SEMESTER : I
CORE COURSE : II

Inst Hour	: 4
Credit	: 4
Code	: 18K1M02

ANALYTICAL GEOMETRY OF 3 - DIMENSIONS AND INTEGRAL CALCULUS

UNIT 1:

Coplanar lines – Shortest distance between two skew lines- Equation of the line of shortest distance.
(Chapter III Sections 7& 8 of Text Book 1)

UNIT 2:

Sphere – Standard equations –Length of tangent from any point–Sphere passing through a given circle – finding the centre and radius of the circle of intersection of a sphere and a plane – Tangent plane.
(Chapter IV Sections 1-8 of Text Book 1)

UNIT 3:

Properties of Definite Integrals– Integration by parts– reduction formula
(Chapter I Sections 11, 12 &13 of Text Book 2)

UNIT 4:

Double integrals – changing the order of Integration – Triple Integrals.
(Chapter V Sections 2.1, 2.2, 4 of Text Book 2)

UNIT 5:

Beta & Gamma functions and the relation between them-Integration using Beta & Gamma functions.
(Chapter VII Sections 2.1, 2.2, 2.3,3, 4 of Text Book 2)

Text Book(s)

[1] T.K.Manickavasagam Pillai , Natarajan, A Text book of Analytical Geometry Part II (Three Dimensions) S.V Publications – 2010 - Revised Edition.

[2] S.Narayanan ,T.K.Manickavasagam Pillai, Calculus Volume II S.V Publications 2015 Edition.

Books for Reference

[1] P.Duraipandian & Laxmi Duraipandian. Analytical Geometry

[2] Shanti Narayanan, Differential & Integral Calculus

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

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HOD of Mathematics

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Analytical Geometry of 3D and Integral calculus

UNIT - I

Coplanar lines

Two straight lines which lie in the same plane are called as coplanar lines. They may be parallel or they may intersect.

* The condition that two given straight lines should be coplanar

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Theorem:

The condition that two given straight lines should be coplanar.

Sol: Let their equations be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \longrightarrow (1)$$

$$\frac{x-x_2}{l_1} = \frac{y-y_2}{m_1} = \frac{z-z_2}{n_1} \longrightarrow (2)$$

The equation to a plane through the first line is,

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \longrightarrow (3)$$

$$Al + Bm + Cn = 0 \longrightarrow (4)$$

If it contains the line (2) then the point (x_2, y_2, z_2) lies on it.

$$A(x_2-x_1) + B(y_2-y_1) + C(z_2-z_1) = 0 \longrightarrow (5)$$

Also the line (2) is perpendicular to the normal to the plane (3)

$$Al_1 + Bm_1 + Cn_1 = 0 \longrightarrow (6)$$

Eliminating A, B, C from equations (5) (4) and (6) we get the condition for the lines to be coplanar.

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Eliminating A, B, C from (3), (4) and (6) we get the equation of the plane passing through the two lines are,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Example: 1

Find the condition for the lines,

$ax+by+cz+d=0=a_1x+b_1y+c_1z+d_1$

$a_2x+b_2y+c_2z+d_2=0=a_3x+b_3y+c_3z+d_3$ to be the coplanar.

Sol: Let the lines intersect at the point (x_1, y_1, z_1)

Then (x_1, y_1, z_1) lies on the planes

$$ax+by+cz+d=0$$
$$a_1x+b_1y+c_1z+d_1=0$$
$$a_2x+b_2y+c_2z+d_2=0$$
$$a_3x+b_3y+c_3z+d_3=0$$

$$\therefore ax_1+by_1+cz_1+d=0$$
$$a_1x_1+b_1y_1+c_1z_1+d_1=0$$
$$a_2x_1+b_2y_1+c_2z_1+d_2=0$$
$$a_3x_1+b_3y_1+c_3z_1+d_3=0$$

Eliminating x_1, y_1, z_1 from the above four equations. we get the condition

$$\begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0$$

Example : 2 : prove that the lines,

$$\frac{x+1}{-3} = \frac{y+10}{8} = \frac{z-1}{2} \; ; \; \frac{x+3}{-4} = \frac{y+1}{7} = \frac{z-4}{1}$$

are coplanar. find also their point of intersection and the plane though them.

Sol : The coordinates of the points on the two lines are respectively of the form.

$$\frac{x+1}{-3} = \frac{y+10}{8} = \frac{z-1}{2} = r \text{ (say)}$$

$x+1 = -3r \quad y+10 = 8r \quad z-1 = 2r$

$x = -3r-1, \quad y = 8r-10, \quad z = 2r+1$ —— ①

$x = -4r_1-3, \quad y = 7r_1-1, \quad z = r_1+4$ —— ②

The lines are coplanar if the lines intersect i, if the three equations

$$-3r-1 = -4r_1-3$$
$$8r-10 = 7r_1-1$$
$$2r+1 = r_1+4$$

are simultaneously true.

$$3r-4r_1 = 2$$
$$8r-7r_1 = 9$$
$$2r-r_1 = 3$$

solving the first two equations we get $r=2$ and $r_1=1$

These values satisfy the third equation also.

$\therefore$ The lines are coplanar.

substituting $r=2$ in eqn ① and $r_1=1$ in eqn ②

The intersecting point is $(-7, 6, 5)$

The equation of the plane containing the lines is

$$\begin{vmatrix} x+1 & y+10 & z-1 \\ -3 & 8 & 2 \\ -4 & 7 & 1 \end{vmatrix} \quad \text{is} \quad 6x+5y-11z+67=0.$$

Thm: The shortest distance between the two given lines

Let the given lines AB and $A'B'$ whose equations are $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

Let the shortest distance between the lines have direction cosines l, m, n.

The shortest distance GH is perpendicular to both the lines. $\therefore ll_1 + mm_1 + nn_1 = 0$

$$ll_2 + mm_2 + nn_2 = 0$$

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - l_1n_2} = \frac{n}{l_1m_2 - l_2m_1} = \frac{1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

Let the point A be (x_1, y_1, z_1) and A' be (x_2, y_2, z_2)

GH = projection of AA' on $GH = (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$

$$= \frac{(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - l_1n_2) + (z_2 - z_1)(l_1m_2 - l_2m_1)}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2 - m_2n_1)^2}$$

The shortest distance is the line of intersection of the plane containing the lines AB and GH and $A'B'$ and GH.

Hence
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0; \quad \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

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Cor: 1 The two lines,

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

are coplanar if the shortest distance between them is zero.

$$\text{i.e., } \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$$

Example: 1 Find the shortest distance between the lines

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}; \quad \frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$$

Sol: Let the D.C of the lines perpendicular to both the lines be l, m, n. then

$$-l+2m+n=0 \qquad \frac{l}{1} = \frac{m}{3} = \frac{n}{-5}$$

$$l+3m+2n=0$$

$$l = \frac{1}{\sqrt{35}}, \quad m = \frac{3}{\sqrt{35}} \quad n = \frac{-5}{\sqrt{35}}$$

The magnitude of the shortest distance is the projection of the line joining the points $(3,4,-2)$ and $(1,-7,-2)$ on the line of shortest distance,

$$\therefore SD = (3-1)\frac{1}{\sqrt{35}} + (4+7)\frac{3}{\sqrt{35}} + (-2+2)\frac{1}{\sqrt{35}}$$

$$= \sqrt{35}$$

Then the equation of the shortest distance between them is

$$\begin{vmatrix} x-3 & y-4 & z+2 \\ -1 & 2 & 1 \\ 1 & 3 & -5 \end{vmatrix} = 0 = \begin{vmatrix} x-1 & y+7 & z+2 \\ 1 & 3 & 2 \\ 1 & 3 & -5 \end{vmatrix}$$

simplyfying we get, $13x+4y+5z-45=0 = 3x-y-10$.

Example : 2 : Find the shortest distance between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and find the equation of the line of shortest distance also.

Sol : Let the direction cosines of the line perpendicular to both the lines l, m, n then

$$l - 2m + n = 0$$
$$7l - 6m + n = 0$$

$$\frac{l}{-2+6} = \frac{m}{7-1} = \frac{n}{-6+14}$$

$$\frac{l}{4} = \frac{m}{6} = \frac{n}{8}$$

$$l = \frac{4}{\sqrt{4^2+6^2+8^2}} = \frac{4}{\sqrt{116}}$$

$$m = \frac{6}{\sqrt{116}} \quad n = \frac{8}{\sqrt{116}}$$

The magnitude of the shortest distance is the projection of the line joining the points $(3,5,7)$ and $(-1,-1,-1)$ on the line of shortest distance.

$$SD = (3+1)\frac{4}{\sqrt{116}} + (5+1)\frac{6}{\sqrt{116}} + (7+1)\frac{8}{\sqrt{116}}$$

$$= \frac{16}{\sqrt{116}} + \frac{36}{\sqrt{116}} + \frac{64}{\sqrt{116}}$$

$$SD = \frac{16+36+64}{\sqrt{116}} = \frac{116}{\sqrt{116}} = \sqrt{116}$$

The equation of the shortest distane between them is

$$\begin{vmatrix} x-3 & y-5 & z-7 \\ 1 & -2 & 1 \\ 4 & 6 & 8 \end{vmatrix} = 0 = \begin{vmatrix} x+1 & y+1 & z+1 \\ 7 & -6 & 1 \\ 4 & 6 & 8 \end{vmatrix}$$

Simplifying, $(x-3)(-16-6) - (y-5)(8-4) + (z-7)(6+8) = 0 =$
$(x+1)(-48-6) - (y+1)(56-4) + (z+1)(42+24)$

$= 22x - 4y + 14z - 12 = 0 = 54x - 52y + 66z.$

Example: 3 : The straight lines

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$; $\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$ are cut by a third whose direction cosines, are λ, μ, ν. Show that the length intercepted on the third line is given by

$$\begin{vmatrix} \alpha-\alpha_1 & \beta-\beta_1 & \gamma-\gamma_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix}$$

and deduce the length of the S.D.

Sol: The general co-ordinate of points on the two lines are respectively

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say) } \quad \frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} = r_1$$

$$\frac{x-\alpha}{l} = r \quad \frac{y-\beta}{m} = r \quad \frac{z-\gamma}{n} = r$$

$$x-\alpha = lr \quad y-\beta = mr \quad z-\gamma = nr$$

$$x = \alpha+lr \quad y = \beta+mr \quad z = \gamma+nr$$

$(\alpha+lr, \beta+mr, \gamma+nr)$ and $(\alpha_1+l_1r_1, \beta_1+m_1r_1, \gamma_1+n_1r_1)$

If d is the intercept on the third line.

If P, Q are the points of intersection of the lines on the third line, projection of PQ on the x-axis

$$d\lambda = (\alpha+lr) - (\alpha_1+l_1r_1)$$

$$d\mu = (\beta+mr) - (\beta_1+m_1r_1)$$

$$d\nu = (\gamma+nr) - (\gamma_1+n_1r_1)$$

i.e. $(\alpha-\alpha_1) + lr - l_1r_1 - d\lambda = 0$

$$\beta-\beta_1 + mr - m_1r_1 - d\mu = 0$$

$$\gamma-\gamma_1 + nr - n_1r_1 - d\nu = 0$$

Hence solving these equations considering them as equations containing r, r_1 and α we get,

$$\frac{r}{\begin{vmatrix} l_1 & \lambda & \alpha-\alpha_1 \\ m_1 & \mu & \beta-\beta_1 \\ n_1 & \nu & \gamma-\gamma_1 \end{vmatrix}} = \frac{r_1}{\begin{vmatrix} \lambda & \alpha-\alpha_1 & l \\ \mu & \beta-\beta_1 & m \\ \nu & \gamma-\gamma_1 & n \end{vmatrix}} = \frac{d}{\begin{vmatrix} \alpha-\alpha_1 & l & l_1 \\ \beta-\beta_1 & m & m_1 \\ \gamma-\gamma_1 & n & n_1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} l & l_1 & \lambda \\ m & m_1 & \mu \\ n & n_1 & \nu \end{vmatrix}}$$

$$\therefore d = \begin{vmatrix} \alpha-\alpha_1 & l & l_1 \\ \beta-\beta_1 & m & m_1 \\ \gamma-\gamma_1 & n & n_1 \end{vmatrix} \div \begin{vmatrix} l & l_1 & \lambda \\ m & m_1 & \mu \\ n & n_1 & \nu \end{vmatrix}$$

This intercept will be the shortest distance if the third line is perpendicular to both lines,

$$l\lambda + m\mu + n\nu = 0$$

$$l_1\lambda + m_1\mu + n_1\nu = 0$$

$$\frac{\lambda}{mn_1 - m_1 n} = \frac{\mu}{nl_1 - ln_1} = \frac{\nu}{lm_1 - ml_1} = \frac{1}{\sqrt{\Sigma (mn_1 - m_1 n)^2}}$$

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix} = \lambda (mn_1 - m_1 n) + \mu (nl_1 - ln_1) + \nu (lm_1 - ml_1)$$

$$= \frac{(mn_1 - m_1 n)^2 + (nl_1 - ln_1)^2 + (lm_1 - ml_1)^2}{\sqrt{\Sigma (mn_1 - m_1 n)^2}}$$

$$SD = \begin{vmatrix} \alpha-\alpha_1 & \beta-\beta_1 & \gamma-\gamma_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} \div \sqrt{\Sigma (mn_1 - m_1 n)^2}$$

UNIT – II

SPHERE

Defn: A sphere is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the centre of the sphere and the constant distance the radius of the sphere.

Equation of the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Centre of the sphere $(-u,-v,-w)$

Radius of the sphere $\sqrt{(-u)^2+(-v)^2+(-w)^2-d}$

Equation of a sphere whose centre C and radius are given,

$$CP^2=r^2$$

$$CP^2=(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

The characteristics of the equation of a sphere are

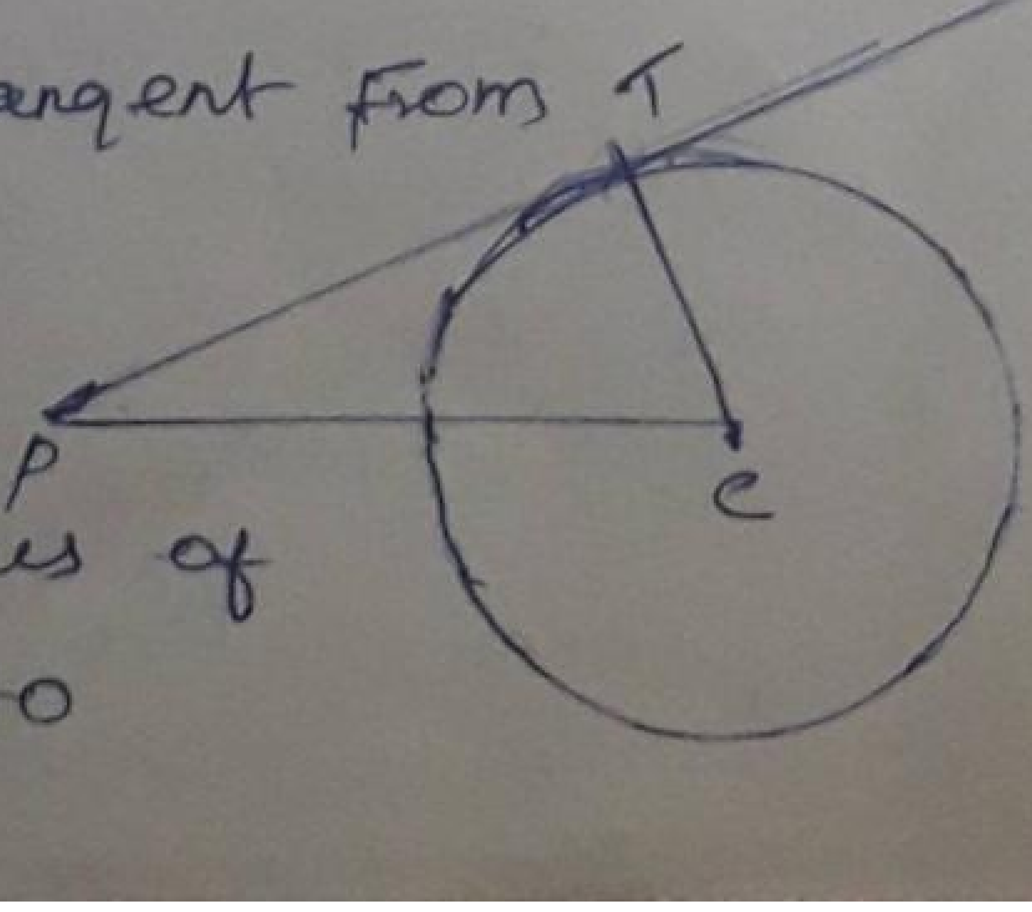
(i) It is of the second degree in x, y, z

(ii) The coefficients of x^2, y^2, z^2 are equal

(iii) The terms xy, yz, zx are absent.

The length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Sol: Let P be the point (x_1,y_1,z_1). C be the centre of the sphere and PT a tangent from P to the sphere.



The coordinates of C are $(-u,-v,-w)$. CT is the radius of the sphere and is equal to $\sqrt{u^2+v^2+w^2-d}$.

CT is perpendicular to PT

$\therefore PC^2 = PT^2 + CT^2$

$(x_1+u)^2 + (y_1+v)^2 + (z_1+w)^2 = PT^2 + u^2+v^2+w^2 - d$

$$\boxed{PT^2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d}$$

Note 1: The value of PT^2 is called the power of P with respect to the circle.

Cor: 1 The point (x_1, y_1, z_1) lies outside, on or inside the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ according as

$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d >=< 0$

Cor: 2 If d is positive the origin lies outside the sphere; if d is negative, the origin lies inside the sphere; if $d=0$ the origin lies on the sphere

Example: 1 : Find the equation of the sphere with centre $(-1, 2, -3)$ and radius 3 units.

Sol: The equation of the sphere is

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ $\quad (-1, 2, -3)$ $r=3$

$(x+1)^2 + (y-2)^2 + (z+3)^2 = 3^2$

$x^2+2x+1+y^2-4y+4+z^2+6z+9 = 9$

$x^2+y^2+z^2+2x-4y+6z+5=0$

Example: 2 Find the co-ordinates of the centre and radius of the sphere

$2x^2+2y^2+2z^2-2x+4y+2z-15=0$

The equation of the sphere can be written in the form

$\div 2$

$x^2+y^2+z^2-x+2y+z-15/2=0$

centre $(-u, -v, -w)$

$u = 1/2 \times$ coefficient of x $\quad v = 1/2$ coefficient of y

$w = 1/2 \times$ coefficient of z

centre $= (-u, -v, -w) = \left(-\frac{1}{2}(-1), -\frac{1}{2}(2), -\frac{1}{2}(1)\right)$

centre $\left(\frac{1}{2}, -1, -\frac{1}{2}\right)$

radius $r = \sqrt{(-u)^2 + (-v)^2 + (-w)^2 - d}$

$= \sqrt{\left(\frac{1}{2}\right)^2 + (-1)^2 + \left(-\frac{1}{2}\right)^2 + \frac{15}{2}}$

$= \sqrt{\frac{1}{4} + 1 + \frac{1}{4} + \frac{15}{2}} = \sqrt{\frac{1}{2} + 1 + \frac{15}{2}}$

$r = \sqrt{\frac{1+2+15}{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

Example: 3: Find the centre and radius of the sphere

$x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$

Sol: Given the sphere is $x^2 + y^2 + z^2 + 2x - 4y - 6z + \overset{d}{5} = 0$

centre $(-u, -v, -w) = \left(-\frac{1}{2} \times \text{coeff. of } x, -\frac{1}{2} \times \text{coeff of } y, -\frac{1}{2} \text{ coeff. of } z\right)$

$= \left(-\frac{1}{2} \times (2), -\frac{1}{2}(-4), -\frac{1}{2}(-6)\right)$

$c = (-1, 2, 3)$

Radius $r = \sqrt{(-u)^2 + (-v)^2 + (-w)^2 - d}$

$= \sqrt{(-1)^2 + 2^2 + 3^2 - 5} = \sqrt{1 + 4 + 9 - 5}$

$r = 3$

Example: 4: Find the equation of the sphere whose centre is at $(1, 1, 1)$ and which passes through the point $(2, 0, 3)$

Sol: The given centre is $(\overset{-u}{1}, \overset{-v}{1}, \overset{-w}{1})$

The equation of the sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$x^2 + y^2 + z^2 + 2(-1)x + 2(-1)y + 2(-1)z + d = 0$

$x^2 + y^2 + z^2 - 2x - 2y - 2z + d = 0 \quad -(1)$

eqn (1) passes through $(2, 0, 3)$

$(2)^2 + (0)^2 + (3)^2 - 2(2) - 2(0) - 2(3) + d = 0$

$4 + 9 - 4 - 6 + d = 0$

$\boxed{d = -3}$

Hence the required sphere equation is

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 3 = 0$$

Example: 5: Find the equation of the sphere on the join of $(1,-1,-1)$ and $(-3,4,5)$ as diameter.

sol: we know that the equation of a sphere whose diameter is the join of (x_1, y_1, z_1) and $(x_2\, y_2\, z_2)$ is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

points $(1, -1, -1)$ $(-3, 4, 5)$
$x_1\ y_1\ z_1$ $x_2\ y_2\ z_2$

$(x-1)(x+3) + (y+1)(y-4) + (z+1)(z-5) = 0$

$x^2 + 2x \cancel{+3} + y^2 - 3y - 4 + z^2 - 4z - 5 = 0$

$x^2 + y^2 + z^2 + 2x - 3y - 4z - 12 = 0$

Example: 6 Find the equation of the sphere which has its centre at the point $(6,-1,2)$ and touch the plane $2x - y + 2z - 2 = 0$

sol: The radius of the sphere is the perpendicular distance from $(6,-1,2)$ to the plane $2x - y + 2z - 2 = 0$

Radius of the sphere $= \dfrac{2(6) - (-1) + 2(2) - 2}{\sqrt{(2)^2 + (-1)^2 + 2^2}}$

$r = \dfrac{12 + 1 + 4 - 2}{\sqrt{4+1+4}} = \dfrac{15}{3} = 5$

Hence the equation of the sphere is

$(x-6)^2 + (y+1)^2 + (z-2)^2 = 5^2$

$x^2 + y^2 + z^2 - 12x + 2y - 4z + 16 = 0$

Example: 7 : find the equation of the sphere which passes through the points (2,3,1) (5,-1,2) (4,3,-1) and (2,5,3)

sol: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (1)$$

eqn (1) passes through the pt (2,3,1) we get

$$2^2+3^2+1^2+2u(2)+2v(3)+2w(1)+d=0$$

$$4u+6v+2w+d=-14 \longrightarrow (2)$$

eqn (1) passes through the pt (5,-1,2) we get.

$$25+1+4+10u-2v+4w+d=0$$

$$10u-2v+4w+d=-30 \longrightarrow (3)$$

eqn (1) passes through the pt (4,3,-1) we get

$$16+9+1+8u+6v-2w+d=0$$

$$8u+6v-2w+d=-26 \longrightarrow (4)$$

eqn (1) passes through the pt (2,5,3)

$$4+25+9+4u+10v+6w+d=0$$

$$4u+10v+6w+d=-38 \longrightarrow (5)$$

solving these equations (2), (3), (4) and (5) we get,

$$u=\frac{-47}{8} \quad v=\frac{-25}{8} \quad w=\frac{-23}{8} \quad d=34$$

Then the equation of the sphere becomes,

$$x^2+y^2+z^2-\frac{47}{4}x-\frac{25}{4}y-\frac{23}{4}z+34=0$$

$$4x^2+4y^2+4z^2-47x-25y-23z+136=0.$$

Example: 8 : find the equation of the sphere of centre at (1,2,3) and touching a plane at (2,1,3)

sol: Since it touches at (2,1,3)

Radius of the sphere $= \sqrt{(2-1)^2+(1-2)^2+(3-3)^2}$

$= \sqrt{1^2+(-1)^2+0^2} = \sqrt{2}$

Hence, the sphere is $(x-1)^2+(y-2)^2+(z-3)^2 = 2$

$$x^2+y^2+z^2-2x-4y-6z+14 = 2$$

$$x^2+y^2+z^2-2x-4y-6z+12 = 0$$

Example: 9 : A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2) = 4k^2$.

Sol: Let the equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

It passes through the origin $\therefore d=0$

Radius of the sphere is k

$$k^2 = u^2+v^2+w^2+d$$

$$k^2 = u^2+v^2+w^2 \quad \text{———} \quad (1).$$

The sphere meets the x-axis at points given by

$$x^2+2ux = 0$$

i, $\cancel{x^2 = -2ux}$ $x(x+2u) = 0$

$$x=0 \quad \text{or} \quad x = -2u$$

$\therefore$ The coordinates of A are $(-2u, 0, 0)$

Illy the coordinates of B and C are $(0, -2v, 0)$ and $(0, 0, -2w)$ respl.

Let the centroid of the triangle ABC be (x_1, y_1, z_1)

$$\therefore \frac{-2u+0+0}{3} = x_1, \quad \frac{0-2v+0}{3} = y_1, \quad \frac{0+0-2w}{3} = z_1$$

$$u = -\frac{3x_1}{2} \quad v = -\frac{3y_1}{2} \quad w = -\frac{3z_1}{2}$$

Substitute these values of u, v, w in (1) we get

$$\frac{9x_1^2}{4}+\frac{9y_1^2}{4}+\frac{9z_1^2}{4} = k^2$$

$\therefore$ Locus of $9x^2+9y^2+9z^2 = 4k^2$

Example: 10: A plane passes through a fixed point (a,b,c) and cuts the axes in A,B,C. Show that the locus of the centre of the sphere OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Sol: Let the equation of the plane be $lx+my+nz=p$

since it passes through the point (a,b,c)

$$la+mb+nc=p \quad \text{——} \; (1)$$

The coordinates of the points where the plane $lx+my+nz=p$ meets the x axis are obtained by putting $y=0$, $z=0$ in that equation.

$\therefore$ The coordinates of that point A are $\left(\frac{p}{l},0,0\right)$ $\left(0,\frac{p}{m},0\right)$ $\left(0,0,\frac{p}{n}\right)$

The sphere OABC passes through the points $(0,0,0)$ $\left(\frac{p}{l},0,0\right)$ $\left(0,\frac{p}{m},0\right)$ and $\left(0,0,\frac{p}{n}\right)$

Hence its equation can easily be found as

$$x^2+y^2+z^2-\frac{p}{l}x-\frac{p}{m}y-\frac{p}{n}z=0$$

Let the centre of this sphere be (x_1,y_1,z_1)

$$x_1=\frac{p}{2l}, \quad y_1=\frac{p}{2m} \quad z_1=\frac{p}{2n}$$

$$l=\frac{p}{2x_1} \quad m=\frac{p}{2y_1}, \quad n=\frac{p}{2z_1}$$

Substituting the values of l,m,n in eqn (1) we ge

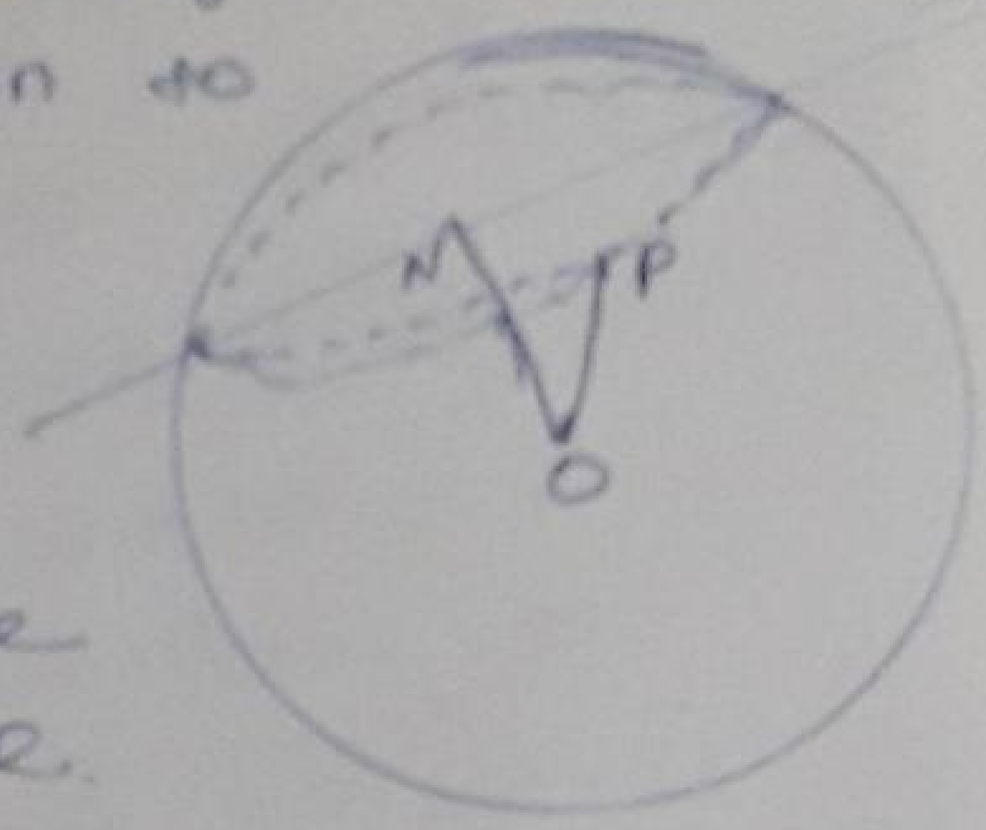
$$\frac{pa}{2x_1}+\frac{pb}{2y_1}+\frac{pc}{2z_1}=p \text{ i, } \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2$$

$\therefore$ Locus of (x_1,y_1,z_1) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$

Section : 5

5 The Plane section of a sphere is a circle

Let O be the centre of the sphere of radius r and P any point common to the sphere and the plane



Then OP = radius of the sphere

Draw ON perpendicular to the plane.

Then $NP^2 = OP^2 - ON^2$

$= r^2 - ON^2$

O and N are fixed points.

$\therefore$ ON is constant.

$\therefore$ NP = constant.

Hence the locus of P is a circle whose centre is N, the foot of the perpendicular from the centre of the sphere to the plane, such a circle is called a Small circle on the sphere

Defn: If the plane passes through the centre of the sphere the circle is of radius r and is called a great circle.

6. Equation of a sphere passing through a given circle

The eqn. $x^2+y^2+z^2+2ux+2vy+2wz+d+k(lx+my+nz-p)=0$ in which k is any constant represents a sphere. Moreover the equation satisfied by the coordinates of any point which is common to the sphere, which

$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$

$lx+my+nz=p.$

7. Intersection of two spheres is a circle

Let the equation of the two spheres be

$S_1 : x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S_2 : x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

$S_1-S_2 = 2x(u-u_1)+2y(v-v_1)+2z(w-w_1)+d-d_1=0$

which is being of the first degree represents a plane. Thus the point of intersection of the two spheres are the same as those of any one of them on this plane and therefore lie on the circle.

Example: 1 Find the equation of the sphere having the circle $x^2+y^2+z^2-2x+4y-6z+7=0$; $2x-y+2z=5$ for a great circle.

Sol: Any sphere through the circle is,

$x^2+y^2+z^2-2x+4y-6z+7+k(2x-y+2z-5)=0$

$x^2+y^2+z^2-2x+4y-6z+7+2kx-ky+2kz-5k=0$

$x^2+y^2+z^2+(2k-2)x+(4-k)y+(2k-6)z+7-5k=0$

Centre $(-u,-v,-w) = \left(-\frac{(2k-2)}{2}, -\frac{(4-k)}{2}, -\frac{(2k-6)}{2}\right)$

Since the centre of a great circle coincides with the centre of the sphere, $\left(\frac{2-2k}{2}, \frac{k-4}{2}, \frac{6-2k}{2}\right)$ lies on the plane $2x-y+2z-5=0$

$\therefore 2\left(\frac{2-2k}{2}\right)-\left(\frac{k-4}{2}\right)+2\left(\frac{6-2k}{2}\right)-5=0$

$2-2k-\frac{k}{2}+2+6-2k-5=0$

$-4k-\frac{k}{2}+1=0$

$-\frac{9k}{2}=-1 \qquad 9k=2$

$k=2/9$

Hence the equation of the required sphere is

$$x^2+y^2+z^2-2x+4y-6z+7+\frac{2}{9}(2x-y-2z-5)=0$$

$$9(x^2+y^2+z^2-2x+4y-6z+7)+2(2x-y-2z-5)=0.$$

Example : 2 find the equation of the sphere which passes through the circle $x^2+y^2+z^2-2x-4y=0$, $x+2y+3z=8$ and touches the plane $4x+3y=25$.

Sol: The equation of the required sphere is of the form.

$$x^2+y^2+z^2-2x-4y+k(x+2y+3z-8)=0. \rightarrow (1)$$

i, $x^2+y^2+z^2-(2-k)x-(4-2k)y+3kz-8k=0$

The centre of the sphere is $\left(\frac{2-k}{2}, \frac{4-2k}{2}, \frac{-3k}{2}\right)$

and the radius is $\left\{\left(\frac{2-k}{2}\right)^2+\left(\frac{4-2k}{2}\right)^2+\left(\frac{-3k}{2}\right)^2+8k\right\}^{1/2}$

i, $\left\{\frac{7k^2+6k+10}{2}\right\}^{1/2}$

If the sphere touches the plane $4x+3y=25$ then the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere

$$\frac{4\left(\frac{2-k}{2}\right)+3\left(\frac{4-2k}{2}\right)-25}{\sqrt{(4)^2+(3)^2}}=\left\{\frac{7k^2+6k+10}{2}\right\}^{1/2}$$

$$\frac{4-2k+6-3k-25}{5}=\left\{\frac{7k^2+6k+10}{2}\right\}^{1/2}$$

$$-(k+3)=\left\{\frac{7k^2+6k+10}{2}\right\}^{1/2}$$

$$2(k+3)^2=7k^2+6k+10$$

$$5k^2-6k-8=0$$

$$k=2 \text{ or } -4/5$$

Substituting the values of K in the equation (1), we get the equations of the required spheres,

They are

$$x^2+y^2+z^2-2x-4y+2(x+2y+2z-8)=0$$

$$x^2+y^2+z^2-2x-4y-\frac{4}{5}(x+2y+3z-8)=0$$

$$x^2+y^2+z^2+6z-16=0 \text{ and.}$$

$$5(x^2+y^2+z^2)-14x-28y+32=0.$$

Example : 3 The Plane ABC, whose equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. find the eqn to the circumcircle of the triangle ABC and obtain the coordinates of its centre and radius.

Sol: The points A, B, C are respectively $(a,0,0)$ $(0,b,0)$ $(0,0,c)$ Let O be the origin,

Then the equation to the sphere OABC can be easily seen as $x^2+y^2+z^2-ax-by-cz=0$

The equation of the circumcircle of the triangle ABC is $x^2+y^2+z^2-ax-by-cz=0;\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

The centre of the sphere is $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

The centre of the circle is the foot of the perpendicular from $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ to the plane.

The equation of the perpendicular to the plane through $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ is $\frac{x-a/2}{1/a}=\frac{y-b/2}{1/b}=\frac{z-c/2}{1/c}$

The coordinates of any point on the line are of the form

$$\left(\frac{a}{2}+\frac{\lambda}{a},\ \frac{b}{2}+\frac{\lambda}{b},\ \frac{c}{2}+\frac{\lambda}{c}\right)$$

If this point lies on the plane, we get

(12)

$$\frac{1}{a}\left(\frac{a}{2}+\frac{\lambda}{a}\right)+\frac{1}{b}\left(\frac{b}{2}+\frac{\lambda}{b}\right)+\frac{1}{c}\left(\frac{c}{2}+\frac{\lambda}{c}\right)=1$$

$$\therefore \lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=-\frac{1}{2}$$

$$\therefore \lambda=-\frac{1}{2(a^{-2}+b^{-2}+c^{-2})}$$

$\therefore$ The coordinates of the circumcentre of the triangle ABC are

$$\frac{\frac{a}{2}(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}},\ \frac{\frac{b}{2}(c^{-2}+a^{-2})}{a^{-2}+b^{-2}+c^{-2}},\ \frac{c/2(a^{-2}+b^{-2})}{a^{-2}+b^{-2}+c^{-2}}$$

Let R be the radius of the circle, d the perpendicular distance from $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and r the radius of the sphere $x^2+y^2+z^2-ax-by-cz=0$

we have $R^2=r^2-d^2$

but $r^2=\frac{a^2+b^2+c^2}{4}$ and $d=\frac{1/2}{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^{1/2}}$

$$R^2=\frac{a^2+b^2+c^2}{4}-\frac{1}{4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

$$=\frac{(a^2+b^2+c^2)(b^2c^2+c^2a^2+a^2b^2)-a^2b^2c^2}{4(b^2c^2+c^2a^2+a^2b^2)}$$

$$=\frac{(b^2+c^2)b^2c^2+(c^2+a^2)c^2a^2+(a^2+b^2)a^2b^2+2a^2b^2c^2}{4(b^2c^2+c^2a^2+a^2b^2)}$$

$$=\frac{(b^2+c^2)(c^2+a^2)(a^2+b^2)}{4(b^2c^2+c^2a^2+a^2b^2)}$$

$$\therefore R=\frac{1}{2}\left\{\frac{(b^2+c^2)(c^2+a^2)(a^2+b^2)}{b^2c^2+c^2a^2+a^2b^2}\right\}^{1/2}$$

Example: 4: Find the equation of a sphere which passes through the point $(1,-2,3)$ and the circle $z=0,\ x^2+y^2+z^2-9=0$.

Sol: Any sphere through the given circle is

$$x^2+y^2+z^2-9+\lambda z=0 \quad \text{---(1)}$$

It passes through $(1,-2,3)$ we must have

$$(1+4+9-9)+\lambda(3)=0$$

$$\lambda=-5/3$$

Sub in (1) we get,

$$(x^2+y^2+z^2-9)-\frac{5}{3}z=0$$

$$3(x^2+y^2+z^2)-5z-27=0$$

which is required equation of the sphere.

8. The equation of the tangent plane for the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at point (x_1,y_1,z_1)

Sol: Let P be the point (x_1,y_1,z_1). P lies on the sphere.

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \text{---(1)}$$

The tangent plane passes through the point (x_1,y_1,z_1) and is perpendicular to OP. where O is the centre of the sphere.

The co-ordinates of O are $-u,-v,-w$

$\therefore$ The direction cosines of OP are propotional to

$$u+x_1,\ v+y_1,\ w+z_1.$$

Hence the equation of the plane is

$$(x-x_1)(u+x_1)+(y-y_1)(v+y_1)+(z-z_1)(w+z_1)=0$$

$$xx_1+yy_1+zz_1+ux+vy+wz=x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$$

But from (1)

$$x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1=-ux_1-vy_1-wz_1-d$$

$\therefore$ The equation of the tangent plane is

$xx_1 + yy_1 + zz_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$

Cor: The equation of the tangent plane to the sphere $x^2+y^2+z^2=r^2$ at the point (x_1, y_1, z_1) is $xx_1+yy_1+zz_1=r^2$

Example: 1 Show that the plane $2x-y-2z=16$ touches the sphere $x^2+y^2+z^2-4x+2y+2z-3=0$ and find the point of contact.

Sol: The centre of the sphere is $(2,-1,-1)$ and its radius 3.

$\therefore$ The length of the perpendicular from $(2,-1,-1)$ to the plane $2x-y-2z-16=0$ is

$$\frac{2(2)-(-1)-2(-1)-16}{\sqrt{(2)^2+(1)^2+2^2}} = \frac{4+1+2-16}{3} = \frac{9}{3} = 3$$

Hence the plane touches the sphere.

Let (x_1, y_1, z_1) be the point of contact.

$\therefore$ The equation of the tangent plane at (x_1, y_1, z_1) is

$xx_1+yy_1+zz_1-2(x+x_1)+(1)(y+y_1)+(1)(z+z_1)-3=0$

i.e. $(x_1-2)x+(y_1+1)y+(z_1+1)z-2x_1+y_1+z_1-3=0$

$$\therefore \frac{x_1-2}{2} = \frac{y_1+1}{-1} = \frac{z_1+1}{-2} = \frac{2x_1-y_1-z_1+3}{16}$$

Solving these equations we get,

$$\frac{x_1-2}{2} = \frac{y_1+1}{-1} = \frac{z_1+1}{-2} = r \text{ (say)}$$

$x_1 = 2r+2 \quad y_1 = -r-1 \quad z_1 = -2r-1 \rightarrow (1)$

eqn (1) lies on $2x-y-2z-16=0$

$2(2r+2)-(-r-1)-2(-2r-1)-16=0$

$4r+4+r+1+4r+2-16=0$

$9r-9=0$

$r=1$

$\therefore$ The point of contact $(4,-2,-3)$

Example: 2 Find the equation of a sphere which touches the sphere $x^2+y^2+z^2-6x+2z+1=0$ at the pt $(2,-2,1)$ and passes through the origin

Sol: The tangent plane to the given sphere at $(2,-2,1)$ is given by

$$2x-2y+z-3(x+2)+(z+1)+1=0$$

i.e. $x+2y-2z+4=0$

The required sphere is therefore of the form

$$x^2+y^2+z^2-6x+2z+1+k(x+2y-2z+4)=0$$

and this passes through the origin if $k=-1/4$

$\therefore$ The equation of the required sphere is

$$x^2+y^2+z^2-6x+2z+1-1/4(x+2y-2z+4)=0$$

$$\therefore 4(x^2+y^2+z^2)-25x-2y+10z=0$$

Example: 3 Find the condition that the line $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$ where $l^2+m^2+n^2=1$ should touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ show that there are two spheres through the points $(0,0,0)$ $(2a,0,0)$ $(0,2b,0)$ which touch the above line and that the distance between their centre is $\frac{2}{n^2}(c^2-(a^2+b^2+c^2)n^2)^{1/2}$

$\therefore$ **Sol:** The coordinates of any pt on the line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \rightarrow (1)$$

are of the form $(a+rl, b+rm, c+rn)$

Hence the points of intersection of this line and the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (2)$

are given by the equation

$$(a+rl)^2+(b+rm)^2+(c+rn)^2+2u(a+rl)+2v(b+rm)$$
$$+2w(c+rn)+d=0$$

Hence the points of the intersection of this line and the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \rightarrow (2)$

are given by the equation,

$(a+rl)^2+(b+rm)^2+(c+rn)^2+2u(a+rl)+2v(b+rl)+2w(c+rn)+d=0$

i.e., $r^2+2r[l(a+u)+m(b+v)+n(c+w)]+a^2+b^2+c^2+2ua+2vb+2wc+d=0$

Hence the line (1) touches the sphere (2) if,

$[l(a+u)+m(b+v)+n(c+w)]^2=a^2+b^2+c^2+2au+2bv+cw+d.$

Let the equation of the sphere passing through the points $(0,0,0)$ $(2a,0,0)$ $(0,2b,0)$ be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

then $d=0$; $4a^2+4au=0$ i.e., $u=-a.$

$4b^2+4bv=0$ i.e., $v=-b$

The equation of the sphere then becomes

$x^2+y^2+z^2-2ax-2by+2wz=0.$

If this sphere touches the line (1) then

$n^2(c+w)^2=a^2+b^2+c^2-2a^2-2b^2+2wc$

$n^2w^2+2(n^2-1)cw+a^2+b^2-c^2+n^2c^2=0 \rightarrow (3)$

It is a quadratic equation in w and so there are two values of w satisfying the equation.

Hence there are two spheres touching the line (1).

The centre of the two spheres are $(a,b,-w_1)$ $(a,b,-w_2)$ where $w_1 w_2$ are the roots of the eqn (3)

$w_2+w_3=-\frac{2(n^2-1)c}{n^2}$; $w_1w_2=\frac{a^2+b^2-c^2+n^2c^2}{n^2}$

Hence the square of the distance between the centres $=(w_1-w_2)^2=(w_1+w_2)^2-4w_1w_2$

$=\frac{4c^2(n^2-1)^2}{n^4}-4\frac{(a^2+b^2-c^2+n^2c^2)}{n^2}$

$=\frac{4}{n^4}(c^2-n^2)(a^2+b^2+c^2)$

distance between the centres $=\frac{2}{n^2}[c^2-n^2(a^2+b^2+c^2)]^{1/2}$

ANALYTICAL GEOMETRY OF 3-DIMENSIONS AND INTEGRAL CALCULUS

UNIT-3 Properties of Definite Integrals

18K1M02

1. Prove that $\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}\,dx = \pi/4$

Solution: Let I be the value of this integral and

$$f(x) = \frac{(\sin x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}$$

$$I = \int_0^{\pi/2} f(x)\,dx. \quad \text{---} (1)$$

$$f(a-x) = \frac{[\sin(\pi/2 - x)]^{3/2}}{(\sin(\pi/2-x))^{3/2}+(\cos(\pi/2-x))^{3/2}}$$

$$= \frac{(\cos x)^{3/2}}{(\cos x)^{3/2}+(\sin x)^{3/2}} \quad \text{as } a = \pi/2$$

Also $I = \int_0^{\pi/2} f(a-x)\,dx$ --- (2)

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}\,dx + \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2}+(\sin x)^{3/2}}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sin x)^{3/2}+(\cos x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}\,dx$$

$$2I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$$

$$I = \pi/4$$

2. Evaluate $\int_0^{\pi/2} \log \sin x \, dx$

Let $I = \int_0^{\pi/2} f(x)\,dx$ --- (1) $f(x) = \log \sin x$.

$f(a-x) = f(\pi/_2 - x) = \log(\tan(\pi/_2 - x))$ Here $a = \pi/_2$

Let $= \log \cot x$.

$\cancel{f(a+x)}\ I = \int_0^{\pi/2} f(a-x)\, dx \longrightarrow (2)$.

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} \log \tan x\, dx + \int_0^{\pi/2} \log \cot x\, dx.$$

$$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\cancel{\tan x} \times \frac{1}{\cancel{\tan x}}\right) dx$$

$$2I = \int_0^{\pi/2} \log(1)\, dx = 0.$$

$$I = 0.$$

3. $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \pi/_8 \log 2$.

Solution: Let $I = \int_0^{\pi/4} f(\theta)\, d\theta$. Here $f(\theta) = \log(1+\tan\theta)$ $\to (1)$

$f(a-\theta) = f(\pi/_4 - \theta) = \log(1+\tan(\pi/_4 - \theta))$ $a = \pi/_4$.

$$= \log\left(1 + \frac{\tan\pi/_4 - \tan\theta}{1+\tan\pi/_4 \tan\theta}\right) = \cancel{\log(a+\ldots)}$$

$$= \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) = \log\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right)$$

$$f(a-\theta) = \log\left(\frac{2}{1+\tan\theta}\right)$$

Let $I = \int_0^{\pi/4} f(a-\theta)\, d\theta \longrightarrow (2)$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\cancel{1+\tan\theta} \times \frac{2}{\cancel{1+\tan\theta}}\right) d\theta$$

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$$I = \int_0^{\pi/4} \log 2 \, d\theta = \log 2 \, (\theta)_0^{\pi/4} = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8} \log 2.$$

Integration by parts.

formula: $\int u\,dv = uv - \int v\,du$

1. $\int x e^x dx.$

Here $u = x$ $dv = e^x dx.$

$du = dx$ $\int dv = \int e^x dx$ $v = e^x$

formula: $\int u\,dv = uv - \int v\,du.$

$$\int x e^x dx = xe^x - \int e^x dx.$$
$$= xe^x - e^x = e^x(x-1)$$

2. $\int x \sin 2x \, dx.$

Here $u = x$ $\int dv = \int \sin 2x \, dx.$ $\int u\,dv = uv - \int v\,du$

$du = dx$ $v = -\frac{\cos 2x}{2}$

$$\int x \sin 2x \, dx = -\frac{x\cos 2x}{2} - \int -\frac{\cos 2x}{2} dx.$$
$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}$$

3. $\int \tan^{-1} x \, dx.$ $u = \tan^{-1}x$ $\int dv = \int dx.$

$du = \frac{1}{1+x^2} dx$ $v = x.$

$$\int \tan^{-1}x \, dx = x\tan^{-1}x - \int x \frac{1}{1+x^2} dx. = x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$
$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

4. $\int (\log x)^2 dx.$ and $\int dv = \int dx$

$u = (\log x)^2$ $v = x.$

$du = 2\log x \cdot \frac{1}{x} dx$

$$\int (\log x)^2 dx = x(\log x)^2 - \int x \cdot 2\log x \cdot \frac{1}{x} dx$$
$$= x(\log x)^2 - 2\int \log x \, dx.$$
$$= x(\log x)^2 - 2\left[x\log x - \int x \frac{1}{x} dx\right]$$
$$= x(\log x)^2 - 2\left[x\log x - x\right]$$
$$= x(\log x)^2 - 2x\log x + 2x$$

$u = \log x$

$du = \frac{1}{x} dx$

$\int dv = \int dx$

$v = x$

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5. $\int \frac{x+\sin x}{1+\cos x}\,dx.$

Sol: $I = \int \frac{x\,dx}{1+\cos x} + \int \frac{\sin x\,dx}{1+\cos x}$

$= \int \frac{x\,dx}{2\cos^2 x/2} + \int \frac{2\sin x/2\,\cos x/2}{2\cos^2 x/2}\,dx.$

$= \int x\sec^2 x/2\,dx + \int \tan x/2\,dx.$

$u = x \quad \int dv = \int \sec^2 x/2\,dx.$

$du = dx \quad v = \tan x/2$

$= x\tan x/2 - \int \tan x/2\,dx + \int \tan x/2\,dx.$

$= x\tan x/2$

6. $\int e^x \frac{x+1}{(x+2)^2}\,dx. = \int e^x \frac{(x+2-1)}{(x+2)^2}\,dx.$

$= \int e^x \frac{x+2}{(x+2)^2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx$

$= \int \frac{e^x}{x+2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx.$

$u = \frac{1}{x+2} \quad \int dv = \int e^x\,dx.$

$du = -\frac{1}{(x+2)^2}\,dx. \quad v = e^x$

$= \int \frac{1}{x+2}\,d(e^x) - \int \frac{e^x}{(x+2)^2}\,dx$

$= \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx$

$= \frac{e^x}{x+2}$

7. $\int e^x(\sin x + \cos x)\,dx. = \int e^x \sin x\,dx + e^x\cos x\,dx.$

$u = \sin x \quad \int dv = \int e^x\,dx \qquad u = \cos x \quad \int dv = \int e^x\,dx.$

$du = \cos x\,dx \quad v = e^x \qquad du = -\sin x\,dx \quad v = e^x$

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$= e^x \sin x - \int e^x \cos x\, dx + \cancel{e^x \cos x} + \int e^x \cos x\, dx.$

$= e^x \sin x.$

<u>Reduction formulae</u>

1. $I_n = \int x^n e^{ax} dx$ where n is a positive integer.

Here $u = x^n$ $\quad \int dv = \int e^{ax} dx$ $\qquad \int u\,dv = uv - \int v\,du$

$du = n x^{n-1} dx$ $\quad v = \dfrac{e^{ax}}{a}$

$$I_n = x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} n x^{n-1} dx.$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx.$$

$$= \frac{e^{ax}}{a} I_n - \frac{n}{a} I_{n-1}$$

2. $I_n = \int x^n \cos ax\, dx$ where n is a positive integer.

Here $u = x^n$ $\quad \int dv = \int \cos ax\, dx$

$du = n x^{n-1} dx$ $\quad v = \dfrac{\sin ax}{a}$

$$= x^n \frac{\sin ax}{a} - \int \frac{\sin ax}{a} n x^{n-1} dx.$$

$u = x^{n-1}$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax\, dx.$$

$du = (n-1) x^{n-2}$

$\int dv = \int \sin ax\, dx$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\left[x^{n-1} \frac{\cos ax}{a} - \int \frac{\cos ax}{a} (n-1) x^{n-2} dx \right.$$

$v = \dfrac{\cos ax}{a}$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a^2} x^{n-1} \cos ax + \frac{n(n-1)}{a^2} \int x^{n-2} \cos ax\, dx$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a^2} x^{n-1} \cos ax + \frac{n(n-1)}{a^2} I_{n-2}$$

3. $\int x^3 e^x dx.$

$u = x^3$ $\quad \int dv = \int e^x dx$

$du = 3x^2 dx$ $\quad v = e^x.$

$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$ $\qquad u = x^2$

$= x^3 e^x - 3\left[x^2 e^x - \int e^x 2x\, dx\right]$ $\qquad du = 2x\,dx$

$\int dv = \int e^x dx$

$= x^3 e^x - 3x^2 e^x + 6 \int e^x x\, dx$ $\qquad v = e^x$

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$= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x dx\right]$ $u = x$ $\int dv = \int e^x dx$ $du = dx$ $v = e^x$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x$

$= e^x(x^3 - 3x^2 + 6x - 6)$

1. $I_n = \int \sin^n x\, dx$ where n is a positive integer.

$I_n = \int \sin^{n-1}x \sin x\, dx = \int \sin^{n-1}x\, d(-\cos x)$ $u = \sin^{n-1}x$ $du = (n-1)\sin^{n-2}x \cos x\, dx$

$= -\sin^{n-1}x \cos x + \int \cos x (n-1)\sin^{n-2}x \cos x\, dx$ $\int dv = \int \sin x\, dx$ $v = -\cos x$

$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$

$= -\sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x (1 - \sin^2 x)\, dx$

$I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$(n-1)I_n + I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$

$nI_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$

<u>Formulae</u>

$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}, \frac{n-1}{n-2} \cdots \frac{1}{2}\frac{\pi}{2}$ when n is even.

$= \frac{n-1}{n}, \frac{n-3}{n-2} \cdots \frac{2}{3}$ when n is odd.

2. $\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$

3. $\int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{48}{105} = \frac{16}{35}$

4. $\int \sin^5 x\, dx$ put $y = \cos x$ $dy = -\sin x\, dx$

$= \int \sin^4 x \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx$

$= \int -(1 - y^2)^2 dy = -\int (1 + y^4 - 2y^2)\, dy = -y + \frac{2y^3}{3} - \frac{y^5}{5}$

$= -\cos x + 2\cos^3 x/3 - \cos^5 x/5$

5. $\int_0^{\pi/2} x(1-x^2)^{1/2}dx$.

put $x=\sin\theta$ $dx=\cos\theta\,d\theta$. when $x=0$ $\theta=0$
$x=1$ $\theta=\pi/2$

$$\int_0^{\pi/2} x(1-x^2)^{1/2}dx = \int_0^{\pi/2} \sin\theta(1-\sin^2\theta)^{1/2}\cos\theta\,d\theta.$$

$$= \int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta = \int_0^{\pi/2}\cos^2\theta\,d(-\cos\theta) = \left(-\frac{\cos^3\theta}{3}\right)_0^{\pi/2} = 1/3.$$

6. $I_n = \int\cos^n x\,dx$. (n is a positive integer)

$$I_n = \int\cos^{n-1}x\cos x.dx. = \int\cos^{n-1}x\,d(\sin x).$$

$u = \cos^{n-1}x$

$du = (n-1)\cos^{n-2}x\cdot(-\sin x)dx$

$\int dv = \int\cos x\,dx$

$V = \sin x.$

$$= \cos^{n-1}x\sin x - \int\sin x\cos^{n-2}x\,(n-1)(-\sin x)\,dx.$$

$$= \cos^{n-1}x\sin x + (n-1)\int\sin^2x\cos^{n-2}x\,dx.$$

$$= \cos^{n-1}x\sin x + (n-1)\int(1-\cos^2x)\cos^{n-2}x\,dx.$$

$$I_n = \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n.$$

$$(n-1)I_n + I_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}.$$

Formulae

$$\int_0^{\pi/2}\cos^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2} \quad n \text{ is even}.$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} \quad \text{if } n \text{ is odd}$$

1. $\int_0^{\pi/2}\cos^8 x\,dx = \frac{7}{8}\times\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$

2. $\int_0^{\pi/2}\cos^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

3. $\int\cos^7x\,dx = \int\cos^6x\cos x\,dx$ put $y=\sin x$
$dy = \cos x\,dx$.

$$= \int(1-\sin^2x)^3\cos x\,dx$$

$$= \int(1-y^2)^3\,dy = \int(1-3y^2+3y^4-y^6)\,dy$$

$$= y - 3y^3/3 + 3y^5/5 - y^7/7$$

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Result: 5   7.

$I_{m,n} = \int \sin^m x \cos^n x \, dx$ (m, n being positive integers)

$(m+n) I_{m,n} = -\cos^{n+1} \sin^{m-1} x + (m-1) I_{m-2,n}$

1. $\int \sin^6 x \cos^3 x \, dx$. put $y = \sin x$ $dy = \cos x \, dx$.

$\int \sin^6 x \cos^3 x \, dx = \int y^6 (1-y^2) \, dy = \frac{y^7}{7} - \frac{y^9}{9} = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}$

2. $\int \sin^9 x \cos^5 x . dx$.   put $y = \sin x$ $dy = \cos x \, dx$.

$\int \sin^9 x \cos^5 x \, dx = \int y^9 (1-y^2)^2 \, dy = \int y^9 (1 + y^4 - 2y^2) \, dy$

$= \frac{y^{10}}{10} + \frac{y^{14}}{14} - \frac{2y^{12}}{126} = \frac{\sin^{10} x}{10} + \frac{\sin^{14} x}{14} - \frac{\sin^{12} x}{12}$

3. $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7} = \frac{8}{693}$

4. $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{512}$

$I_n = \int \tan^n x \, dx$. ($n$ being a positive integer)

$= \int \tan^{n-2} x \tan^2 x \, dx$.

$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$.

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \int \tan^{n-2} x \, d(\tan x) - \int \tan^{n-2} x \, dx$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

1. $\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx$ put $n=4$ in formula

$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\,dx.$

$= \frac{\tan^3 x}{3} - \tan x + x$

2. $\int_0^{\pi/4} \tan^3 x\,dx = \left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx$ put $n=3$

$= 1/2 + (\log \cos x)_0^{\pi/4} = \frac{1}{2} + \log\frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \log 2)$

$I_n = \int \sec^n x\,dx$

$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$

$= \int \sec^{n-2} x\,d(\tan x)$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx.$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}.$

1. $\int \sec^3 x\,dx = \int \sec x\,d(\tan x)$

$= \sec x \tan x - \int \tan^2 x \sec x\,dx.$

$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx,$

$= \sec x \tan x - \int \sec^3 x + \int \sec x\,dx$

$I = \sec x \tan x - I + \log(\sec x + \tan x)$

$2I = \sec x \tan x + \log(\sec x + \tan x)$

$\int \sec^6 x\,dx = \int \sec^4 x\, d(\tan x) = \int (1+t^2)^2 dt$ put $t = \tan x$

$= \int (1+2t^2+t^4)\,dt = t + \frac{2t^3}{3} + \frac{t^5}{5}$

$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5}$

$I_{m,n}$ $\int x^m (\log x)^n dx$

$I_{m,n} = \int (\log x)^n d\left(\frac{x^{m+1}}{m+1}\right)$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$

$\int x^4 (\log x)^3 dx = \int (\log x)^3 d\left(\frac{x^5}{5}\right)$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 x^4 dx$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 d\left(\frac{x^5}{5}\right)$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\int x^4 (\log x)\,dx$

$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\left[\frac{x^5}{5}\log x - \frac{x^5}{25}\right]$

$= \frac{x^5}{5}\left[(\log x)^3 - \frac{3}{5}(\log x)^2 + \frac{6}{25}\log x - \frac{6}{125}\right]$

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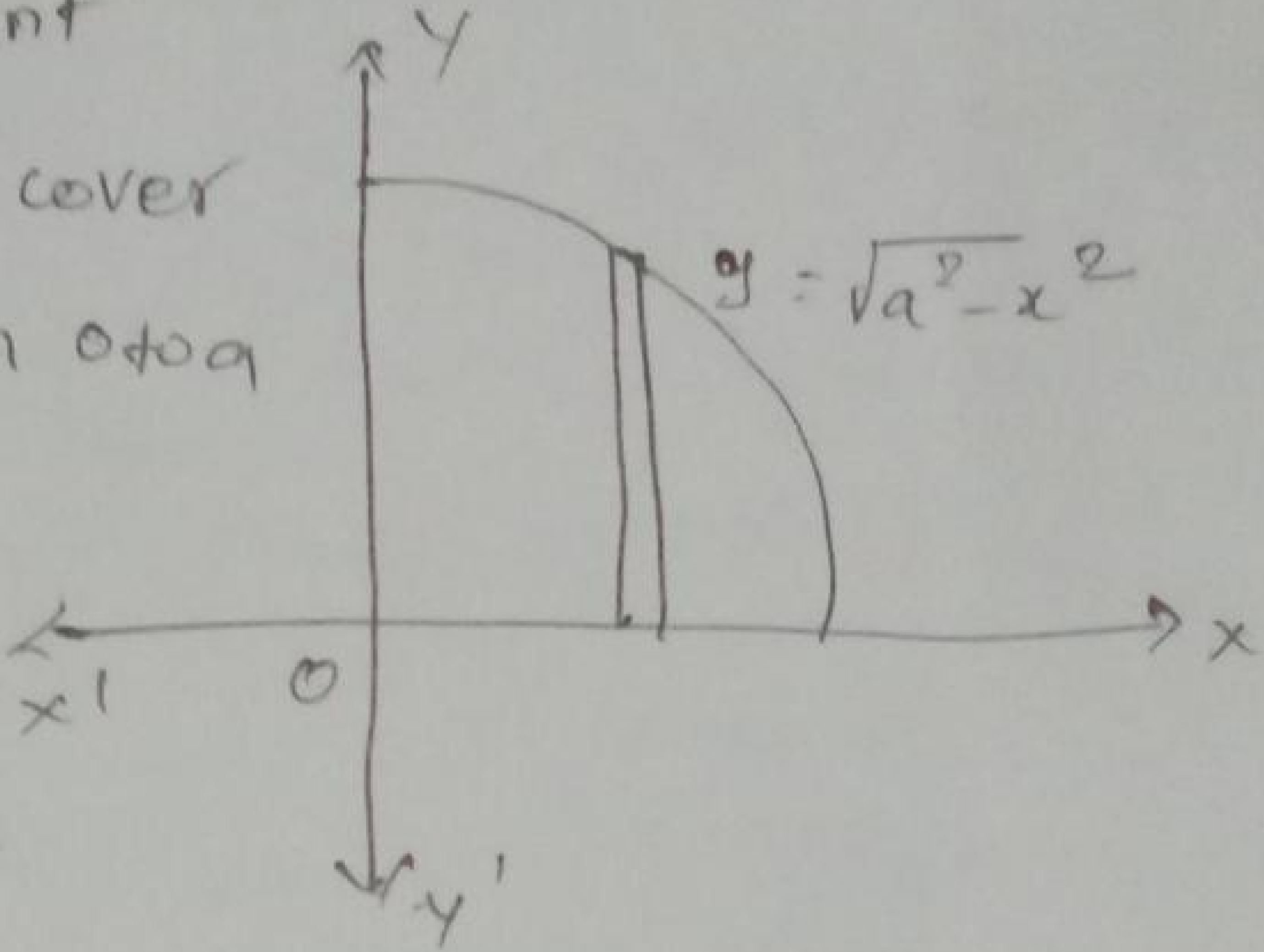
UNIT - IV Double Integral

1. Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$

Solution: If we keep x as constant Y varies from 0 to $\sqrt{a^2-x^2}$. To cover the whole area. x varies from 0 to a



$$\therefore \iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a \left[x\,\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

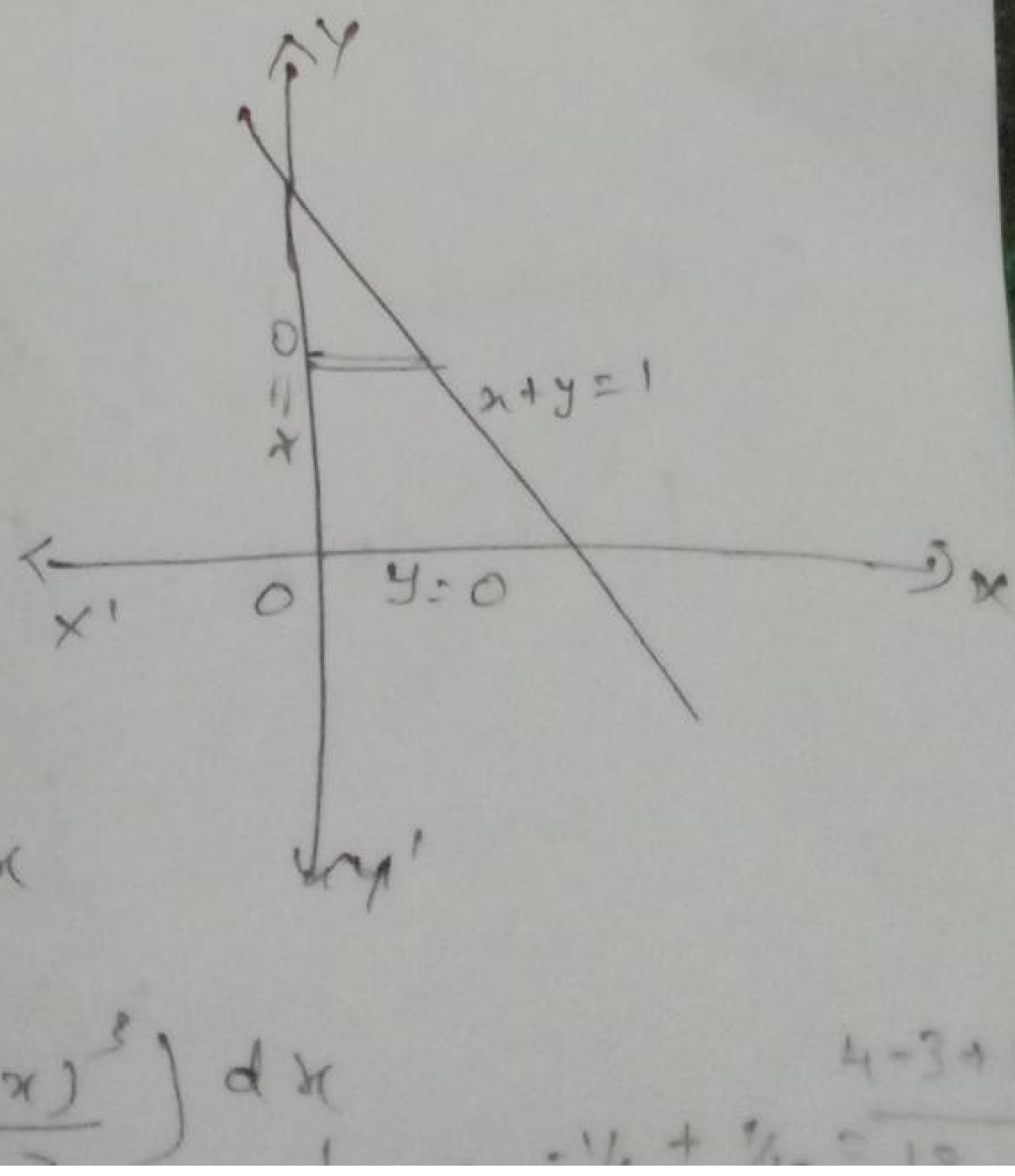
$$= \int_0^a \frac{x(a^2-x^2)}{2}\,dx = \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{a^4}{8}$$

2. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x, y area each ≥ 0 and $x+y \leq 1$

Solution: This region is the region triangle formed by the lines,

$x=0,\ y=0,\ x+y=1$



$$\iint (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) dx$$

4-3+1

3. Evaluate $\int_0^1 \int_1^2 (x^2+y^2)\, dx\, dy$

$\int_0^1 \int_1^2 (x^2+y^2)\, dx\, dy = \int_0^1 \left(\frac{x^3}{3} + y^2 x \right)_1^2 dy$

$= \int_0^1 \left[\left(\frac{8}{3} + 2y^2 \right) - \left(\frac{1}{3} + y^2 \right) \right] dy$

$= \int_0^1 \left(\frac{7}{3} + y^2 \right) dy$

$= \left(\frac{7}{3} y + \frac{y^3}{3} \right)_0^1$

$= 8/3$

4. Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta$

$\int_0^{\pi} \int_0^{\sin\theta} r\, dr\, d\theta = \int_0^{\pi} \left(\frac{r^2}{2} \right)_0^{\sin\theta} = 1/2 \int_0^{\pi} \sin^2\theta\, d\theta$

$= \frac{1}{2} \int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta$ $\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$

$= 1/4 \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{\pi}$ $\sin n\pi = 0$

$= \pi/4$

5. Evaluate $\int_0^1 \int_0^x dy\, dx.$

$\int_0^1 \int_0^x dy\, dx = \int_0^1 (y)_0^x\, dx$

$= \int_0^1 x\, dx$

$= \left(\frac{x^2}{2} \right)_0^1$

$= 1/2$

6. Evaluate : $\int_0^{\pi/2} \int_0^{\pi/2} (\sin\theta + \varphi)\, d\theta\, d\varphi$

$$\int_0^{\pi/2} \int_0^{\pi/2} (\sin\theta + \varphi) d\theta d\varphi = \int_0^{\pi/2} \left(-\cos(\theta + \varphi)\Big|_0^{\pi/2}\right) d\varphi$$

$$= \int_0^{\pi/2} \left(-\cos(\pi/2 + \varphi) + \cos\varphi\right) d\varphi$$

$\cos \pi/2 + \theta = \sin\theta$

$$= \left[-\sin(\pi/2 + \varphi) + \sin\varphi\right]_0^{\pi/2}$$

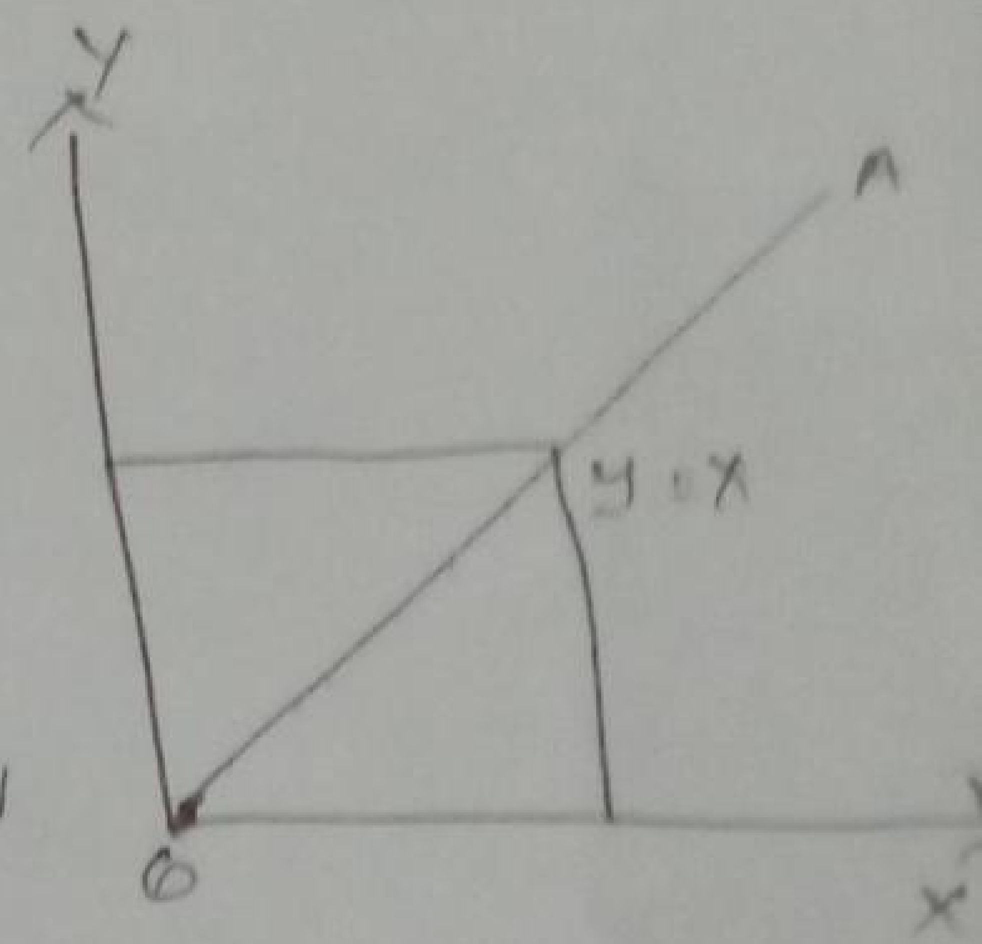
$$= \left(-\sin\pi + \sin\pi/2\right) - \left(-\sin\pi/2 + \sin 0\right)$$

$$= 1 - (-1) = 2$$

$\sin n\pi = 0$
$\sin \pi/2 = 1$
$\sin 0 = 0$

7. By changing the order of integration evaluate

$$\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y} dx dy$$



$$\text{Let } I = \int_0^{\infty} dx \int_x^{\infty} \frac{e^{-y}}{y} dy$$

Integrate with respect to y from x to ∞ and, then with respect to x from 0 to ∞.

Let OA be the straight line OA, Y = X

Region of integration is R above OA

Reverse the order of integration

keep Y constant, X varies from 0 to Y

Then allow y to vary from 0 to ∞ to cover R.

$$= \int_0^{\infty} \frac{e^{-y}}{y} \, dy \, (x)_0^y = \int_0^{\infty} e^{-y} dy$$

$$= (-e^{-y})_0^{\infty}$$

$$= 1$$

Triple Integrals

1. Evaluate $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) \, dx \, dy \, dz$

$$\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) dx \, dy \, dz = \int_0^a \int_0^b \left(\frac{x^3}{3} + xy^2 + xz^2\right]_0^c dy \, dz$$

$$= \int_0^a \int_0^b \left(\frac{c^3}{3} + cy^2 + cz^2\right) dy \, dz.$$

$$= c\int_0^a \int_0^b \left(\frac{c^2}{3} + y^2 + z^2\right) dy \, dz$$

$$= c\int_0^a \left(\frac{c^2}{3} y + \frac{y^3}{3} + z^2 y\right)_0^b dz$$

$$= c\int_0^a \left(\frac{c^2}{3} b + \frac{b^3}{3} + z^2 b\right) dz$$

$$= bc\int_0^a \left(\frac{c^2}{3} + \frac{b^2}{3} + z^2\right) dz$$

$$= bc\left(\frac{c^2}{3} z + \frac{b^2}{3} z + \frac{z^3}{3}\right)_0^a$$

$$= bc\left(\frac{c^2}{3} a + \frac{b^2}{3} a + \frac{a^3}{3}\right)$$

$$= abc\left(\frac{c^2}{3} + \frac{b^2}{3} + \frac{a^2}{3}\right)$$

$$= \frac{abc}{3}(a^2+b^2+c^2)$$

2. Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{(a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{(\sqrt{a^2-x^2-y^2})^2 - z^2}}\, dz \right) dy\, dx \qquad \text{put } a = \sqrt{a^2-x^2-y^2}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^a \frac{dz}{\sqrt{a^2-z^2}}\, dy\, dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{a}\right)\right]_0^a dy\, dx \qquad \int \frac{dz}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (\sin^{-1}(1) - \sin^{-1}(0)]\, dy\, dx$$

$$= \pi/2 \int_0^a \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx$$

$$= \pi/2 \int_0^a (y)_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \pi/2 \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \pi/2 \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \pi/2 \left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0 \right]$$

$$= \pi/2 \left[\frac{a^2}{2}\, \pi/2 \right]$$

$$= \frac{a^2\pi^2}{8}$$

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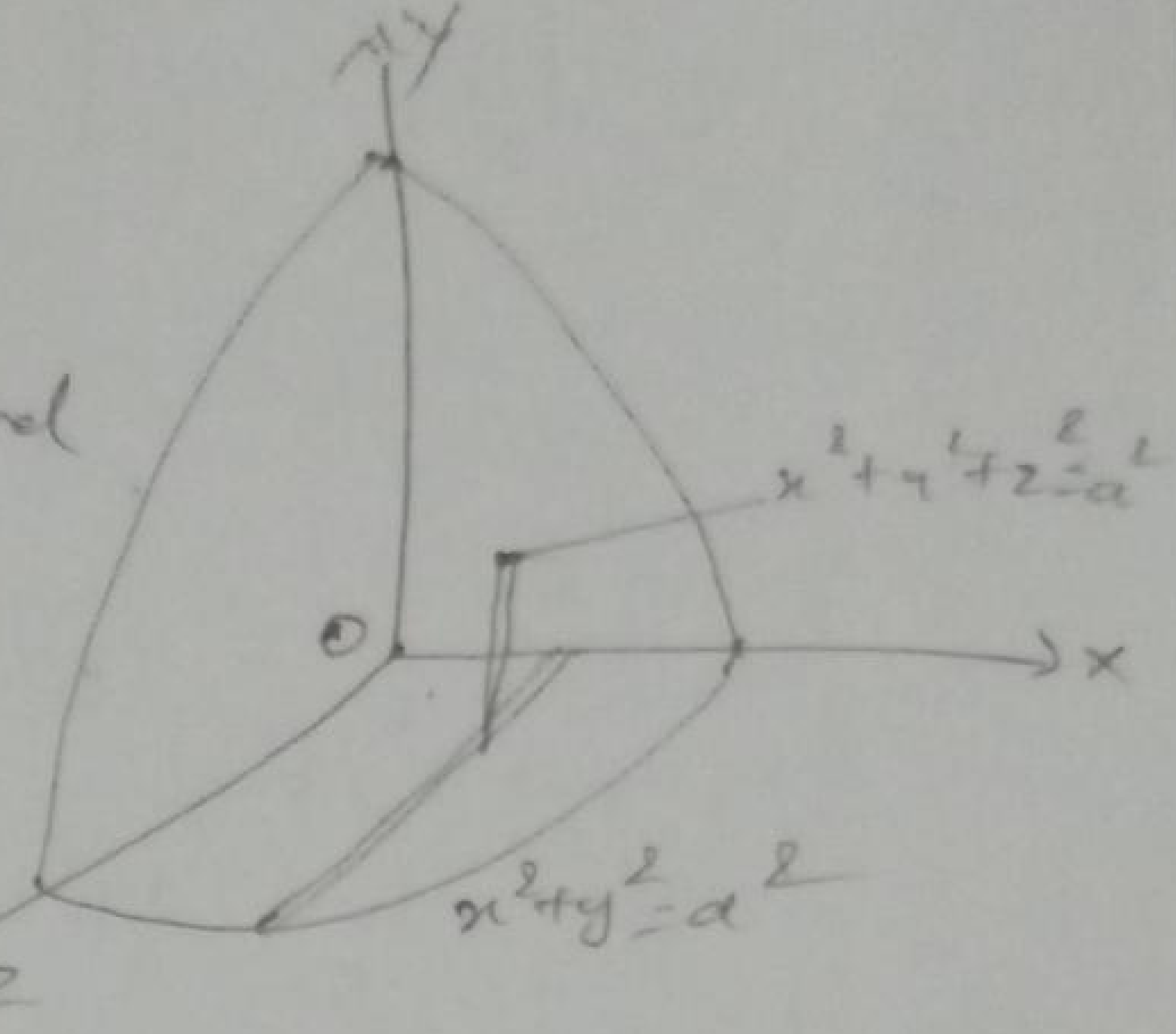
3. Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$

Sol: To cover the whole positive octant of the sphere $x^2+y^2+z^2=a^2$



z varies from 0 to $\sqrt{a^2-x^2-y^2}$

Y varies from 0 to $\sqrt{a^2-x^2}$ and

x varies from 0 to a.

Hence the required integral is

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\, dz\, dy\, dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(xy\frac{z^2}{2} \right)_0^{\sqrt{a^2-x^2-y^2}} dy\, dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\frac{(a^2-x^2-y^2)}{2}\, dy\, dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\, dy\, dx.$$

$$= \frac{1}{2} \int_0^a \left(\frac{xy^2a^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a \left[\frac{x(a^2-x^2)a^2}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4} \right].$$

$$= \frac{1}{2} \left[\frac{a^4x^2}{4} - \frac{x^4a^2}{8} - \frac{x^4a^2}{8} + \frac{x^6}{12} - \frac{x^2a^4}{8} - \frac{x^6}{24} + \frac{a^2x^4}{8} \right]_0^a$$

$$= \frac{a^6}{48}$$

4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ find the volume of the ellipse

Sol: Volume $V = 8\iiint dx\,dy\,dz$

z varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

x varies from 0 to a

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$

$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$\text{volume} = 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dy\,dx.$$

$$= 8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left(c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right) dy\,dx$$

$$= 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2-y^2}\,dy\,dx$$

$$= \frac{8c}{b}\int_0^a\left(\int_0^a\sqrt{a^2-y^2}\,dy\right)dx.$$

put $a=b\sqrt{1-\frac{x^2}{a^2}}$

$$= \frac{8c}{b}\int_0^a\left[\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\left(y/a\right)\right]_0^a dx$$

$$= \frac{8c}{b}\int_0^a\frac{a^2}{2}\sin^{-1}(1)\,dx$$

$$= \frac{8c}{b}\frac{\pi}{4}\int_0^a a^2\,dx.$$

$$= \frac{2\pi \cancel{b} c}{b} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2\pi bc \left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi bc \left[a - \frac{a^{\cancel{3}}}{3a^{\cancel{2}}}\right]$$

$$= 2\pi bc \left[a - \frac{a}{3}\right]$$

$$= 2\pi bc \, \frac{2a}{3}$$

$$= \frac{4\pi abc}{3}$$

5. Evaluate $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx$

Sol: $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx$

$$= \int_1^3 \int_{1/x}^1 xy \left(\frac{z^2}{2}\right)_0^{\sqrt{xy}} dy \, dx$$

$$= \int_1^3 \int_{1/x}^1 xy \left(\frac{xy}{2}\right) dy \, dx$$

$$= 1/2 \int_1^3 x^2 \left(\frac{y^3}{3}\right)_{1/x}^1 dx$$

$$= 1/6 \int_1^3 x^2 \left(1 - 1/x^3\right) dx$$

$$= 1/6 \int_1^3 \left(x^2 - 1/x\right) dx$$

$$= 1/6 \left[\frac{x^3}{3} - \log x\right]_1^3$$

$$= 1/6 \left[9 - \log 3 - 1/3\right]$$

$$= 1/6 \left[\frac{26}{3} - \log 3\right]$$

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6. Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Sol) $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$= \int_0^{\log a}\int_0^{x}\left[e^{x+y+z}\right]_0^{x+y} dy\,dx$

$= \int_0^{\log a}\int_0^{x}\left[e^{2(x+y)} - e^{x+y}\right] dy\,dx$

$= \int_0^{\log a}\left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^{x} dx$

$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - e^{2x}\right) - \left(\frac{e^{2x}}{2} - e^{x}\right) dx$

$= \left[\frac{e^{4x}}{8} - \frac{3}{2}\cdot\frac{e^{2x}}{2} + e^{x}\right]_0^{\log a}$

$= \left(\frac{e^{4\log a}}{8} - \frac{3}{4}e^{2\log a} + e^{\log a}\right) - \left(1/8 - 3/4 + 1\right)$

$= \left(e^{\log a^4} - 3/4\, e^{\log a^2} + a\right) - \left(\frac{1-6+8}{8}\right)$

$= \frac{a^4}{8} - \frac{3}{4}a^2 + a - \frac{3}{8}$

7. $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dx\,dy\,dz$

Sol: $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dx\,dy\,dz = \int_0^1\int_0^{1-x} (e^z)_0^{x+y}\,dx\,dy$

$$= \int_0^1 \int_0^{1-x} (e^{x+y} - 1)\, dy\, dx$$

$$= \int_0^1 (e^{x+y} - y)_0^{1-x}\, dx$$

$$= \int_0^1 [e^{x+1-x} - ((1-x))] - [e^{x}]\, dx$$

$$= \int_0^1 (e^1 - 1 + x - e^{x})\, dx$$

$$= \left(e^{x} - x + \frac{x^2}{2} - e^{x}\right)_0^1$$

$$= e - 1 + 1/2 - e + 1$$

$$= 1/2$$

UNIT – V Beta and Gamma functions

Definitions

1. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for $m>0$, $n>0$ is known as Beta function and is denoted by $\beta(m,n)$

2. $\int_0^\infty x^{n-1} e^{-x}\,dx$ for $n>0$ is known as Gamma function and is denoted by $\Gamma(n)$.

Convergence of Γn.

$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\,dx$ this integral exists if $n>0$.

$\Gamma n = \int_0^1 x^{n-1} e^{-x}\,dx + \int_1^\infty x^{n-1} e^{-x}\,dx$.

The first limit is $\lim_{\epsilon \to 0} \int_\epsilon^1 x^{n-1} e^{-x}\,dx$ if this limit exists.

The second integral certainly exists for $e^x > \frac{x^r}{r!} > \frac{x^{n+1}}{r!}$

Hence $x^{n-1}e^{-x} < \frac{r!}{x^2}$ $\qquad r > n+1$

$\therefore \int_1^\infty e^{-x} x^{n-1}\,dx$ does not exceed a constant multiple of $\int_1^\infty \frac{dx}{x^2}$ which converges.

$\therefore \Gamma n$ converges for $n>0$.

formula:

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$\beta(m,n)$ exists if $m>0$ & $n>0$

Recurrence formula of Gamma function

$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx \quad (n > -1)$

Integrating it by parts, taking $u = x^n$ and $dv = e^{-x} dx$

$$\Gamma(n+1) = \left[-e^{-x} x^n\right]_0^\infty - n\int_0^\infty -e^{-x} x^{n-1} dx$$

$$\lim_{x\to 0} e^{-x} x^n = 0 \text{ if } n > 0$$

$$\lim_{x\to\infty} e^{-x} x^n = \lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

$$\therefore \Gamma(n+1) = n\int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

This recurrence formula is true only when $n > 0$

Cor: $\Gamma(n+1) = n\Gamma(n)$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\ldots\ldots 1\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx$$

$$= \int_0^\infty e^{-x} dx = (-e^{-x})_0^\infty$$

$$\Gamma(1) = 1$$

$$\therefore \Gamma(n+1) = n!$$

Cor: $\Gamma(n+a) = (n+a-1)(n+a-2)\ldots a\,\Gamma(a)$

Properties of Beta functions

1. $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

putting $x = 1-y$ we have

$$\beta(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= \beta(n,m)$$

2. $\beta(m,n)$ can be expressed as a definite integral with $0, \infty$ as limits.

$$\text{In } \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $x = \frac{y}{1+y}$ when $x=0,\ y=0$ when $x=1\ y=\infty$

$$\text{Hence } \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}} \cdot \frac{1}{(1+y)^{n-1}} \cdot \frac{dy}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

(iii) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x\, dx$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $x = \sin^2\theta$ we have

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

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$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

which can be written as $2 I_{2m-1,2n-1}$

$$I_{m,n} = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

<u>Relation between Beta and Gamma functions</u>

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma m = \int_0^{\infty} x^{m-1} e^{-x} dx$$

put $x = t^2$ we have

$$\Gamma(m) = \int_0^{\infty} (t^2)^{m-1} e^{-t^2} 2t\, dt$$

we can take as $2\int_0^{\infty} x^{2m-1} e^{-x^2} dx$

$$\Gamma(n) = 2\int_0^{\infty} y^{2n-1} e^{-y^2} dy$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} x^{2m-1} e^{-x^2} dx \; y^{2n-1} e^{-y^2} dy$$

put $x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

we transform cartesian into polar coordinates

r varies from 0 to ∞.

θ varies from 0 to $\pi/2$

$$\therefore \Gamma(m)\,\Gamma(n) = 4\int_0^{\infty}\int_0^{\pi/2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} e^{-r^2} r\,dr\,d\theta$$

$$= 4\int_0^{\infty}\int_0^{\pi/2} r^{2n+2m-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr\,d\theta$$

$$\int_0^{\alpha} e^{-r^2} r^{2n+2m-1}\, dr = \frac{1}{2}\int_0^{\alpha} t^{m+n-1} e^{-t}\, dt \quad \text{put } t = r^2$$

$$= \tfrac{1}{2}\,\Gamma(m+n)$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = I_{2m-1},\ I_{2n-1}$$

$$= \tfrac{1}{2}\,\beta(m,n)$$

$$\therefore \Gamma(m)\,\Gamma(n) = 4\ \tfrac{1}{2}\,\Gamma(m+n)\ \tfrac{1}{2}\,\beta(m,n)$$

$$= \Gamma(m+n)\ \beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

cor: $\Gamma\tfrac{1}{2} = \sqrt{\pi}$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \text{put } m = n = \tfrac{1}{2}$$

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})}{\Gamma 1}$$

$$\beta(\tfrac{1}{2}\ \tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = \pi$$

$$\Gamma 1 = 1$$

$$\left[\Gamma\tfrac{1}{2}\right]^2 = \pi$$

$$\therefore \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Cor: $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$ is often expressed in the following form.

Sol: putting $2m = p$ and $2n = q$.

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p}{2}, \frac{q}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(q/2)}{\Gamma\left(\frac{p+q}{2}\right)} \quad \text{—(1)}$$

if we put $q=1$ in eqn (1)

$$\int_0^{\pi/2} \sin^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma(p/2)\,\Gamma(1/2)}{\Gamma\left(\frac{p+1}{2}\right)} \quad \text{—(2)}$$

if we put $p=q$ in eqn (1) we get

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\{\Gamma(p/2)\}^2}{\Gamma(p)}$$

$$\frac{1}{2^{p-1}} \int_0^{\pi/2} \sin^{p-1} 2\theta \, d\theta = \frac{\{\Gamma(p/2)\}^2}{2\,\Gamma(p)}$$

put $2\theta = \varphi$ we get

$$\frac{1}{2^{p-1}} \int_0^{\pi} \sin^{p-1}\varphi \, d\varphi = \frac{\{\Gamma(p/2)\}^2}{\Gamma(p)}$$

using (2) we get

$$\frac{\sqrt{\pi}}{2^{p-1}\,\Gamma\left(\frac{p+1}{2}\right)} = \frac{\Gamma(p/2)}{\Gamma(p)}$$

where $\Gamma(p/2)\,\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\,\Gamma(p)$ — (3)

put $p = 2n$ we have

$$\Gamma(n)\,\Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

put $n = 1/4$; $\Gamma(1/4)\,\Gamma(3/4) = \frac{\sqrt{\pi}\,\Gamma(1/2)}{2^{-1/2}}$

$$= \sqrt{2}\,\pi$$

1. Evaluate $\int_0^1 x^m (\log 1/x)^n \, dx$

put $\log(1/x) = t \quad x = e^{-t}$

$dx = -e^{-t}\,dt$

$$\int_0^1 x^m (\log 1/x)^n dx = \int_0^\infty (e^{-t})^m t^n (-e^{-t})\,dt$$

$$= \int_0^\infty e^{-(m+1)t}\, t^n \, dt$$

put $(m+1)t = y \quad dt = 1/_{m+1}\,dy$

Then the integral on this sustitution becomes

$$\int_0^\infty \frac{e^{-y} y^n}{(m+n)^n} \frac{1}{m+n}\,dy = \frac{1}{(m+1)^{n+1}}\,\Gamma(n+1)$$

2. $\int_0^\infty e^{-x^2} dx \quad$ put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{1}{2\sqrt{t}}\,dt.$

$$\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}}\,dt$$

$$= 1/2 \int_0^\infty t^{-1/2} e^{-t}\,dt$$

$= \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{2}$

3. Evaluate $\int_0^1 x^7 (1-x)^8 \, dx$

$$\int_0^1 x^7 (1-x)^8 \, dx = \beta(8, 9)$$

$$= \frac{\Gamma(8)\,\Gamma(9)}{\Gamma(17)} = \frac{7!\; 8!}{16!}$$

4. $$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2}\,\beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{2}\,\beta(4, 3)$$

$$= \frac{1}{2}\,\frac{\Gamma(4)\;\Gamma(3)}{\Gamma(7)}$$

$$= \frac{1}{2}\,\frac{3!\; 2!}{6!}$$

$$= \frac{1}{120}$$

5. $$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2}\,\frac{\Gamma\left(\frac{11}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{10}{2}+\frac{1}{2}\right)}$$

$n = 9/2$

$n+1 = 11/2$

$$= \frac{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 6}$$

$$= \frac{9\cdot 7\cdot 5\cdot 3\,(\sqrt{\pi})^2}{5!\; 2^6} = \frac{63\pi}{512}$$

6. $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$

$$= \frac{1}{2} \frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(3/4+1/4)}$$

$$= \frac{1}{2}\Gamma(3/4)\,\Gamma(1/4)$$

$$= \frac{1}{2}\Gamma(1-1/4)\,\Gamma(1/4)$$

$$= \frac{\pi}{2\sin\pi/4}$$

$$= \frac{\pi}{\sqrt{2}}$$

7. Express $\int_0^1 x^m (1-x^n)^p\, dx$ in terms of Gamma functions and evaluate the integral $\int_0^1 x^5 (1-x^3)^{10}\, dx$.

Sol. put $x^n = y$ then $nx^{n-1}dx = dy$

$$\therefore \int_0^1 x^m (1-x^n)^p\, dx = \int_0^1 y^{m/n} (1-y)^p \frac{dy}{n \cdot y^{\frac{n-1}{n}}}$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}} (1-y)^p\, dy$$

$$= \frac{1}{n}\beta\left(\frac{m-n+1}{n}+1,\ p+1\right)$$

$$= \frac{1}{n}\beta\left(\frac{m+1}{n},\ p+1\right)$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

$$\int_0^1 x^5 (1-x^3)^{10}\, dx = \frac{1}{3}\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)}$$

$$= \frac{1}{3} \frac{\Gamma(2)\,\Gamma(7/1)}{\Gamma(11/3)}$$

$$= \frac{1}{396}$$

8. prove that $\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}$

Put $\tan\theta = t$ the integral reduces to

$$\int_0^{\infty} \frac{t^{2n-1}\,dt}{(a+bt^2)^{m+n}}$$

put $\sqrt{b}\,t = \sqrt{a y}$ the above integral reduces to

$$\frac{1}{2a^m b^n} \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}.$$