Inst Hour	: 4
Credit	: 3
Code	: 18K2CH/PAM3

:R : II
:OURSE : III - Mathematics

DIFFERENTIAL EQUATIONS AND TRANSFORMS

(For B. Sc., Physics & Chemistry Major)

s of first order but of higher degree – Equations solvable for dy/dx, Equations solvable for ions Solvable for y, Clairaut's form (simple cases only) –Linear equations with constant nts – Definitions – The Operator D, Complementary Function of a Linear Equation with Coefficients, General Method of finding Particular Integral, Special Methods for finding ar Integral.

:r 4 Sec: 1, 2, 2.1, 2.2, 3.1, of Text Book1 & Chapter 5 Sec 1-4, 4.1, 4.2of Text Book1)

2 :

Differential Equations- Definition – Derivation of Partial Differential Equations: By ating arbitrary constants, By the elimination of arbitrary functions – Different Integrals of Differential Equations – Solutions of Partial Differential equations in some simple cases – ard type of First Order Equations – Standard 1, Standard 2, Standard 3, Standard 4 – Clairaut's

.

ding : Differential equations - Chapter 6 Sec: 1, 2, 2.1,2.2, 3, 4, 5, 5.1-5.4 of Text Book 2)

T 3 :

.ace Transform –Definition –L(e^{at}), L(cos(at)), L (sin (at)), L(t^n) , where n is a positive integer. $L[e^{-at}\ cosbt]$, $L[e^{-at}\ sinbt]$, ıe general theorems in Laplace Transforms (formula only) – $^{-at} f(t)]$, L [f'(t)],L[f''(t)], L [f^n(t)] and related Problems

eading : Differential equations - Chapter 4: Sec1, 2, 3 Text Book2)

NIT 4:

verse Laplace Transforms relating to the above standard forms – Modified results to get the Inverse ansforms of Functions - Solving Ordinary Differential Equations with constant coefficients using aplace Transforms.

Heading : Differential equations - Chapter 4 Sec4, 5, 6 Text Book2)

JNIT 5:

Fourier series- definition-Finding Fourier series expansion of periodic functions with Period 2π – Odd & Even functions in Fourier series – Half Range Fourier Series – Development in Sine series – Development in cosine series.

(Heading : Integral calculus - Chapter 4 Sec1, 2, 3, 3.1, 3.2, 4, 5.1, 5.2 Text Book2)

Text Books :

[1] S.Narayanan, T.K.Manickavasagom Pillai, Differential Equations, Viswanatham Publishers, 2001
[2] S.Narayanan, T.K.Manickavasagom Pillai, Ancillary Mathematics Book2.

Reference Books:

[1] S.Arumugam , Issac ,Trigonometry & Fourier Series
[2] B.R.Subramanian, Laplace Transform

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
... 3 out of 5. One Question from each Unit.

18K2CH/PAM3

1

DIFFERENTIAL EQUATIONS AND TRANSFORMS

UNIT - I

Differential Equations of First order and higher degree

We know that a differential Equation of first order in which the differential coefficient $\frac{dy}{dx}$ occur is of the form.

$$f(x, y, \frac{dy}{dx}) = 0 \quad \text{——} (1)$$

If we denote $P = \frac{dy}{dx}$ eqn(1) can be written as

$$f(x, y, P) = 0$$

This differential Equation is also called a differential equation of first order and first degree.

Equation solvable for $\frac{dy}{dx}$ (or) P

we shall denote $\frac{dy}{dx}$ hereafter by P.

Let the equation of the first order and of the n^{th} degree be

$$P^n + P_1 P^{n-1} + P_2 P^{n-2} + \dots + P_n = 0 \quad \text{——} (1)$$

where $P_1 P_2 \dots P_n$ denote functions of x and y

Hence the solution of eqn (1) is

$$\phi(x, y, c), \phi_2(x, y, c) \dots \phi_n(x, y, c) = 0$$

Example: 1

Solve $P^2 - 3P + 2 = 0$

Proof: Let us solve the given equation for P, since the given eqn is Quadratic in P, we have

$$P^2 - 3P + 2 = 0$$

$$P = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 2 \text{ or } 1$$

$P = 2$	$P = 1$
$\frac{dy}{dx} = 2$	$\frac{dy}{dx} = 1$
$\int dy = \int 2\,dx$	$\int dy = \int dx$
$y = 2x + c_1$	$y = x + c_2$
$y - 2x - c_1 = 0$	$y - x - c_2 = 0$

Hence the solution is

$$(y - 2x - c_1)(y - x - c_2) = 0$$

Example: 2

solve $x^2 P^2 + 3xyP + 2y^2 = 0$

proof $x^2 P^2 + 3xyP + 2y^2 = 0$

$$x^2 P^2 + xyP + 2xyP + 2y^2 = 0$$

$3xyP$	$2x^2y^2P^2$
xyP	$2xyP$

$xP(xP+y) + 2y(xP+y) = 0$

$(xP+y)(xP+2y) = 0$

$xP+y = 0 \qquad xP+2y = 0$

$xP = -y \qquad xP = -2y$

$P = -y/x \qquad P = -2y/x$

$\frac{dy}{dx} = -y/x \qquad \frac{dy}{dx} = -2y/x$

$\int \frac{dy}{y} = \int \frac{-dx}{x} \qquad \int \frac{dy}{y} = \int -2\, dx/x$

$\log y = -2\log x + \log c_2$

$\log y = -\log x + \log c_1, \qquad \log y + \log x^2 = \log c_2$

$\log y + \log x = \log c_1 \qquad \log y x^2 = \log c_2$

$\log xy = \log c_1 \qquad yx^2 - c_2 = 0.$

$xy - c_1 = 0$

$\therefore$ The solution is $(xy - c_1)(yx^2 - c_2) = 0$

Example : 3

Solve $p^2 + (x+y-\frac{2y}{x})p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$

Proof : $p^2 + (x+y-2y/x)p + xy + y^2/x^2 - y - y^2/x = 0.$

$$p = \frac{-(x+y-2y/x) \pm \sqrt{(x+y-2y/x)^2 - 4\cdot 1\cdot (xy + \frac{y^2}{x^2} - y - y^2/x)}}{2\cdot 1}$$

$$= \frac{-(x+y-2y/x) \pm \sqrt{x^2+y^2+\cancel{\frac{4y^2}{x^2}} + 2xy - \cancel{\frac{4y^2}{x}} - \cancel{4y} - 4xy - \cancel{4y^2/x^2} + \cancel{4y} + \cancel{4y^2/x}}}{2}$$

$$= \frac{-(x+y-2y/x) \pm \sqrt{x^2+y^2-2xy}}{2} = \frac{-(x+y-\frac{2y}{x}) \pm \sqrt{(x-y)^2}}{2}$$

$$= \frac{-(x+y-\frac{2y}{x}) \pm (x-y)}{2} = \frac{-\cancel{x}-y+\frac{2y}{x}+\cancel{x}-y}{2}, \frac{-x-\cancel{y}+\frac{2y}{x}-x+\cancel{y}}{2}$$

$$p = \frac{-2y+\frac{2y}{x}}{2}, \frac{-2x+\frac{2y}{x}}{2} = y/x - y, \quad y/x - x$$

$P = y/x - y$

$\frac{dy}{dx} = \frac{y}{x} - y$

$\frac{dy}{dx} = y(1/x - 1)$

$\int \frac{dy}{y} = \int (1/x - 1)\, dx$

$\log y = \log x - x + \log c$

$p = y/x - x$

$\frac{dy}{dx} = y/x - x$

$\frac{dy}{dx} - \frac{y}{x} = -x$

$\frac{dy}{dx} + Py = Q.$

$e^{\int P dx} = e^{\int -1/x\, dx} = e^{-\log x} = 1$

general solution

$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

$y/x = \int -x \cdot 1/x \, dx + c$

$y/x = -x + c$

$y = -x^2 + xc$

3 $\quad y + x^2 - xc = 0$

$\ln y - \ln x = -x + \ln c.$

$\ln y/x = -x + \ln c$

$\ln y/x = \ln(ce^{-x})$

$y = cxe^{-x}$

$\therefore$ The solution is $\quad y = cxe^{-x}, \quad y + x^2 - xc = 0.$

Equation solvable for x

$f(x, y, P) = 0$ can be put in the form $y = f(x, P)$

Example: 1

solve $x = y + a \log P$ —①

Proof: $x = y + a \log P$

Diff. w.r.to y we get,

$$\frac{dx}{dy} = 1 + a\frac{1}{P} \cdot \frac{dP}{dy}$$

$$\frac{1}{P} - 1 = \frac{a}{P}\frac{dP}{dy}$$

$$\frac{1-P}{P} = \frac{a}{P}\frac{dP}{dy}$$

$$\int dy = -a\int \frac{dP}{P-1}$$

$$y = -a \log(P-1) + c \quad \text{—②}$$

sub ② in ① we get,

$$x = c - a\log(P-1) + a\log P$$

$$x = c + a\log\frac{P}{P-1}$$

$$y = c - a\log P - 1$$

Example: 2

Solve $x = P^2 + y$

Proof: Given $x = P^2 + y$

Diff. w. to y we get,

$$\frac{dx}{dy} = 2P \cdot \frac{dP}{dy} + 1$$

$$\frac{1}{P} = 2P\frac{dP}{dy} + 1$$

$$2P\frac{dP}{dy} = 1/P - 1 = \frac{1-P}{P}$$

$$\frac{2P^2}{1-P}\frac{dP}{dy} = 1$$

$$\int \frac{2P^2}{1-P}\, dP = \int dy$$

$$\int\left(-P - 1 + \frac{1}{1-P}\right) dP = \int dy$$

$$\text{Sing}\quad 2\int\left(-P-1+\frac{1}{1-P}\right) \quad -2\int(P+1+1/(P-1))\, dP = y$$

$$-2\left[\frac{P^2}{2} + P + \log(P-1)\right] = $$

$$y = c - 2\left[\frac{P^2}{2} + P + \log(\right.$$

$$x = P^2 + y.$$

Solve $x = 1 - \frac{p}{\sqrt{p^2+1}}$

Proof:

Given $x = 1 - \frac{p}{\sqrt{p^2+1}}$ —— (1)

Diff w.to y we get,

$$\frac{dx}{dy} = \frac{-\sqrt{p^2+1}\,\frac{dp}{dy} + p\,\frac{1}{2\sqrt{p^2+1}}\,2p\cdot\frac{dp}{dy}}{p^2+1}$$

$$= \frac{dp}{dy}\left[\frac{-(p^2+1)+p^2}{\sqrt{p^2+1}\,(p^2+1)}\right]$$

$$\frac{1}{p} = \frac{-dp}{dy}\,\frac{1}{(p^2+1)^{3/2}}$$

$$\int dy = -\int \frac{p}{(p^2+1)^{3/2}}\,dp$$

Put $p^2+1 = t^2$

$2p\,dp = 2t\,dt$. $p\,dp = t\,dt$

$$\int dy = -\int \frac{t\,dt}{t^3} = -\int \frac{dt}{t^2} = \frac{1}{t} + c$$

$$y = 1/t + c$$

$$y = \frac{1}{\sqrt{p^2+1}} + c$$

$$(y-c)^2 = \frac{1}{p^2+1}$$

from (1), $x - 1 = -\frac{p}{\sqrt{p^2+1}}$

$$(x-1)^2 = \frac{p^2}{p^2+1}$$

$$(x-1)^2 + (y-c)^2 = 1$$

Equation Solvable for y

Example : 1

Solve $y = 3x + \log p$ —— (1)

Diff. w. to x

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\frac{dp}{dx} = p(p-3)$$

$$\int \frac{dp}{p(p-3)} = \int dx$$

$$\int\left(\frac{-1/3}{p} + \frac{1/3}{p-3}\right)dp = \int dx$$

$$-\frac{1}{3}\log p + \frac{1}{3}\log(p-3) = x + c$$

$$\frac{p-3}{p} = e^{3x+c_1}$$

$$1 - \frac{3}{p} = e^{3x+c_1}$$

$$1 - e^{3x}\cdot e^{c_1} = \frac{3}{p}$$

$$1 - c_2 e^{3x} = \frac{3}{p}$$

$$p = \frac{3}{1-c_2 e^{3x}}$$ —— (2)

sub (2) in (1)

$$y = 3x + \log \frac{3}{1-c_2 e^{3x}}$$

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1. Evaluate $\frac{d^2y}{dx^2} - 4y = 6e^{5x}$

Solution: The given equation is

$$\frac{d^2y}{dx^2} - 4y = 6e^{5x}$$

$$D^2 - 4 = 0$$

The Auxillary equation is

$$m^2 - 4 = 0$$

$$m^2 = 4$$

$$m = \pm 2$$

The complementary equation $y_c = Ae^{2x} + Be^{-2x}$

The particular equation $PI = y_p = \frac{1}{D^2-4} 6e^{5x}$

$$= 6 \frac{1}{D^2-4} e^{5x}$$

$$= 6 \frac{1}{5^2-4} e^{5x}$$

$$= \frac{6}{21} e^{5x}$$

$$= \frac{2}{7} e^{5x}$$

The general solution

$$y = y_c + y_p = Ae^{2x} + Be^{-2x} + \frac{2}{7} e^{5x}$$

2. Evaluate $(D^2 - 5D + 4)y = 0$

Solution: The given equation is $D^2 - 5D + 4 = 0$

The Auxillary equation is $m^2 - 5m + 4 = 0$

$$(m-1)(m-4) = 0$$

$$m = 1, 4$$

Complementry equation $y_c = c_1 e^x + c_2 e^{4x}$

The general solution is $y = c_1 e^x + c_2 e^{4x}$

3. Evaluate $(D^2-3D+2)y = e^{3x}$

Solution: The given equation is $(D^2-3D+2)y = e^{3x}$

The Auxillary equation is $m^2-3m+2=0$.

$$(m-1)(m-2)=0.$$

$$m=1,2.$$

complementry function $y_c = c_1e^{x} + c_2e^{2x}$

particular equation $y_p = \frac{1}{D^2-3D+2}e^{3x}$

$$= \frac{1}{3^2-3(3)+2}e^{3x}$$

$$= \frac{1}{9-9+2}e^{3x}$$

$$= \frac{e^{3x}}{2}$$

The general solution is

$$y = y_c + y_p = c_1e^{x} + c_2e^{2x} + \frac{e^{3x}}{2}$$

4. Evaluate $(D^2-6D+13)y = 5e^{2x}$

Solution: The given equation is $(D^2-6D+13)y = 5e^{2x}$

The Auxillary equation is $m^2-6m+13=0$

$$m = \frac{6 \pm \sqrt{36-4(1)(13)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-52}}{}$$

$= \frac{6 \pm i2\sqrt{3}}{2}$

$= 3 \pm i\sqrt{3}$

The complementry equation $= e^{5x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$

particular equation $= \frac{1}{(D^2 - 6D + 13)} 5e^{2x}$

$= 5 \frac{1}{2^2 - 6(2) + 13}$

$= \frac{5}{4 - 12 + 13} e^{2x}$

$= \frac{5e^{2x}}{5}$

$= e^{2x}$.

The general solution

$y = y_c + y_p$

$y = e^{3x}(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + e^{2x}$

Evaluate $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

olution: The given equation is $(D^2 + 5D + 4)y = x^2 + 7x + 9$

The Auxillary equation is $m^2 + 5m + 4 = 0$

$(m+4)(m+1) = 0$

$m = -4, -1$

The complementry function $y_c = Ae^{-4x} + Be^{-x}$

particular equation $y_p = \frac{1}{D^2 + 5D + 4} x^2 + 7x + 9$

$= \frac{1}{4\left(1 + \frac{D^2 + 5D}{4}\right)^{-1}} (x^2 + 7x + 9)$

xe^{-x}

$$= \frac{1}{4}\left[1+\left(\frac{D^2+5D}{4}\right)\right]^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2 - \cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{5D}{4}-\frac{D^4}{16}+\frac{25D^2}{16}\cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{D^2}{4}(x^2+7x+9)+\frac{5}{4}D(x^2+7x+9)\cdots\right]$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{2}-\frac{10}{4}x-\frac{35}{4}+\frac{50}{16}\right]$$

$$= \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

$$= \frac{1}{32}\left[8x^2+36+23x\right]$$

The general solution is

$$y = y_c + y_p = Ae^{-4x} + Be^{-x} + \frac{1}{32}(8x^2+36+23x).$$

6. Evaluate $(D^2-4D+3)y = \sin 3x \cos 2x$.

Solution: The given equation is $(D^2-4D+3)y = \sin 3x \cos 2x$

The Auxillary equation is $m^2-4m+3=0$

$$(m-3)(m-1)=0$$

$$m = 3, 1$$

The complementry equation $y_c = C_1e^{x} + C_2e^{3x}$

particular equation $y_p = \frac{1}{D^2-4D+3}\sin 3x \cos 2x$

$\therefore \sin 3x \cos 2x = \frac{\sin 5x + \sin x}{2}$ $\quad \sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$

$$= \frac{\sin 5x}{2} + \frac{\sin x}{2}$$

$$= \frac{1}{2}\sin 5x + \frac{1}{2}\sin x$$

$$PI_1 = \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4D+22} \times \frac{4D-22}{4D-22}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{16D^2-484}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{-400-484}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{2(2D-11)}{884}\sin 5x\right]$$

$$= \frac{2D-11}{884}\sin 5x$$

$$= \frac{2D(\sin 5x)-11\sin 5x}{884}$$

$$PI_1 = \frac{10\cos 5x-11\sin 5x}{884}$$

$$PI_2 = \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4D-2}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{(4D+2)}{16D^2-4}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-2(2D+1)}{-20}\sin x\right]$$

$$= \frac{2D+1}{20}\sin x.$$

$$= \frac{2D(\sin x)+\sin x}{20.}$$

$$PI_2 = \frac{2\cos x+\sin x.}{20.}$$

The general solution is

$$y = y_c + y_p = Ae^x + Be^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

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7. Example $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

Solution Let $x = e^z$

$\log x = z$

$(D(D-1) + 4D + 2)y = e^x$

$(D^2 - D + 4D + 2)y = e^x$

$(D^2 + 3D + 2)y = e^x$

The Auxillary equation is $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = 2, -1$

complementary function $y_c = Ae^{-x} + Be^{-2x}$

Particular equation $y_p = \frac{1}{(\theta+1)(\theta+2)} e^x$ here $D = \theta = x$

$= \frac{1}{(\theta+1)(\theta+2)} e^x$

$= \left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right]$

$= x^{-1}\int e^x dx - x^{-2}\int xe^x dx$

$= x^{-1}e^x - x^{-2}(xe^x - e^x)$

$= x^{-2}e^x$

$\therefore y = Ae^{-x} + Be^{-2x} + x^{-2}e^x$

UNIT - III

Partial Differential Equations

Partial Differential Equations are those which involve one or more partial derivatives. The order of a partial differential equation is determined by the highest order of the partial derivatives occurring in it. Partial differential equations involving one dependent variable z and only two independent variables x and y.

$$p=\frac{\partial z}{\partial x} \quad q=\frac{\partial z}{\partial y} \quad s=\frac{\partial^2 z}{\partial x \partial y} \quad r=\frac{\partial^2 z}{\partial x^2} \quad t=\frac{\partial^2 z}{\partial y^2}$$

Derivation of partial differential Equations

Partial differential equations can be derived by the elimination of (i) arbitrary constants from a relation between x,y,z (ii) arbitrary functions of these variables.

By Eliminating arbitrary constants

Consider the function $f(x,y,z,a,b)=0$ containing two [illegible] arbitrary constants a and b.

Example: 1 Eliminate a and b from $z=(x+a)(y+b)$ —①

Sol: Differentiating partially with respect x and y.

$$\frac{\partial z}{\partial x}=y+b \qquad \frac{\partial z}{\partial y}=x+a$$

$$p=y+b \qquad q=x+a$$

Sub. p and q in eqn ①, $\boxed{z=pq}$

Example: 2 obtain the partial differential equation of all spheres whose centres lie on the plane $z=0$ and whose radius is constant and equal to r.

Proof: The equation of the given sphere is

$$(x-a)^2+(y-b)^2+z^2=r^2 \quad —①$$

where a and b are independent arbitrary constants and r is a fixed given constant.

Diff. partially w.r.to x and y

$$2(x-a)+2z\frac{\partial z}{\partial x}=0 \qquad 2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$(x-a)+pz=0 \qquad (y-b)+qz=0$$

$$(x-a)=-pz \qquad (y-b)=-qz$$

$$(-pz)^2+(-qz)^2+z^2=r^2$$

$$p^2z^2+q^2z^2+z^2=r^2$$

$$\boxed{z^2(p^2+q^2+1)=r^2}$$

By the elimination of arbitrary functions:

Let u and v be any two functions of x, y, z and be connected by an arbitrary relation $\phi(u,v)=0$

Example: 1 Eliminate the arbitrary function from $z=f(x^2+y^2)$

Proof: The given eqn is $z=f(x^2+y^2)$

Diff. partially w.r.to x and y.

$$\frac{\partial z}{\partial x}=f'(x^2+y^2)\,2x \qquad \frac{\partial z}{\partial y}=f'(x^2+y^2)\,2y$$

$$\frac{p}{2x}=f'(x^2+y^2) \qquad \frac{q}{2y}=f'(x^2+y^2)$$

$$\therefore \frac{p}{\cancel{2}x}=\frac{q}{\cancel{2}y} \qquad \boxed{py=qx}$$

Example: 2 Eliminate the arbitrary function f and ϕ from the relation $z=f(x+ay)+\phi(x-ay)$

Proof: The given equation is $z=f(x+ay)+\phi(x-ay)$

Diff. partially w.r.to x and y.

$$\frac{\partial z}{\partial x}=f'(x+ay)+\phi'(x-ay), \qquad \frac{\partial z}{\partial y}=f'(x+ay)a+\phi'(x-ay)(-a)$$

$$p=f'(x+ay)+\phi'(x-ay), \qquad q=af'(x+ay)-a\phi'(x-ay)$$

Again Diff. w.r.to x and y.

$$\frac{\partial^2 z}{\partial x^2}=f''(x+ay)+\phi''(x-ay) \qquad \frac{\partial^2 z}{\partial y^2}=a^2f''(x+ay)+a^2\phi''(x-ay)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2}=a^2\frac{\partial^2 z}{\partial x^2}}$$

Solutions of Partial differential equation in some simple cases.

Example: 3 $\frac{\partial^2 z}{\partial x\partial y}=0$

Proof: $\frac{\partial^2 z}{\partial x\partial y}=0 \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

Hence $\frac{\partial z}{\partial y}=f(y)$ when f is an arbitrary function

$$\int\frac{\partial z}{\partial y}=\int f(y)\,dy+\phi(x)$$

$$z=F(y)+\phi(x)$$

Hence $F(y)$ and $\phi(x)$ are arbitrary functions

Different Integrals of partial differential Equation.

Complete Integral

A solution containing as many arbitrary constants as there are independent variables is called a complete Integral.

Particular Integral:

A solution obtained by giving particular values to the arbitrary constants in a complete Integral is called particular Integral.

singular Integral.

Let $f(x, y, z, p, q) = 0$. be the partial differential Equation whose complete integral is

$$\phi(x, y, z, a, b) = 0.$$

we eliminate a and b between $\frac{\partial\phi}{\partial a} = 0,\ \frac{\partial\phi}{\partial b} = 0$ when it exists is called. the singular Integral.

General Integral.

The complete Integral $\phi(x, y, z, a, f(a)) = 0$. Put $b = f(a)$

Diff. partially w.r.to a. $\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} f'(a) = 0.$ and if it

we eliminate a between these two equations. integral. is exists is called. the general

Example: Solve $\frac{\partial^2 z}{\partial y} = \sin y$

Proof: The given eqn is $\frac{\partial^2 z}{\partial y} = \sin y$

$$\int \text{ing} \quad \frac{\partial z}{\partial y} = -\cos y + f(x)$$

$$\text{Again } \int \text{ing} \quad z = -\sin y + y f(x) + \phi(x)$$

where f and ϕ are arbitrary functions.

when $x=0$.

Example: 5 Solve $x + y\frac{\partial z}{\partial x} = 0$.

Proof: the given equation is $x + y\frac{\partial z}{\partial x} = 0$

$$\frac{\partial z}{\partial x} = \frac{-x}{y}$$

Sing
$$z = \frac{-x^2}{2y} + \varphi(y)$$

Example: 6 Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

Proof: The given equation is $\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$

Sing $\frac{\partial z}{\partial x} = x^2 y + \frac{y^3}{3} + \varphi(x)$

Again Sing $z = \frac{x^3 y}{3} + \frac{x y^3}{3} + F(x) + f(y)$

Example: 7 Solve $x\frac{\partial z}{\partial x} = 2x + y + 3z$.

Proof: The given equation is $x\frac{\partial z}{\partial x} = 2x + y + 3z$

This eqn. can be written in the form.

$$\frac{\partial z}{\partial x} = \frac{2x}{x} + y/x + \frac{3z}{x}.$$

$$\frac{\partial z}{\partial x} - 3\frac{z}{x} = 2 + y/x.$$

This is a linear equation.

The integrating factor. $e^{\int P dx} = e^{\int -3/x \, dx} = e^{-3\int 1/x \, dx}$

$$= e^{-3\log x} = e^{\log x^{-3}} = 1/x^3.$$

Hence $1/x^3\left(\frac{\partial z}{\partial x} - \frac{3z}{x}\right) = \frac{2}{x^3} + \frac{y}{x^4}$

$$\frac{\partial}{\partial x}\left(\frac{z}{x^3}\right) = \frac{2}{x^3} + y/x^4$$

$$\frac{z}{x^3} = -1/x^2 - y/3x^3 + \varphi(y)$$

$$z = -x - y/3 + x^3\varphi(y)$$

Example : 8 Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$.

proof: $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

If z is a function of x alone, the solution would be,

$$z = Ae^{ax} + Be^{-ax}.$$

where a and b are constants.

Here z is a function of x and y. hence the solution of the equation is

$$z = f(y) e^{ax} + \phi(y) e^{-ax}.$$

$$\frac{\partial z}{\partial x} = f(y) a e^{ax} - a\phi(y) e^{-ax}.$$

$$\frac{\partial z}{\partial y} = f'(y) e^{ax} + \phi'(y) e^{-ax}$$

when $x = 0$ $\frac{\partial z}{\partial x} = a \sin y$.

$$a f(y) - a \phi(y) = a \sin y$$

$$f(y) - \phi(y) = \sin y \quad \text{—} (1)$$

when $x = 0$, $\frac{\partial z}{\partial y} = 0$.

$$f'(y) + \phi'(y) = 0. \quad \text{—} (2)$$

Diff. (1) we get,

$$f'(y) - \phi'(y) = \cos y \quad \text{—} (3)$$

from (2) and (3) $f'(y) - \frac{1}{2} \cos y$.

$$\phi'(y) = -\frac{1}{2} \cos y.$$

$$f(y) = \frac{1}{2} \sin y + A \qquad \phi'(y) = -\frac{1}{2} \sin y + B. \quad \text{—} (4)$$

from (1) and (4) $A = B$,

$$z = \frac{1}{2} \sin y \, e^{ax} - \frac{1}{2} \sin y \, e^{-ax} + Ae^{ax} + Ae^{-ax}.$$

$$z = \sin y \sinh ax + 2A\cosh ax.$$

Standard type of first order equations:

Standard : 1

The variables x, y, z do not occur explicity.

Such equations are of the form.

$$f(p, q) = 0 \quad \text{where} \quad p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}.$$

we can easily verify that $z = ax + by + c$ is the solution of the equation $f(p, q) = 0$ provided. $f(a, b) = 0$.

solving $b = F(a)$, then we get complete and singular Integral.

Example : 1 solve $p^2 + q^2 = npq$.

Proof : The given equation is $p^2 + q^2 = npq$.

The general solution is $z = ax + by + c$

where $a^2 + b^2 = nab$. put $p = a$ $q = b$

$$\text{solving} \quad b = \frac{a(n \pm \sqrt{n^2 - 4})}{2}.$$

The complete Integral is.

$$y = ax + \frac{ay}{2}(n \pm \sqrt{n^2 - 4}) + c.$$

To find general Integral. put $c = f(a)$

$$z = ax + \frac{ay}{2}(n \pm \sqrt{n^2 - 4}) + f(a)$$

Diff. partially w.r.to a.

$$0 = x + \frac{y}{2}(n \pm \sqrt{n^2 - 4}) + f'(a)$$

The eliminate of a between these equations gives the general Integral.

Standard - 2

The only one of the variables x, y, z occurs explicity.

$F(x, p, q) = 0$, $F(y, p, q) = 0$; $F(z, p, q) = 0$.

Example : 1 $q = xp + p^2$

Let $q = a$. then $a = xp + p^2$

$$p^2 + xp - a = 0.$$

$$p = \frac{-x \pm \sqrt{x^2 - 4a}}{2}.$$

$$dz = \frac{-x \pm \sqrt{x^2 - 4a}}{2} dx + a\,dy.$$

Sing

$$z = \int \frac{-x \pm \sqrt{x^2 - 4a}}{2} dx + ay + b$$

$$= -\frac{x^2}{4} \pm \left\{ \frac{x}{4}\sqrt{4a + x^2} + a \sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) \right.$$
$$+ ay + b.$$

Example : 2

$$p = y^2 q^2$$

Let $p = a^2$ $q = \pm a/y$.

$$dz = a^2 dx \pm \frac{a}{y} dy.$$

Sing $z = a^2 x \pm a \log y + b.$

Example : 3

$$p(1 + q^2) = q(z - 1)$$

Let $q = ap$. then.

$$p(1 + a^2 p^2) = ap(z - 1)$$

$$1 + a^2 p^2 = a(z - 1)$$

$$p = \pm \frac{\sqrt{az - a - 1}}{a}.$$

Example: Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Proof:

$$dz = \frac{\pm\sqrt{ax-a-1}}{a}\,dx \pm \sqrt{az-a-1}\,dy.$$

i.e., $$\pm \frac{a\,dz}{\sqrt{a\cancel{x}-a-1}} = dx + a\,dy.$$

i.e., $$\pm \int \frac{a\,dz}{\sqrt{az-a-1}} = x + ay + b.$$

$$\pm 2\sqrt{az-a-1} = x + ay + b.$$

Standard: 3

Equation of the form $f_1(x,p) = f_2(y,q)$

Example: 4 Solve the equation $p+q = x+y$.

we can write the equation in the form $p-x = y-q$.

Let $p-x = a$ then $y-q = a$.

Hence $p = x+a$ $q = y-a$.

$$dz = (x+a)\,dx + (y-a)\,dy.$$

$$\text{Sly } z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b.$$

Standard: 4 (Clairaut's form)

This is of the form $z = px + qy + f(p,q)$.

Example: Solve $z = px + qy + \sqrt{1+p^2+q^2}$.

The complete integral is

$$z = ax + by + \sqrt{1+a^2+b^2}$$

to find the singular integral, diff. partially with respect to a and b.

$$x + \frac{a}{\sqrt{1+a^2+b^2}} = 0.$$

$$y + \frac{b}{\sqrt{1+a^2+b^2}} = 0.$$

Eliminating a and b the singular integral is.

$$x^2+y^2+z^2 = 1.$$

To find general integral assume $b = f(a)$ where f is arbitrary.

then $z = ax + f(a)y + \sqrt{1+a^2+(f(a))^2}$

Diff. partially with respect to a and eliminate a between the two equations.

UNIT - III

LAPLACE TRANSFORM

Definition: If a function $f(t)$ is defined for all positive values of the variable t and if $\int_0^\infty e^{-st} f(t)$ exists and is equal to $F(s)$ then $F(s)$ is called the Laplace Transform of $f(t)$ and is denoted by the symbol $L\{f(t)\}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

The operator L that transform $f(t)$ into $F(s)$ is called the Laplace operator.

RESULTS: (i) $L\{f(t)+g(t)\} = L\{f(t)\} + L\{g(t)\}$

$$L\{f(t)+g(t)\} = \int_0^\infty (e^{-st} f(t) + g(t))\,dt.$$
$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt.$$
$$= L\{f(t)\} + L\{g(t)\}$$

(ii) $L\{f(t)\} = c\,L\{f(t)\}$ where c is a constant.

$$L\{c f(t)\} = \int_0^\infty e^{-st} c\, f(t)\,dt.$$
$$= c\int_0^\infty e^{-st} f(t)\,dt.$$
$$= c\,L\{f(t)\}$$

(iii) $L\{f'(t)\} = s\,L\{f(t) - f(0)\}$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt.$$
$$= \{f(t)e^{-st}\}_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt.$$

($\because$ By integrating By parts)

$$= -f(0) + s\int_0^\infty f(t)e^{-st}\,dt.$$
$$= s\,L\{f(t)\} - f(0)$$

(iv) $L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$

$$L\{f''(t)\} = L\{F'(t)\} \quad \text{when } F(t) = f'(t)$$
$$= s\,L[F(t)] - F(0)$$
$$= s\,L[f'(t)] - f'(0)$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$
$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

(1)

$$1 \quad L(e^{-at}) = \frac{1}{s+a} \text{ provided } s+a > 0.$$

Soln

Corollory: 1 $\quad L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt.$

$$= \int_0^\infty e^{-(s+a)t} dt.$$

$$= \left[-\frac{1}{s+a} e^{-(s+a)t} \right]_0^\infty$$

$$L(e^{-at}) = 1/s+a.$$

$$\text{IIIly} \quad L(e^{at}) = \frac{1}{s-a} \text{ provided } s-a > 0.$$

Corollory: 2 $\quad L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2} \right)$

$$= 1/2\, L(e^{at}) + 1/2\, L(e^{-at})$$

$$= 1/2 \frac{1}{s-a} + 1/2 \frac{1}{s+a}$$

$$L(\cosh at) = \frac{s}{s^2 - a^2}$$

(2)

$$L(\sinh at) = \frac{\cancel{s}}{s^2 - a^2}$$

Result: $\quad L(t^n) = \frac{n!}{s^{n+1}}$ where n is a positive integer.

We have, $L(t^n) = \int_0^\infty e^{-st} t^n dt.$

Put $st = x$ then $dt = 1/s\, dx$

$$L(t^n) = \int_0^x \left(\frac{x}{s}\right)^n e^{-x} 1/s\, dx.$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

Integrating by parts, we can easily show that,

$$\int_0^\infty x^n e^{-x} dx = \int_0^\infty x^{n-1} e^{-x} dx$$

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$$\int_0^\infty x^n e^{-x}\,dx = \int_0^\infty x^{n-1} e^{-x}\,dx.$$

$$= n(n-1)\int_0^\infty x^{n-2} e^{-x}\,dx.$$

$$= n(n-1)\cdots 2\cdot 1\int_0^\infty e^{-x}\,dx.$$

$$= n(n-1)\cdots 2\cdot 1\left[e^{-x}\right]_0^\infty$$

$$= n(n-1)\cdots 2\cdot 1$$

$$= n!$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Corollary $L(t) = 1/s^2$

$$L(t^2) = 2/s^3$$

$$L(1) = 1/s$$

Example: 1 Find $L(t^2+2t+3)$

proof: $L(t^2+2t+3) = L(t^2) + 2L(t) + 3L(1)$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$$

Example: 2 Find $L(\sin^2 2t)$

proof! $L(\sin^2 2t) = L\left(\frac{1-\cos 4t}{2}\right)$

$$= \frac{1}{2}L(1) - \frac{1}{2}L(\cos 4t)$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\frac{s}{s^2+4^2}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$= \frac{8}{s(s^2+16)}$$

Example 3: Find $L(e^{-at}\sin bt)$

Proof: $L(e^{-at}\sin bt) = \int_0^\infty e^{-at}\sin bt\, e^{-st}\,dt$

$$= \int_0^\infty e^{-(s+a)t}\sin bt\,dt$$

$$= \left[e^{-(s+a)t}\frac{-(s+a)\sin bt - b\cos bt}{(s+a)^2+b^2}\right]_0^\infty$$

$$= \frac{b}{(s+a)^2+b^2}$$

More generally, $L\{e^{-at}f(t)\} = F(s+a)$

where $F(s)$ is $L\{f(t)\}$.

Example 4: $L\{f(t)\} = f(s)$ then $L[F(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$ put $at = y$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{sy}{a}} f(y)\,dy$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right)$$

Examples:

1. $L(\cos at) = \frac{1}{a}F\left(\frac{s}{a}\right)$ where

$L(\cos t) = F(s) = \frac{s}{s^2+1}$

$L(\cos at) = \frac{1}{a}\frac{s/a}{\frac{s^2}{a^2}+1} = \frac{s}{s^2+a^2}$

$L(\sinh at) = \frac{1}{a}F\left(\frac{s}{a}\right)$

$L(\sinh t) = F(s) = \frac{1}{s^2-1}$

$L(\sinh at) = \frac{1}{a}\frac{1}{\frac{s^2}{a^2}-1} = \frac{a}{s^2-a^2}$

$L(1) = 1/s$

$L(e^{-at}) = \frac{1}{s+a}$

$L(\cos bt) = \frac{s}{s^2+b^2}$

$L(e^{-at}\cos bt) = \frac{s+a}{(s+a)^2+b^2}$

$L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$

$L(\sin bt) = \frac{b}{s^2+b^2}$

$L(e^{-at}\sin bt) = \frac{b}{(s+a)^2+b^2}$

$L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$

$L(t^n) = \frac{n!}{s^{n+1}}$ if n is positive integer.

$L(e^{-at}t^n) = \frac{n!}{(s+a)^{n+1}}$

$L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$

$L(t\sin at) = L\left\{ t\frac{(e^{at}-e^{-at})}{2i} \right\}$

$= \frac{1}{2i}\left\{ L(te^{at}) - L(te^{-at}) \right\}$

$= \frac{1}{2i}\left\{ \frac{1}{(s-ai)^2} - \frac{1}{(s+ai)^2} \right\}$

$= \frac{1}{2i}\left\{ \frac{(s+ai)^2-(s-ai)^2}{(s+a^2)^2} \right.$

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$$= \frac{1}{2i}\frac{4sai}{(s^2+a^2)^2} = \frac{2as}{(s^2+a^2)^2}$$

$$L(t^2\cos at) = L\left\{t^2\left(\frac{e^{at}+e^{-at}}{2}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{(s-ai)^3}+\frac{2}{(s+ai)^3}\right\}$$

$$= \frac{(s+ai)^3+(s-ai)^3}{(s^2+a^2)^2}$$

$$L(te^{-t}\sin t) = L(e^{-t}\,t\sin t)$$

$$= \frac{2(s+1)}{\{(s+1)^2+1^2\}^2} \quad \text{since } L(t\sin t) = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}$$

Problems:

$a=7$

1. $L(\sin 7t) =$

$$L[\sin at] = \frac{a}{s^2+a^2} = \frac{7}{s^2+7^2}$$

2. $L(\sin\sqrt{10}\,t) \quad a=\sqrt{10}$

$$L(\sin at) = \frac{a}{s^2+a^2} = \frac{\sqrt{10}}{s^2+(\sqrt{10})^2} = \frac{\sqrt{10}}{s^2+10.}$$

3. $L(\sin 3t\cos t) = L\left[\frac{\sin 4t+\sin 2t}{2}\right]$

$$= \tfrac{1}{2}L(\sin 4t) + \tfrac{1}{2}L(\sin 2t)$$

$$= \tfrac{1}{2}\left(\frac{4}{s^2+4}+\frac{2}{s^2+4}\right)$$

① Find $L[\cos^4 t]$

Solu:- $\cos^4 t = (\cos^2 t)^2 = \left(\frac{1+\cos 2t}{2}\right)^2$

$= \frac{1}{4}[1+2\cos 2t+\cos^2 2t]$

$= \frac{1}{4}\left[1+2\cos 2t+\frac{1+\cos 4t}{2}\right]$

$= \frac{1}{4}\left[1+2\cos 2t+\frac{1}{2}+\frac{\cos 4t}{2}\right]$

$= \frac{1}{4}\left[\frac{3}{2}+2\cos 2t+\frac{\cos 4t}{2}\right]$

$L[\cos^4 t] = \frac{1}{4}L\left[\frac{3}{2}+2\cos 2t+\frac{\cos 4t}{2}\right]$

$= \frac{1}{4}\left\{L\left[\frac{3}{2}\right]+L[2\cos 2t]+L\left(\frac{\cos 4t}{2}\right)\right\}$

$= \frac{1}{4}\left[\frac{3}{2s}+2\frac{s}{s^2+4}+\frac{1}{2}\frac{s}{s^2+16}\right]$

Find $L(7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2)$

Solu:- $= L(7e^{2t})+L(9e^{-2t})+L(5\cos t)$
$+L(7t^3)+L(5\sin 3t)+L(2)$

$= \frac{7}{x-2}+\frac{9}{s+2}+5\frac{s}{s^2+1}+7\frac{3!}{s^4}+5\frac{3}{s^2+9}+\frac{2}{s}$

Find $L[(t+1)^2]$

Solu:- $L[(t+1)^2] = L[t^2+2t+1] = L(t^2)+L(2t)+L(1)$

$= \frac{2}{s^3}+2\frac{1}{s^2}+\frac{1}{s} = \frac{2+2x+s^2}{s^3}$

④ find $L(\cosh at + \sin 2t + t^3)$

Solu:- $L(\cosh at + \sin 2t + t^3) = L(\cosh at) + L(\sin 2t) + L(t^3)$

$$= \frac{s}{s^2-a^2} + \frac{2}{s^2+4} + \frac{3!}{s^4}$$

⑤ find $L[t^2 + e^{-5t} + 8 + \sinh 5t]$

$$= L(t^2) + L(e^{-5t}) + L[8] + L[\sinh 5t]$$

$$= \frac{2!}{s^3} + \frac{1}{s+5} + \frac{8}{s} + \frac{5}{s^2-25}$$