SEMESTER : I & II
ALLIED COURSE : II - Mathematics

Inst Hour	: 2 +(3)
Credit	: 3
Code	: 18K2CH/PAM2

ALGEBRA, ANALYTICAL GEOMETRY (3D) AND TRIGONOMETRY

(For B.Sc., Physics & Chemistry Major)

UNIT 1:

Binomial – Application of Binomial Theorem – Approximation and Limits, Exponential and Logarithmic series (Formulae Only) – Summation & Approximation related problems only.
(Chapter2 Sec: 1, 2, 3, Chapter3 Sec1-5of Text Book 1)

UNIT 2:

Eigen Values, Eigen Vectors – Cayley Hamilton Theorem – An Important Application of Cayley Hamilton Theorem.
(Chapter2 Sec: 16, 16.3, 16.4 of Text Book 2)

UNIT 3:

Finding the Shortest distance between two skew lines and the equation of the plane containing them– Tangent plane – Plane section of a sphere – The Intersection of Plane and Sphere is a Circle.
(Chapter3 Sec: 7, Chapter4 Sec 4-6Text Book 4)

UNIT 4 :

Expansion of $\sin n\theta$, $\cos n\theta$ (n being a positive integer) - Expansion of $\sin^n \theta$, $\cos^n\theta$ in a series of sines & cosines of multiples of θ (θ - given in radians) – Expansion of $Sin\theta, Cos\theta$ in ascending *powers of θ - simple problems.*
(Chapter 3 Sec: 1, 4, 4.1,5 of Text Book 3)

UNIT 5 :

Euler's formula for $e^{i\theta}$ - Definition of Hyperbolic functions –Formulae involving Hyperbolic functions -Relation between Hyperbolic & circular functions – Expansion of sinh x , cosh x , tanh x in powers of x- Expansion of Inverse hyperbolic functions $\sinh^{-1}x$,$\cosh^{-1}x$ and $\tanh^{-1}x$.
(Chapter 4 Sec: 1, 2, 2.1-2.3 of Text Book 3)

Text Books :

[1] S .Narayanan, T.K.Manicavachagom Pillai and others, Ancillary Mathematics BookI
[2] T.K.Manicavachagom Pillai & others,Algebra Volume II
[3] S.Narayanan &T.K. Manicavachagom Pillai, Trigonometry
[4] Narayanan and Hanumantharao, Ancillary Mathematics Book4

Reference Books:

[1] A.Singaravelu,Allied Mathematics Paper II, 1998
[2] P.Duraipandian, Laxmi Duraipandian & Muhilan, Analytical Geometry of 3D –Emerald Publication

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

9.3.18

Unit-II

Matrix

A matrix is defined as be a rectangular array of numbers arranged into rows and columns. It is written as follows:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Types of matrices

i) A row matrix is a matrix with only one row

eg $[2, 1, 3]$

ii) A column matrix is a matrix with only one column. eg $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

iii) A square matrix is one in which the number of rows is equal to the number of columns.

iv) A square matrix whose elements above the principal diagonal (or below the principal diagonal) are all zero is called a triangular matrix.

eg

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

v) Diagonal matrix is a square matrix of any order with zero elements everywhere except on the main diagonal

eg $\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

vi) Scalar matrix is a diagonal matrix in which all the elements along the main diagonal are equal.

eg $\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$

vii) Unit matrix is a Scalar matrix in which all the elements along the main diagonal are unity.

eg I_2 is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, I_3 is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

vii) Null or zero matrix: If all the elements in a matrix are zeros, it is called a null or zero matrix and is denoteg by O.

eg (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, ii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viii) If rows and columns are interchanged in a matrix A, we obtain a Second matrix that is called the transpose of the original matrix and is denoted by A^t (or) A'.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ then $A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$

ix) If A is a square matrix and A is the transpose of its conjugate such a matrix is called a Hermitian matrix
If A is Hermitian matrix, then $A = (\bar{A})^t$ (or) $A^t = \bar{A}$.

x) If A is a square matrix and if A is the negative of the transpose of its conjugate such a matrix is called the skew Hermitian matrix

xi) If A is a square matrix
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
then the determinant
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
is called the determinant of the matrix A and it is denoted by $|A|$ or det A. We have determinant only for square matrix.

xii) The sum of the diagonal elements of a square matrix is called the trace of a matrix.

xiii) If the $|A| = 0$, then the matrix is called a singular matrix. If $|A| \neq 0$ then the matrix is called a non-singular matrix.

xiv) A square matrix A (with real elements) is said to be <u>orthogonal</u> if

$$AA' = A'A = I$$

ie, a matrix is orthogonal if $A^{-1} = A'$

xv) <u>Unitary matrices</u>: A square matrix A is said to be unitary if $(\bar{A})'A = A(\bar{A})' = I$

ie., $(\bar{A})' = A^{-1}$

xvi) <u>Rank of a matrix</u>:

The minor of a given matrix A is the determinant composed of elements of the matrix left after striking out certain rows and columns.

<u>Eigenvalues and Eigenvectors</u>

Suppose we have a vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where

$x_1^2 + x_2^2 + x_3^2 \neq 0$

If we are transforming the vector X by means of a matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$, we get a

vector $Y = AX$. ——— (1)

Parallel to the vector X, we have

$Y = \lambda X$ (λ is scalar) ——— (2)

From (1) & (2), we have

$\boxed{AX = \lambda X}$ ——— (3)

then the vector X is termed the eigen vector of matrix A or the eigen vector of the given linear transformation; the scalar λ is termed the eigenvalue associated with this eigenvector. Other names for eigenvectors are characteristic vectors, proper vectors and latent vectors; eigenvalues are also known as characteristic values, proper values and latent roots.

Computing eigenvalues and eigenvectors

Equation (3) can also be written as

$$(A-\lambda I)X = 0 \qquad \text{—————} \quad (4)$$

where I is the unit matrix, the order is the same as that of A.

In (4) if we write $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{12} & \cdots & a_{nn} \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ (order n)

and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

we get $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{m} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\left.\begin{aligned}(a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0\\ \cdots\cdots\\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0\end{aligned}\right\} \quad \text{--- (5)}$$

Equation (5) is a system of n simultaneous linear algebraic equations for the n unknowns $x_1, x_2, \ldots, x_n$. The necessary and sufficient condition for the existence of a non-trivial solution to such a system of homogeneous equations is the vanishing of the determinant of the co-efficients.

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{12} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \text{--- (6)}$$

Equation (6) is called the characteristic equation

Eg:

1) Find the eigen values and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Solution:-

Given, $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I| = 0$ --- (7)

$$A-\lambda I = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix}$$

$$① \Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - (1)(3) = 0$$

$$8 - 4\lambda - 2\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0 \quad \text{———} ②$$

$+5$

$-1 \quad -5$

factorising this equation, we get

$$(\lambda-1)(\lambda-5) = 0$$

$$\therefore \lambda = 1 \text{ or } 5.$$

$\therefore$ The eigenvalues are 1 and 5

To find eigenvectors:

When $\lambda = 1$, the equation

$(A - \lambda I)X = 0$ becomes

$$\begin{bmatrix} 4-1 & 1 \\ 3 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0 \quad \text{———} ③$$

$$3x_1 + x_2 = 0 \quad \text{———} ④$$

$$x_2 = -3x_1$$

Whatever be the value of x_1, the value of x_2 is -3 times of it.

Therefore the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} k \\ -3k \end{bmatrix}$ where k is a constant.

put $k=1$, we get $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and is the simplest eigenvector.

When $\lambda=5$, the equation $(A-\lambda I)X=0$ becomes

$$\begin{bmatrix} 4-5 & 1 \\ 3 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1+x_2=0 \Rightarrow \boxed{x_1=x_2}$$

$$3x_1-3x_2=0$$

$$\Rightarrow 3x_1=3x_2 \Rightarrow \boxed{x_1=x_2}$$

Here x_1 and x_2 takes the same value. Put $x_2=k$ then $x_1=k$. So we get different eigenvectors for different values of k.

The general eigenvector is $\begin{bmatrix} k \\ k \end{bmatrix}$

The simplest eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda=1$ the eigen vector is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and

for $\lambda=5$ the eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2) Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution:-

Given, $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$.

To find the eigenvalues :-

$$|A-\lambda I| = 0 \quad \text{——— (1)}$$

$$A-\lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$(1) \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right] = 0$$

$$(1-\lambda)\left[1-\lambda-\lambda+\lambda^2-1\right] = 0$$

$$(1-\lambda)\left[\lambda^2-2\lambda\right] = 0$$

$$(1-\lambda)\lambda(\lambda-2) = 0$$

$$\lambda=1;\ \lambda=0;\ \lambda=2$$

Therefore the eigenvalues are $\lambda = 0, 1, 2$.

To find eigenvectors :-

The eigenvector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution of the equation $(A-\lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (1-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{——— (2)}$$

By using ② we can find eigenvectors corresponding to each eigenvalue.

Case-i:-

When $\lambda = 0$, the equation set ② becomes

$$x_1 + 0x_2 + 0x_3 = 0 \Rightarrow x_1 = 0$$

$$0x_1 + x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$$

$$0x_1 + x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$$

Putting $x_3 = k$, weget $x_1 = 0$, $x_2 = -k$, $x_3 = k$

The general eigenvector is $X = \begin{bmatrix} 0 \\ -k \\ k \end{bmatrix}$

The Simplest eigenvector is $X = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Case-ii:-

When $\lambda = 1$, the equation set ② becomes

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = 0}$$

$$0x_1 + x_2 + 0x_3 = 0 \Rightarrow \boxed{x_2 = 0}$$

Substitituting $x_2 = 0 = x_3$ in $0x_1 + 0x_2 + 0x_3 = 0$,
weget $0 \cdot x_1 = 0 \Rightarrow x_1$ may take any value (k).

The general eigenvector is $X = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

The Simplest eigenvector is $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case iii:-

When $\lambda = 2$, the equation set ② becomes

$$-1 \cdot x_1 + 0x_2 + 0x_3 = 0 \Rightarrow -x_1 = 0 \Rightarrow \boxed{x_1 = 0}$$

$$0x_1 - x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = x_3}$$

$$0x_1 + x_2 - x_3 = 0 \Rightarrow \boxed{x_2 = x_3}$$

Thus x_2 and x_3 takes same values (say k) and $x_1 = 0$.

The general eigenvector is $X = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$

The simplest eigenvector is $X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$\therefore$ The eigenvector corresponding to $\lambda = 0$ is $(0,-1,1)$, $\lambda = 1$ is $(1,0,0)$ and $\lambda = 2$ is $(0,1,1)$.

3) Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$$

Solution:-

Given, $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)(-3-\lambda) - 2] - 2[2(-3-\lambda) + 7] = 0$$

$$(2-\lambda)[-3 - \lambda + 3\lambda + \lambda^2 - 2] - 2[-6 - 2\lambda + 7] = 0$$

$$(2-\lambda)[\lambda^2 + 2\lambda - 5] - 2[1 - 2\lambda] = 0$$

$$2\lambda^2 + 4\lambda - 10 - \lambda^3 + 5\lambda - 2\lambda^2 - 2 + 4\lambda = 0$$

$$-\lambda^3 + 13\lambda - 12 = 0$$

$\boxed{\lambda^3 - 13\lambda + 12 = 0}$ we have to solve this eqn/..

$$(\lambda-1)(\lambda^2+\lambda-12) = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -13 & 12 \\ & 0 & \underset{(+)}{\overset{-}{}}1 & \underset{(+)}{\overset{-}{}}1 & \underset{(-)}{+12} \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda+4)(\lambda-3) = 0$$

$\lambda = 1$; $\lambda = -4$; $\lambda = 3$

-12: 4, -3

The eigenvalues are $1, 3, -4$.

<u>To find eigenvectors:-</u>

The eigenvector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution of the equation $(A-\lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + 0x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 + x_3 &= 0 \\ -7x_1 + 2x_2 - (3+\lambda)x_3 &= 0 \end{aligned}\right\} \text{——— (1)}$$

Substituting the eigenvalues to (1), we can find the corresponding eigenvector.

<u>case i:-</u>

When $\lambda = 1$, the equation set (1) becomes

$$x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 0x_2 + x_3 = 0$$

$$-7x_1 + 2x_2 - 4x_3 = 0$$

Since these equations are linearly dependent, it is enough to consider any two equations, Let us consider the first two equations and using cross rule method, we get

$$x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 + 0x_2 + x_3 = 0.$$

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}} = k$$

$$\frac{x_1}{2-0} = \frac{x_2}{-(1-0)} = \frac{x_3}{0-4} = k$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4} = k$$

Thus $x_1 = 2k$; $x_2 = -k$; $x_3 = -4k$.

The general eigenvector is $X = \begin{bmatrix} 2k \\ -k \\ -4k \end{bmatrix}$

The simplest eigenvector is $X = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

Case ii:-

Put $\lambda = 3$, the equation ① becomes

$$-x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 - 2x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 - 6x_3 = 0$$

These equations are linearly dependent, so we can take any two of them.

Let us consider the first two equations and using cross rule method, we find the values of x_1, x_2, x_3.

$$-x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 - 2x_2 + x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}} = k$$

$$\frac{x_1}{2-0} = \frac{x_2}{-(-1+0)} = \frac{x_3}{2-4} = k$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k$$

Thus $x_1 = 2k$; $x_2 = k$; $x_3 = -2k$

The general eigenvector is $X = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix}$

The simplest eigenvector is $X = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case iii :-

Put $\lambda = -4$, the equation set ① becomes,

$$6x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 + x_3 = 0.$$

The equations are linearly dependent. Therefore we consider any two of them,

$$6x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0$$

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} 6 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix}} = k.$$

$$\frac{x_1}{2-0} = \frac{x_2}{-(6-0)} = \frac{x_3}{(30-4)} = k$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26} = k$$

Thus, $x_1 = 2k$; $x_2 = -6k$; $x_3 = 26k$.

The general eigenvector is $X = \begin{bmatrix} 2k \\ -6k \\ 26k \end{bmatrix}$

The simplest eigenvector is $X = \begin{bmatrix} 2 \\ -6 \\ 26 \end{bmatrix}$

$\therefore$ The eigenvector corresponding to $\lambda = 1$ is $(2, -1, -4)$, $\lambda = 3$ is $(2, 1, -2)$ and $\lambda = -4$ is $(2, -6, 26)$.

4) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution

Given $A = \begin{bmatrix} -2 & +2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$.

$$A - \lambda I = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left((1-\lambda)(-\lambda) - (12)\right) - 2(-2\lambda - 6) - 3\left[-4 + (1-\lambda)\right] = 0$$

$$(-2-\lambda)\left[-\lambda + \lambda^2 - 12\right] + 4\lambda + 12 - 3(-3 - \lambda) = 0$$

$$+2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 9 + 3\lambda = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\boxed{\lambda^3 + \lambda^2 - 21\lambda - 45 = 0} \quad \text{———} \ (1)$$

We get the eigenvalues by Solving equation ①

$$(\lambda+3)(\lambda^2-2\lambda-15)=0$$

$$(\lambda+3)(\lambda-5)(\lambda+3)=0.$$

$$\lambda=-3\,;\quad \lambda=+5\,;\quad \lambda=-3$$

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -21 & -45 \\ & 0 & \overset{(-)}{3} & \overset{(+)}{-6} & \overset{(+)}{-45} \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{-15} \\ -5 & 3 \end{array}$$

∴ The eigenvalues are $-3, -3, 5$.

<u>To find the eigen vectors:-</u>

The eigenvector $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution of the equation $(A-\lambda I)X=0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (-2-\lambda)x_1+2x_2-3x_3&=0 \\ 2x_1+(1-\lambda)x_2-6x_3&=0 \\ -x_1-2x_2-\lambda x_3&=0 \end{aligned}\right\} \text{ — ②}$$

<u>Case (i)</u>

Put $\lambda=-3$, in ②, we get

$$x_1+2x_2-3x_3=0$$

$$2x_1+4x_2-6x_3=0 \Rightarrow x_1+2x_2-3x_3=0$$

$$-x_1-2x_2+3x_3=0 \Rightarrow x_1+2x_2-3x_3=0$$

The above equations are equivalent to one equation

$$x_1+2x_2-3x_3=0$$

Hence, giving arbitrary values to any two variables we get the third variable in terms of other two.

Let us give $x_2=k_1$ and $x_3=k_2$ then $x_1=3k_2-2k_1$

The general eigenvector is $x = \begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix}$

Putting $k_1 = 0, k_2 = 1$, the eigen vector is $x = \begin{bmatrix} 3-0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Putting $k_1 = 1, k_2 = 0$, the eigen vector is $x = \begin{bmatrix} 0-2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

There are two eigenvectors corresponding to $\lambda = -3$ are $(3, 0, 1)$ and $(-2, 1, 0)$.

Case (ii):-

Put $\lambda = 5$ in ②, we get

$$-7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 - 5x_3 = 0.$$

To find the values of x_1, x_2, x_3, Let us consider any two equations and use cross rule method.

Consider first two equations $-7x_1 + 2x_2 - 3x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & +2 \\ +2 & -4 \end{vmatrix}} = k$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-(42+6)} = \frac{x_3}{28-4} = k$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24} = k$$

Thus $x_1 = -24k$; $x_2 = -48k$; $x_3 = 24k$.

The general eigen vector is $x = (-24k, -48k, 24k)$

The simplest eigen vector is $x = (+1, 2, -1)$

Properties of eigenvalues:-

1. Sum of eigenvalues is equal to the sum diagonal elements.
2. Product of eigenvalues is equal to it determinant value.

Properties of eigenvectors:-

1. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues corresponding to a square matrix A of order n, and $X_1, X_2, \ldots, X_n$ be the corresponding eigenvectors. Then $X_1, X_2, \ldots, X_n$ are linearly independent:, that is eigen vectors corresponding to distinct eigenvalues are linearly independent.

2. If X is a characteristic vector of a matrix A corresponding to a characteristic root λ; then KX for every non-zero scalar K, is also a characteristic vector of A corresponding to λ.

3. A eigen vector X of a matrix A cannot correspond to more than one eigenroot of A.

Diagonalisation of a matrix:-

A square matrix A of order n with n linearly independent eigenvectors can be diagonalised by a similarly transformation $D = P^{-1}AP$, where P is the matrix whose columns are eigenvectors of A.

Note:-

"To diagonalise the matrix, the eigenvalues of the given matrix should be different."

The process of finding P such that $P^{-1}AP = D$ is called diagonalising the matrix A.

Calculation of the powers of a matrix:-

Let A be the given matrix of order 3.

We know that $D = B^{-1}AB$

$$D^2 = (B^{-1}AB)^2$$

$$= B^{-1}ABB^{-1}AB$$

$$= B^{-1}AIAB \qquad [\because BB^{-1} = I]$$

$$= B^{-1}AAB$$

$$D^2 = B^{-1}A^2B$$

$$D^3 = D^2 \cdot D$$

$$= (B^{-1}A^2B)(B^{-1}AB)$$

$$= B^{-1}A^2IAB$$

$$D^3 = B^{-1}A^3B$$

In general $D^n = B^{-1}A^nB$ [n is a positive integer]

Now $BD^nB^{-1} = BB^{-1}A^nBB^{-1}$

$$= A^n$$

$$\therefore A^n = BD^nB^{-1}$$

$$A^n = B\begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}B^{-1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the given matrix A.

Cayley-Hamilton Theorem:-

Statement:-

Every matrix satisfies its own characteristic equation.

Proof Let A be a matrix of order n.

The matrix $[A-\lambda I]$ is $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$

Let $|A-\lambda I|$ be $a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots + a_n\lambda^n$

we have $\{adj[A-\lambda I]\}[A-\lambda I] = |A-\lambda I| I$ since

$[(adj A) A = |A| I]$

Hence $adj[A-\lambda I]$ is of the form

$B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$

where $B_0, B_1, B_2, \ldots B_{n-1}$ are matrices.

$\therefore (B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1})[A-\lambda I]$

$= (a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n) I$

Equating the same powers of λ on both sides, we get

$$B_0 A = a_0 I$$
$$B_1 A - B_0 = a_1 I$$
$$B_2 A - B_1 = a_2 I$$
$$\vdots$$
$$B_{n-1} A - B_{n-2} = a_{n-1} I$$
$$-B_{n-1} = a_n I$$

Multiplying these equations successively by $I, A, A^2, \ldots, A^{n-1}, A^n$ and adding we get

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Hence A satisfies its characteristic equation.

*) An important application of the cayley-Hamilton theorem is to express the inverse of a matrix in terms of powers of A.

Method of finding inverse of a matrix

Let A be a non-singular matrix ie, $|A| \neq 0$

From the cayley-Hamilton theorem, we have

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0 \quad \text{--- (1)}$$

Premultiplying equation (1) by A^{-1}, we get

$$a_0 A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_n A^{-1} = 0 \qquad [\because A^{-1} I = A^{-1}]$$

$$a_n A^{-1} = -(a_0 A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-2} A + a_{n-1} I)$$

$$A^{-1} = \frac{-1}{a_n}(a_0 A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I).$$

Examples:

1) Diagonalise the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence find A^4

Solution:

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[(1-\lambda) - 3] + 3[1 - 3(5-\lambda)] = 0$$

$$(1-\lambda)[5 - 5\lambda - \lambda + \lambda^2 - 1] - [1 - \lambda - 3] + 3[1 - 15 + 3\lambda] = 0$$

$$(1-\lambda)(\lambda^2 - 6\lambda + 4) + 2 + \lambda - 42 + 9\lambda = 0$$

$$\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + 2 + \lambda - 42 + 9\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0 \Rightarrow \boxed{\lambda^3 - 7\lambda^2 + 36 = 0}$$

By solving this equation we get the eigenvalues.

$$(\lambda+2)(\lambda^2-9\lambda+18)=0$$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$(\lambda+2)(\lambda-3)(\lambda-6)=0$$

$$18 \to -3,\ -6$$

$$\lambda=-2;\ \lambda=3;\ \lambda=6$$

The eigenvalues are $-2, 3, 6$.

To find the eigenvectors substitute the values of λ in the equation $(A-\lambda I)X=0$.

Putting $\lambda=-2$, we get

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1+x_2+3x_3=0$$
$$x_1+7x_2+x_3=0$$
$$3x_1+x_2+3x_3=0$$

Consider the first two equations and use cross rule method.

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}}=\frac{x_2}{-\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}=k$$

$$\frac{x_1}{1-21}=\frac{x_2}{-(3-3)}=\frac{x_3}{21-1}=k$$

$$\frac{x_1}{-20}=\frac{x_2}{0}=\frac{x_3}{20}=k$$

$$x_1=-20k;\ x_2=0;\ x_3=20k.$$

The general eigenvector is $X=(-20k, 0, 20k)$.

The simplest eigenvector is $X=(-1, 0, 1)$ —— (1)

Putting $\lambda = 3$, we get

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0$$

Consider the first two equations to find x_1, x_2, x_3,

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}} = k$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} = k$$

The general eigenvector is $X = (-5k, 5k, -5k)$

The simplest eigenvector is $X = (1, -1, 1)$ —— (2)

Putting $\lambda = 6$, we get

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$

Consider the first two equations to find x_1, x_2, x_3,

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}} = k$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} = k$$

The general eigenvector is $X = (4k, 8k, 4k)$

The simplest eigenvector is $X = (1, 2, 1)$ —— (3)

From 1, 2 & 3, we get the matrix

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

whose columns are the eigenvectors of the matrix A.

<u>To find B^{-1}</u>

$$|B| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -1(-1-2)-1(0-2)+1(0+1)$$

$$= -1(-3)-1(-2)+1(1)$$

$$= 3+2+1$$

$$\boxed{|B| = 6}$$

Since $|B| \neq 0$ B^{-1} exists.

$$B^{-1} = \frac{1}{|B|}(\text{adj } B).$$

$$\text{adj } B = [\text{cofactor matrix } B]^T$$

$$= \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}^T$$

$$\text{adj } B = \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \quad \text{——— } (4)$$

Let D be the diagonal matrix of A, then

$$\boxed{D = B^{-1}AB} \quad \text{——— } (5)$$

Now $AB = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix}$$

(5) $\Rightarrow$ $D = B^{-1}AB = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix}$

$$= \begin{bmatrix} -1+0-1 & \frac{-3}{2}+0+\frac{3}{2} & \frac{-6}{2}+0+\frac{6}{2} \\ \frac{2}{3}+0-\frac{2}{3} & 1+1+1 & \frac{6}{3}-\frac{12}{3}+\frac{6}{3} \\ \frac{2}{6}+0-\frac{2}{6} & \frac{3}{6}-\frac{3}{3}+\frac{3}{6} & 1+\frac{12}{3}+1 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

whose elements in the leading diagonal are the eigenvalues of the given matrix A.

<u>To find A^4:-</u>

$$B^{-1}A^4B = D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & (3)^4 & 0 \\ 0 & 0 & (6)^4 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$$

$$BB^{-1}A^4BB^{-1} = BD^4B^{-1}$$

$$A^4 = BD^4B^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -8+0+0 & 0+0+0 & 8+0+0 \\ 0+27+0 & 0+(-27)+0 & 0+27+0 \\ 0+0+216 & 0+0+512 & 0+0+216 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix}$$

$$= \begin{bmatrix} 8+27+216 & 0-27+512 & -8+27+216 \\ 0-27+512 & 0+27+1024 & 0-27+512 \\ -8+27+216 & 0-27+512 & 8+27+216 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}$$

2) Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Hence find its inverse.

<u>Solution</u>

Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((1-\lambda)^2 - 1) + 3(-2-(1-\lambda)) = 0$$

$$(1-\lambda)(1-2\lambda+\lambda^2-1)+3(-2-1+\lambda)=0$$

$$(1-\lambda)(\lambda^2-2\lambda)+3(\lambda-3)=0$$

$$\lambda^2-2\lambda-\lambda^3+2\lambda^2+3\lambda-9=0$$

$$-\lambda^3+3\lambda^2+\lambda-9=0$$

$$\lambda^3-3\lambda^2-\lambda+9=0. \text{———} (1)$$

To verify cayley-Hamilton Theorem, we claim that

$$A^3-3A^2-A+9I=0 \text{———} (2)$$

$$A^2=\begin{bmatrix}1&0&3\\2&1&-1\\1&-1&1\end{bmatrix}\begin{bmatrix}1&0&3\\2&1&-1\\1&-1&1\end{bmatrix}=\begin{bmatrix}1+0+3&0+0-3&3+0+3\\2+2+(-1)&0+1+1&6-1-1\\1-2+1&0+(-1)-1&3+1+1\end{bmatrix}$$

$$A^2=\begin{bmatrix}4&-3&6\\3&2&4\\0&-2&5\end{bmatrix} \text{———} (3)$$

$$A^3=A^2A=\begin{bmatrix}4&-3&6\\3&2&4\\0&-2&5\end{bmatrix}\begin{bmatrix}1&0&3\\2&1&-1\\1&-1&1\end{bmatrix}=\begin{bmatrix}4-6+6&0+(-3)-6&12+3+6\\3+4+4&0+2-4&9-2+4\\0-4+5&0-2-5&0+2+5\end{bmatrix}$$

$$A^3=\begin{bmatrix}4&-9&21\\11&-2&11\\1&-7&7\end{bmatrix} \text{———} (4)$$

Substituting A, A^2 & A^3 in (2), we get

$$=\begin{bmatrix}4&-9&21\\11&-2&11\\1&-7&7\end{bmatrix}-3\begin{bmatrix}4&-3&6\\3&2&4\\0&-2&5\end{bmatrix}-\begin{bmatrix}1&0&3\\2&1&-1\\1&-1&1\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}4&-9&21\\11&-2&11\\1&-7&7\end{bmatrix}+\begin{bmatrix}-12&9&-18\\-9&-6&-12\\0&6&-15\end{bmatrix}+\begin{bmatrix}-1&0&-3\\-2&-1&1\\-1&1&-1\end{bmatrix}+\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence cayley-Hamilton theorem is verifies

To find A^{-1}

We have $A^3 - 3A^2 - A + 9I = 0$

Premultiplying by A^{-1}, we get

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$9A^{-1} = -A^2 + 3A + I$$

$$A^{-1} = \frac{+1}{9}(-A^2 + 3A + I)$$

$$A^{-1} = \frac{-1}{9}(A^2 - 3A - I)$$

$$A^2 - 3A - I = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^2 - 3A - I = \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & \frac{-7}{9} \\ \frac{1}{3} & \frac{-1}{9} & \frac{-1}{9} \end{bmatrix}.$$

3) Verify Cayley-Hamilton theorem and hence find inverse for $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

<u>Solution</u>

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 3[4(1-\lambda) - 3] + 7[8 - (2-\lambda)] = 0$$

$$(1-\lambda)[2 - 2\lambda - \lambda + \lambda^2 - 6] - 3[4 - 4\lambda - 3] + 7[8 - 2 + \lambda] = 0$$

$$(1-\lambda)[\lambda^2 - 3\lambda - 4] - 3[-4\lambda + 1] + 7[\lambda + 6] = 0$$

$$\lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda + 12\lambda - 3 + 7\lambda + 42 = 0$$

$$-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$$

$$\boxed{\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0} \quad \text{——— } (1)$$

To verify Cayley-Hamilton theorem we claim that

$$A^3 - 4A^2 - 20A - 35I = 0 \quad \text{——— } (2)$$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \quad \text{——— } (3)$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+94+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} \quad \text{--- (4)}$$

$\therefore A^3 - 4A^2 - 20A - 35I \Rightarrow$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} + \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley - Hamilton theorem is verified.

<u>To find A^{-1}</u>

We have that $A^3 - 4A^2 - 20A - 35 = 0$

Premultiplying by A^{-1}, we get

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$35A^{-1} = A^2 - 4A - 20I$$

$$A^{-1} = \frac{1}{35}\left(A^2 - 4A - 20I\right) \quad \text{--- (5)}$$

$$A^2-4A-20=\begin{bmatrix}20 & 23 & 23\\ 15 & 22 & 37\\ 10 & 9 & 14\end{bmatrix}+\begin{bmatrix}-4 & -12 & -28\\ -16 & -8 & -12\\ -4 & -8 & -4\end{bmatrix}+\begin{bmatrix}-20 & 0 & 0\\ 0 & -20 & 0\\ 0 & 0 & -20\end{bmatrix}$$

$$A^2-4A-20=\begin{bmatrix}-4 & 11 & -5\\ -1 & -6 & 25\\ 6 & 1 & -10\end{bmatrix}$$

$$A^{-1}=\frac{1}{35}\begin{bmatrix}-4 & 11 & 5\\ -1 & -6 & 25\\ 6 & 1 & -10\end{bmatrix}$$

4) If $A=\begin{bmatrix}4 & 2\\ 3 & 3\end{bmatrix}$ determine A^n in terms of A

Solution:-

Given, $A=\begin{bmatrix}4 & 2\\ 3 & 3\end{bmatrix}$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}4-\lambda & 2\\ 3 & 3-\lambda\end{vmatrix}=0$$

$$(4-\lambda)(3-\lambda)-6=0$$

$$12-4\lambda-3\lambda+\lambda^2-6=0$$

$$\lambda^2-7\lambda+6=0 \quad \text{——— (1)}$$

Substituting the matrix A in place of λ in (1), we get

$$A^2-7A+6=0. \quad \text{——— (2)}$$

$$A^2=\begin{bmatrix}4 & 2\\ 3 & 3\end{bmatrix}\begin{bmatrix}4 & 2\\ 3 & 3\end{bmatrix}=\begin{bmatrix}16+6 & 8+6\\ 12+9 & 6+9\end{bmatrix}=\begin{bmatrix}22 & 14\\ 21 & 15\end{bmatrix}$$

$$② \Rightarrow \begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix} - 7\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 22 & 14 \\ 21 & 15 \end{bmatrix} + \begin{bmatrix} -28 & -14 \\ -21 & -21 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence A satisfies the equation ①

Let $\lambda^n = f(\lambda)(\lambda^2 - 7\lambda + 6) + p\lambda + q$ — ③

when $\lambda = 1$ or 6, $\lambda^2 - 7\lambda + 6 = 0$.

putting $\lambda = 1$ in ③ $\Rightarrow$ $1^n = p + q$ — ④

" $\lambda = 6$ in ③ $\Rightarrow$ $6^n = 6p + q$ — ⑤

Solving ④ & ⑤

$$\begin{aligned} p + q &= 1^n \\ 6p + q &= 6^n \\ (-) \quad (-) &\quad (-) \\ \hline -5p &= 1 - 6^n \\ 5p &= 6^n - 1 \end{aligned}$$

$$\boxed{p = \frac{6^n - 1}{5}}$$

Substituting the value of p in ④ gives,

$$1^n = \frac{6^n - 1}{5} + q \Rightarrow q = 1^n - \frac{(6^n - 1)}{5}$$

$$q = \frac{5 - 6^n + 1}{5}$$

$$\boxed{q = \frac{6 - 6^n}{5}}$$

Substituting the values of P & q in ③, we get

$$\therefore \lambda^n = f(\lambda)(\lambda^2-7\lambda+6) + \frac{(6^n-1)\lambda + (6-6^n)I}{5}$$

Hence $A^n = f(A)(A^2-7A+6I) + \frac{(6^n-1)A+(6-6^n)I}{5}$

$$= \frac{1}{5}\left[(6^n-1)A + (6-6^n)I\right] \quad [\because A^2-7A+6I=0]$$

$$A^n = \frac{6^n-1}{5}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} + \frac{6-6^n}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5) Calculate A^4 when $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Solution:-

Given, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

The characteristic equation of the matrix A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 6 = 0$$

$$4-\lambda-4\lambda+\lambda^2-6 = 0$$

$$\lambda^2 - 5\lambda - 2 = 0 \quad \text{——— (1)}$$

Substituting A instead of λ in (1), we get

$$A^2 - 5A - 2I = 0 \quad \text{——— (2)}$$

$$A^2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 3+12 \\ 2+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$$

(2) =>
$$\begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} + \begin{bmatrix} -5 & -15 \\ -10 & -20 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

A ~~sta~~ matrix satisfies the equation.

Hence $A^2 = 5A + 2I$

$\therefore A^4 = (5A+2I)(5A+2I)$

$= 25A^2 + 10A + 10A + 4I$

$= 25A^2 + 20A + 4I$

$= 25(5A+2I) + 20A + 4I$

$= 125A + 50I + 20A + 4I$

$= 145A + 54I$

$= 145\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 145 & 435 \\ 290 & 580 \end{bmatrix} + \begin{bmatrix} 54 & 0 \\ 0 & 54 \end{bmatrix}$

$A^4 = \begin{bmatrix} 199 & 435 \\ 290 & 634 \end{bmatrix}$ //

Unit - III

Skew lines:-

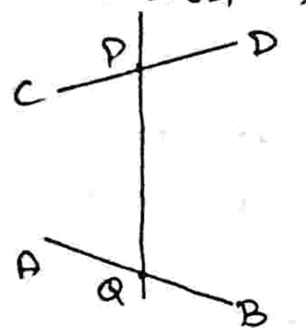
Any two lines which donot lie on the same plane and also not parallel are called skew lines

Shortest distance between the lines:-

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{——— (1)}$$

and $$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{——— (2)}$$

Let D be the point (x_1, y_1, z_1), CD be the line (1), B be the point (x_2, y_2, z_2) and AB be the line (2).



The shortest distance between them is the intercept PQ which they make on their common perpendicular LM.

Let the direction cosines of LM be l, m, n.

PQ is the projection of DB on LM

$$\therefore PQ = (x_2 - x_1) l + (y_2 - y_1) m + (z_2 - z_1) n.$$

LM is perpendicular to CD.

$$\therefore l l_1 + m m_1 + n n_1 = 0 \quad \text{——— (1)}$$

LM is perpendicular to AB

$$\therefore l l_2 + m m_2 + n n_2 = 0 \quad \text{——— (2)}$$

$$\therefore \frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$

Hence the each ratio is equal to

$$\frac{1}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

If this ratio is $\frac{1}{\lambda}$, the direction cosines of LM are

$$\frac{m_1n_2-m_2n_1}{\lambda},\ \frac{n_1l_2-n_2l_1}{\lambda},\ \frac{l_1m_2-l_2m_1}{\lambda}$$

$$(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)$$

$$\therefore PQ = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

Corollary:

If the two lines are coplanar, the shortest distance between them is zero.

Hence the condition for coplanar lines is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Example:-

1) Find the shortest distance between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{——} (1)$$

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} \quad \text{——} (2)$$

Determine also its equations.

Solution:-

Given $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ ——(1)

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} \quad \text{——} (2)$$

The line ① passes through the point (3, 8, 3) and the line ② passes through the point (−3, −7, 6).
Let the direction cosines of the shortest distance (S.D) be l, m, n

Since S.D is perpendicular to both ① & ②

$$ll_1 + mm_1 + nn_1 = 0$$

$$3l - m + n = 0 \quad \text{——— ③}$$

$$ll_2 + mm_2 + nn_2 = 0$$

$$-3l + 2m + 4n = 0 \quad \text{——— ④}$$

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$= \frac{1}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}$$

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3} = \frac{1}{\sqrt{(-4-2)^2 + (-3-12)^2 + (6-3)^2}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{\sqrt{(-6)^2 + (-15)^2 + (3)^2}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{\sqrt{36 + 225 + 9}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{\sqrt{270}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{\sqrt{9 \times 30}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{3\sqrt{30}}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = \frac{1}{3\sqrt{30}}$$

Hence the direction cosines of S.D are $\frac{-6}{3\sqrt{30}}, \frac{-15}{3\sqrt{30}}, \frac{3}{3\sqrt{30}}$

S.D = Projection of the line joining (3,8,3) and (−3,−7,6) on S.D.

$$= (x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$$

$$= (-3-3)\frac{(-6)}{3\sqrt{30}} + (-7-8)\frac{(-15)}{3\sqrt{30}} + (6-3)\frac{3}{3\sqrt{30}}$$

$$= \frac{(-6)(-6)}{3\sqrt{30}} + (-15)\frac{(-15)}{3\sqrt{30}} + \frac{3(3)}{3\sqrt{30}}$$

$$= \frac{36+225+9}{3\sqrt{30}} = \frac{270}{3\sqrt{30}} = \frac{9\times 30}{3\sqrt{30}} = \frac{3\times 3\times\sqrt{30}\times\sqrt{30}}{3\sqrt{30}}$$

$$\boxed{S.D = 3\sqrt{30}}$$

The equation of the shortest distance is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ -6 & -15 & 3 \end{vmatrix} = 0 = \begin{vmatrix} x+8 & y+7 & z-6 \\ -3 & 2 & 4 \\ -6 & -15 & 3 \end{vmatrix}$$

$$(x-3)(-3+15) - (y-8)(9+6) + (z-3)(-45-6) = 0 = (x+8)(6+60) - (y+7)(-9+24) + (z-6)(45+12)$$

$$(x-3)(12) - (y-8)(15) + (z-3)(-51) = 0 = (x+8)66 - (y+7)(15) + (z-6)(57)$$

$$12x - 36 - 15y + 120 - 51z + 153 = 0 = 66x + 198 - 15y - 105 + 57z - 342$$

$$12x - 15y - 51z + 237 = 0 = 66x - 15y + 57z - 249$$

($\div$ by 3)

$$4x - 5y - 17z + 79 = 0 = 22x - 5y + 19z - 88.$$

2) Find the S.D between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

Solution:

Given, $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ —— (1)

$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ —— (2)

The line (1) passes through the origin and the line (2) passes through the point $(2, 1, -2)$.

Let the direction cosines of the shortest distance be l, m, n.

Since S.D is perpendicular to both (1) & (2)

$$ll_1 + mm_1 + nn_1 = 0$$

$$2l - 3m + n = 0 \quad \text{—— (3)}$$

$$ll_2 + mm_2 + nn_2 = 0$$

$$3l - 5m + 2n = 0 \quad \text{—— (4)}$$

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\frac{l}{-6+5} = \frac{m}{3-4} = \frac{n}{-10+9}$$

$$\frac{l}{-1} = \frac{m}{-1} = \frac{n}{-1} = \frac{1}{\sqrt{(-1)^2 + (-1)^2 + (-1)^2}}$$

$$\frac{l}{-1} = \frac{m}{-1} = \frac{n}{-1} = \frac{1}{\sqrt{3}}$$

$$\therefore l = \frac{-1}{\sqrt{3}}; \quad m = \frac{-1}{\sqrt{3}}; \quad n = \frac{-1}{\sqrt{3}}$$

(1,0)

The magnitude of the shortest distance is the projection of the line joining the points $(0,0,0)$ and $(2,1,-2)$ on the line of S.D.

$$S.D = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$

$$= (2-0)\frac{(-1)}{\sqrt{3}} + (1-0)\frac{(-1)}{\sqrt{3}} + (-2+0)\frac{(-1)}{\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}$$

$$= \frac{-2-1+2}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

$$\boxed{S.D = \frac{1}{\sqrt{3}}}$$

The equation of the shortest distance between them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ -1 & -1 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$x(3+1) - y(-2+1) + z(-2-3) = 0 = (x-2)(5+2) - (y-1)(-3+2) + (z+2)(-3-5)$$

$$4x + y - 5z = 0 = 7x - 14 + y - 1 - 8z - 16$$

$\boxed{4x + y - 5z = 0 = 7x + y - 8z - 31}$ is the required equation of the shortest distance between the two given lines.

Sphere:-

A sphere is the locus of a point which moves so that its distance from a fixed point is constant. The fixed point is called the centre and the constant distance is called the radius of the sphere.

Equation of the sphere with center (a,b,c) and radius r: is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

General equation of the sphere is

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ where $(-u,-v,-w)$ is centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius.

Equation of a tangent plane to a sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at the point (x_1,y_1,z_1)

The co-ordinates of the centre C of the sphere are $(-u,-v,-w)$

Hence the direction cosines of CP are proportional to

$x_1+u,\ y_1+v,\ z_1+w$

Let $Q(x,y,z)$ be any point in the tangent plane. The direction cosines of PQ are proportional to $x-x_1, y-y_1, z-z_1$.

Since PQ is perpendicular to CP

$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$

i.e., $xx_1+yy_1+zz_1+ux+vy+wz = x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$

Since P lies on the sphere

$\therefore x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$

Hence the equation of the tangent plane becomes

$xx_1+yy_1+zz_1+ux+vy+wz=-ux_1-vy_1-wz_1-d$

i.e., $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)$
$+w(z+z_1)+d=0.$

Condition for the plane $lx+my+nz=p$ to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

The required condition is that the perpendicular distance of the centre of a sphere from the plane is equal to the radius of the sphere.

The centre of the sphere is $(-u,-v,-w)$

Radius is $\sqrt{u^2+v^2+w^2-d}$

The perpendicular distance from $(-u,-v,-w)$ to $lx+my+nz=p$ is $\pm \dfrac{lu+mv+nw+p}{\sqrt{(l^2+m^2+n^2)}}$

$$\therefore \frac{(lu+mv+nw+p)}{\sqrt{(l^2+m^2+n^2)}} = \sqrt{u^2+v^2+w^2-d}$$

ie., $(lu+mv+nw+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d)$

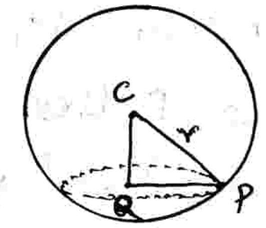
Intersection of a plane and a sphere is a circle:-

Let C be the centre a sphere whose radius is r. Let P be any point common to the ~~plane~~ sphere and the plane. Now CP = Radius of the sphere. Draw CQ perpendicular to the plane. Clearly $\angle CQP = 90°$.

From the right angled triangle CQP, we get

$CQ^2 = QP^2 + CP^2$

ie., $QP^2 = CP^2 - CQ^2 = r^2 - CQ^2$



Since C and Q are fixed points, CQ is constant. Hence QP is constant.

Hence the locus of P is a circle whose centre is Q. (Q is the foot of the perpendicular from the centre of the sphere to the plane).

Intersection of two spheres is a circle:-

A plane section of a sphere is a circle, the curve of intersection of two spheres is a circle.

Let $S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

be the two given spheres.

Consider the equation $S_1-S_2=0$.

It is satisfied by all points which satisfy the equations $S_1=0$, $S_2=0$ i.e., by the common points of the two spheres.

Now $S_1-S_2 = 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$.

clearly this is the first degree equation in x, y, z which represent a plane. Hence the circle of intersection of two spheres is the same as the intersection of the spheres with the common plane of intersection, $S_1-S_2=0$.

Examples:-

1) Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.

Solution:-

Given, plane is $2x-2y+z+12=0$

sphere is $x^2+y^2+z^2-2x-4y+2z=3$.

centre of the given sphere is $(1,2,-1)$

$$\text{Radius} = \sqrt{1^2+(2)^2+(-1)^2-(-3)}$$

$$= \sqrt{1+4+1+3} = \sqrt{9} = 3 \quad \text{---} (1)$$

Length of the perpendicular from $(1,2,-1)$ to the plane $2x-2y+z+12=0$ is $\dfrac{2(1)-2(2)+(-1)+12}{\sqrt{2^2+2^2+1^2}}$

$$= \frac{2-4-1+12}{\sqrt{4+4+1}} = \frac{9}{\sqrt{9}} = \frac{9}{3} = 3. \quad \text{——} (2)$$

The given plane $2x-2y+z+12=0$ touches the given ~~plane~~ sphere if the radius of the sphere is equal to the length of the perpendicular from the centre of the sphere to the plane.

From (1) & (2) we have

Radius of the sphere = Length of the perpendicular

Hence the plane touches the sphere.

To find the point of contact

Let the point of contact be $P(x,y,z)$.

The direction ratios of the line joining the centre $(1,2,-1)$ of the sphere and $P(x,y,z)$ are $x-1, y-2, z+1$

The direction ratios of the normal to the plane are $2,-2,1$

Since the line CP and noramal to the plane are parallel, we have

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r \text{ (we say)}$$

$x = 2r+1$; $y = -2r+2$; $z = r-1$

Any point on this line is $(2r+1, -2r+2, r-1)$ —— (3)

Since this point lies on the plane ~~as~~ $2x-2y+z+12=0$ we get

$2(2r+1) - 2(-2r+2) + (r-1) + 12 = 0$

$4r+2+4r-4+r-1+12 = 0$

$9r+9 = 0$

$9r = -9$

$\boxed{r=-1}$

Substituting $r=-1$ in (3) we get $(-1,4,-2)$ which is the point of contact.

2) Find the points of contact of the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-4=0$ which are parallel to the plane $2x-y+2z=3$.

Solution:

given sphere $x^2+y^2+z^2-4x+2y-4=0$

plane $2x-y+2z=3$

The equation of the tangent plane which is parallel to the given plane is of the form

$$2x-y+2z=k.$$

Centre of the sphere is $(2,-1,0)$

Radius of the sphere $= \sqrt{2^2+(-1)^2+(0)^2+4}$

$= \sqrt{4+1+4} = \sqrt{9}$

$= 3$ ——— (1)

So, the perpendicular distance from $(2,-1,0)$ to the plane $2x-y+2z=k$ is 3.

$$\therefore \pm \frac{2(2)-1(-1)+2(0)-k}{\sqrt{2^2+(-1)^2+2^2}} = 3$$

$$\pm \frac{4+1-k}{\sqrt{9}} = 3.$$

$$\pm \frac{5-k}{3} = 3$$

$$\pm 5-k = 9$$

$+5-k=9$; $-5+k=9$

$-k=9-5$; $k=9+5$

$-k=4$; $\boxed{k=14}$

$\boxed{k=-4}$

Hence the tangent planes are $2x-y+2z=-4$ ——— (2)

$2x-y+2z=14$ ——— (3)

Let the point of contact of (3) with the sphere be (x_1,y_1,z_1)

The straight line joining this point with the centre is perpendicular to the plane

$$\therefore \frac{x_1-2}{2} = \frac{y_1+1}{-1} = \frac{z_1}{2} = r$$

$$x_1 = 2r+2 \; ; \; y_1 = -r-1 \; ; \; z_1 = 2r$$

This point lies on the plane ③

$$\therefore 2(2+2r) - (-1-r) + 2(2r) = 14$$

$$4+4r+1+r+4r = 14$$

$$9r+5 = 14$$

$$9r = 14-5$$

$$9r = 9$$

$$\boxed{r=1}$$

Hence the point of contact is $(4,-2,2)$

Let the point of contact of ② with sphere be (x_2, y_2, z_2)

The straight line joining this point with the centre is perpendicular to the plane

$$\therefore \frac{x_2-2}{2} = \frac{y_2+1}{-1} = \frac{z_2}{2} = r_2$$

$$x_2 = 2r_2+2 \; ; \; y_2 = -r_2-1 \; ; \; z_2 = 2r_2$$

The point lies on the plane ②

$$\therefore 2(2+2r_2) - (-r_2-1) + 2(2r_2) = -4$$

$$4+4r_2 + r_2 + 1 + 4r_2 = -4$$

$$9r_2 + 5 = -4$$

$$9r_2 = -4-5$$

$$9r_2 = -9$$

$$\boxed{r_2 = -1}$$

Hence the point of contact is $(0,0,-2)$.

3) Find the equation of the sphere which has its centre on the plane $5x-y-4z+3=0$ and passing through the circle $x^2+y^2+z^2-3x+4y-2z+8=0$; $4x-5y+3z-3=0$.

Solution:-

given plane $5x-y-4z+3=0$.

The equation of the required sphere is of the form

$$x^2+y^2+z^2-3x+4y-2z+8+\lambda(4x-5y+3z-3)=0$$

ie, $x^2+y^2+z^2+(4\lambda-3)x+(-5\lambda+4)y+(3\lambda-2)z+8-3\lambda=0$ —(1)

The centre of the sphere is $\left(\frac{-(4\lambda-3)}{2}, \frac{-(-5\lambda+4)}{2}, \frac{-(3\lambda+2)}{2}\right)$

$$=\left(\frac{3-4\lambda}{2}, \frac{5\lambda-4}{2}, \frac{2-3\lambda}{2}\right)$$

This point lies on the plane $5x+y-4z+3=0$

$$\therefore \frac{5(3-4\lambda)}{2}+\frac{(5\lambda-4)}{2}-\frac{4(2-3\lambda)}{2}+3=0$$

$$\frac{15-20\lambda+5\lambda-4-8+12\lambda+6}{2}=0$$

$$-3\lambda+9=0$$

$$-3\lambda=-9$$

$$\boxed{\lambda=3}$$

Hence the equation of the sphere is

$$x^2+y^2+z^2-3x+4y-2z+8+3(4x-5y+3z-3)=0$$

$$x^2+y^2+z^2-3x+4y-2z+8+12x-15y+9z-9=0$$

ie, $\boxed{x^2+y^2+z^2+9x-11y+7z-1=0.}$

4) Show that the intersection of two spheres

$$S_1=x^2+y^2+z^2-2x-4y+6z-2=0$$

$$S_2=x^2+y^2+z^2-4x-6y+4z+4=0$$

is a circle lying in the plane $x+y+z=3$. Find its centre and radius

Solution:

The intersecting circle of the two spheres lies on the plane

$$S_1 - S_2 = 0$$

ie., $(x^2+y^2+z^2-2x-4y+6z-2)-(x^2+y^2+z^2-4x-6y+4z+4)=0$

$$2x+2y+2z-6=0$$

ie., $\boxed{x+y+z-3=0}$

The centre of the sphere S_1 is $C(1,2,-3)$ and its radius is 4.

The foot of the perpendicular from $(1,2,-3)$ to the plane $x+y+z=3$ is the centre of the required circle.

The direction cosines of the perpendicular to the plane $x+y+z=3$ are proportional to $1,1,1$.

Hence the equations of the line through $(1,2,-3)$, perpendicular to the plane $x+y+z-3$ are

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z+3}{1}.$$

Hence any point on this line has co-ordinates $(1+r, 2+r, -3+r)$.

If this point lies on the plane $x+y+z=3$, we have

$$1+r+2+r-3+r=3$$

$$3r=3$$

$$\boxed{r=1}$$

Hence the foot of the perpendicular P has co-ordinates $(2,3,-2)$.

Let r be the radius of this circle.

Then $r^2 = (\text{radius of the sphere})^2 - CP^2$

$$=(4)^2-(\sqrt{3})^2 = 16+3 = 13$$

$$\boxed{r=\sqrt{13}}$$

5) Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at $(1,1,-1)$ and passing through $(2,0,1)$.

Solution:-

given sphere $x^2+y^2+z^2-x+3y+2z-3=0$

Tangent plane at $(1,1,-1)$ to the given sphere is

$$x(1)+y(1)+z(-1)-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+(z-1)-3=0$$

$$x+y-z-\frac{x}{2}-\frac{1}{2}+\frac{3y}{2}+\frac{3}{2}+z-1-3=0$$

$$\frac{2x+2y-2z-x-1+3y+3+2z-2-6}{2}=0$$

$$x+5y-6=0$$

$$x+5y=6 \quad \text{——— (1)}$$

As the required sphere touches the given sphere at $(1,1,-1)$, it has the same tangent plane as (1)

Hence the equation of the required sphere is of the form

$$x^2+y^2+z^2-x+3y+2z-3+k(x+5y-6)=0 \quad \text{——— (2)}$$

where k is arbitrary constant.

As this is to pass through $(2,0,1)$

$$4+0+1-2+0+2-3+2k+0-6k=0$$

$$2-4k=0$$

$$-4k=-2$$

$$\boxed{k=\frac{1}{2}}$$

Substituting the value of k in (2), we get

$$x^2+y^2+z^2-x+3y+2z-3+\frac{1}{2}(x+5y-6)=0$$

$$2x^2+2y^2+2z^2-2x+6y+4z-6+x+5y-6=0$$

$$\boxed{2x^2+2y^2+2z^2-x+11y+4z-12=0}$$

6) Show that the spheres $x^2+y^2+z^2+6y+2z+8=0$ and $x^2+y^2+z^2+6x+8y+4z+20=0$ intersect at right angles. Find their plane of intersection.

<u>Solution:-</u>

Let $S_1 = x^2+y^2+z^2+6y+2z+8=0$

$S_2 = x^2+y^2+z^2+6x+8y+4z+20=0$

The condition for orthogonality is

$$2u_1u_2+2v_1v_2+2w_1w_2 = d_1+d_2$$

Here

$u_1=0$ $u_2=-3$

$v_1=-3$ $v_2=-4$

$w_1=-1$ $w_2=-2$

$d_1=8$ $d_2=20$

$\therefore$ $2(u_1u_2+v_1v_2+w_1w_2) = d_1+d_2$

$2(0(-3)+(-3)(-4)+(-1)(-2)) = 8+20$

$2(0+12+2) = 28$

$2(14) = 28$

$28 = 28$

Hence the two given spheres intersect at right angles.

Their plane of intersection is given by

$S_1-S_2=0$

$(x^2-y^2+z^2+6y+2z+8)-(x^2+y^2+z^2+6x+8y+4z+20)=0$

$-6x-2y-2z-12=0$

$$\boxed{3x+y+z+6=0}$$

UNIT - IV

Expansion of $\cos n\theta$ and $\sin n\theta$ in powers of $\sin\theta$ and $\cos\theta$, n being a positive integer:-

By Binomial theorem, we have

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + nc_1\cos^{n-1}\theta(i\sin\theta) + nc_2\cos^{n-2}\theta(i^2\sin^2\theta) + nc_3\cos^{n-3}\theta(i\sin\theta)^3 + \cdots$$

$$= \cos^n\theta + i\,nc_1\cos^{n-1}\theta\sin\theta + i^2nc_2\cos^{n-2}\theta\sin^2\theta + i^3nc_3\cos^{n-3}\theta\sin^3\theta + i^4nc_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$= \cos^n\theta + i\,nc_1\cos^{n-1}\theta\sin\theta - nc_2\cos^{n-2}\theta\sin^2\theta - i\,nc_3\cos^{n-3}\theta\sin^3\theta + nc_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$[\because i^2 = -1,\ i^3 = -i,\ i^4 = 1, \ldots]$

But $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ [By De Moivre's Theorem]

Therefore,

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,nc_1\cos^{n-1}\theta\sin\theta - nc_2\cos^{n-2}\theta\sin^2\theta - i\,nc_3\cos^{n-3}\theta\sin^3\theta + nc_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$= (\cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-4}\theta\sin^4\theta - \cdots) + i(nc_1\cos^{n-1}\theta\sin\theta - nc_3\cos^{n-3}\sin^3\theta + \cdots)$$

Equating real and imaginary parts on both sides we get,

$$\cos n\theta = \cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$\sin n\theta = nc_1\cos^{n-1}\theta\sin\theta - nc_3\cos^{n-3}\theta\sin^3\theta + \cdots$$

Expansion of $\tan n\theta$:-

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{nc_1\cos^{n-1}\theta\sin\theta - nc_3\cos^{n-3}\theta\sin^3\theta + \cdots}{\cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-4}\theta\sin^4\theta - \cdots}$$

Dividing both numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{nc_1\tan\theta - nc_3\tan^3\theta + nc_5\tan^5\theta - \cdots}{1 - nc_2\tan^2\theta + nc_4\tan^4\theta - \cdots}$$

Formulae for any number of angles:-

We have that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

Also $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1+i\tan\theta_1)$

$\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1+i\tan\theta_2)$

$\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots$

$$\therefore \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$
$$= \cos\theta_1\cos\theta_2\cdots\cos\theta_n[(1+i\tan\theta_1)(1+i\tan\theta_2)\cdots(1+i\tan\theta_n)]$$
$$= \cos\theta_1\cos\theta_2\cdots\cos\theta_n[1+i(\tan\theta_1+\tan\theta_2+\cdots+\tan\theta_n) + i^2(\tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3+\cdots) + i^3(\tan\theta_1\tan\theta_2\tan\theta_3+\cdots)+\cdots]$$
$$= \cos\theta_1\cos\theta_2\cdots\cos\theta_n[1+is_1 - s_2 - is_3 + s_4 + \cdots]$$

where $s_1 = \tan\theta_1 + \tan\theta_2 + \cdots + \tan\theta_n$

$s_2 = \tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3 + \cdots$

$s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3$

$\vdots$

etc.

Equating real and imaginary parts, we get

$$\cos(\theta_1+\theta_2+\cdots+\theta_n) = \cos\theta_1\cos\theta_2\cdots\cos\theta_n(1-s_2+s_4-\cdots)$$
$$\sin(\theta_1+\theta_2+\cdots+\theta_n) = \cos\theta_1\cos\theta_2\cdots\cos\theta_n(s_1-s_3+s_4-\cdots)$$

Dividing, we get

$$\tan(\theta_1+\theta_2+\cdots+\theta_n) = \frac{s_1-s_3+s_5-\cdots}{1-s_2+s_4-s_6+\cdots}$$

Examples:

1) Express $\cos 5\theta$ in terms of $\cos\theta$.

Solution:-

$$\cos n\theta = \cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-4}\theta\sin^4\theta - \cdots \quad \text{—①}$$

Put $n=5$ in ①

$$\cos 5\theta = \cos^5\theta - 5c_2\cos^{5-2}\theta\sin^2\theta + 5c_4\cos^{5-4}\theta\sin^4\theta$$
$$= \cos^5\theta - \frac{5!}{(5-2)!\,2!}\cos^3\theta\sin^2\theta + \frac{5!}{(5-4)!\,4!}\cos\theta\sin^4\theta.$$
$$= \cos^5\theta - \frac{5\times4\times3!}{3!\,2!}\cos^3\theta\sin^2\theta + \frac{5\times4!}{1!\,4!}\cos\theta\sin^4\theta$$

$$= \cos^5\theta - \frac{5\times4}{2}\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta(\sin^2\theta)^2.$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

2) Express $\cos 6\theta$ in terms of $\cos\theta$.

Solution:-

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\sin^2\theta + nC_4\cos^{n-4}\theta\sin^2\theta \cdots \quad (1)$$

Put $n=6$ in (1),

$$\cos 6\theta = \cos^6\theta - 6C_2\cos^4\theta\sin^2\theta + 6C_4\cos^2\theta\sin^4\theta - 6C_6\cos^0\theta\sin^6\theta$$

$$= \cos^6\theta - \frac{6\times5}{2}\cos^4\theta\sin^2\theta + \frac{6\times5}{2}\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta(\sin^2\theta)^2 - (\sin^2\theta)^3$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3.$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1-2\cos^2\theta+\cos^4\theta)$$
$$-(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta).$$

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta$$
$$-1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta.$$

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

3) Express $\sin 7\theta$ interms of $\sin\theta$

Solution:-

$$\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta \cdots \quad (1)$$

Put $n=7$ in (1),

$$\sin 7\theta = 7C_1\cos^6\theta\sin\theta - 7C_3\cos^4\theta\sin^3\theta + 7C_5\cos^2\theta\sin^5\theta$$
$$- 7C_7\cos^0\theta\sin^7\theta.$$

$$= 7\cos^6\theta\sin\theta - \frac{7\times6\times5}{3\times2}\cos^4\theta\sin^3\theta + \frac{7\times6}{2}\cos^2\theta\sin^5\theta$$
$$- \sin^7\theta$$

$$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$= 7(\cos^2\theta)^3\sin\theta - 35(\cos^2\theta)^2\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$$

$$= 7\sin\theta(1-\sin^2\theta)^3 - 35\sin^3\theta(1-\sin^2\theta)^2 + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta$$

$$= 7\sin\theta(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35\sin^3\theta(1-2\sin^2\theta+\sin^4\theta) - \sin^7\theta + 21\sin^5\theta - 21\sin^7\theta$$

$$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta - \sin^7\theta + 21\sin^5\theta - 21\sin^7\theta$$

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

4) Show that $\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$

Solution:

$$\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta \ldots \text{—①}$$

Put $n=6$ in ①,

$$\sin 6\theta = 6C_1\cos^5\theta\sin\theta - 6C_3\cos^3\theta\sin^3\theta + 6C_5\cos\theta\sin^5\theta$$

$$= 6\cos^5\theta\sin\theta - \frac{6\times5\times4}{3\times2}\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$$

Dividing both sides by $\sin\theta$, we get

$$\frac{\sin 6\theta}{\sin\theta} = \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\sin\theta}$$

$$= \frac{6\cos^5\theta\sin\theta}{\sin\theta} - \frac{20\cos^3\theta\sin^3\theta}{\sin\theta} + \frac{6\cos\theta\sin^5\theta}{\sin\theta}$$

$$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta$$

$$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$$

$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$

$$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta - 12\cos^3\theta + 6\cos^5\theta$$

$$\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$$

Hence proved.

5) Prove that $\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$.

Proof:-

$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta \ldots \ldots$ — ①

Put $n=7$ in ①

$\cos 7\theta = \cos^7\theta - 7C_2\cos^{7-2}\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta - 7C_1\cos\theta\sin^6\theta$

$= \cos^7\theta - \frac{7\times 6}{2}\cos^5\theta\sin^2\theta + \frac{7\times 6\times 5}{3\times 2}\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

Dividing bothsides by $\cos\theta$, we get

$\frac{\cos 7\theta}{\cos\theta} = \frac{\cos^7\theta}{\cos\theta} - \frac{21\cos^5\theta\sin^2\theta}{\cos\theta} + \frac{35\cos^3\theta\sin^4\theta}{\cos\theta} - \frac{7\cos\theta\sin^6\theta}{\cos\theta}$

$= \cos^6\theta - 21\cos^4\theta\sin^2\theta + 35\cos^2\theta(\sin^2\theta)^2 - 7\cancel{\cos\theta}(\sin^2\theta)^3$

$= \cos^6\theta - 21\cos^4\theta(1-\cos^2\theta) + 35\cos^2\theta(1-\cos^2\theta)^2 - 7(1-\cos^2\theta)^3$

$= \cos^6\theta - 21\cos^4\theta + 21\cos^6\theta + 35\cos^2\theta(1-2\cos^2\theta+\cos^4\theta) - 7(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$= \cos^6\theta - 21\cos^4\theta + 21\cos^6\theta + 35\cos^2\theta - 70\cos^4\theta + 35\cos^6\theta - 7 + 21\cos^2\theta - 21\cos^4\theta + 7\cos^6\theta$

$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$.

6) Prove that $\frac{\cos 5\theta}{\cos\theta} = 1 - 12\sin^2\theta + 16\sin^4\theta$.

Proof:-

$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta \ldots$

Put $n=5$,

$\cos 5\theta = \cos^5\theta - 5C_2\cos^3\theta\sin^2\theta + 5C_4\cos\theta\sin^4\theta$

$= \cos^5\theta - \frac{5\times 4}{2}\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Dividing bothsides by $\cos\theta$, we get

$\frac{\cos 5\theta}{\cos\theta} = \frac{\cos^5\theta}{\cos\theta} - \frac{10\cos^3\theta\sin^2\theta}{\cos\theta} + \frac{5\cos\theta\sin^4\theta}{\cos\theta}$

$= \cos^4\theta - 10\cos^2\theta \sin^2\theta + 5\sin^4\theta$

$= (1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + 5\sin^4\theta$

$= 1 - 2\sin^2\theta + \sin^4\theta \cancel{+10} + 10\sin^4\theta + 5\sin^4\theta - 10\sin^2\theta$

$\frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 12\sin^2\theta + 1.$

7) Prove that $\frac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta$

Proof:

$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + \cdots$

Put $n = 6$.

$\sin 6\theta = 6c_1 \cos^5\theta \sin\theta - 6c_3 \cos^3\theta \sin^3\theta + 6c_5 \cos\theta \sin^5\theta$

$= 6\cos^5\theta \sin\theta - \frac{6\times5\times4}{3\times2}\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$

Divide both sides by $\cos\theta$.

$\frac{\sin 6\theta}{\cos\theta} = \frac{6\cos^5\theta\sin\theta}{\cos\theta} - \frac{20\cos^3\theta\sin^3\theta}{\cos\theta} + \frac{6\cos\theta\sin^5\theta}{\cos\theta}$

$= 6\cos^4\theta \sin\theta - 20\cos^2\theta\sin^3\theta + 6\sin^5\theta.$

$= (\sin\theta)6(\cos^2\theta)^2 - 20\cos^2\theta\sin^3\theta + 6\sin^5\theta$

$= (\sin\theta)6(1-\sin^2\theta)^2 - 20\sin^3\theta(1-\sin^2\theta) + 6\sin^5\theta$

$= (\sin\theta)6(1 - 2\sin^2\theta + \sin^4\theta) - 20\sin^3\theta + 20\sin^5\theta + 6\sin^5\theta$

$= 6\sin\theta - 12\sin^3\theta + 6\sin^5\theta - 20\sin^3\theta + 20\sin^5\theta + 6\sin^5\theta$

$\frac{\sin 6\theta}{\cos\theta} = 32\sin^5\theta - 32\sin^3\theta + 6\sin\theta$

Hence proved//.

Expansion of $\sin^n\theta$ and $\cos^n\theta$ interms of sines and cosines of multiples of θ, n being a positive integer.

Let $\quad x = \cos\theta + i\sin\theta$ —— ①

$\frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$

$\frac{1}{x} = \cos\theta - i\sin\theta$ —— ②

Now, $\quad x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ —— ③

$\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ —— ④

from ① & ②, we get

$x + \frac{1}{x} = 2\cos\theta$ —— ⑤

$x - \frac{1}{x} = 2i\sin\theta$ —— ⑥

from ③ & ④ we get

$x^n + \frac{1}{x^n} = 2\cos n\theta$ —— ⑦

$x^n - \frac{1}{x^n} = 2i\sin n\theta$ —— ⑧

To get expansion of $\cos^n\theta$, we have to consider

⑤ ⇒ $\quad 2\cos\theta = x + \frac{1}{x}$

$(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$

$= x^n + nc_1 x^{n-1}\frac{1}{x} + nc_2 x^{n-2}\frac{1}{x^2} + \cdots + nc_{n-1}\frac{x}{x^{n-1}} + nc_n\frac{1}{x^n}$

$= \left(x^n + \frac{1}{x^n}\right) + nc_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nc_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots$

$= 2\cos n\theta + nc_1(2\cos(n-2)\theta) + nc_2(2\cos(n-4)\theta) + \cdots$

$2^n\cos^n\theta = 2(\cos n\theta + nc_1\cos(n-2)\theta + nc_2\cos(n-4)\theta + \cdots)$

$\cos^n\theta = \frac{2}{2^n}(\cos n\theta + (nc_1\cos(n-2)\theta) + nc_2\cos(n-4)\theta + \cdots)$

$\cos^n\theta = \frac{1}{2^{n-1}}(\cos n\theta + nc_1\cos(n-2)\theta + nc_2\cos(n-4)\theta + \cdots)$

To get expansion of $\sin^n\theta$, we have to consider

(6) $\Rightarrow$ $2i\sin\theta = x - \frac{1}{x}$

$$(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$$

$$= x^n - nc_1 x^{n-1}\frac{1}{x} + nc_2 x^{n-2}\frac{1}{x^2} - \cdots\cdots + nc_{n-1} x \frac{1}{x^{n-1}} - nc_n \frac{1}{x^n} \text{ (n is odd)}$$

$$= \left(x^n - \frac{1}{x^n}\right) - nc_1\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + nc_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \cdots$$

$$= 2i\sin n\theta - nc_1\, 2i\sin(n-2)\theta + nc_2\, 2i\sin(n-4)\theta - \cdots$$

$$(2i\sin\theta)^n = 2i\left(\sin n\theta - nc_1\sin(n-2)\theta + nc_2\sin(n-4)\theta - \cdots\right)$$

$$\sin^n\theta = \frac{2i}{(2i)^n}\left(\sin n\theta - nc_1\sin(n-2)\theta + nc_2\sin(n-4)\theta - \cdots\right)$$

$$\sin^n\theta = \frac{1}{(2i)^{n-1}}\left(\sin n\theta - nc_1\sin(n-2)\theta + nc_2\sin(n-4)\theta - \cdots\right)$$

To get the expansion of $\sin^m\theta\cos^n\theta$, we have to consider

$$(2i\sin\theta)^m(2\cos\theta)^n = \left(x - \frac{1}{x}\right)^m\left(x + \frac{1}{x}\right)^n$$

Expanding this we get the result.

Examples:-

1) Prove that $\sin^6\theta = \frac{-1}{32}[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10]$

Proof:

Let $x = \cos\theta + i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$; $\left(x^n + \frac{1}{x^n}\right) = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$(2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6$$

$$= x^6 - 6c_1 x^5 \cdot \frac{1}{x} + 6c_2 x^4 \frac{1}{x^2} - 6c_3 x^3 \cdot \frac{1}{x^3} + 6c_4 x^4 \frac{1}{x^4} - 6c_5\, x \cdot \frac{1}{x^5} + 6c_6 \cdot \frac{1}{x^6}$$

$$= x^6 + \frac{1}{x^6} - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$2^6 i^6 \sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$64 i^2 i^4 \sin^6\theta = 2(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta) - 20$$

$$-32 \sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$

$$\sin^6\theta = \frac{-1}{32}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

Hence proved.

2) Prove that $\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$

Proof:

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$.

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$.

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$.

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$$(2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5$$

$$= x^5 - 5c_1 x^4 \cdot \frac{1}{x} + 5c_2 x^3 \cdot \frac{1}{x^2} - 5c_3 x^2 \cdot \frac{1}{x^3} + 5c_4\, x \cdot \frac{1}{x^4} - 5c_5 \frac{1}{x^5}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$2^5 i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$32\, i^4 i \sin^5\theta = 2i(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$\sin^5\theta = \frac{2i}{32i}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

Hence proved.

3) Prove that $\cos^8\theta = \frac{1}{2^7}\left[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\right]$

Proof:

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$$(2\cos\theta)^8 = \left(x + \frac{1}{x}\right)^8$$

$$= x^8 + 8c_1 x^7 \cdot \frac{1}{x} + 8c_2 x^6 \frac{1}{x^2} + 8c_3 x^5 \frac{1}{x^3} + 8c_4 x^4 \frac{1}{x^4} + 8c_5 x^3 \frac{1}{x^5} + 8c_6 x^2 \frac{1}{x^6} + 8c_7 x \frac{1}{x^7} + 8c_8 \frac{1}{x^8}$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$= 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$$2^8\cos^8\theta = 2\left[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\right]$$

$$\cos^8\theta = \frac{1}{2^7}\left[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\right]$$

Hence proved.

4) Prove that $\cos^7\theta = \frac{1}{2^6}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]$

Proof:

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$(2\cos\theta)^7 = \left(x+\frac{1}{x}\right)^7$$

$$= x^7 + 7C_1 x^6\frac{1}{x} + 7C_2 x^5\frac{1}{x^2} + 7C_3 x^4\frac{1}{x^3} + 7C_4 x^3\frac{1}{x^4} + 7C_5 x^2\frac{1}{x^5} + 7C_6 x\frac{1}{x^6} + 7C_7\frac{1}{x^7}$$

$$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$= \left(x^7+\frac{1}{x^7}\right) + 7\left(x^5+\frac{1}{x^5}\right) + 21\left(x^3+\frac{1}{x^3}\right) + 35\left(x+\frac{1}{x}\right)$$

$$2^7\cos^7\theta = 2\cos7\theta + 7(2\cos5\theta) + 21(2\cos3\theta) + 35(2\cos\theta).$$

$$2^7\cos^7\theta = 2(\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta).$$

$$\cos^7\theta = \frac{1}{2^6}(\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta).$$

5) Prove that $\cos^5\theta\sin^4\theta = \frac{1}{2^8}\left[\cos9\theta + \cos7\theta - 4\cos5\theta - 4\cos3\theta + 6\cos\theta\right]$

Solution:-

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x+\frac{1}{x} = 2\cos\theta$; $x^n+\frac{1}{x^n} = 2\cos n\theta.$

$x-\frac{1}{x} = 2i\sin\theta$; $x^n-\frac{1}{x^n} = 2i\sin n\theta.$

$$(2\cos\theta)^5(2i\sin\theta)^4 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^4$$

$$= \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^4$$

$$= \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^4$$

$$= \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^4$$

$$= \left(x+\frac{1}{x}\right)\left[x^8 - 4C_1(x^2)^3\frac{1}{x^2} + 4C_2(x^2)^2\frac{1}{(x^2)^2} - 4C_3 x^2\frac{1}{(x^2)^3} + 4C_4\frac{1}{x^8}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^8 - 4x^4 + 6 - \frac{4}{x^4} + \frac{1}{x^8}\right]$$

$$= x^9 + x^7 - 4x^5 - 4x^3 + 6x + \frac{6}{x} - \frac{4}{x^3} - \frac{4}{x^5} + \frac{1}{x^7} + \frac{1}{x^9}$$

$$2^5\cos^5\theta\, 2^4 i^4 \sin^4\theta = \left(x^9+\frac{1}{x^9}\right)+\left(x^7+\frac{1}{x^7}\right)-4\left(x^5+\frac{1}{x^5}\right)-4\left(x^3+\frac{1}{x^3}\right)+6\left(x+\frac{1}{x}\right)$$

$$2^9\cos^5\theta\sin^4\theta = 2\cos 9\theta + 2\cos 7\theta - 4(2\cos 5\theta) - 4(2\cos 3\theta) + 6(\cos\theta \cdot 2)$$

$$2^9\cos^5\theta\sin^4\theta = 2[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta]$$

$$\cos^5\theta\sin^4\theta = \frac{1}{2^8}[\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta].$$

Hence proved

6) Prove that $\sin^4\theta\cos^2\theta = \frac{1}{2^5}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

Proof:

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$(2i\sin\theta)^4(2\cos\theta)^2 = \left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^2$$

$$= \left(x-\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^2\left(x+\frac{1}{x}\right)^2$$

$$= \left(x-\frac{1}{x}\right)^2\left[\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\right]^2$$

$$= \left(x-\frac{1}{x}\right)^2\left[x^2-\frac{1}{x^2}\right]^2$$

$$= \left(x^2-2+\frac{1}{x^2}\right)\left[x^4-2+\frac{1}{x^4}\right]$$

$$= x^6 - 2x^4 + x^2 - 2x^2 + 4 - \frac{2}{x^2} + \frac{1}{x^2} - \frac{2}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6+\frac{1}{x^6}\right) - 2\left(x^4+\frac{1}{x^4}\right) - \left(x^2+\frac{1}{x^2}\right) + 4$$

$$2^4 i^4 \sin^4\theta\, 2^2\cos^2\theta = 2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4$$

$$2^6\sin^4\theta\cos^2\theta = 2[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$$

$$\sin^4\theta\cos^2\theta = \frac{1}{2^5}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$$

Hence proved.

7) Prove that $\cos^4\theta \sin^3\theta = \frac{-1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$

<u>Proof:</u>

Let $x = \cos\theta + i\sin\theta$; $x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$; $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x + \frac{1}{x} = 2\cos\theta$; $x^n + \frac{1}{x^n} = 2\cos n\theta$

$x - \frac{1}{x} = 2i\sin\theta$; $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$(2\cos\theta)^4 (2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4 \left(x-\frac{1}{x}\right)^3$$

$$= \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$= \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3$$

$$= \left(x+\frac{1}{x}\right)\left[x^2 - \frac{1}{x^2}\right]^3$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^4\frac{1}{x^2} + 3x^2\frac{1}{x^4} - \frac{1}{x^6}\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - 3x^2 + 3\frac{1}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 + x^5 - 3x^3 - 3x + \frac{3}{x} + \frac{3}{x^3} - \frac{1}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$

$$2^4\cos^4\theta\, 2^3 i^3 \sin^3\theta = 2i\sin 7\theta + 2i\sin 5\theta - 3(2i\sin 3\theta) - 3(2i\sin\theta)$$

$$2^7 i^2 i \cos^4\theta \sin^3\theta = 2i\left(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right)$$

$$-2^7 i\cos^4\theta\sin^3\theta = 2i\left(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right)$$

$$-2^7\cos^4\theta\sin^3\theta = 2\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$$\cos^4\theta\sin^3\theta = \frac{-1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

Hence proved//.

Expansion of $\sin\theta$ and $\cos\theta$ in ascending powers of θ.

We have that

$$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + \cdots$$

$$\sin n\theta = n \cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{3\times 2} \cos^{n-3}\theta \sin^3\theta + \cdots \quad \text{--- (1)}$$

Put $n\theta = \alpha$ in (1), we get

$$\sin\alpha = \frac{\alpha}{\theta} \cdot \cos^{n-1}\theta \sin\theta - \frac{\left\{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\right\}}{1\cdot 2\cdot 3} \cos^{n-3}\theta \sin^3\theta + \cdots$$

Let $\theta \to 0$ and $n \to \infty$ such that $n\theta = \alpha$ is finite. As $\theta \to 0$, $\frac{\sin\theta}{\theta} \to 1$ and $\cos\theta \to 1$ and therefore every power of these quantities tends to unity

$$\therefore \sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots$$

Also we have that

$$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta - \cdots$$

$$= \cos^n\theta - \frac{n(n-1)}{1\cdot 2} \cos^{n-2}\theta \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \cos^{n-4}\theta \sin^4\theta - \cdots$$

put $n\theta = \alpha$, we get

$$\cos\alpha = \cos^n\theta - \frac{\left(\frac{\alpha}{\theta}\right)\left(\frac{\alpha}{\theta}-1\right)}{1\cdot 2} \cos^{n-2}\theta \sin^2\theta + \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{1\cdot 2\cdot 3\cdot 4} \cos^{n-4}\theta \sin^4\theta \cdots$$

$$\cos\alpha = \cos^n\theta - \frac{\alpha(\alpha-\theta)}{1\cdot 2} \cos^{n-2}\theta \left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{1\cdot 2\cdot 3\cdot 4} \cos^{n-4}\theta \left(\frac{\sin\theta}{\theta}\right)^4 \cdots$$

Let $\theta \to 0$ and $n \to \infty$, then $n\theta = \alpha$.

We know that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$ and therefore every power of these quantities tends to unity. Hence

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots$$

Expansion of $\tan\theta$ in ascending powers of θ:-

We have that $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}$$

$$= \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\left[1 - \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)\right]^{-1}$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)\left[1 + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right) + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right)^2 + \cdots\right]$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)\left[1 + \frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots\right]$$

$$= \theta + \frac{\theta^3}{2} - \frac{\theta^5}{24} + \frac{\theta^5}{4} - \frac{\theta^3}{6} - \frac{\theta^5}{12} + \frac{\theta^5}{120} + \text{ the terms containing } \theta^6 \text{ and so on.}$$

$$= \theta + \theta^3\left(\frac{1}{2} - \frac{1}{6}\right) + \theta^5\left(\frac{-1}{24} + \frac{1}{4} - \frac{1}{12} + \frac{1}{120}\right) + \theta^6(\ldots) + \cdots$$

$$= \theta + \frac{2}{6}\theta^3 + \frac{16}{120}\theta^5 + \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \cdots$$

<u>Examples:-</u>

1) If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, Prove that the angle θ is $1^\circ 58'$ nearly.

<u>Proof:-</u>

We have, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$$= \theta\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right)$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots \quad \text{——— ①}$$

Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046} = \frac{5046-1}{5046}$

$$\frac{\sin\theta}{\theta} = 1 - \frac{1}{5046} \quad (\text{that is nearly } 1)$$

Hence θ must be very small. Therefore omitting θ^4 and higher powers we get,

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{5046}$$

$$1 - \frac{\theta^2}{6} = 1 - \frac{1}{5046}$$

$$\frac{\theta^2}{6} = \frac{1}{5046}$$

$$\theta^2 = \frac{6}{5046} = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians}$$

$$= \frac{1}{29} \times \frac{180}{\pi} \text{ degrees}$$

$$= \frac{1}{29} \times \frac{180}{\frac{22}{7}}$$

$$= \frac{7\times180}{29\times22} = \frac{1260}{638} = (1\cdot97) \text{ degrees}$$

$$\frac{.97\times60}{58.2}$$

$$\theta = 1^\circ 58'$$

Hence proved//

2) Solve approximately $\sin\left(\frac{\pi}{6}+\theta\right) = 0.51$

Solution :-

Here 0.51 is nearly equal to 1/2

we know $\sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$.

Hence θ is very small.

$$\therefore \sin\left(\frac{\pi}{6}+\theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta$$

$$= \frac{1}{2}\cdot\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

$$= \frac{1}{2}\left(1-\frac{\theta^2}{2!}+\cdots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!}+\cdots\right)$$

$$\sin\left(\frac{\pi}{6}+\theta\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\theta \quad \text{[omitting higher powers]} \quad \text{--- (1)}$$

Given $\sin\left(\frac{\pi}{6}+\theta\right) = 0.51$ --- (2)

from (1) & (2) we get

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$0.5 + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$\frac{\sqrt{3}}{2}\theta = 0.51 - 0.5 = 0.01$$

$$\frac{\sqrt{3}}{2}\theta = \frac{1}{100}$$

$$\theta = \frac{2}{\sqrt{3}\times100} = \frac{1}{\sqrt{3}\times50} \text{ radians.}$$

$$\theta = \frac{3}{\sqrt{3}\times150} = \frac{\sqrt{3}\times\sqrt{3}}{150} \text{ radians}$$

$$\theta = \frac{\sqrt{3}}{150} \text{ radians}$$

$$\theta = \frac{\sqrt{3}}{150} \times \frac{180}{\pi} \text{ degrees}$$

$$\theta = \frac{\sqrt{3}}{150} \times \frac{180}{22/7}$$

$$\theta = \frac{7\times\sqrt{3}\times 180}{150\times 22} = \frac{7\times\sqrt{3}\times 18}{15\times 22} = \frac{7\times\sqrt{3}\times 9}{15\times 11}$$

$$\theta = \frac{63\times 1.73}{165} = \frac{108.99}{165}$$

$$\theta = (0.66) \text{ degrees}$$

$$\frac{0.66\times 60}{39.6}$$

$$\theta = 39'6''$$

3) Find θ approximately to the nearest minute if $\cos\theta = \frac{1681}{1682}$

Solution:-

We have $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ —— (1)

Given $\cos\theta = \frac{1681}{1682}$ —— (2)

From (1) & (2) we get

$$\frac{1681}{1682} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\frac{1682-1}{1682} = 1 - \frac{\theta^2}{2!} \quad \text{(Neglecting higher powers)}$$

$$1 - \frac{1}{1682} = 1 - \frac{\theta^2}{2}$$

$$\frac{1}{1682} = \frac{\theta^2}{2}$$

$$\theta^2 = \frac{2}{1682} = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians}$$

$$\theta = \frac{1}{29} \times \frac{180}{\pi} \text{ degrees.}$$

$$\theta = \frac{7\times 180}{29\times 22} = \frac{1260}{638} = (1.97) \text{ degees}$$

$$\frac{.97\times 60}{58.2}$$

$$\boxed{\theta = 1^\circ 58'}$$

4) Find $\lim\limits_{x\to 0} \frac{x-\sin x}{x^3}$

Solution:-

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{+\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^3}$$

$$= \lim_{x\to 0} \frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{3!} - \frac{x^2}{5!} + \cdots$$

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{1}{3!} = \frac{1}{6}$$

5) Evaluate $\lim\limits_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\theta^3}$

Solution:-

$$\lim_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\theta^3} = \lim_{\theta\to 0} \frac{\left(\theta+\frac{\theta^3}{3}+\frac{2\theta^5}{15}+\cdots\right)-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)}{\theta^3}$$

$$= \lim_{\theta\to 0} \frac{\frac{\theta^3}{3}+\frac{\theta^3}{3!}+\frac{2\theta^5}{15}-\frac{\theta^5}{5!}+\cdots \text{higher powers of } \theta}{\theta^3}$$

$$= \lim_{\theta\to 0} \frac{\theta^3\left(\frac{1}{3}+\frac{1}{3!}\right)+\theta^5\left(\frac{2}{15}-\frac{1}{5!}\right)}{\theta^3} + \text{higher powers of } \theta.$$

$$= \lim_{\theta\to 0} \left(\frac{1}{3}+\frac{1}{3!}\right)+\theta^2\left(\frac{2}{15}-\frac{1}{5!}\right) + \text{higher powers of } \theta$$

$$= \frac{1}{3}+\frac{1}{3!} = \frac{1}{3}+\frac{1}{6}$$

$$= \frac{2+1}{6} = \frac{3}{6}$$

$$\lim_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\theta^3} = \frac{1}{2}$$

———x———x———