I - CS

SEMESTER : I & II

ALLIED COURSE : II - Mathematics

Inst Hour	: 2 + (3)
Credit	: 3
Code	: 18K2CSAM2

INTEGRAL CALCULUS, DIFFERENTIAL EQUATIONS AND TRANSFORMS

(For B. Sc., Computer Science Major)

UNIT 1:

Properties of Definite Integrals – Integration by parts - Multiple integrals. (Simple problems only)

(Chapter 1 section 11,12, Chap5: 2.1,2.2,3,3.1,3.2,4 of Text Book1)

UNIT 2:

Fourier series for functions of period 2π– odd and even functions – Half range sine and cosine series and problems to the relevant concepts only.

(Chapter6:sec 1,2,3,3.1,3.2,4,5.1,5.2, of Text Book2)

UNIT 3:

First order first degree ordinary differential equations – Linear equations – Bernoulli's equations.

(Chapter 1: Sec 1.1-2.5 of Text Book2)

UNIT 4:

Equations of first order but of higher degree – simultaneous linear differential equations – second order differential equations with constant coefficients.

(Chapter 1: Sec 5, 5.1-7.3, Chapter 2: Sec 1-4 of Text Book2)

UNIT 5:

Laplace Transforms – Conditions for the existence of the Laplace Transforms – General theorems – Inverse transforms – Solving the second order ordinary differential equations with constant coefficients using the Laplace transforms (simple problems only).

(Chapter5: Sec 1, 1.1, 1.2, 2-12 of Text Book2)

Text Books:

[1] S.Narayanan , T.K.M.Pillai, Calculus Volume II , S.Viswanathan Publication 2015

[2] S.Narayanan , T.K.M.Pillai, Calculus Volume III, S.Viswanathan Publication 2015

Reference Books:

[1] A.Singaravelu, Calculus

[2] M.D.Raisinghania, Ordinary & Partial Differential Equations

[3] M.L.Khanna, Differential Equations

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.

Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.

Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

9.3.18

HOD

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Unit - 1

Properties of definite Integrals

The following list of formulae for integrals is based directly on the results of differentiation

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for all values of n

2. $\int e^x dx = e^x$

3. $\int \sin x \, dx = -\cos x$

4. $\int \cos x \, dx = \sin x$

5. $\int \sec^2 x \, dx = \tan x$

6. $\int \operatorname{cosec}^2 x \, dx = -\cot x$

7. $\int \sec x \tan x \, dx = \sec x$

8. $\int \cosh x \, dx = \sinh x$

9. $\int \sinh x \, dx = \cosh x$

10. $\int \frac{dx}{1+x^2} = \tan^{-1} x$ (or) $-\cot^{-1} x$

11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$ (or) $-\cos^{-1} x$

12. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x \, . \, \log(x+\sqrt{x^2-1})$

13. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x \, , \, \log(x+\sqrt{x^2+1})$

(x) 2m

Definite Integral

Let $\int f(x)\,dx = F(x)+C$ where C is arbitrary constant. The value of integral

when $x = b$ is $F(b)+C$

when $x = a$ is $F(a)+C$

Subtracting

$F(b) - F(a)$ = The value of the integral when $x = b$ − The value of the integral when $x = a$.

$\therefore \int_a^b f(x)\,dx$ denotes the value of the integral when $x = b$, minus the value of the integral when $x = a$

$$\boxed{\therefore F(b) - F(a) = \int_a^b f(x)\,dx}$$ is called the definite Integral.

a and b are called the limits of Integrating

Property : 1

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Proof :

$$\text{L.H.S} = \int_a^b f(x)\,dx$$

$$= \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a) \longrightarrow (1)$$

$$R.H.S = -\int_b^a f(x)\,dx = -\left[F(x)\right]_b^a$$

$$= -\left[F(a) - F(b)\right]$$

$$= F(b) - F(a) \longrightarrow (2)$$

From (1) and (2) we get,

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.}$$

Hence the result.

Property : 2

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Where c is same value of x between a and b

<u>Proof</u>

$$L.H.S = \int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$

$$= F(b) - F(a) \longrightarrow (1)$$

$$R.H.S = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \left[F(x)\right]_a^c + \left[F(x)\right]_c^b$$

$$= \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

$$= F(b) - F(a) \longrightarrow (2)$$

From (1) and (2) we get,

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.}$$

Hence the Result.

Property : 3

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

If $f(x)$ is an even function of x.

Proof :

If $f(x)$ is even, $f(x) = f(-x)$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \longrightarrow (1)$$

In the first Integral on the right apply on first

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)\,dx.$$

Put $-x = y \Rightarrow y = -x$

When $x = -a$, $y = a$	
$x = 0$, $y = 0$	

$$\frac{dy}{dx} = -1$$

$$dy = -dx$$

$$dx = -dy$$

$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(y)\,(-dy)$$

$$= -\int_{a}^{0} f(y)\,dy$$

$$= \int_{0}^{a} f(y)\,dy$$

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)\,dx. \longrightarrow (2).$$

Substitute (2) in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.}$$

Hence the results

Property : A

If $f(x)$ is an odd function of x,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Proof :

If $f(x)$ is an odd, $f(x) = -f(-x)$

We know that,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx. \longrightarrow (1)$$

In the first integral on the right. apply odd ~~even~~ function.

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} -f(-x)\,dx$$

Put $-x = y$

$y = -x$

$dy/dx = -1$

$dy = -dx$

$dx = -dy$

When $x = -a,\ y = a$

$x = 0,\ y = 0$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(y)\,(-dy)$$

$$= \int_{a}^{0} f(y)\,dy$$

$$= -\int_{0}^{a} f(y)\,dy$$

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \longrightarrow (2)$$

Substitute (2) in (1) we get,

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

$$\int_{-a}^{a} f(x)\,dx = 0$$

Hence the results.

Multiple Integrals

- Double integral – surface
- Triple integral – volume

1) 2 variable : $\int_{x=} \int_{y=} f(x,y)\,dx\,dy$

Dependent / Independent

ex : $\int_{x=0}^{2} \int_{y=1}^{\sqrt{x}} f(x,y)\,dx\,dy$ ex : $\int_{x=0}^{1} \int_{y=0}^{1} f(x,y)\,dx\,dy.$

Dependent variable:

1. $\int_{x=0}^{1} \int_{y=0}^{x} x\,dx\,dy = \int_{x=0}^{1} x\,dx \left(\int_{y=0}^{x} dy \right)$

$= \int_{0}^{1} x\,dx\,(y)_0^x$

$= \int_{0}^{1} x(x)\,dx$

$= \left(\frac{x^3}{3}\right)_0^1 = 1/3$

" y has x term "

* Solve → y first

Independent variable

$\int_{x=0}^{1} \int_{y=0}^{2} xy\,dx\,dy = \int_{x=0}^{1} x\,dx \cdot \int_{y=0}^{2} y\,dy$

$= \int_{x=0}^{1} x\,dx\,\left[y^2/2\right]_0^2$

$$= \left(\frac{x^2}{2}\right)_0^1 \cdot \left(\frac{y^2}{2}\right)_0^2$$

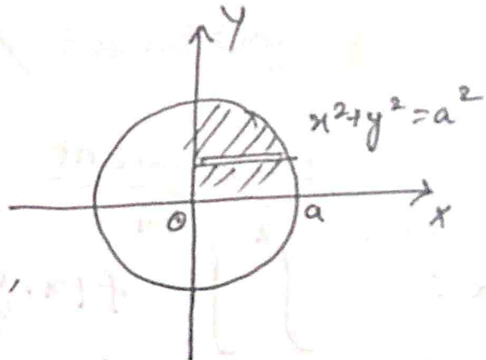
$$= \frac{1}{2} \cdot \frac{4}{2} = 1$$

Examples:

① Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$

solution:

Given $\iint xy\,dx\,dy$



If be y as constant, x varies from 0 to $\sqrt{a^2-y^2}$ to cover the whole area.

y varies from 0 to a

$$\iint xy\,dx\,dy = \int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_{y=0}^{y=a} \left[\int_{x=0}^{x=\sqrt{a^2-y^2}} x^2\,dx\right] y\,dy$$

$$= \int_{y=0}^{y=a} \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} y\,dy$$

$$= \int_{y=0}^{y=a} \frac{1}{2}\left[a^2-y^2-0\right] y\,dy$$

$$= \frac{1}{2}\int_0^a (a^2-y^2)\,y\,dy$$

$$= \frac{1}{2}\int_0^a (a^2y-y^3)\,dy$$

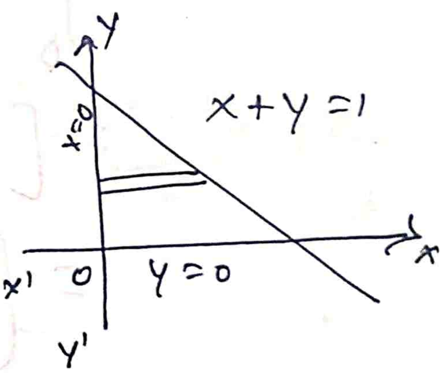
$$= \frac{1}{2}\left[a^2y^2/2 - y^4/4\right]_0^a$$

$$= \frac{1}{2}\left[a^2\cdot a^2/2 - a^4/4\right]$$

$$= 1/2\left[\frac{a^4}{4}\right] = a^4/8 \,//.$$

Ex: 2 Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x, y are earch ≥ 0 and $x+y \leq 1$.

solution



Given $x>0$, $y>0$ and

$x+y \leq 1$

$\Rightarrow y = 1-x$

The region is form by the lines $x=0$, $y=0$ and $x+y=1$.

Here x varries from 0 to 1 and y varries from 0 to $1-x$.

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (x^2+y^2)\,dy\,dx.$$

$$= \int_0^1 \left[x^2 y + y^3/3 \right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[x^3 - x^3 + 1/3 \left(1 - 3x + 3x^2 - x^3 \right) \right] dx$$

$$= \int_0^1 \left(x^2 + x^3 - x + x^2 - x^3/3 + 1/3 \right) dx$$

$$= \int_0^1 \left(2x^2 - x^3 (1 + 1/3) - x + 1/3 \right) dx$$

$$= \int_0^1 \left(2x^2 - 4/3\, x^3 - x + 1/3 \right) dx$$

$$= \left[2x^3/3 - 4/3 \frac{x^4}{4} - x^2/2 + 1/3 \right]_0^1$$

$$= \left[2 \cdot \frac{1}{3} - 4/3 \cdot 1/4 - 1/2 + 1/3 \right] - 0$$

$$= 2/3 - \cancel{1/3} - 1/2 + \cancel{1/3}$$

$$= 2/3 - 1/2$$

$$= \frac{4-3}{6}$$

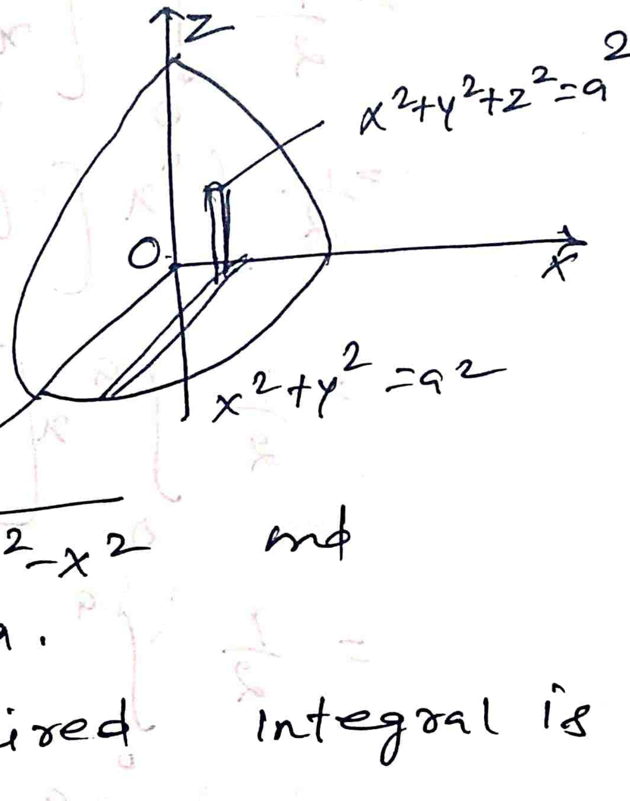
$$= 1/6$$

Section: 4 Triple Integrals

$$\int_R f(x,y,z)\,dV = \int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\varphi_2(y,z)}^{\varphi_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

① Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$

$\longrightarrow$



To cover the whole positive octant of the sphere $x^2+y^2+z^2=a^2$, z varies from 0 to $\sqrt{a^2-x^2-y^2}$, and y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

Hence the required integral is

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx.$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{xy}{2} \left(\sqrt{a^2-x^2-y^2}\right)^2 dy\, dx.$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,(a^2-x^2-y^2)\, dy\, dx.$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} x \left[(a^2-x^2)y - y^3\right] dy\, dx$$

$$= \frac{1}{2} \int_0^a \left[x\left[(a^2-x^2)\,y^2/2 - y^4/4\right]\right] dx$$

$$= \frac{1}{2} \int_0^a x \left[\frac{(a^2-x^2)}{2}(a^2-x^2) - \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2} \int_0^a x\left[\frac{(a^2-x^2)^2}{2} - \frac{(a^2-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2} \int_0^a x/4\, (a^2-x^2)^2\, dx$$

$$= \frac{1}{8} \int_0^a x\,(a^4+x^4-2a^2x^2)\, dx$$

$$= \frac{1}{8} \int_0^a (a^4x + x^5 - 2a^2x^3)\, dx$$

$$= \frac{1}{8}\left[a^4 x^2/2 + x^6/6 - \frac{2a^2x^4}{4}\right]_0^a$$

$$= \frac{1}{8}\left[\frac{a^4a^2}{2} + \frac{a^6}{6} - \frac{a^4a^2}{2}\right]$$

$$= \frac{1}{8}\left[\cancel{\frac{a^6}{2}} + \frac{a^6}{6} - \cancel{\frac{a^6}{2}}\right]$$

$$= \frac{a^6}{6}(1/8)$$

$$= \frac{a^6}{48}.$$

Exercise:

1. Evaluate $\iiint (x-y+z)\,dx\,dy\,dz$ where R given by $1 \le x \le 2$, $2 \le y \le 3$. $1 \le z \le 3$.

$$\rightarrow \int_1^2 \int_2^3 \int_1^3 (x-y+z)\,dx\,dy\,dz$$

$$= \int_1^3 \int_2^3 \int_1^2 (x-y+z)\,dx\,dy\,dz$$

$$= \int_1^3 \int_2^3 \left(x^2/2 + (-y+z)(x)\right)_1^2 dy\,dz$$

$$= \int_1^3 \int_2^3 \left[4/2 + (-y+z)(2)) - (1/2 + (-y+z)\right] dy\,dz$$

$$= \int_1^3 \int_2^3 \left[2 - 2y + 2z - \left(\tfrac{1}{2} - y + z \right) \right] dy\, dz$$

$$= \int_1^3 \int_2^3 \left[2 - 2y + 2z - \tfrac{1}{2} + y - z \right] dy\, dz$$

$$= \int_1^3 \int_2^3 \left[-y + z + \frac{4-1}{2} \right] dy\, dz$$

$$= \int_1^3 \int_2^3 \left(-y + z + 3/2 \right) dy\, dz$$

$$= \int_1^3 \left(-y^2/2 + (z + 3/2)\, y \right]_2^3 dz$$

$$= \int_1^3 \left[\left(-9/2 + (z + 3/2)(3) \right) - \left(-4/2 + (z + 3/2)(2) \right) \right] dz$$

$$= \int_1^3 \left[\left(-9/2 + 3z + 9/2 \right) - \left[-4/2 + 2z + 6/2 \right) \right] dz$$

$$= \int_1^3 \left[3z - \left(-2 + 2z + 3 \right) \right] dz$$

$$= \int_1^3 \left[3z - (2z + 1) \right] dz$$

$$= \int_1^3 \left[3z - 2z - 1 \right] dz$$

$$= \int_1^3 (z-1)\,dz$$

$$= \left[\frac{z^2}{2} - z\right]_1^3$$

$$= \left[\frac{9}{2} - 3\right] - \left[\frac{1}{2} - 1\right]$$

$$= \left[\frac{9-6}{2}\right] - \left[\frac{1-2}{2}\right]$$

$$= \frac{3}{2} - \left(\frac{-1}{2}\right)$$

$$= \frac{3}{2} + \frac{1}{2}$$

$$= \frac{3+1}{2} = \frac{4}{2}$$

$$= 2$$

② . Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ where R is the region bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ and $z=0$, $z=1$

$\rightarrow$ $$\int_0^1 \int_0^1 \int_0^1 (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^1 \left(\frac{x^2}{2} + (y+z)(x)\right)_0^1 dy\,dz$$

$$= \int_0^1 \int_0^1 \left(\frac{1}{2} + (y+z)(1)\right) dy\,dz$$

$$= \int_0^1 \int_0^1 \left(\tfrac{1}{2} + y + z\right) dy\, dz$$

$$= \int_0^1 \left(\tfrac{y^2}{2} + \left(z + \tfrac{1}{2}\right) y \right)_0^1 dz$$

$$= \int_0^1 \left(\tfrac{1}{2} + \left(z + \tfrac{1}{2}\right)(1) \right) dz$$

$$= \int_0^1 \left(\tfrac{1}{2} + z + \tfrac{1}{2} \right) dz$$

$$= \int_0^1 (z + 1)\, dz$$

$$= \left(\tfrac{z^2}{2} + z \right)_0^1 = \left(\tfrac{1}{2} + 1 \right) = \frac{1+2}{2}$$

$$= \tfrac{3}{2}$$

<u>H.W</u> Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ where $D: 1 \le x \le 2,\ 2 \le y \le 3,\ 1 \le z \le 2$

UNIT- 2

Fourier Series

Def :

A function $f(x)$ is said to have a period T if for all x. $f(x+T) = f(x)$, where T is a positive constant. The least value of $T>0$ is called the period of $f(x)$.

Fourier series

If $f(x)$ is a period function and satisfies Dirichlet condition, then it can be represented by an infinite series called fourier series as.

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \rightarrow (1).$$

where a_0, a_n, b_n are called fourier co-efficients

Dirichlet conditions

Suppose that

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(-l, l)$

(ii) $f(x)$ is periodic with period $2l$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-l, l)$. Then the above series ①

a) $f(x)$ if x is a point of discontinuity

b) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity.

Determining the fourier coefficients a_0, a_n & b_n

The fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx\, dx$$

The value of a_0, a_n, b_n are known as Euler's formulas.

①. Show that $x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$ in the interval $(-\pi \le x \le \pi)$.

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots = \pi^2/12$

(ii) $1/1^2 + 1/2^2 + 1/3^2 + \dots = \pi^2/6$

(iii) $1/1^2 + 1/3^2 + 1/5^2 + \dots\dots = \pi^2/8$

Soln: -

Here $f(x) = x^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right] = \frac{1}{\pi}\left[\pi^3/3 + \pi^3/3\right]$$

$$\boxed{a_0 = 2\pi^2/3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(\cancel{\frac{\sin nx}{n}}\right)^{0} - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\cancel{\frac{-\sin nx}{n^3}}\right)^{0}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[2x\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$\{ \because \sin n\pi = 0$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{\cos n\pi}{n^2}\right) - \left[(-2\pi)\frac{\cos(-n\pi)}{n^2}\right]\right]$$

$\because \cos(-n\pi) = \cos n\pi \}$

$$= \frac{1}{\pi n^2}\left[2\pi(-1)^n - \left[-2\pi(-1)^n\right]\right]$$

$$= \frac{1}{\pi n^2}\left[2\pi(-1)^n + 2\pi(-1)^n\right]$$

$$= \frac{4\cancel{\pi}(-1)^n}{\cancel{\pi} n^2}$$

$$\therefore \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

When n is odd, $a_n = -4/n^2$

When n is even $a_n = 4/n^2$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\cancel{\sin nx}^{\,0}}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{n}(-1)^n} - \cancel{\pi^2\frac{(-1)^n}{n}} + \cancel{\frac{2(-1)^n}{n^3}} - \cancel{\frac{2(-1)^n}{n^3}}\right]$$

$$\boxed{b_n = 0}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$x^2 = \frac{\cancel{2}\pi^2}{\cancel{2}\times 3} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}4\cos nx + 0\right)$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

When $x=0$, we have,

$$0 = \pi^2/3 + 4\sum\frac{(-1)^n}{n^2}$$

$$= \pi^2/3 + 4\left\{-1/1^2 + 1/2^2 - 1/3^2 + \dots\right\}$$

$$\boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{12}} \quad (1).$$

Put $x=\pi$, we have,

$$\pi^2=\frac{\pi^2}{3}+4\sum\frac{(-1)^n\cos n\pi}{n^2}$$

$$\pi^2=\frac{\pi^2}{3}+4\sum\frac{1}{n^2}$$

$$\pi^2-\frac{\pi^2}{3}=4\sum\frac{1}{n^2}$$

$$\frac{3\pi^2-\pi^2}{3}=4\sum\frac{1}{n^2}$$

$$\frac{\cancel{2}\pi^2}{3\times\cancel{4}}=\sum\frac{1}{n^2} \Rightarrow \pi^2/6=\sum\frac{1}{n^2}$$

$$\boxed{\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\cdots} \quad (2)$$

From Adding (1) and (2), we get the resuet

$$\left(\frac{1}{1^2}-\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cdots\right)+\left(\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cdots\right)=\frac{\pi^2}{12}+\frac{\pi^2}{6}.$$

$$\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)+\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\frac{\pi^2+2\pi^2}{12}$$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\frac{\cancel{3}\pi^2}{\cancel{12}}$$

$$\cancel{2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\pi^2/\cancel{4}$$

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\cdots=\pi^2/8}$$

② $f(x) = \frac{1}{2}(\pi - x)$ as a fourier series with Period 2π, to be valid in the interval 0 to 2π.

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} (\pi - x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) dx$$

$$= \frac{1}{2\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\pi (2\pi) - \frac{(2\pi)^2}{2} \right]$$

$$= \frac{1}{2\pi} \left[\frac{4\pi^2 - 4\pi^2}{2} \right]$$

$$\boxed{a_0 = 0}$$

$$a_n = 1/\pi \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} (\pi - x) \cos nx\, dx.$$

$$= \frac{1}{2\pi} \left\{ (\pi - x) \left(\cancel{\frac{\sin nx}{n}}^{0} \right) - (-1) \left(\frac{-\cos nx}{n^2} \right) \right\}_0^{2\pi}$$

$$= \frac{1}{2\pi} \left(\frac{-\cos nx}{n^2} \right)_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{-\cos 2n\pi}{n^2} + \frac{\cos 0}{n^2} \right]$$

$$a_n = \frac{1}{2\pi n^2} (-1+1)$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} (\pi - x) \sin nx \, dx.$$

$$= \frac{1}{\pi} \left[\frac{1}{2} (\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \left(\cancel{\frac{-\sin nx}{n^2}}^{\,0} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[-(\pi - x) \left(\frac{\cos nx}{n} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi n} \left\{ \left[-(\pi - 2\pi)(\cos n 2\pi) \right] + (\pi - 0) \cos n0 \right\}$$

$$= \frac{1}{2\pi n} \left\{ -(-\pi)(1) + \pi(1) \right\}$$

$$= \frac{1}{2\pi n} \left\{ \pi + \pi \right\} = \frac{1}{2\pi n} \left\{ 2\pi \right\}$$

$$\boxed{b_n = \frac{1}{n}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\frac{1}{2}(\pi - x) = 0 + \sum_{n=1}^{\infty} \left(0 \cos nx + \frac{1}{n} \sin nx \right)$$

$$\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin nx \right)$$

$$\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots$$

HW 1. $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \le x < \pi \end{cases}$

(2) $f(x) = 1+x$, $-1+x$

2. $f(x) = e^x$ for $(-\pi, \pi)$

In this series if we put $x = \pi/2$, we get the well known result

$$\frac{1}{2}(\pi - \pi/2) = \sin\pi/2 + \frac{1}{2}\sin 2\pi/2 + 1/3 \sin 3\pi/2 + \cdots$$

$$\frac{1}{2}\left(\frac{2\pi - \pi}{2}\right) = 1 - 1/3 + 1/5 - 1/7 + \cdots$$

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$$

(3). A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relations

$$f(x) = \begin{cases} x & (0, \pi) \\ (2\pi - x) & (\pi, 2\pi) \end{cases}$$

Express $f(x)$ as a fourier series in the range $(0, 2\pi)$.

soln

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - x^2/2\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi^2/2 + 4\pi^2 - \cancel{\frac{4\pi^2}{2}} - \left(2\pi^2 - \pi^2/2\right)\right]$$

$$= \frac{1}{\pi}\left[\pi^2/2 + 4\pi^2 - 2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\pi^2/2 + 2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 + 4\pi^2 - 3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{5\pi^2 - 3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{2}\pi^2}{\cancel{2}}\right]$$

$$= \frac{\pi^{\cancel{2}}}{\cancel{\pi}}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\cancel{\sin nx}^{\,0}}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\cancel{\sin nx}^{\,0}}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right) + \left(\frac{-\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$\boxed{a_n = \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right]}$$

lly

It can be show that $b_n = 0$.

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^n - 2}{n^2\pi} \cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx.$$

When n is even, $1-(-1)^n = 0$ and

When n is odd, $1-(-1)^n = 2$.

$$\therefore f(x) = \pi/2 - 4/\pi\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$$

When $x = 0$, $f(x) = 0$

$$\therefore 0 = \pi/2 - 4/\pi\left(1 + 1/3^2 + 1/5^2 + \ldots\right)$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \pi^2/8$$

If we put $x = \pi$, we get the same result.

Even and odd functions:

If $f(x) = f(-x)$ then $f(x)$ is said to be even function.

If $f(x) = -f(-x)$ then $f(x)$ is said to be an odd function.

Examples:

The functions $x^2, x^4 + 3x^2 + 2\cos x, \ldots$ are examples of even functions and $x^3, 2x^3 + 3x, \sin 2x \ldots$ are examples of odd functions.

Properties of odd and even functions

i) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

ii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if $f(x)$ is even

Proof:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

In the first integral on the right side, Put $x = -y \Rightarrow dx = -dy$ $\begin{cases} \text{When } x = -a \\ \boxed{y = a} \\ \text{When } x = 0 \\ \boxed{y = 0} \end{cases}$

Then, $\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)(-dy)$

$$= -\int_{a}^{0} f(-y)\,dy$$

$$= \int_{0}^{a} f(-y)\,dy$$

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]\,dx$$

If $f(x)$ is odd, $f(-x) = -f(x)$

Hence if $f(x)$ is odd, $\int_{-a}^{a} f(x)\,dx = 0$

If $f(x)$ is even, $f(-x) = f(x)$

$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even.

odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = 2/\pi \int_{0}^{\pi} f(x) \sin nx\,dx$$

Even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx$$

Example: 1

Express $f(x) = x \ (-\pi < x < \pi)$ as a fourier series with period 2π.

Soln:

Here $f(x) = x$ is an odd function

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx\, dx.$$

$$b_n = 2/\pi \int_0^{\pi} x \sin nx\, dx$$

$$= 2/\pi \left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\cancel{\frac{-\sin nx}{n^2}}\right)\right]_0^{\pi}$$

$$= \cancel{2/\pi} \left[\frac{-\cancel{\pi} \cos n\pi}{n}\right] = -\frac{2(-1)^n}{n}$$

$$\boxed{\therefore b_n = \frac{2(-1)^{n+1}}{n}}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

Example : 2

If $f(x) = -x$ in $-\pi < x < 0$

x in $0 \leq x < \pi$

Expand $f(x)$ as a fourier series in the interval $-\pi$ to π. Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

Solution :

If $f(x)$ ∴ is an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi} x\, dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \cancel{2}/\cancel{\pi}\left(\frac{\pi^{\cancel{2}}}{\cancel{2}}\right)$$

$$\boxed{a_0 = \pi}$$

$$a_n = 2/\pi \int_0^{\pi} x \cos nx\, dx$$

$$= 2/\pi \left[x\left(\frac{\cancel{\sin nx}^{\,0}}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= 2/\pi \left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= 2/\pi \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

When n is odd, $a_n = \frac{-4}{n^2\pi}$

When n is even, $a_n = 0$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=odd}^{\infty} \frac{\cos nx}{n^2}$$

$$= \pi/2 - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right].$$

When $x = 0$, $f(x) = 0$.

$$\therefore 0 = \pi/2 - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \pi/2$$

$$\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \pi/2 \times \pi/4$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

§ 4. Half range fourier series

Development in Cosine series

Let $f(x)$ be expanded as a series containing cosines only and let,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx.$$

Development in sine series

Let $f(x)$ be expanded as a series only and let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx.$$

Ex: 1

Find a sine series for $f(x) = c$ in the range 0 to π.

soln:

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$= 2/\pi \int_0^{\pi} c \sin nx \, dx \quad = \frac{2c}{\pi} \int_0^{\pi} \sin nx \, dx$$

$$= \frac{2c}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \quad = \frac{2c}{\pi} \left[-\frac{\cos n\pi}{n} + \frac{\cos 0}{n} \right]$$

$$= \frac{2c}{\pi n} \left[1 - (-1)^n \right]$$

When n is even, $b_n = 0$

When n is odd, $b_n = \frac{4c}{n\pi}$

Hence $\cancel{c} = \frac{4\cancel{c}}{\pi} \left[\sin x + 1/3 \sin 3x + \frac{1}{5} \sin 5x \dots \right]$

$\therefore \sin x + 1/3 \sin 3x + \frac{1}{5} \sin 5x = \pi/4.$

Ex: 2

If $f(x) = x$ when $0 < x < \pi/2$

$= \pi - x$ when $x > \pi/2$

Expand $f(x)$ as a sine series in the Interval $(0, \pi)$.

Soln

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx.$

$= 2/\pi \int_0^{\pi/2} x \sin nx \, dx + 2/\pi \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx.$

$= 2/\pi \left\{ \left[x \left(\cancel{\frac{-\cos nx}{n}} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi/2} + \frac{2}{\pi} \left\{ (\pi - x) \left(\cancel{\frac{-\cos nx}{n}} \right) - (-1) \left(\frac{-\sin nx}{n^2} \right) \right\}_{\pi/2}^{\pi} \right\}$

$\{ \because \cos \pi/2 = 0 \}$

$= 2/\pi \left\{ \frac{\sin nx}{n^2} \right\}_0^{\pi/2} - 2/\pi \left\{ \frac{\sin nx}{n^2} \right\}_{\pi/2}^{\pi}$

$= \frac{2}{n^2 \pi} \{ \sin n\pi/2 - 0 \} + \frac{2}{\pi n^2} \{ 0 - \sin n\pi/2 \}$

$= \frac{2}{n^2 \pi} \sin n\pi/2 + \frac{2}{\pi n^2} \sin n\pi/2$

$$\boxed{b_n = \frac{4}{n^2 \pi} \sin \frac{n\pi}{2}}$$

UNIT- 3

Differential Equations of the first oder

§ 1.1 Definitions

A differential equation is an equation which involves derivatives.

Eg

1. $y' = \sin x$

2. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 3y = x^3$

3. $\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2+y^2$

Differential equation are of two types namely.

(i) Ordinary
(ii) partial

Ordinary diff. eqn:

In a diff eqn if there is a single independent variable and derivatives are ordinary derivatives then it is called an ordinary diff eqn.

eg. $y''' + (3y'')^2 + y' = e^x$.

Partial diff eqn

If there are two (or) more independent variables and derivatives are the partial derivatives, ~~are~~ then it is called a partial diff eqn.

Eg

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2$$

Defn (order)

The order of diff eqn is the order of the highest derivative appearing in it

Eg $(y'')^2 + (y')^3 + 3y = \sin x$

Here the order is 2.

Defn (Degree)

The degree of the diff eqn is the "degree of the highest derivative" occuring in it when the differential co-efficient are free radicals and fraction.

Ex 1. $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/4} \Rightarrow \left[\frac{d^2y}{dx^2}\right]^4 = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^3$

order : (highest derivative) = 2

Degree : 4.

2. $(1+y')^2 = y'^2 \Rightarrow 1+y'^2+2y' = y'^2$

$\Rightarrow 1+2y' = 0$

$\therefore$ order $= 1$

degree $= 1$

Formation of differential equation:

Given equ in the variables x and y contains 'n' arbitrary constants. We differentiate it 'n' times to get n additional equs involving n arbitrary constant.

We eliminate the n abitrary constants from the above $n+1$ equation to obtain a diff equ of n^{th} order.

eg ① $y = mx + c$

D.w.r. to x

$$\frac{dy}{dx} = m$$

again d.w.r. to x

$$\frac{d^2y}{dx^2} = 0$$

② $y = ax^2$ —— ①

d.w.r. to $x \Rightarrow y' = a(2x)$

$$a = \frac{y'}{2x} \text{ —— ②}$$

sub ② in eqn ①

$$y = \frac{y'}{2x} \times x^2$$

$$= \frac{y'}{2} x$$

$$xy' = 2y.$$

③. Form the diff eqn by eliminating α and β from $(x-\alpha)^2 + (y-\beta)^2 = r^2$.

soln

Differentiate $(x-\alpha)^2 + (y-\beta)^2 = r^2$ w.r. to x —①

then,

$$2(x-\alpha) + 2(y-\beta)\frac{dy}{dx} = 0$$

$$(x-\alpha) + (y-\beta)\frac{dy}{dx} = 0 \quad \text{—②}$$

Again Diff w.r. to x then

$$\Rightarrow 1 + (y-\beta).\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = 0$$

$$(y-\beta).\frac{d^2y}{dx^2} + \left[\frac{dy}{dx}\right]^2 = -1 \quad \text{—③}$$

from ② $\Rightarrow$

$$x-\alpha = -(y-\beta)\frac{dy}{dx}$$

sub this in ① we have

$$\left(-(y-\beta).\frac{dy}{dx}\right)^2 + (y-\beta)^2 = r^2$$

$$(y-\beta)^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = r^2 \quad \text{—④}$$

$$③ \Rightarrow (y-\beta)\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -1$$

$$\Rightarrow (y-\beta) = -\left(\frac{dy}{dx}\right)^2 - 1 \Big/ \frac{d^2y}{dx^2}$$

$$(y-\beta)^2 = \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^2 \Big/ \left(\frac{d^2y}{dx^2}\right)^2$$

$$④ \Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 \left[1+\left(\frac{dy}{dx}\right)^2\right] \Big/ \left(\frac{d^2y}{dx^2}\right)^2 = r$$

$$\Rightarrow r^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 \Big/ \left(\frac{d^2y}{dx^2}\right)^2$$

$$r^2\left[\frac{d^2y}{dx^2}\right]^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

④. Form the diff eqn that represents all parabolas each of which has latus rectum 4a and whose axes are parallel to the x-axis.

Soln

Since the vertex of the parabola is not at the origin. the vertex of the parabola is (α, β).

The required eqn- $(y-\beta)^2 = 4a(x-\alpha)$ —— ①

D.w.r.to x. $\Rightarrow$ $\cancel{2}(y-\beta)\cdot\frac{dy}{dx} = \cancel{4}a\cdot 1$

$$(y-\beta)\frac{dy}{dx} = 2a \quad \text{—— ②}$$

($a \neq$ constant)

again D. w.r.to x then

$$(y-\beta).\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = 0 \longrightarrow ③$$

$$\Rightarrow (y-\beta).\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \longrightarrow ③$$

$$② \Rightarrow (y-\beta) = 2a \Big/ \frac{dy}{dx}$$

$$③ \Rightarrow 2a.\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

This is the required diff eqn.

Home work:

1. Form the diff equation of all circle passing through the orgin and having their centre on the x-axis.

§.2. Equation of the first order and the first degree

§.2.1. Variable separable

In this type the variables are separable one equation is of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ which can be written

$$g(y)\,dy = f(x)\,dx$$

hence. $\int g(y)\,dy = \int f(x)\,dx + c$

Example: 1.

Solve $\frac{dy}{dx} + \left[\frac{1-y^2}{1-x^2}\right]^{1/2} = 0$

soln

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$$

Integrating both side,

$$\sin^{-1}y = -\sin^{-1}x + c$$

$$\therefore \boxed{\sin^{-1}x + \sin^{-1}y = c}$$

②. Solve $\tan y \sec^2 x\,dx + \tan x \sec^2 y\,dy = 0$.

Separating x & y, $\frac{\sec^2 x\,dx}{\tan x} = -\frac{\sec^2 y}{\tan y}\,dy$

Integrating both sids.

$$\int \frac{\sec^2 x\,dx}{\tan x} = -\int \frac{\sec^2 y}{\tan y}\,dy$$

$$\log(\tan x) = -\log(\tan y) + \log c$$

$$\log(\tan x) + \log(\tan y) = \log c$$

$$\boxed{\tan x \tan y = c}$$

Home work:

1. $x\sqrt{1+y^2} + y\sqrt{1+x^2}\,y' = 0$

2. $y' = e^{x-y} + x^2 e^{-y}$

3. $y.dx - x\,dy + 3x^2y^2e^{x^3}.dx = 0.$

8.22 Homogeneous equation:

Defn

A diff eqn of first order and first degree is said to be homogenous if it can be put in form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ —①.

Working Rule: Let $\frac{y}{x} = v$ (ie) $y = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

eqn ① becomes $v + x\frac{dv}{dx} = f(v)$

$$x\frac{dv}{dx} = f(v) - v$$

Separating x and v $\frac{dx}{x} = \frac{dv}{f(v)-v}$.

Integrating $\Rightarrow \log x = \int \frac{dv}{dx-v} + C$

after integrating replace v by $\frac{y}{x}$

Examples

① Solve : $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

(ie) $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ——— ①

Put $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$

① ⇒ $v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2} = \frac{v^2}{v-1}$

⇒ $x\frac{dv}{dx} = \frac{v^2}{v-1} - v$

$= \frac{v^2 - v(v-1)}{v-1}$

$= \frac{v}{v-1}$

Separating v & x.

$\frac{v-1}{v}\,dv = \frac{dx}{x}$

Integrating both side

$\int \frac{v-1}{v}\,dv = \int \frac{dx}{x}$

$v - \log v = \log x + \log c$

$v = \log x + \log v + \log c$

$v = \log(cvx)$ [put $v = y/x$]

$\frac{y}{x} = \log cx\left(\frac{y}{x}\right)$

$y/x = \log cy$

$\boxed{y = ce^{y/x}}$

Home work:

1. $x\,dy - y\,dx = \sqrt{x^2+y^2}\;.dx$

2. $\frac{dy}{dx} = \frac{x-y}{x+y}$.

Linear Differential equation:

Defn:

A diff eqn is said to be linear if the dependent variables and its derivatives appear only in the first degree.

form: $\frac{dy}{dx} + Py = Q$ where P and Q are constants

Integrating factor $I.f = e^{\int P dx}$

Solution : $y e^{\int P dx} = \int Q e^{\int P dx} . dx + c$

(i.e) $y\,(I.F) = \int Q\,(I.F)\,dx + c$.

Note:

Solution of the linear diff eqn $\frac{dy}{dx} + Py = 0$ (i.e) $y e^{\int P dx} = c$

Example:

1. Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

Soln

$$\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$$

$P = \cos x$
$Q = \frac{1}{2}\sin 2x$

Integrating factor.

(i) $I.F = e^{\int p\,dx}$

$= e^{\int \cos x \cdot dx}$

$= e^{\sin x}$

Solution $\Rightarrow$ $y \cdot e^{\int p\,dx} = \int Q e^{\int p\,dx} \cdot dx + c$

$$y \cdot e^{\sin x} = \int \frac{1}{2}\sin 2x \cdot e^{\sin x}\, dx + c \quad \text{—} ①$$

Put $\sin x = z$ then $\frac{dz}{dx} = \cos x \Rightarrow dz = \cos x\, dx$.

$$y \cdot e^{z} = \frac{1}{\cancel{2}} \int e^{z} \cdot \cancel{2} \sin x \cdot \cos x\, dx + C$$

$u = z$ $\int dv = \int e^{z} dz$
$u_1 = 1$ $v = e^{z}$
$u_2 = 0$ $v_1 = e^{z}$.

$$y \cdot e^{z} = \int e^{z} \cdot z \cdot dz + c$$

$$ye^{z} = [uv - u_1 v_1] \Rightarrow e^{z} \cdot z - e^{z} + c$$

$$\Rightarrow ye^{\sin x} = e^{\sin x} \cdot \sin x - e^{\sin x} + c$$

$$\Rightarrow ye^{\sin x} = \boxed{e^{\sin x}(\sin x - 1) + c}.$$

2. Solve $x\frac{dy}{dx} + y\log x = e^x x^{1-\frac{1}{2}\log x}$.

Soln

$$x\frac{dy}{dx} + y\log x = e^x x^{1-\frac{1}{2}\log x}$$

$\div x \Rightarrow$

$$\frac{dy}{dx} + y\frac{\log x}{x} = e^x x^{-\frac{1}{2}\log x}$$

$P = \frac{\log x}{x}$

$Q = e^x x^{-\frac{1}{2}\log x}$

I.F

$$\int P dx = \int \frac{\log x}{x} dx$$

$u = \log x$

$du = \frac{1}{x}dx$

$$= \int u \cdot du$$

$$= \frac{u^2}{2}$$

$$= (\log x)^2/2$$

$$\therefore \text{I.F } e^{\int P dx} = e^{(\log x)^2/2}$$

$$= e^{(\log x)\log x/2}$$

$$= x^{\log x/2}$$

Soln

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$$

$$y \cdot x^{\frac{\log x}{2}} = \int e^x \cancel{x^{-\frac{1}{2}\log x}} \cdot \cancel{x^{\frac{\log x}{2}}} \cdot dx + C$$

$$= \int e^x \cdot dx + C$$

$$\boxed{y \cdot x^{\frac{\log x}{2}} = e^x + C}$$

3. Solve. $\frac{dy}{dx} + y \tan x = \cos^3 x$

Soln

$$\frac{dy}{dx} + y \tan x = \cos^3 x \qquad P = \tan x$$

$$Q = \cos^3 x.$$

$$I.F = e^{\int P dx}$$

$$= e^{\int \tan x \cdot dx}$$

$$= e^{\log \sec x}$$

$$= \sec x.$$

Solution

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} \cdot dx + C$$

$$y \cdot \sec x = \int \sec x \cdot \cos^3 x \, dx + C$$

$$= \int \cos^2 x \cdot dx$$

$$= \int \left(\frac{1 + \cos 2x}{2} \right) \cdot dx$$

$$y \cdot \sec x = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] + C.$$

Home work:

1. Solve $(1 - x^2) \frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$.

2. Solve $(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$.

Bernoulli's equation

Form of the Bernoulli's Equation is

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are constant or functions of x alone (or y alone) and 'n' is constant except 0 and 1.

Solving Method:

It can be reduced to linear form

$$\frac{dy}{dx} + Py = Qy^n$$

$\div y^n$

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q.$$

Let $v = y^{1-n}$

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$y^{-1}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}.$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$\frac{dv}{dx} + P(1-n)v = Q(1-n)$$

This is linear in v

Example:

1. Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

Soln

$$\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$$

$\Rightarrow \times y^2$

$$y^2\frac{dy}{dx} - y^3\tan x = \frac{\sin x \cos^2 x}{6}$$

$$\frac{1}{3}\frac{dz}{dx} - z\tan x = \sin x\cos^2 x$$

$y^3 = z$

$\frac{dz}{dx} = 3y^2 \cdot \frac{dy}{dx}$

$y^2\frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx}$

$$\frac{dz}{dx} - 3z\tan x = 3\sin x\cos^2 x$$

$$\int p\,dx = -3\int \tan x \cdot dx = 3\log\cos x$$

$P = -3\tan x$

$Q = 3\sin x\cos^2 x$

$$e^{\int p\,dx} = e^{\log\cos^3 x}$$

$$= \cos^3 x.$$

$\therefore$ Soln

$$z\cos^3 x = \int 3\sin x\cos^2 x\cos^3 x\,dx + c$$

$$= 3\int \sin x\cos^5 x\,dx + c$$

$u = \cos^6 x$

$du = 6\cos^5 x.(-\sin x).dx$

$$= -\frac{1}{2}\int du$$

$$= -\frac{u}{2}$$

$$= -\frac{1}{2}\cos^6 x + c$$

② Solve $(x+1)\frac{dy}{dx}+1 = 2e^{-y}$

Soln

$$(x+1)\frac{dy}{dx}+1 = 2e^{-y}$$

Multiple with e^{y}

$$e^{y}(x+1)\frac{dy}{dx}+e^{y} = 2 \quad \text{——— ①}$$

Put $e^{y}=z$ Then $\frac{dz}{dx} = e^{y}\cdot\frac{dy}{dx}$

$$① \Rightarrow (x+1)\cdot\frac{dz}{dx}+z = 2$$

$P = \frac{1}{x+1}$, $Q = \frac{2}{x+1}$

$$\Rightarrow \frac{dz}{dx}+\frac{z}{x+1} = \frac{2}{x+1}$$

$$\int p\,dx = \int\frac{dx}{x+1}$$

$$= \log(x+1)$$

$$e^{\int p dx} = e^{\log(x+1)}$$

$$= x+1$$

Solution

$$z\cdot e^{\int p dx} = \int Q\cdot e^{\int p dx}\,dx+c$$

$$e^{y}\cdot(x+1) = \int\frac{2}{\cancel{(x+1)}}\cdot\cancel{(x+1)}\,dx+C$$

$$e^{y}\cdot(x+1) = 2x+c$$