x - 1/x = 2i\sin\theta$.

$(2\cos\theta)^5 (2i\sin\theta)^3 = (x + 1/x)^5 (x - 1/x)^3$

$-2^8 i \cos^5\theta \sin^3\theta = (x + 1/x)^5 (x - 1/x)^3$

$= [(x + 1/x)^2] [(x + 1/x)(x - 1/x)]^3$

$= (x + 1/x)^2 [x^2 - 1/x^2]^3$

$= x^2 + 2 + 1/x^2 [x^6 - 1/x^6 - 3x^2 + 3/x^2]$

$= x^8 - 1/x^4 - 3x^4 + 3 + 2x^6 - 2/x^6 - 6x^2 + 6/x^2$

$+ x^4 - 1/x^8 - 3 + 3/x^4$.

$= (x^8 - 1/x^8) + 2(x^6 - 1/x^6) - 2(x^4 - 1/x^4) -$

$6(x^2 - 1/x^2)$

$= 2i\sin 8\theta + 2 \times 2i\sin 6\theta - 2 \times 2i\sin 4\theta -$

$6 \times 2i \sin 2\theta$

$\div$ by $2i$

$\cos^5\theta \sin^3\theta = \frac{-1}{2^7} [\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta -$

$6 \sin 2\theta]$.

Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in powers of θ.

Theorem.

(i) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

(ii) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

(iii) $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$

Problems :-

① Find approximately the value of θ in radians if $\frac{\sin\theta}{\theta} = \frac{863}{864}$.

Soln :-

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{863}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$1 - \frac{1}{864} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}$$

$$\frac{1}{864} = \frac{\theta^2}{3!} - \frac{\theta^4}{5!} + \cdots$$

$$\frac{\theta^2}{3!} = \frac{1}{864}$$ (Neglecting higher powers of θ).

$$\theta^2 \cong \frac{6}{864} = \frac{1}{144}$$

$$\theta = 1/12 \text{ radians.}$$

② Show that if x is small

$$\cos(\alpha + x) = \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha.$$

Soln :- $\cos(\alpha + x) = \cos\alpha\cos x - \sin\alpha\sin x$

$$= \cos\alpha\left[1 - \frac{x^2}{2!} + \cdots\right] - \sin\alpha\left[x - \frac{x^3}{3!} + \cdots\right]$$

$$= \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha$$

(Neglecting higher powers)

(3) Solve approximately $\cos(\pi/3 + \theta) = 0.49$.

Soln :-

$$\cos(\pi/3 + \theta) = 0.49$$

$$\cos \pi/3 \cos\theta - \sin\pi/3 \sin\theta = 0.49$$

$$\tfrac{1}{2}\left(1 - \theta^2/2! + \cdots\right) - \sqrt{3}/2\left(\theta - \frac{\theta^3}{3!} + \cdots\right) = 0.49$$

$$1/2 - \frac{\sqrt{3}}{2}\theta = 0.49 \quad \text{(Neglecting higher powers of } \theta\text{)}.$$

$$\sqrt{3}/2\, \theta = \frac{1}{2} - 0.49 = \frac{1}{100}$$

$$\theta = \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{150} = \frac{1.732}{150} = 0.115 \text{ (radian)}$$

$\therefore \theta = 40$ minutes.

---

(4) Evaluate $\sin 3^\circ$ correct to three places of decimals.

Soln :-

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$180^\circ = \pi \text{ radian} \Rightarrow 1^\circ = \pi/180 \text{ radian}$$

$$3^\circ = 3\pi/180 \text{ radian} = \pi/60 \text{ radian}.$$

$$\sin 3^\circ = \sin(\pi/60) = \pi/60 - 1/3!\,(\pi/60)^3 + \cdots$$

$$= \pi/60 \quad \text{(Neglecting higher powers)}$$

$$= 22/7 \times 1/60 = 0.052.$$

---

(5) If $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!}$ & $y = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots$

prove that $x^2 = y$.

Soln :-

$$x = \frac{1+1}{1!} - \frac{3+1}{3!} + \frac{5+1}{5!} - \frac{7+1}{7!} + \cdots$$

$$= \left(\frac{1}{1!} - \frac{3}{3!} + \frac{5}{5!} - \frac{7}{7!}\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$$= \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!}\right) + \left(\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)$$

$$= \cos 1 + \sin 1$$

$$x^2 = (\cos 1 + \sin 1)^2 = \cos^2 1 + \sin^2 1 + 2\sin 1\cos 1$$

$$x^2 = 1 + \sin 2 = 1 + \left(\frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\right) = y.$$

Hence the Result.

(6) If θ is small prove that $\theta \cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$ (appl.)

Soln :-

$$\theta \cot\theta = \frac{\theta}{\tan\theta} = \frac{\theta}{\theta + \theta^3/3 + 2\theta^5/15}$$

$$= \frac{1}{1 + \theta^2/3 + 2\theta^4/15} = \left(1 + \theta^2/3 + 2\theta^4/15\right)^{-1}.$$

$$= 1 - \left(\theta^2/3 + 2\theta^4/15\right) + \left(\theta^2/3 + 2\theta^4/15\right)^2 - \dots$$

$$= 1 - \theta^2/3 - \left(2/15 - 1/9\right)\theta^4 \text{ (appl.)}$$

$$= 1 - \theta^2/3 - \theta^4/45 \text{ (appl.)}$$

---

(7) Show that $\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x - \sin x} = 24$.

Soln :-

$$\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x - \sin x} = \lim_{x\to 0}\left[\frac{3\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - \left(\frac{3x}{1!} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots\right)}{x - \left(x/1! - x^3/3! + x^5/5! - \dots\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(-x^3/2 + 27x^3/6\right) - \left(3x^5/5! - (3x)^5/5!\right) + \dots}{x^3/3! - x^5/5! + \dots}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(-1/2 + 9/2\right) - \left(3/5! - \frac{35x^2}{5}\right) + \dots}{1/3! - x^2/5! + \dots}\right] = 24.$$

---

(8) show that $\lim_{x\to 0}\left(\frac{\cos^2 ax - \cos^2 bx}{1 - \cos cx}\right) = 2\left(\frac{b^2 - a^2}{c^2}\right)$

Soln :-

$$\lim_{x\to 0}\left(\frac{\cos^2 ax - \cos^2 bx}{1 - \cos cx}\right) = \lim_{x\to 0}\left(\frac{(\cos ax)^2 - (\cos bx)^2}{1 - \cos cx}\right)$$

$$= \lim_{x\to 0}\left[\frac{\left(1 - \frac{a^2x^2}{2} + \dots\right)^2 - \left(1 - \frac{b^2x^2}{2!} + \dots\right)^2}{1 - \left(1 - \frac{c^2x^2}{2!} + \dots\right)}\right]$$

$$= \lim_{x \to 0} \left[\frac{\left(1 - \frac{2a^2x^2}{2!} + \frac{a^4x^4}{(2!)^2} + \cdots\right) - \left(1 - \frac{2b^2x^2}{2!} + \frac{b^4x^4}{2!} + \cdots\right)}{c^2x^2/2!} \right]$$

$$= \lim_{x \to 0} \left[\frac{0 + \frac{(-2a^2 + 2b^2)x^2}{2!}}{c^2x^2/2!} \right]$$

$$= \lim_{x \to 0} \left[\frac{\frac{2(b^2 - a^2)x^2}{2!} + \cdots}{c^2x^2/2 + \cdots} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^2(b^2 - a^2)}{x^2(c^2/2)} + \cdots \right]$$

$$= \frac{b^2 - a^2}{c^2/2}$$

$$= \frac{2(b^2 - a^2)}{c^2}$$

UNIT - IV

HYPERBOLIC FUNCTIONS.

Definitions :-

The hyperbolic functions are defined by

(1) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ (2) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

(3) $\tanh x = \dfrac{\sinh x}{\cosh x}$ (4) $\coth x = \dfrac{\cosh x}{\sinh x}$

(5) $\operatorname{cosech} x = \dfrac{1}{\sinh x}$ (6) $\operatorname{sech} x = \dfrac{1}{\cosh x}$.

Result

(1) $\cosh^2 x - \sinh^2 x = 1$

proof :-

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4}$$

$$= 4/4 = 1.$$

---

(2) $2 \sinh x \cosh x = \sinh 2x$.

proof :-

$$2 \sinh x \cosh x = 2\left[\left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{x} + e^{-x}}{2}\right)\right]$$

$$= \frac{2}{4}\left[e^{x}e^{x} + e^{x}e^{-x} - e^{-x}e^{x} - e^{-x}e^{-x}\right]$$

$$= 1/2\left[e^{2x} - e^{-2x}\right] = \sinh 2x.$$

---

(3) $\cosh^2 x + \sinh^2 x = \cosh 2x$.

proof :-

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^{x} + e^{-x}}{2}\right)^2 + \left(\frac{e^{x} + e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left[(e^{2x}+e^{-2x}+2e^{x}e^{-x})+(e^{2x}+e^{-2x}+2e^{x}e^{-x})\right]$$

$$= \frac{1}{4}\left[2e^{2x}+2e^{-2x}\right] \Rightarrow = \frac{2}{4}\left[e^{2x}+e^{-2x}\right]$$

$$= \frac{e^{2x}+e^{-2x}}{2} = \cosh 2x .$$

---

(4) From Result (1) and (3), we get following Result.

Result 1 $\Rightarrow$ $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x = 1+\sinh^2 x \rightarrow (1)$$

Result 3 $\Rightarrow$ $\cosh^2 x + \sinh^2 x = \cosh 2x \rightarrow (2)$

Sub/- (1) in (2)

$$1+\sinh^2 x + \sinh^2 x = \cosh 2x$$

$$2\sinh^2 x = \cosh 2x - 1$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}.$$

from Result (1) $\Rightarrow$ $\sinh^2 x = \cosh^2 x - 1 \rightarrow (3)$

subl- (3) in (2)

$$\cosh^2 x + \cosh^2 x - 1 = \cosh 2x$$

$$2\cosh^2 x = \cosh 2x + 1$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}.$$

Relation between Hyperbolic functions and Trigonometric function.

Theorem :-

(i) $\sin(ix) = i\sinh x$

(ii) $\cos(ix) = \cosh x$

(iii) $\tan(ix) = i\tanh x$.

Proof :-

(i) $\sin(ix) = (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!}$

$$= i\left[x - \frac{i^2x^3}{3!} + \frac{i^4x^5}{5!}\right]$$

$$= i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!}\right]$$

$$\sin(ix) = i\sinh x.$$

(ii) $\cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix)^6}{6!}$

$= 1 - \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} - \frac{i^6x^6}{6!}$

$= 1 + x^2/2! + x^4/4! + x^6/6!$

$\cos(ix) = \cosh x$.

(iii) ~~tan(ix)~~

(iii) $\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x$.

Corollary :-

① $\sinh x = (1/i)(\sin(ix)) = -i\sin(ix)$

② $\cosh x = \cos ix$

③ $\tanh x = -i\tan(ix)$.

Inverse Hyperbolic functions :-

① $\sinh^{-1} x = \log_e(x + \sqrt{x^2+1})$

Proof :-

Let $y = \sinh^{-1} x \Rightarrow x = \sinh y$.

$x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - e^{-y} \Rightarrow 2x = e^y - \frac{1}{e^y}$

$\Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y + 1 = 0$.

$\Rightarrow e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} \Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$= \frac{2x \pm 2(\sqrt{x^2+1})}{2} = \frac{2(x \pm \sqrt{x^2+1})}{2}$

$e^y = x \pm \sqrt{x^2+1}$

Taking log on both sides

$\log e^y = \log_e(x \pm \sqrt{x^2+1})$

$y = \log_e(x \pm \sqrt{x^2+1})$.

Simillarly :

② $\cosh^{-1} x = \log_e (x + \sqrt{x^2-1})$.

---

③ $\tanh^{-1} x = 1/2 \log \left(\frac{1+x}{1-x}\right)$.

Proof :-

$y = \tanh^{-1} x \Rightarrow x = \tanh y \Rightarrow x = \frac{\sinh y}{\cosh y}$

$$x = \frac{\frac{e^y - e^{-y}}{2}}{\frac{e^y + e^{-y}}{2}} \Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$x(e^y + e^{-y}) = e^y - e^{-y}$

$xe^y + xe^{-y} = e^y - e^{-y} \Rightarrow e^{-y}(1+x) = e^y - xe^y$

$1 + x = e^{2y}(1-x) \Rightarrow e^{2y} = \frac{1+x}{1-x}$.

Taking log on both sides.

$\log e^{2y} = \log_e \left(\frac{1+x}{1-x}\right) \Rightarrow 2y = \log_e \left(\frac{1+x}{1-x}\right)$

$\Rightarrow y = 1/2 \log_e \left(\frac{1+x}{1-x}\right)$.

Problems :-

① $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

Soln :-

$(\cosh x + \sinh x)^n = (\cos ix + \sin ix)^n$

$= \cos (ix) - i \sin (ix)$

$= \cos i(nx) - i \sin i(nx)$

$= \cosh nx + \sinh nx$.

---

② Prove that $\frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x$.

Soln :-

$$\frac{1+\tanh x}{1-\tanh x} = \frac{1+\left(\frac{\sinh x}{\cosh x}\right)}{1-\left(\frac{\sinh x}{\cosh x}\right)} = \frac{\cosh x + \sinh x}{\cosh x - \sinh x}$$

$$= \left(\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right)\left(\frac{\cosh x + \sinh x}{\cosh x + \sinh x}\right)$$

$$= \frac{(\cosh x + \sinh x)^2}{\cosh^2 x - \sinh^2 x} = \frac{\cosh^2 x + \sinh^2 x + 2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x}$$

$$\Rightarrow \quad = \cosh 2x + \sinh 2x$$

---

(3) If $\tan(a+ib) = x+iy$ prove that $\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$

Soln :-

$$x+iy = \tan(a+ib) \Rightarrow = \frac{\sin(a+ib)}{\cos(a+ib)}$$

$$= \frac{\sin(a+ib)}{\cos(a+ib)} \times \frac{\cos(a-ib)}{\cos(a-ib)} = \frac{\sin 2a + \sin(i2b)}{\cos 2a + \cos(i2b)}$$

$$= \frac{\sin 2a + i\sinh 2b}{\cos 2a + \cosh 2b}$$

Equating Real & I-part.

$$x = \frac{\sin 2a}{\cos 2a + \cosh 2b} \qquad y = \frac{\sinh 2b}{\cos 2a + \cosh 2b}.$$

$$\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}$$

---

(4) If $x+iy = \tan(A+iB)$ P.T $x^2+y^2+2x\cot 2A = 1$

Soln :- let $x+iy = \tan(A+iB)$; $x-iy = \tan(A-iB)$

$$\cot 2A = \frac{1}{\tan 2A} = \frac{1}{\tan[(A+iB)+(A-iB)]}$$

$$= \frac{1-\tan(A+iB)\tan(A-iB)}{\tan(A+iB)+\tan(A-iB)}$$

$$= \frac{1-(x+iy)(x-iy)}{(x+iy)+(x-iy)} = \frac{1-(x^2+y^2)}{2x}$$

$$2x\cot 2A = 1-x^2-y^2$$

$$\Rightarrow x^2+y^2+2x\cot 2A = 1$$

---

(5) If $x+iy = \sin(A+iB)$. Prove that

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$$

Soln :-

$x + iy = \sin(A + iB)$

$= \sin A \cos iB + \cos A \sin iB$

$= \sin A \cosh B + i \cos A \sinh B$

Equating Real & I.P

$x = \sin A \cosh B \Rightarrow \cosh B = \frac{x}{\sin A} \rightarrow (1)$

$y = \cos A \sinh B \Rightarrow \sinh B = \frac{y}{\cos A} \rightarrow (2)$

Squaring & subt/ (1) & (2)

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$$

---

(6) If $\cos(x+iy) = \cos\theta + i\sin\theta$. P.T $\cos 2x + \cosh 2y = 2$.

Soln :-

$\cos\theta + i\sin\theta = \cos(x+iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

Equating Real & I.P

$\cos\theta = \cos x \cosh y \rightarrow (1)$; $\sin\theta = -\sin x \sinh y \rightarrow (2)$

Squaring & adding (1) & (2)

$\cos^2\theta + \sin^2\theta = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x \cosh^2 y + (1 - \cos^2 x)\sinh^2 y$

$= \cos^2 x \cosh^2 y + \sinh^2 y - \cos^2 x \sinh^2 y$

$= \cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y$

$\cos^2\theta + \sin^2\theta = \cos^2 x + \sinh^2 y$

$$1 = \frac{1 + \cos 2x}{2} + \frac{\cosh 2y - 1}{2}$$

$2 = 1 + \cos 2x + \cosh 2y - 1 \Rightarrow 2 = \cos 2x + \cosh 2y.$

---

(7) If $\cos\alpha + i\sin\alpha = \cos(\theta + i\phi)$. P.T $\sin^2\theta = \pm\sin\alpha$.

Soln :-

Let $\cos\alpha + i\sin\alpha = \cos(\theta + i\phi)$

$= \cos\theta \cosh\phi - \sin\theta \, i \sinh\phi$

$\cos\alpha = \cos\theta \cosh\phi \quad ; \quad \sin\alpha = -\sin\theta \sinh\phi$

$\Rightarrow \cosh\phi = \dfrac{\cos\alpha}{\cos\theta} \quad ; \Rightarrow \sinh\phi = -\dfrac{\sin\alpha}{\sin\theta}$

$\cosh^2\phi = \dfrac{\cos^2\alpha}{\cos^2\theta} \rightarrow ① \qquad \sinh^2\phi = \dfrac{\sin^2\alpha}{\sin^2\theta} \rightarrow ②$

① − ②

$\cosh^2\phi - \sinh^2\phi = \dfrac{\cos^2\alpha}{\cos^2\theta} - \dfrac{\sin^2\alpha}{\sin^2\theta}$

$1 = \dfrac{\cos^2\alpha \sin^2\theta - \sin^2\alpha \cos^2\theta}{\cos^2\theta \sin^2\theta}$

$\cos^2\alpha \sin^2\theta - \sin^2\alpha \cos^2\theta = \cos^2\theta \sin^2\theta$

$(1-\sin^2\alpha)\sin^2\theta - \sin^2\alpha(1-\sin^2\theta) = (1-\sin^2\theta)\sin^2\theta$

$\sin^2\theta - \sin^2\theta\sin^2\alpha - \sin^2\alpha + \sin^2\theta\sin^2\alpha =$
$\sin^2\theta - \sin^4\theta$

$\sin^2\theta - \sin^2\alpha - \sin^2\theta = -\sin^4\theta$

$-\sin^2\alpha = -\sin^4\theta$

$\therefore \sin^2\theta = \pm \sin\alpha$.

---

⑧ $\cosh^7 x = \dfrac{1}{2^6}\left[\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x\right]$.

Soln :-

$\cosh^7 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^7$

$= \dfrac{1}{2^7}\Big[e^{7x} + (7c_1 e^{6x} e^{-x}) + (7c_2 e^{5x} e^{-2x}) + (7c_3 e^{4x} e^{-3x}) + (7c_4 e^{3x} e^{-4x}) + (7c_5 e^{2x} e^{-5x}) + (7c_6 e^{x} e^{-6x}) + (7c_7 e^{-7x})\Big]$

$= \dfrac{1}{2^7}\left[(e^{7x}+e^{-7x}) + 7(e^{5x}+e^{-5x}) + 21(e^{3x}+e^{-3x}) + 35(e^x+e^{-x})\right]$

$= \dfrac{1}{2^6}\left[\dfrac{e^{7x}+e^{-7x}}{2} + \dfrac{7(e^{5x}+e^{-5x})}{2} + \dfrac{21(e^{3x}+e^{-3x})}{2} + \dfrac{35(e^x+e^{-x})}{2}\right]$

$= \dfrac{1}{2^6}\left[\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x\right]$

(3) Separate into Real & I. parts.

(i) $\sinh(\alpha+i\beta)$

Soln :-

let $x+iy = \sinh(\alpha+i\beta) = -i\sin i(\alpha+i\beta)$

$x+iy = -i\sin(i\alpha-\beta) = -i(\sin i\alpha\cos\beta - \cos i\alpha\sin\beta)$

$= -i(i\sinh\alpha\cos\beta - \cosh\alpha\sin\beta)$

$x+iy = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

$x = \sinh\alpha\cos\beta \;;\; y = \cosh\alpha\sin\beta$

$R.p = \sinh\alpha\cos\beta \;;\; I.p = \cosh\alpha\sin\beta.$

(ii) $\tanh(1+i)$

Soln :-

let $x+iy = \dfrac{\sinh(1+i)}{\cosh(1+i)} = -i\dfrac{\sin i(1+i)}{\cos i(1+i)} = -\dfrac{i\sin(i-1)}{\cos(i-1)}$

$= -\dfrac{i\,2\sin(i-1)\cos(i+1)}{2\cos(i-1)\cos(i+1)} = \dfrac{-i[\sin 2i - \sin 2]}{\cos 2i + \cos 2}$

$= \dfrac{-i[i\sinh 2 - \sin 2]}{\cosh 2 + \cos 2} = \dfrac{\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}$

$\therefore R.p = \dfrac{\sinh 2}{\cosh 2 + \cos 2} \qquad I.p = \dfrac{\sin 2}{\cosh 2 + \cos 2}$

(iii) $\tan^{-1}(x+iy)$

Soln :-

let $\tan^{-1}(x+iy) = A+iB$

$\tan(A+iB) = x+iy \;;\; \tan(A-iB) = x-iy$

$\tan 2A = \tan(A+iB) + A-iB)$

$= \dfrac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)} = \dfrac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$

$= \dfrac{2x}{1-(x^2+y^2)}$

$\therefore 2A = \tan^{-1}\left(\dfrac{2x}{1-(x^2+y^2)}\right)$

$\therefore$ Real part is $\frac{1}{2}\tan^{-1}\left(\dfrac{2x}{1-(x^2+y^2)}\right)$

Similarly, Imaginary part is $\frac{1}{2}\tanh^{-1}\left[\dfrac{2y}{1+x^2+y^2}\right]$.

(10) If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$. P.T
(i) $\theta = \frac{1}{2}n\pi + \frac{1}{4}\pi$ (ii) $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

Soln :-
$\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$; $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

(i) Now, $2\theta = (\theta + i\phi) + (\theta - i\phi)$

$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha - i^2\sin^2\alpha)} = \frac{2\cos\alpha}{1 - 1} = \frac{2\cos\alpha}{0} = \infty = \tan\frac{\pi}{2}$$

$2\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{2} + \frac{\pi}{4}$.

(ii) $\tan(2i\phi) = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{(\cos\alpha + i\sin\alpha) - (\cos\alpha - i\sin\alpha)}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$= \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1 + (\cos^2\alpha - i^2\sin^2\alpha)} = \frac{2i\sin\alpha}{1+1} = i\sin\alpha$$

$i\tanh 2\phi = i\sin\alpha \Rightarrow \tanh 2\phi = \sin\alpha$.

$2\phi = \tanh^{-1}(\sin\alpha) \Rightarrow 2\phi = \frac{1}{2}\log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$

$$= \frac{1}{2}\log\left[\frac{1 + \frac{2\tan\alpha/2}{1+\tan^2\alpha/2}}{1 - \frac{2\tan\alpha/2}{1+\tan^2\alpha/2}}\right] = \frac{1}{2}\log\left[\frac{(1+\tan\alpha/2)^2}{(1-\tan\alpha/2)^2}\right]$$

$$= \frac{2}{2}\log\left[\frac{1+\tan\alpha/2}{1-\tan\alpha/2}\right] = \log\left[\frac{\tan\pi/4 + \tan\alpha/2}{1 - \tan\pi/4\tan\alpha/2}\right]$$

$2\phi = \log\left[\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right] \Rightarrow \phi = \frac{1}{2}\log\left[\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]$.

(11) If $\tan x/2 = \tanh x/2$. P.T $\cos x \cosh x = 1$

Soln :-

$$\cosh x = \frac{1 + \tanh^2 x/2}{1 - \tanh^2 x/2} = \frac{1 + \tan^2 x/2}{1 - \tan^2 x/2}$$

$$= \frac{1+\frac{\sin^2 x/2}{\cos^2 x/2}}{1-\frac{\sin^2 x/2}{\cos^2 x/2}} = \frac{\frac{\cos^2 x/2+\sin^2 x/2}{\cos^2 x/2}}{\frac{\cos^2 x/2-\sin^2 x/2}{\cos^2 x/2}}.$$

$$\therefore \cosh x = \frac{1}{\cos^2 x/2 - \sin^2 x/2} = \frac{1}{\cos 2\cdot x/2} = \frac{1}{\cos x}$$

$$\therefore \cosh x \cos x = 1.$$

---

(12) prove ~~of~~ $u = \log_e \tan(\pi/4 + \theta/2)$ iff $\cosh u = \sec\theta$.

Soln :-

let $\cosh u = \sec\theta \Rightarrow u = \cosh^{-1}(\sec\theta)$

$$u = \log_e(\sec\theta + \sqrt{\sec^2\theta - 1}) = \log_e(\sec\theta + \sqrt{\tan^2\theta})$$

$$= \log_e(\sec\theta + \tan\theta) = \log_e\left(1/\cos\theta + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \log_e\left(\frac{1+\sin\theta}{\cos\theta}\right) = \log_e\left(\frac{1+2\tan\theta/2 / 1+\tan^2\theta/2}{1-\tan^2\theta/2 / 1+\tan^2\theta/2}\right)$$

$$= \log_e\left(\frac{(1+\tan\theta/2)^2}{1-\tan^2\theta/2}\right] = \log_e\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$= \log_e\left[\frac{\tan\pi/4 + \tan\theta/2}{1-\tan\pi/4\tan\theta/2}\right] = \log_e[\tan(\pi/4+\theta/2)]$$

Conversely,

let $u = \log_e \tan(\pi/4 + \theta/2)$

$$e^u = \tan(\pi/4+\theta/2) \Rightarrow e^{u/2}e^{u/2} = \tan(\pi/4+\theta/2)$$

$$\frac{e^{u/2}}{e^{-u/2}} = \frac{\tan\pi/4 + \tan\theta/2}{1-\tan\pi/4\tan\theta/2} = \frac{1+\tan\theta/2}{1-\tan\theta/2}.$$

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{1+\tan\theta/2 - (1-\tan\theta/2)}{1+\tan\theta/2 + (1-\tan\theta/2)}$$

$$= \frac{2\tan\theta/2}{2} = \tan\theta/2.$$

$$\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \tan\theta/2.$$

$\tanh(u/2) = \tan(\theta/2)$

$$\cosh u = \frac{1+\tanh^2 u/2}{1-\tanh^2 u/2} = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1+\frac{\sin^2\theta/2}{\cos^2\theta/2}}{1-\frac{\sin^2\theta/2}{\cos^2\theta/2}}$$

$$= \frac{\cos^2\theta/2 + \sin^2\theta/2}{\cos^2\theta/2 - \sin^2\theta/2} = \frac{1}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$= \frac{1}{\cos 2\,\theta/2} = \frac{1}{\cos\theta}.$$

$\cosh u = \sec\theta$.

---

(13) If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$

Prove that $y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$.

Soln :-

$\cos(x+iy) = \cos x \cos iy - \sin x \sin(iy)$

$= \cos x \cosh y - \sin x \, i \sinh y$.

$\cos x \cosh y - \sin x \, i \sinh y = r(\cos\alpha + i\sin\alpha)$.

Equating Real & I.P.

$r\cos\alpha = \cos x \cosh y \rightarrow (1)$

$r\sin\alpha = -\sin x \sinh y \rightarrow (2)$

divide (2) by (1)

$$\frac{r\sin\alpha}{r\cos\alpha} = -\frac{\sin x \sinh y}{\cos x \cosh y} \Rightarrow \tan\alpha = -\tan x \tanh y$$

$$\tanh y = -\frac{\tan\alpha}{\tan x} \Rightarrow y = \tan^{-1}h\left(-\frac{\tan\alpha}{\tan x}\right)$$

$$y = \frac{1}{2}\log\left[\frac{1-\tan\alpha/\tan x}{1+\tan\alpha/\tan x}\right] = \frac{1}{2}\log\left[\frac{\tan x - \tan\alpha}{\tan x + \tan\alpha}\right]$$

$$= \frac{1}{2}\log\left[\frac{\sin x/\cos x - \sin\alpha/\cos\alpha}{\sin x/\cos x + \sin\alpha/\cos\alpha}\right]$$

$$y = \frac{1}{2}\log\left[\frac{\sin x\cos\alpha - \sin\alpha\cos x}{\sin x\cos\alpha + \sin\alpha\cos x}\right]$$

$$y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right].$$

Unit - 5

Logarithm of a Complex Number :-

Definition :-

Let $z = r(\cos\theta + i\sin\theta)$ be a non zero Complex number, we define $\log z = \log r + i\theta$.

principal Value :-

If we take θ to be the principal value of the amplitude of z, the corresponding value of $\log z$ is called its principal value.

$$\text{Log } z = \log z + i(2n\pi), \; n \in z$$

$$(\text{i.e}) \; \text{Log } z = \log r + i(\theta + 2n\pi), \; n \in z.$$

Definition :-

If z and ω are any two complex numbers then we define $z^{\omega} = e^{\omega \log z}$.

problems :-

① prove that $\text{Log } i = i(4n+1)(\pi/2)$

Soln :- $i = \cos(\pi/2) + i\sin(\pi/2)$

$$\therefore \text{Log } i = \log 1 + i[(\pi/2) + 2n\pi]$$

$$= i[(\pi/2) + 2n\pi]$$

$$= i(4n+1)(\pi/2)$$

② Find (i) $\text{Log}(1+i)$ (ii) $\text{Log}(-e)$

Soln :-

(i) $1+i = \sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)]$

$$\therefore \text{Log}(1+i) = \log\sqrt{2} + i(\pi/4) + 2n\pi i$$

$$= \tfrac{1}{2}\log 2 + i(8n+1)(\pi/4)$$

(ii) $-e = e(\cos\pi + i\sin\pi)$

$$\text{Log}(-e) = \log e + i(\pi + 2n\pi)$$

$$= 1 + i(2n+1)\pi.$$

(13) prove that $i^i = e^{-(4n+1)\pi/2}$

Soln :-

$$i^i = e^{i \operatorname{Log} i}$$

$$= e^{i\, i (4n+1)\pi/2}$$

$$= e^{-(4n+1)\pi/2}.$$

(14) If $i^{a+ib} = a+ib$ prove that $a^2+b^2 = e^{-(4n+1)\pi b}$.

Soln :-

$$i^{a+ib} = a+ib \rightarrow (1)$$

Now, $i^{a+ib} = e^{(a+ib) \operatorname{Log} i}$

$$= e^{(a+ib) i (4n+1)(\pi/2)}$$

$$= e^{(ai-b)(4n+1)(\pi/2)}$$

$$= e^{ia(4n+1)(\pi/2)} e^{-b(4n+1)(\pi/2)}$$

$$= e^{-b(4n+1)(\pi/2)} [\cos\theta + i\sin\theta]$$

$$a+ib = e^{-b(4n+1)(\pi/2)} [\cos\theta + i\sin\theta]$$

By (1) $a+ib = e^{-b(4n+1)(\pi/2)} [\cos\theta + i\sin\theta]$

Equating Real & Imaginary parts.

$$a = e^{-b(4n+1)(\pi/2)} \cos\theta$$

$$b = e^{-b(4n+1)(\pi/2)} \sin\theta$$

$$\therefore a^2+b^2 = e^{-b(4n+1)\pi} = e^{-(4n+1)\pi b}.$$

Summation of Trigonometric Series :-

problem :-

(1) Sum the series

$$\frac{1}{\sin\theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} + \dots n \text{ terms (or)}$$

$$\operatorname{cosec}\theta \operatorname{cosec} 2\theta + \operatorname{cosec} 2\theta \operatorname{cosec} 3\theta + \dots n \text{ terms}.$$

Soln :-

Let $S_n = \dfrac{1}{\sin\theta \sin 2\theta} + \dfrac{1}{\sin 2\theta \sin 3\theta} + \dots n$ terms

Here, the n^{th} term

$$T_n = \frac{1}{\sin n\theta \sin(n+1)\theta} = \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\sin n\theta \sin(n+1)\theta}\right]$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin[(n+1)\theta - n\theta]}{\sin n\theta \sin(n+1)\theta}\right]$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta\cos n\theta - \cos(n+1)\theta \sin n\theta}{\sin n\theta \sin(n+1)\theta}\right]$$

$$= \operatorname{cosec}\theta\left[\cot n\theta - \cot(n+1)\theta\right]$$

Now, $T_1 = \operatorname{cosec}\theta[\cot\theta - \cot 2\theta]$

$T_2 = \operatorname{cosec}\theta[\cot 2\theta - \cot 3\theta]$

$T_3 = \operatorname{cosec}\theta[\cot 3\theta - \cot 4\theta]$

$\vdots$

$T_n = \operatorname{cosec}\theta[\cot n\theta - \cot(n+1)\theta]$

Adding we get,

$S_n = \operatorname{cosec}\theta[\cot\theta - \cot(n+1)\theta]$.

---

(2) Find

$$S_n = \frac{\sin\theta}{\cos\theta + \cos 2\theta} + \frac{\sin 2\theta}{\cos\theta + \cos 4\theta} + \frac{\sin 3\theta}{\cos\theta + \cos 6\theta} + \dots n \text{ terms}$$

Soln:- $T_n = \dfrac{\sin n\theta}{\cos\theta + \cos 2n\theta}$.

$$= \frac{1}{2\sin(\theta/2)}\left[\frac{2\sin(\theta/2)\sin n\theta}{2\cos\left(\frac{(2n+1)\theta}{2}\right)\cos\left(\frac{(2n-1)\theta}{2}\right)}\right]$$

$$= \frac{1}{2\sin(\theta/2)}\left[\frac{\cos(n-1/2)\theta - \cos(n+1/2)\theta}{2\cos\left(\frac{(2n+1)\theta}{2}\right)\cos\left(\frac{(2n-1)\theta}{2}\right)}\right]$$

$$= 1/4 \operatorname{cosec}(\theta/2)\left[\frac{\cos\left(\frac{(2n-1)\theta}{2}\right) - \cos\left(\frac{(2n+1)\theta}{2}\right)}{\cos\left(\frac{(2n+1)\theta}{2}\right)\cos\left(\frac{(2n-1)\theta}{2}\right)}\right]$$

$$= \frac{1}{4} \operatorname{cosec}(\theta/2)\left[\sec\left(\frac{(2n+1)\theta}{2}\right) - \sec\left(\frac{(2n-1)\theta}{2}\right)\right]$$

$$\therefore T_1 = \frac{1}{4} \operatorname{cosec}(\theta/2)\left[\sec(3\theta/2) - \sec(\theta/2)\right]$$

$$T_2 = 1/4 \operatorname{cosec}(\theta/2)\left[\sec(5\theta/2) - \sec(3\theta/2)\right]$$

$$\vdots$$

$$T_n = 1/4 \operatorname{cosec}(\theta/2)\left[\sec\left(\frac{(2n+1)\theta}{2}\right) - \sec\left(\frac{(2n-1)\theta}{2}\right)\right]$$

Adding we get,

$$S_n = \frac{1}{4} \operatorname{cosec}(\theta/2)\left[\sec\left(\frac{(2n+1)\theta}{2}\right) - \sec(\theta/2)\right].$$

---

(3) Find

$$S_n = \frac{\sin 2\theta}{\cos\theta \cos 3\theta} + \frac{\sin 4\theta}{\cos 3\theta \cos 5\theta} + \frac{\sin 6\theta}{\cos 5\theta \cos 7\theta} + \dots n \text{ terms.}$$

Soln :-

$$T_n = \frac{\sin 2n\theta}{\cos(2n-1)\theta \cos(2n+1)\theta}$$

$$= \frac{1}{2\sin\theta}\left[\frac{2\sin\theta \sin 2n\theta}{\cos(2n-1)\theta \cos(2n+1)\theta}\right]$$

$$= \frac{1}{2\sin\theta}\left[\frac{\cos(2n-1)\theta - \cos(2n+1)\theta}{\cos(2n-1)\theta \ \cos(2n+1)\theta}\right]$$

$$= \frac{1}{2} \operatorname{cosec}\theta\left[\sec(2n+1)\theta - \sec(2n-1)\theta\right]$$

$$T_1 = \frac{1}{2} \operatorname{cosec}\theta\left[\sec 3\theta - \sec\theta\right]$$

$$T_2 = \frac{1}{2} \operatorname{cosec}\theta\left[\sec 5\theta - \sec 3\theta\right]$$

$$\vdots$$

$$T_n = \frac{1}{2} \operatorname{cosec}\theta\left[\sec(2n+1)\theta - \sec(2n-1)\theta\right]$$

Adding we get,

$$S_n = \frac{1}{2} \operatorname{cosec}\theta\left[\sec(2n+1)\theta - \sec\theta\right].$$

(4) Find

$$S_n = \frac{\sin 2\theta}{\sin\theta \sin 3\theta} - \frac{\sin 4\theta}{\sin 3\theta \sin 5\theta} + \frac{\sin 6\theta}{\sin 5\theta \sin 7\theta} + \dots n \text{ terms}$$

Soln :-

$$T_n = (-1)^{n+1}\left[\frac{\sin 2n\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$= \frac{(-1)^{n+1}}{2\cos\theta}\left[\frac{2\cos\theta \sin 2n\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$= \frac{(-1)^{n+1}}{2\cos\theta}\left[\frac{\sin(2n+1)\theta + \sin(2n-1)\theta}{\sin(2n-1)\theta \sin(2n+1)\theta}\right]$$

$$= (-1)^{n+1}(\tfrac{1}{2}\sec\theta)\left[\operatorname{cosec}(2n-1)\theta + \operatorname{cosec}(2n+1)\theta\right]$$

$$T_1 = \tfrac{1}{2}\sec\theta\left[\operatorname{cosec}\theta + \operatorname{cosec} 3\theta\right]$$

$$T_2 = -\tfrac{1}{2}\sec\theta\left[\operatorname{cosec} 3\theta + \operatorname{cosec} 5\theta\right]$$

$$\vdots$$

$$T_n = (-1)^{n+1}(\tfrac{1}{2}\sec\theta)\left[\operatorname{cosec}(2n-1)\theta + \operatorname{cosec}(2n+1)\theta\right]$$

Adding we get,

$$S_n = \tfrac{1}{2}\sec\theta\left[\operatorname{cosec}\theta + (-1)^{n+1}\operatorname{cosec}(2n+1)\theta\right].$$

---

(5) Find

$$S_n = \tan\theta \sec 2\theta + \tan 2\theta \sec 4\theta + \tan 4\theta \sec 8\theta + \dots n \text{ terms}$$

Soln :-

$$T_n = \tan(2^{n-1}\theta)\sec(2^n\theta)$$

$$= \frac{\sin(2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos(2^n\theta)}$$

$$= \frac{\sin(2^n\theta - 2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos(2^n\theta)}$$

$$= \frac{\sin(2^n\theta)\cos(2^{n-1}\theta) - \cos(2^n\theta)\sin(2^{n-1}\theta)}{\cos(2^{n-1}\theta)\cos(2^n\theta)}.$$

$$T_n = \tan(2^n\theta) - \tan(2^{n-1}\theta)$$

$$T_1 = \tan 2\theta - \tan\theta$$

$$T_2 = \tan 4\theta - \tan 2\theta$$

$$\vdots$$

$$T_n = \tan(2^n\theta) - \tan(2^{n-1}\theta)$$

Adding we get,

$$S_n = \tan(2^n\theta) - \tan\theta .$$

---

(6) Find $S_n = \tan x \tan(x+y) + \tan(x+y)\tan(x+2y) + \tan(x+2y)\tan(x+3y) + \dots$ n terms.

Soln :-

$$T_n = \tan[x+(n-1)y]\tan(x+ny)$$

We notice that,

$$\tan y = \tan[\overline{x+ny} - \overline{x+(n-1)y}]$$

$$= \frac{\tan(x+ny) - \tan(x+(n-1)y)}{1+\tan(x+ny)\tan[x+(n-1)y]}$$

$$1+\tan(x+ny)\tan[x+(n-1)y] = \cot y[\tan(x+ny) - \tan(x+(n-1)y]].$$

$$1+T_n = \cot y[\tan(x+ny) - \tan\{x+(n-1)\}]$$

$$T_n = \cot y[\tan(x+ny) - \tan\{x+(n-1)y\}]$$

$$T_1 = \cot y[\tan(x+y) - \tan x] - 1$$

$$T_2 = \cot y[\tan(\tan(x+2y)) - \tan(x+y)] - 1$$

$$\vdots$$

$$T_n = \cot y[\tan(x+ny) - \tan(x+(n-1)y)] - 1$$

Adding we get,

$$S_n = \cot y[\tan(x+ny) - \tan x] - n$$

---

(7) Find, $S_n = \tan^{-1}(1/3) + \tan^{-1}(1/7) + \tan^{-1}(1/13) + \dots$ n terms.

Soln :-

$$S_n = \tan^{-1}\left(\frac{1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{1}{1+2\cdot 3}\right) + \tan^{-1}\left(\frac{1}{1+3\cdot 4}\right) + \dots \text{ n terms}$$

$\therefore T_n = \tan^{-1}\left[\frac{1}{1+n(n+1)}\right] = \tan^{-1}\left[\frac{(n+1)-n}{1+n(n+1)}\right]$

$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$

$T_1 = \tan^{-1}(2) - \tan^{-1}(1)$

$T_2 = \tan^{-1}(3) - \tan^{-1}(2)$

$\vdots$

$T_n = \tan^{-1}(n+1) - \tan^{-1}(n)$.

Adding we get,

$S_n = \tan^{-1}(n+1) - \tan^{-1}(1) = \tan^{-1}\left[\frac{(n+1)-1}{1+(n+1)}\right]$

$= \tan^{-1}\left[\frac{n}{n+2}\right]$.

---

(8) Find

$S_n = \tan^{-1}\left(\frac{1}{1+1+1^2}\right) + \tan^{-1}\left(\frac{1}{1+2+2^2}\right) + \tan^{-1}\left(\frac{1}{1+3+3^2}\right)$

$+ \dots n$ terms.

Soln :-

$T_n = \tan^{-1}\left(\frac{1}{1+n+n^2}\right) = \tan^{-1}\left(\frac{1}{1+n(n+1)}\right)$

$= \tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right)$

$T_n = \tan^{-1}(n+1) - \tan^{-1} n$.

$\therefore S_n = \tan^{-1}\left(\frac{n}{n+2}\right)$.

---

(9) Find

$\operatorname{cosec} \alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 2^2\alpha + \dots n$ terms.

Soln :-

$T_1 = \frac{1}{\sin\alpha} = \frac{\sin\alpha/2}{\sin\alpha \sin\alpha/2} = \frac{\sin(\alpha - \alpha/2)}{\sin\alpha \sin\alpha/2}$

$= \frac{\sin\alpha \cos\alpha/2 - \cos\alpha \sin\alpha/2}{\sin\alpha \sin\alpha/2}$.

$T_1 = \cot\alpha/2 - \cot\alpha$.

$T_2 = \cot\alpha - \cot 2\alpha$.

$T_3 = \cot 2\alpha - \cot 2^2\alpha$

$\vdots$

$T_n = \cot(2^{n-2}\alpha) - \cot(2^{n-1}\alpha)$

Adding we get,

$S_n = \cot \alpha/2 - \cot(2^{n-1}\alpha)$.

Angles in Arithmetic progression :-

Problems

① Find $S_n = \sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots n$ terms.

Soln :-

$2\sin(\alpha/2) S_n = 2\sin\alpha \sin(\alpha/2) + 2\sin 2\alpha \sin(\alpha/2) + \dots + 2\sin n\alpha \sin(\alpha/2)$

$= [\cos(\alpha/2) - \cos(3\alpha/2)] + [\cos(3\alpha/2) - \cos(5\alpha/2)] + \dots + \left[\cos\frac{(2n-1)\alpha}{2} - \cos\frac{(2n+1)\alpha}{2}\right]$

$= \cos(\alpha/2) - \cos[(2n+1)\alpha/2]$

$= 2\sin\left[\frac{(n+1)\alpha}{2}\right]\sin\left(\frac{n\alpha}{2}\right)$.

$$S_n = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin(\alpha/2)}.$$

---

② prove that $\tan n\alpha = \dfrac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \dots n \text{ terms}}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots n \text{ terms}}$.

Soln :-

$$R.H.S = \frac{\sin\left[\alpha + \frac{(n-1)2\alpha}{2}\right]\sin\left(\frac{n2\alpha}{2}\right)\sin\alpha}{\cos\left[\alpha + \frac{(n-1)2\alpha}{2}\right]\sin\left(\frac{n2\alpha}{2}\right)\sin\alpha}$$

$$= \frac{\sin[\alpha + (n-1)\alpha]\sin(n\alpha)}{\cos[\alpha + (n-1)\alpha]\sin(n\alpha)}$$

$$= \frac{\sin^2 n\alpha}{\cos n\alpha \sin n\alpha}$$

$= \tan n\alpha$.

(3) Find $S_n = \sin\alpha - \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \dots n$ terms.

Soln :-

$\sin(\pi + 2\alpha) = -\sin 2\alpha$

$\sin(2\pi + 3\alpha) = \sin 3\alpha$; $\sin(3\pi + 4\alpha) = -\sin 4\alpha$ ..etc.

$\therefore S_n = \sin\alpha + \sin(\pi + 2\alpha) + \sin(2\pi + 3\alpha) + \sin(3\pi + 4\alpha) + \dots n$ terms

$= \sin\alpha + \sin[\alpha + (\alpha + \pi)] + \sin[\alpha + 2(\alpha + \pi)] + \dots n$ terms

This is a sum of series of sines of angles which are in A.P. with first term α and Common difference $\alpha + \pi$.

$$S_n = \frac{\sin\left[\alpha + \frac{(n-1)(\pi+\alpha)}{2}\right] \sin\left[\frac{n(\pi+\alpha)}{2}\right]}{\sin\left(\frac{\pi+\alpha}{2}\right)}$$

$$= \frac{\sin\left[\alpha + \frac{(n-1)(\pi+\alpha)}{2}\right] \sin\left[\frac{n(\pi+\alpha)}{2}\right]}{\cos(\alpha/2)}$$

(4) Find $S_n = \cos\alpha\cos 3\alpha + \cos 3\alpha \cos 5\alpha + \cos 5\alpha \cos 7\alpha + \dots n$ terms.

Soln :-

$S_n = (1/2)(\cos 4\alpha + \cos 2\alpha) + (1/2)(\cos 8\alpha + \cos 2\alpha) + \dots n$ terms

$= (1/2)[n\cos 2\alpha + (\cos 4\alpha + \cos 8\alpha + \cos 12\alpha + \dots n \text{ terms})]$

$$= \frac{1}{2}\left[n\cos 2\alpha + \frac{\cos\left(4\alpha + \frac{(n-1)4\alpha}{2}\right)\sin\left(\frac{4n\alpha}{2}\right)}{\sin 2\alpha}\right]$$

$$= \frac{1}{2}\left[n\cos 2\alpha + \frac{\cos[2(n+1)\alpha]\sin[2n\alpha]}{\sin 2\alpha}\right]$$

(5) Find $S_n = \cos^3\alpha + \cos^3 2\alpha + \cos^3 3\alpha + \dots n$ terms.

Soln :-

$S_n = \frac{1}{4}[\cos 3\alpha + 3\cos\alpha] + \frac{1}{4}[\cos 6\alpha + 3\cos 2\alpha] + \dots n$ terms.

$= (1/4)[\cos 3\alpha + \cos 6\alpha + \dots n \text{ terms}] + (3/4)[\cos\alpha + \cos 2\alpha + \dots n \text{ terms}]$

$$= \frac{1}{4}\left[\frac{\cos\left(3\alpha + \frac{(n-1)3\alpha}{2}\right)\sin\left(\frac{n3\alpha}{2}\right)}{\sin(3\alpha/2)}\right] +$$

$$\frac{3}{4}\left[\frac{\cos\left(\alpha + \frac{(n-1)\alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin(\alpha/2)}\right]$$

$$= \frac{1}{4}\left[\frac{\cos[(3/2)(n+1)\alpha]\sin[(3/2)n\alpha]}{\sin(3/2)\alpha}\right] +$$

$$\frac{3}{4}\left[\frac{\cos[(n+1)\alpha/2]\sin(n\alpha/2)}{\sin(\alpha/2)}\right]$$

Gregory's Series :-

Theorem :-

If $-1/4\,\pi \le \theta \le \pi/4$, then $\theta = \tan\theta - 1/3\tan^3\theta + 1/5\tan^5\theta + \dots$

Proof :-

Consider, $1 + i\tan\theta = 1 + i\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta + i\sin\theta}{\cos\theta}$.

$1 + i\tan\theta = (\sec\theta)e^{i\theta}$

$\log(1 + i\tan\theta) = \log(\sec\theta) + \log(e^{i\theta})$

$= \log\sec\theta + i\theta \rightarrow (1)$

Since, $-1/4\,\pi \le \theta \le \pi/4$, $|\tan\theta| \le 1$, we get

$|i\tan\theta| = |i|\,|\tan\theta| \le 1$.

Hence $\log(1 + i\tan\theta)$ can be expanded in logarithmic Series and hence from (1)

$(i\tan\theta) - 1/2\,(i\tan\theta)^2 + 1/3\,(i\tan\theta)^3 - 1/4\,(i\tan\theta)^4 + \dots = \log(\sec\theta) + i\theta$

$i\tan\theta + 1/2\tan^2\theta - 1/3\,i\tan^3\theta - 1/4\tan^4\theta + \dots = \log(\sec\theta) + i\theta$

Equating imaginary parts,

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta + \dots$$

Problems :-

① When x lies between $-\pi/4$ & $\pi/4$ show that

$$\tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} \dots = \tanh x + \frac{\tanh^3 x}{3} + \frac{\tanh^5 x}{5} + \dots$$

Soln :- put $\tanh x = y$

$$R.H.S = y + \frac{y^3}{3} + \frac{y^5}{5} \dots = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) = \tanh^{-1} y$$

$$= \tanh^{-1}(\tanh x) = x \rightarrow ①$$

Put $\tan x = z$.

$$L.H.S = z - \frac{z^3}{3} + \frac{z^5}{5} \dots = \tan^{-1} z = \tan^{-1}(\tan x) = x \rightarrow ②$$

From ① & ②

LHS = RHS.

② prove that $\pi = 2\sqrt{3}\left[1 - \tfrac{1}{3^2} + \tfrac{1}{5}\cdot\tfrac{1}{3^2} - \tfrac{1}{7}\cdot\tfrac{1}{3^2} + \dots\right]$

Soln :-

$$R.H.S = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^2} + \dots\right]$$

$$= 2\sqrt{3}\sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3^{5/2}} + \frac{1}{5}\cdot\frac{1}{3^{5/2}} - \dots\right]$$

$$= 6\left[\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^5 - \dots\right]$$

$$= 6\tan^{-1}(1/\sqrt{3}) = 6(\pi/6) = \pi$$

RHS = LHS.

③ If θ lies between $\tfrac{1}{4}\pi$ and $\tfrac{3}{4}\pi$. prove that

$$\theta = \tfrac{1}{2}\pi - \cot\theta + \tfrac{1}{3}\cot^3\theta - \tfrac{1}{5}\cot^5\theta + \dots$$

Soln :- Given, $\tfrac{1}{4}\pi \leq \theta \leq \tfrac{3}{4}\pi$

$$\therefore \tfrac{1}{4}\pi - \tfrac{1}{2}\pi \leq \theta - \tfrac{1}{2}\pi \leq \tfrac{3}{4}\pi - \tfrac{1}{2}\pi$$

$$\therefore -\tfrac{1}{4}\pi \leq \theta - \tfrac{1}{2}\pi \leq \tfrac{1}{4}\pi$$

By Gregory's series, for $\theta - \tfrac{1}{2}\pi$, we have.

$\theta - \frac{1}{2}\pi = \tan(\theta - \frac{1}{2}\pi) - \frac{1}{3}\tan^3(\theta - \frac{1}{2}\pi) + \frac{1}{5}\tan^5(\theta - \frac{1}{2}\pi) - \cdots$

$= -\cot\theta + \frac{1}{3}\cot^3\theta - \frac{1}{5}\cot^5\theta + \cdots$

$\theta = \frac{1}{2}\pi - \cot\theta + \frac{1}{3}\cot^3\theta - \frac{1}{5}\cot^5\theta + \cdots$

---

(5) Prove that $(1-3^{-1/2}) - \frac{1}{3}(1-3^{-3/2}) + \frac{1}{5}(1-3^{-5/2}) - \cdots = \frac{\pi}{12}$.

Soln :-

$S = (1-3^{-1/2}) - \frac{1}{3}(1-3^{-3/2}) + \frac{1}{5}(1-3^{-5/2}) - \cdots$

$= \left(1-\frac{1}{\sqrt{3}}\right) - \frac{1}{3}\left(1-\frac{1}{3\sqrt{3}}\right) + \frac{1}{5}\left(1-\frac{1}{3^2\sqrt{3}}\right) - \cdots$

$= \left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right] - \left[\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \cdots\right]$

$= \tan^{-1}(1) - \tan^{-1}(1/\sqrt{3})$

$= \pi/4 - \pi/6$

$= \pi/12$

$\therefore S = \pi/12$.
