SEMESTER : II
CORE COURSE : III

Inst Hour	: 5
Credit	: 4
Code	: 18K2M03

THEORY OF EQUATIONS AND LINEAR ALGEBRA

UNIT 1:

Relations between the roots and coefficients of equations - Symmetric function of the roots –Sum of the powers of the roots – Newton's Theorem on the sum of the powers of the roots.
(Chapter 6: Sections 11-14 of Text Book 1)

UNIT 2:

Transformations of Equations – Reciprocal equations of all types - Diminishing, Increasing and multiplying the roots by a constant – Forming equations with the given roots – Removal of terms – Descarte's rule of Signs (Statement only) – simple problems.
(Chapter 6: Sections 15 to 20 & 24 of Text Book 1)

UNIT 3:

Definition and simple properties of a vector space – subspaces and quotient spaces- sums and direct sums- linear independence – basis and dimension.
(Chapter 6: sec 6.1- 6.5 of Text Book 2).

UNIT 4:

Homomorphism – dual spaces- algebra of linear transformations- Eigen value and Eigen vectors- algebra of matrices – triangular form- trace and transpose- rank of a matrix.
(Chapter 6: sec 6.6, 6.7, Chapter 7: Sections 7.1 - 7.6 of Text Book 2)

UNIT 5:

Matrices – Rank of a Matrix – Eigen Values, Eigen Vectors – Cayley's Hamilton Theorem - Verification of Cayley's Hamilton theorem.
(Chapter 2 :Sections 1-14,16-16.5 of Text Book 3)

Text Book(s)

[1] T.K. Manickavasagom Pillai ,T.Natarajan, K.S.Ganapathy , Algebra Volume I , S.V Publications - 2016

[2] M.L. Santiago, Modern Algebra, Tata McGraw Hill Publishing Company, New Delhi, 2002

[3] T.K.Manickavasagom Pillai & others, Algebra Volume II S.V. Publications -2015

Books for Reference

[1]Classical Algebra, A.Singaravelu, R.Ramaa.

[2]V.Krishnamoorthy, V.P.Mainra, J.L.Arora, An Introduction to Linear Algebra, Affiliated East – West Press.

[3] Frank Ayres ,Matrices - Schaum's Outline Series.

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

9.3.18

9/3/18

Department of Mathematics
N. GOVERNMENT ARTS COLLEGE

UNIT - 1

RELATION BETWEEN THE ROOTS AND CO-EFFICIENTS OF EQUATIONS:

Let the equation be

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots\dots + P_{n-1} x + P_n = 0$$

If this equation has the roots $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$

Then we have

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots P_{n-1} x + P_n = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

$$= x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} + \dots + (-1)^n \alpha_1\alpha_2 \dots \alpha_n \quad \text{—} (1)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} + \dots + (-1)^n S_n$$

where S_r is the sum of the products of the quantities $\alpha_1, \alpha_2 \dots \alpha_n$ taken r at a time,

Equating the co-efficients of like powers on both the sides we have

$-P_1 = S_1 =$ sum of the roots

$(-1)^2 P_2 = S_2 =$ sum of the products of the roots taken two at a time

$(-1)^3 P_3 = S_3 =$ sum of the products of the roots taken three at a time

$(-1)^n P_n = S_n =$ Product of the roots

If the equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} \dots + a_n = 0 \rightarrow (2)$$

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} \dots + \frac{a_n}{a_0} = 0 \rightarrow (2)$$

comparing the equations (1) and (2).

$$\Sigma\alpha_1 = -\frac{a_1}{a_0}$$

$$\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = \frac{a_3}{a_0}$$

$$\alpha_1\alpha_2 \cdots \alpha_n = (-1)^n\frac{a_n}{a_0}$$

(1) show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in A.P if $2p^3 - 9pq + 27r = 0$. show that the above condition is satisfied by the equation. $x^3 - 6x^2 + 13x - 10 = 0$ Hence or otherwise solve the equation.

Soln

Let the roots of the equations $x^3 + px^2 + qx + r = 0$ be $\alpha - \delta, \alpha, \alpha + \delta$. From the relation of roots and co-efficients we have,

$$\alpha - \delta + \alpha + \alpha + \delta = -p \text{ [sum of roots]}$$

$$3\alpha = -p \rightarrow (1). \quad \Sigma\alpha_1 = -\frac{a_1}{a_0}$$

$$\boxed{\alpha = -\frac{p}{3}} \rightarrow (1)$$

$(\alpha-\delta)\alpha+\alpha(\alpha+\delta)+(\alpha-\delta)(\alpha+\delta)=q$ [product of roots]

$\alpha^2-\alpha\delta+\alpha^2+\alpha\delta+\alpha^2-\delta^2=q$ $\quad \sum\alpha_1\alpha_2=\dfrac{a_2}{a_0}.$

$$3\alpha^2-\delta^2=q$$

$$3\alpha^2-q=\delta^2$$

$$3\left(\frac{-p}{3}\right)^2-q=\delta^2$$

$$\boxed{\frac{p^2}{3}-q=\delta^2} \rightarrow (2).$$

$$(\alpha-\delta)\alpha(\alpha+\delta)=-r$$

$$(\alpha^2-\delta^2)\alpha=-r$$

$$\boxed{\alpha^3-\alpha\delta^2=-r} \rightarrow (3).$$

Sub (1) and (2) in (3).

$$\left(\frac{-p}{3}\right)^3-\left(\frac{-p}{3}\right)\left(\frac{p^2}{3}-q\right)=-r$$

$$\frac{-p^3}{27}+\frac{p^3}{9}-\frac{pq}{3}=-r$$

$$\frac{-p^3+3p^3-9pq}{27}=-r$$

$$2p^3-9pq=-27r$$

$$\boxed{2p^3-9pq+27r=0}$$ Hence proved.

Given equation $\Rightarrow x^3+px^2+qx+r=0$

Also $\quad x^3-6x^2+13x-10=0$

$$\boxed{p=-6 \quad q=13 \quad r=-10}$$

$2p^3 - 9pq + 27r = 0$.

$2(-6)^3 - 9(-6)(13) + 27(-10)$

$2(216) + 54 \times 13 - 270 = 0$

$-432 + 702 - 270$

$-702 + 702 = 0$

$\therefore$ The condition is satisfied and so the roots of the equation are in A.P.

In this case, eqns ① & ② (&) ③ become

$3\alpha = -\frac{p}{3}$

$= +\frac{6}{3}$

$\boxed{\alpha = +2}$.

② $\Rightarrow$ ~~kt~~ $3\alpha^2 - q = \delta^2$

$3(4) - 13 = \delta^2$

$\boxed{\delta^2 = -1}$

$\boxed{\delta = \pm i}$

$\therefore$ The roots are

$= \alpha - \delta, \alpha, \alpha + \delta$

$= 2 - i, 2, 2 + i$ [substituting the values of α, δ)

and $2 + i, 2, 2 - i$

2. Solve $x^3 - 12x + 39x$ ...

in A.P

Let the roots be $\alpha-\delta, \alpha, \alpha+\delta$

Sum of the roots

$\Sigma\alpha_1 = \frac{+12}{1} = 12$

$3\alpha = 12$

$\boxed{\alpha = 4}$

$\Sigma\alpha_1\alpha_2 = \frac{39}{1} = 39$

$\Sigma\alpha_1\alpha_2\alpha_3 = 28$

Product of the roots

$(\alpha-\delta)\alpha + \alpha(\alpha+\delta) + (\alpha-\delta)(\alpha+\delta) = 39$

$\alpha^2 - \cancel{\alpha\delta} + \alpha^2 + \cancel{\alpha\delta} + \alpha^2 - \delta^2 = 39$

$3\alpha^2 - \delta^2 = 39$

$3(16) - \delta^2 = 39$

$48 - \delta^2 = 39$

$\delta^2 = 9$

$\boxed{\delta = 3}$

Therefore the roots are

$= \alpha-\delta, \alpha, \alpha+\delta$

$= 4-3, 4, 4+3$

$= 1, 4, 7$

3. Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in G.P.

olve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots in G.P

Soln

Let the roots of the equation be $\frac{K}{r}, K, Kr$

Sum of the roots

$\frac{K}{r} + K + Kr$ Given equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

$\div a$

$$x^3 + \frac{3b}{a}x^2 + \frac{3c}{a}x + \frac{d}{a} = 0$$

Sum of the roots

$$\boxed{\frac{K}{r} + K + Kr = -\frac{3b}{a}} \rightarrow (1) \qquad \boxed{K\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}} \rightarrow (1)$$

Sum of the products of the roots, taken two at a time

$$\frac{K^2}{r} + K^2 r + K^2 = \frac{3c}{a}$$

$$\boxed{K^2\left(\frac{1}{r} + r + 1\right) = \frac{3c}{a}} \rightarrow (2)$$

$$(2) \div (1) \Rightarrow \frac{K^2\left(\frac{1}{r} + r + 1\right)}{K\left(\frac{1}{r} + 1 + r\right)} = -\frac{3c}{a} \times \frac{a}{3b}$$

$$\boxed{\Rightarrow K = -\frac{c}{b}}$$

Sum of the products of the roots taken 3 at a time

$$\frac{K}{r} \cdot K \cdot Kr = -\frac{d}{a}$$

$$\boxed{K^3 = -\frac{d}{a}} \rightarrow (3)$$

Substituting the value of K

$$\frac{-c^3}{b^3} = \frac{-d}{a}$$

$$\cancel{-}ac^3 = \cancel{-}b^3d$$

$\boxed{ac^3 = b^3d}$ is the required condition.

Given $27x^3 + 42x^2 - 28x - 8 = 0$

$\div\ 27$

$$x^3 + \frac{42}{27}x^2 - \frac{28}{27}x - \frac{8}{27} = 0$$

$$\Sigma\alpha_1 = \frac{K}{r} + K + Kr = \frac{-42}{27} \rightarrow (4).$$

$$\Sigma\alpha_1\alpha_2 = \frac{K^2 r}{r} + K^2 + K^2 r = \frac{-28}{27} \rightarrow (5).$$

$$K^3 = \frac{8}{27}$$

$$\boxed{\therefore K = \frac{2}{3}}$$

Substituting the values of K in (4)

$$\frac{2}{3}\ \frac{K}{r} + K + Kr = -\frac{42}{27}$$

$$K\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$$

$$\frac{2}{3}\left(\frac{1+r+r^2}{r}\right) = \frac{-42}{27}$$

$$\frac{1+r+r^2}{r} = -\frac{\overset{7}{\cancel{\overset{21}{\cancel{42}}}}}{\underset{9_3}{\cancel{27}}} \times \frac{\cancel{3}^1}{\cancel{2}_1}$$

$$\frac{1+r+r^2}{r} = \frac{-7}{3}$$

$$\frac{7}{10}$$

$$3+3r+3r^2 = -7r$$

$$3r^2+3r+7r+3=0$$

$$3r^2+10r+3=0$$

$$(r+3)\left(r+\frac{1}{3}\right)=0$$

$$\boxed{r=-3 \quad r=\frac{-1}{3}}$$

The roots are

$$\frac{k}{r}, k, kr$$

when $r=-3$ $\quad k=\frac{2}{3}$

$$= \frac{\frac{2}{3}}{-3} \qquad = \frac{2}{3} \times \frac{-1}{3}$$

$$= \frac{-2}{9}, \frac{2}{3}, -2$$

when $r=\frac{-1}{3}$ $\quad K=\frac{2}{3}$

$$, \frac{\frac{2}{3}}{\frac{-1}{3}}, \frac{2}{3}, \frac{2}{3}\left(\frac{-1}{3}\right)$$

$$= -2, \frac{2}{3}, \frac{-2}{9}$$

$\therefore$ the roots are $-\frac{2}{9}, \frac{2}{3}, -2, -2, \frac{2}{3}, \frac{-2}{9}$

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SYMMETRIC FUNCTION OF THE ROOTS

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ be the roots of the equation

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$$

we learned that $s_1 = \Sigma \alpha_1 = -p$

$$s_2 = \Sigma \alpha_1 \alpha_2 = p_2$$

$$s_3 = \Sigma \alpha_1 \alpha_2 \alpha_3 = -p_3.$$

$$\dots\dots\dots\dots$$

=) without knowing the values of the roots separately in terms of the co-effecients, by using

=) the above relation b/n the co-efficients and the roots of the equation. we can express any symmetric function of the roots in terms of the co-efficients of the equations.

Ex 1:

If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, express the value of $\Sigma \alpha^2 \beta$ in terms of the co-effecients.

Soln

We have.

$$\alpha+\beta+\gamma = -p$$
$$\alpha\beta+\beta\gamma+\alpha\gamma = q$$
$$\alpha\beta\gamma = -r$$

$$\Sigma\alpha^2\beta = \alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta$$
$$= (\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma) - 3\alpha\beta\gamma$$
$$= q(-p) - 3(-r)$$
$$\Sigma\alpha^2\beta = 3r - pq.$$

Ex 2 :

If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $x^4+px^3+qx^2+rx+s$. Find (i) $\Sigma\alpha^2$ (ii) $\Sigma\alpha^2\beta\gamma$ (iii) $\Sigma\alpha^2\beta^2$ (iv) $\Sigma\alpha^3\beta$ (v) $\Sigma\alpha^4$

Sum of the roots

$$\boxed{\alpha+\beta+\gamma+\delta = -p} \rightarrow (1)$$

Sum of product of roots taken two at a time

$$\boxed{\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta+\alpha\gamma+\beta\delta = q} \rightarrow (2)$$

Sum of product of roots taken 3 at a time

$$\boxed{\alpha\beta\gamma+\beta\gamma\delta+\delta\alpha\beta+\alpha\gamma\delta = -r} \rightarrow (3)$$

Sum of product of roots taken 4

$$\boxed{\alpha\beta\gamma\delta = -s} \rightarrow (4).$$

(i) $\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta+\alpha\gamma+\beta\delta)$

$= (-p)^2 - 2(q)$

$$\boxed{\Sigma\alpha^2 = p^2 - 2q}$$

$\cancel{(ii)\ \Sigma\alpha^2\beta\gamma =}$

$\cancel{= (\alpha\beta\gamma + \alpha\beta}$

(ii) $\Sigma\alpha^2\beta\gamma = \alpha^2\beta\delta + \alpha^2\beta\gamma + \alpha^2\gamma\delta + \beta^2\gamma\delta + \beta^2\alpha\delta + \beta^2\alpha\gamma + \gamma^2\alpha\beta + \gamma^2\alpha\delta + \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma.$

$= (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)(\alpha+\beta+\gamma+\delta) - 4\alpha\beta\gamma\delta$

$= (-r)(-p) - 4s$

$\Rightarrow$ $$\boxed{\Sigma\alpha^2\beta\gamma = pr - 4s}$$

$q^2 - 2pr + 2s$

$\Rightarrow$ $\cancel{(iii)\ \Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\delta^2 + \beta^2}$

(iii) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$

$= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2 - 2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma + \beta\gamma\delta + \delta\alpha\beta + \alpha\gamma\delta) + 2(\alpha\beta\gamma\delta)$

(iii) $\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$

$= (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$

$= q^2 - 2(pr - 4s) - 6s$

$\Sigma\alpha^2\beta^2 = q^2 - 2pr + 2s$

NEWTON'S THEOREM ON THE SUM OF THE POWERS OF THE ROOTS : 10m

Let $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$, be the roots of the equation. $f(x)=0$

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$$ and ~~but~~ let

$S_r = \alpha_1^r + \alpha_2^r + \dots \alpha_n^r$ so that $s_0 = n$

$$f(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

Taking log on both sides and differentiating

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \dots + \frac{1}{x-\alpha_n}$$

(ie) $$f'(x) = \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \dots + \frac{f(x)}{x-\alpha_n}$$

By actual division we obtain

$$f(x) = x^{n-1} + (\alpha_1 + p_1)x^{n-2} + (\alpha_1^2 + p_1\alpha_1 + p_2)x^{n-1} + \dots + (\alpha_1^{n-1} + p_1\alpha_1^{n-2} + \dots + p_{n-1})$$

$$\frac{f(x)}{x-\alpha_2} = x^{n-1} + (\alpha_2 + p_1)x^{n-2} + (\alpha_n^2 + p_1\alpha_n + p_2)x^{n-3} + \dots + (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \dots + p_{n-1})$$

Adding all these fractions, we get

$$f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1 s_1 + np_2)x^{n-3} + \dots + (s_{n-1} + p_1 s_{n-2} + \dots np_{n-1})$$

But $f'(x)$ is also equal to

$$nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \dots + 2p_{n-2} + p_{n-1}$$

Equating the co-efficients in the two values of $f'(x)$ we get

$$s_1 + p_1 = 0$$

$$s_2 + p_1 s_1 + 2p_2 = 0$$

$$s_3 + p_1 s_2 + p_2 s_1 + 3p_3 = 0$$

$$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0$$

$$\vdots$$

$$s_{n-1} + p_1 s_{n-2} + p_2 s_{n-3} + \dots p_{n-2} s_1 + (n-1)p_{n-1} = 0$$

From these $(n-1)$ relations we can calculate in succession the values of $s_1, s_2, s_3 \dots s_{n-1}$ in terms of co-efficients $p_1, p_2, p \dots p_{n-1}$. We can extend our result to the sum of all positive powers of the roots viz, $s_n, s_{n+1} \dots s_r$ where $r > n$ we have $x^{r-n} f(x) = x^r + p_1 x^{r-1} + p_2 x^{r-2} + \dots p_n x^{r-n}$

Replacing in this identity, x by the roots $\alpha_1, \alpha_2 \cdots \alpha_n$ in successor and adding we have

$$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \cdots + p_n s_{r-n} = 0$$

Now giving $r \Rightarrow$ values $n, n+1, n+2$ successively and observing that $s_0 = n$ we obtain from the last equation

$$s_n + p_1 s_{n-1} + p_2 s_{n-2} + \cdots + n p_n = 0$$

$$s_{n+1} + p_1 s_n + p_2 s_{n-1} + \cdots + p_n s_1 = 0$$

$$s_{n+2} + p_1 s_{n+1} + p_2 s_n + \cdots + p_n s_2 = 0$$

and so on

Thus we get

$$s_r + p_r s_{r-1} + p_2 s_{r-2} + \cdots r p_r = 0 \quad \text{if } r < n$$

and $s_r + p_1 s_{r-1} + p_2 s_{r-2} + \cdots + p_n s_{r-n} = 0$ if $\geqslant n$

NEWTON'S THEOREM :

Let $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$ be the roots of the equation

$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$ and let

$S_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$ and

$S_0 = n$.

$$f(x) = (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_n)$$

Differentiating and taking log on both the sides

$$f'(x) \times \frac{1}{f(x)} = \frac{1}{(x-\alpha_1)} \times \frac{1}{(x-\alpha_2)} \times \cdots \frac{1}{(x-\alpha_n)}$$

$$\therefore f'(x) = \frac{f(x)}{(x-\alpha_1)} + \frac{f(x)}{(x-\alpha_2)} + \dots \frac{f(x)}{(x-\alpha_n)}$$

$$= \frac{f(x)}{(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)}$$

$$= \frac{x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n}{(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)}$$

$$\frac{f(x)}{(x-\alpha_1)} = \frac{x^n}{(x-\alpha_1)} + \frac{p_1 x^{n-1}}{(x-\alpha_1)} + \frac{p_2 x^{n-2}}{(x-\alpha_1)} + \dots + \frac{p_n}{(x-\alpha_1)}$$

$$= \frac{x^n}{x\left(1-\frac{\alpha_1}{x}\right)} + \frac{p_1 x^{n-1}}{x\left(1-\frac{\alpha_1}{x}\right)} + \frac{p_2 x^{n-2}}{x\left(1-\frac{\alpha_1}{x}\right)} + \dots + \frac{p_n}{x\left(1-\frac{\alpha_1}{x}\right)}$$

$$= x^{n-1}\left(1-\frac{\alpha_1}{x}\right)^{-1} + p_1 x^{n-2}\left(1-\frac{\alpha_1}{x}\right)^{-1} + p_2 x^{n-3}\left(1-\frac{\alpha_1}{x}\right)^{-1} + \dots + p_n x^{-1}\left(1-\frac{\alpha_1}{x}\right)$$

$$= x^{n-1}\left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}\right) + p_1 x^{n-2}\left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}\right) + p_2 x^{n-3}\left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}\right) + \dots + p_n x^{-1}\left(1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}\right)$$

$$= x^{n-1}\left(1+\alpha_1 x^{-1}+\alpha_1^2 x^{-2}\right) + p_1 x^{n-2}\left(1+\alpha_1 x^{-1}+\alpha_1^2 x^{-2}\right) + p_2 x^{n-3}\left(1+\alpha_1 x^{-1}+\alpha_1^2 x^{-2}\right) + \dots + p_n x^{-1}\left(1+\alpha_1 x^{-1}+\alpha_1^2 x^{-2}\right)$$

$$= x^{n-1} + \alpha_1 x^{n-2} + \alpha_1^2 x^{n-3} + p_1 x^{n-2} + p_1 \alpha_1 x^{n-3} + p_1 \alpha_1^2 x^{n-4} + p_2 x^{n-3} + p_2 \alpha_1 x^{n-4} + p_2 \cancel{x^{n-3}} \alpha_1^2 x^{n-5} + \dots + p_n x^{-1} + p_n \alpha_1 x^{n-2} + p_n \alpha_1^2 x^{n-3}$$

$$\frac{f(x)}{(x-\alpha_1)} = x^{n-1} + (\alpha_1 + p_1)x^{n-2} + (\alpha_1^2 + p_1\alpha_1 + p_2)x^{n-3} + (p_1\alpha_1^2 + p_2\alpha_1)x^{n-4} + \dots\dots + (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \dots + p_{n-1})$$

$$\frac{f(x)}{(x-\alpha_2)} = x^{n-1} + (\alpha_2 + p_1)x^{n-2} + (\alpha_2^2 + p_1\alpha_1 + p_2)x^{n-3} + \dots + (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \dots + p_{n-1}).$$

Adding all these fractions,

$$\frac{f(x)}{(x-\alpha_1)} + \frac{f(x)}{(x-\alpha_2)} + \frac{f(x)}{(x-\alpha_3)} + \dots + \frac{f(x)}{(x-\alpha_n)} = f'(x)$$

$$f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1 s_1 + np_2)x^{n-3} + \dots + (s_{n-1} + p_1 s_{n-2} + \dots \overset{np_{n-1}x^0}{np x^n_{n-1}}).$$

Actually ~~$f'(x)$~~ =

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$$

$$f'(x) = nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \dots + 2p_{n-2} + p_{n-1}$$

Equating the coefficients of x^{n-2} from $f'(x)$, we get

$$s_1 + np_1 = (n-1)p_1$$

$$s_1 + np_1 = np_1 - p_1$$

$$\boxed{s_1 + p_1 = 0}$$

Equating the co-efficients of x^{n-3} from $f'(x)$

$$s_2 + p_1 s_1 + np_2 = (n-2)p_2$$

$$s_2 + p_1 s_1 + np_2 = np_2 - 2p_2$$

$$\boxed{s_2 + p_1 s_1 + 2p_2 = 0}$$

Equating the co-efficients of $x^{(n-4)}$

$$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0$$

$$\vdots$$

$$s_{n-1} + p_1 s_{n-2} + p_2 s_{n-3} + \dots + p_{n-2} s_1 + (n-1) p_{n-1} = 0$$

From these $(n-1)$ relations, we can calculate in successions the values of $s_1, s_2, s_3 \dots s_{n-1}$ in terms of co-efficients $p_1, p_2, p_3 \dots p_{n-1}$ we can extend our result to the sum of all positive powers of the roots. $s_n, s_{n+1} \dots s_r$ where $r \geq n$

we have,

$$x^{r-n} f(x) = x^r + p_1 x^{r-1} + p_2 x^{r-2} + \dots + p_n x^{r-n}$$

$$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \dots + p_n s_{r-n} = 0$$

Now $r = n, n+1, n+2$; $s_0 = n$

$r = n \Rightarrow \quad s_n + p_1 s_{n-1} + p_2 s_{n-2} + \dots + n p_n = 0$

$r = n+1 \Rightarrow s_{n+1} + p_1 s_n + p_2 s_{n-1} + \dots + p_n s_1 = 0$

$r = n+2 \Rightarrow s_{n+2} + p_1 s_{n+1} + p_2 s_n + \dots + p_n s_2 = 0$

Thus we get,

$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \dots + r p_r = 0 \quad$ if $r < n$

$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \dots + p_n s_{r-n} = 0 \quad$ if $r \geq n$

Hence the theorem.

Show that the sum of the eleventh powers of the roots of $x^7+5x^4+1=0$ is zero.

Soln

$n=7 \quad r=11 \quad r \geq n$

Since 11 is greater than 7, the degree of the equation we have to use the Newton's theorem

Here $n=7$

$r=11$

$r>n$

$S_r + P_1 S_{r-1} + P_2 S_{r-2} \dots rP_1$ if $r<n$

$$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \cdots P_n S_{r-n} = 0 \text{ if } r \geq n$$

$\therefore$ when $n=7$ $r=11$

$$S_{11} + P_1 S_{10} + P_2 S_9 + P_3 S_8 + P_4 S_7 + P_5 S_6 + P_6 S_5 + P_7 S_4 = 0 \rightarrow (1)$$

Expanding the given equation

$$x^7 + P_1 x^6 + P_2 x^5 + P_3 x^4 + P_4 x^3 + P_5 x^2 + P_6 x + P_7 = 0$$

$$P_1 = P_2 = P_4 = P_5 = P_6 = 0$$

$$P_3 = 5 \quad P_7 = 1$$

$$S_{11} + 5S_8 + S_4 = 0 \rightarrow (2).$$

$$S_8 + P_1 S_7 + P_2 S_6 + P_3 S_5 + P_4 S_4 + P_5 S_3 + P_6 S_2 + P_7 S_1 = 0$$

$$S_8 + 5S_5 + S_1 = 0 \rightarrow (3).$$ $n=8$ $5<7$ $n=7$

$$S_5 + P_1 S_4 + P_2 S_3 + P_3 S_2 + P_4 S_1 + P_5 S_0 = 0$$ $n=5$ $r=n$

$$\boxed{S_5 + 5S_2 = 0} \rightarrow (4)$$

$$\therefore S_2 + P_1 S_1 + 2P_2 = 0$$ $n=2$ $r=11$

$$\boxed{\therefore S_2 = 0} \rightarrow (5)$$ sub (4) in (3)

$$\boxed{S_5 = 0} \rightarrow (6).$$

Put $r=1$, $s_1 + 1p_1 = 0$ $\boxed{s_5 = 0}$ in (4)

$\boxed{\therefore s_1 = 0} \to$ (3) (7) $\Rightarrow \boxed{s_2 = 0}$

$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0$

$s_4 + 3 s_1 = 0$

$\boxed{\therefore s_4 = 0}$ in $\to$ (6)

Substitute in (1).

$\boxed{\therefore S_{11} = 0}$.

Ex : 02 If $a+b+c+d=0$ show that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \times \frac{a^3+b^3+c^3+d^3}{3}$$

Soln

$a+b+c+d=0$, we can consider that a, b, c, d

$\Rightarrow$ are the roots of the equation $x^4+p_1x^3+p_2x^2+p_3x+p_4=0$

where $p_1 = 0$ $n=4$ $r=5$

$\Rightarrow$ From Newton's theorem, on the sums of powers of the roots, we get

$s_5 + p_1 s_4 + p_2 s_3 + p_3 s_2 + p_4 s_1 = 0 \to$ (1)

$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0 \to$ (2) $r=4$

$s_3 + p_1 s_2 + p_2 s_1 + 3p_3 = 0 \to$ (3)

$s_2 + p_1 s_1 + 2p_2 = 0 \to$ (4)

$s_1 + p_1 = 0 \to$ (5)

From (5) we get $s_1 = 0$

From (4) $s_2 = -2p_2$

From (3) $s_3 = -3p_3$

From (1)

$$S_5 - 3P_2P_3 - 2P_3P_2 = 0$$

(ie) $S_5 = 5P_2P_3$

$$\therefore \frac{S_5}{5} = \frac{S_2}{2} \cdot \frac{S_3}{3} \qquad \because P_2P_3 = \left(-\frac{S_2}{2}\right)\left(-\frac{S_3}{3}\right)$$ [from the value of S_2 and S_3

(ie) $$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \times \frac{a^3+b^3+c^3+d^3}{3}$$

Ex : 03

Find $\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5}$ where α, β, γ are the roots of the equation $x^3 + 2x^2 - 3x - 1 = 0$

Soln

Put $x = \frac{1}{y}$ the eqn becomes

$$\frac{1}{y^3} + \frac{2}{y^2} - \frac{3}{y} - 1 = 0$$

(ie) $y^3 + 3y^2 - 2y - 1 = 0$

$\therefore$ the roots of the eqn are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = S_5$$

For the equation $y^3 + 3y^2 - 2y - 1 = 0$

From Newton's theorem on the sum of powers of roots of eqns we get

$$S_5 + 3S_4 - 2S_3 - S_2 = 0 \quad (2)$$

$$S_4 + 3S_3 - 2S_2 - S_1 = 0$$

$$S_3 + 3S_2 - 2S_1 - S_0 = 0$$

$$S_2 + 3S_1 - 4 = 0$$

$$S_1 + 3 = 0$$

UNIT-2

TRANSFORMATION OF EQUATIONS

If an eqn given, it is possible to transform this eqn, into another whose roots bear with the roots of the original eqn a given relation Such a transformation of an helps us to solve equations easily or to the discuss the nature of the roots of the eqn. we shall explain here the most important elementary transformation of equation.

Roots with Signs changed :.

TO transform an equation into another whose roots are numerically the same as those of the given equation But opposite in sign. Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be the roots of the equation.

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots + P_n = 0.$$

Then we have,

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \ldots P_n = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Changing x into $-x$ we have.

$$(-x)^n + P_1 (-x)^{n-1} + P_2 (-x)^{n-2} + \ldots + P_n$$

$$\ldots (-x-\alpha_1)(-x-\alpha_2)\ldots(-x-\alpha_n)$$

The roots of equation.

$$x^n - P_1 x^{n-1} + P_2 x^{n-2} \mp \ldots \pm P_n = 0$$ are $-\alpha_1, -\alpha_2 \ldots -\alpha_n$.

[odd powers]

$\therefore$ TO effect the required transformation we have to substitute $(-x)$ for (x) in given equation. that is to take the sign of change a alternative term of the given equation begining with the second.

ROOTS MULTIPLIED BY A GIVEN NUMBER :

To transform an equation into another whose roots are m times that of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = (x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_n)$$

Instead of x, substitute y/m, we get

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \dots + p_n$$

$$= \left(\frac{y}{m} - \alpha_1\right)\left(\frac{y}{m} - \alpha_2\right) \dots \left(\frac{y}{m} - \alpha_n\right)$$

Multiply both sides by m^n

$$y^n + m p_1 y^{n-1} + m^2 p_2 y^{n-2} + \dots + m^n p_n = (y - m\alpha_1)(y - m\alpha_2) \dots (y - m\alpha_n)$$

The equation

$$y^n + m p_1 y^{n-1} + m^2 p_2 y^{n-2} + \dots + m^n p_n = 0$$

has the roots

$$m\alpha_1, m\alpha_2, \dots m\alpha_n$$

Hence to effect the transformation we have to multiply the successive terms beginning with the second by

$$m, m^2 \dots m^n$$

This transformation is useful for the purpose of removing the co-efficients of the first term of an equation, when it is other than unity and generally for removing the fractional co-efficients from an equation.

EXAMPLE 1 :

change the equation $2x^4 - 3x^3 + 3x^2 - x + 2 = 0$ into another the co-efficient of whose highest term will be unity.

Soln

Multiply the roots by 2, then the transformed equation becomes

$$2x^4 - 3\times 2 x^3 + 3\times 2^2 x^2 - 1\times 2^3 x + 2\times 2^4 = 0$$

$\div 2$

$$x^4 - 3x^3 + 6x^2 - 4x + 16 = 0$$

2. Remove the fractional co-efficient from

$$x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$$

$$x^3 - \frac{1}{4}\times 12\, x^2 + \frac{1}{3}\times 12^2 x - 12^3 = 0$$

$$x^3 - 3x^2 + 48x - 12^3 = 0.$$

$$x^3 - 3x^2 + 48x - 1728 = 0$$

3. Remove the fractional co-efficient from

$$x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0$$

$$= x^3 + \frac{1}{4}\times 144\, x^2 - \frac{1}{16}\times 144\times 144\, x + \frac{1}{72}\times 144\times 144\times 144$$

$$= x^3 + 36x^2 - 1296x + \cancel{15552}\ 41472$$

EXAMPLE I :

Find the roots of the eqn $x^5 + 4x^4 + 3x^3 + 3x$
$4x + 1 = 0$

This is a reciprocal eqn of odd degree with like signs

$\therefore (x+1)$ is a factor of $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x$

$x^5 + x^4 + 3x^4 + 3x^3 + 3x^2 + 3x + x + 1 = 0.$

(ie) $x^4(x+1) + 3x^3(x+1) + 3x(x+1) + 1(x+1) = 0$

(ie) $(x+1)(x^4 + 3x^3 + 3x + 1) = 0$

$\therefore x+1 = 0$ or $x^4 + 3x^3 + 3x + 1 = 0$

Dividing by x^2 we get

$$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$$

Put $x + \frac{1}{x} = z$, $\therefore x^2 + \frac{1}{x^2} = z^2 - 2$

$\therefore z^2 - 2 + 3z = 0$

$$z = \frac{-3 \pm \sqrt{17}}{2}$$

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

(ie) $2x^2 + (3+\sqrt{17})x + 2 = 0$

or $2x^2 + (3-\sqrt{17})x + 2 = 0$

From these eqns x can be found

EXAMPLE 2 :

Solve : $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$

This is a reciprocal eqn of odd degree with unlike signs. Hence $(x-1)$ is a factor of the L.H.S

The eqn can be written as

$$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^3 + 38x^2 + 5x^2 - 5x + 6x - 6 = 0$$

(ie) $6x^4(x-1) + 5x^3(x-1) - 38x^2(x-1) + 5x(x-1) + 6(x-1) = 0$

(ie) $(x-1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore x = 1$ or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

we have to solve the eqn $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$

Dividing by x^2,

$$6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$$

(ie) $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$

Put $x + \frac{1}{x} = z$ $\therefore x^2 + \frac{1}{x^2} = z^2 - 2$

the eqn becomes

$6(z^2-2)+5z-38=0$

(ie) $6z^2+5z-50=0$

$(2z-5)(3z+10)=0$

$\therefore z=\frac{5}{2}$ or $-\frac{10}{3}$

$\boxed{-15\times 20=-300}$

(ie) $x+\frac{1}{x}=\frac{5}{2}$ or $x+\frac{1}{x}=-\frac{10}{3}$

(ie) $2x^2-5x+2=0$ or $3x^2+10x+3=0$

(ie) $(2x-1)(x-2)=0$ or $(3x+1)(x+3)=0$

(ie) $x=\frac{1}{2}$ or 2 or $-\frac{1}{3}$ or -3.

$\therefore$ The roots of the eqn are $\frac{1}{2}, 2, -\frac{1}{3}$ & -3

Ex: 3

Solve the equation

$6x^6-35x^5+56x^4-56x^2+35x-6=0$

There is no mid term and this is a reciprocal eqn of even degree with unlike signs; we can easily see that x^2-1 is a factor of the expression on L.H.S

The eqn can be written as

$6(x^6-1)-35x(x^4-1)+56x^2(x^2-1)=0$

(ie) $6(x^2-1)(x^4+x^2+1)-35x(x^2-1)(x^2+1)$
$+56x^2(x^2-1)=0$

(ie) $(x^2-1)(6x^4-35x^3+62x^2-35x+6)=0$

(ie) $x=1$ or $x=-1$ or $6x^4-35x^3+62x^2-35x+6=0$

Dividing by x^2 we get $6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)$

$+62=0$

Put $x+\frac{1}{x}=z$ then $x^2+\frac{1}{x^2}=z^2-2$

$\therefore 6(z^2-2)-35z+62=0$

(ie) $6z^2-35z+50=0$

(ie) $(3z-10)(2z-5)=0$

(ie) $z=\frac{10}{3}$ or $\frac{5}{2}$

$\therefore x+\frac{1}{x}=\frac{10}{3}$ or $x+\frac{1}{x}=\frac{5}{2}$

(ie) $3x^2-10x+3=0$ or $2x^2-5x+2=0$

(ie) $(x-3)(3x-1)=0$ or $(x-2)(2x-1)=0$

(ie) $x=3$ or $\frac{1}{3}$ or 2 or $\frac{1}{2}$

The roots of the eqn are $1, -1, 3, \frac{1}{3}, 2$ and $\frac{1}{2}$

Ex : 1 Find the quotient and remainder when $3x^3+8x^2+8x+12$ is divided by $x-4$

The calculation is arranged as follows,

4	3	8	8	12
		12	80	352
	3	20	88	364

The quotient is $3x^2+20x+88$ and the remainder is 364.

Ex : 2 Find the quotient and remainder when $2x^6+3x^5-15x^2+2x-4$ is divided by $x+5$

-5	2	3	0	0	-15	2	-4
		-10	35	-175	875	-4300	21490
	2	-7	35	-175	860	-4298	21486

The quotient is $2x^5-7x^4+35x^3-175x^2+860x$ -4298 and the remainder is

21, 486.

REMOVAL OF TERMS :

Let the given equation be,

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_{n-1}x + a_n = 0$$

Then if $y = x - h$, we obtain the new equation

$$a_0(x+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \dots + a_n = 0$$

which when arranged in descending powers of y, becomes

$$a_0y^n + (na_0h) + a_1)y^{n-1} + \left\{\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_0h + a_2\right\}y^{n-2} + \dots = 0$$

If the term to be removed is second, we get

$$na_0h + a_1 = 0$$

so that $h = \dfrac{-a_1}{na_0}$

If the term to be removed is third, we get,

$$\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_1h + a_2 = 0$$

and so we obtain a quadratic to find h, and similarly we may removed any other assigned term.

Ex 1:

Find the relation between the co-effecients in the eqns $x^4 + px^3 + qx^2 + rx + s = 0$ in order that the co-efficients of x^3 and x may be removable by the same transformation.

soln: Let us reduce the roots of the eqn by h.

instead of x, Substitute $(x+h)$. the ~~transformed~~

transformed eqn is

$$\Rightarrow (x+h)^4 + p(x+h)^3 + q(x+h)^2 + r(x+h) + s = 0$$

$$\Rightarrow x^4 + 4C_1 x^3 h + 4C_2 x^2 h^2 + 4C_3 x h^3 + 4C_4 h^4 + p(x^3 + 3x^2 h + 3xh^2 + h^3)$$

$$+ q(x^2 + 2xh + h^2) + rx + rh + s = 0$$

$$x^4 + 4x^3 h + 6x^2 h^2 + 4xh^3 + h^4 + px^3 + 3px^2 h + 3xph^2 + ph^3 + qx^2 +$$

$$2qxh + qh^2 + rx + rh + s = 0$$

$$x^4 + x^3(4h+p) + x^2(6h^2 + 3ph + q) + x(4h^3 + 3ph^2 + 2qh + r)$$

$$+ h^4 + ph^3 + qh^2 + rh + s = 0$$

The co-efficient of x^3 and x in the tranformed eqn are zeros.

$$4h + p = 0, \quad 4h^3 + 3ph^2 + 2qh + r = 0$$

Eliminating function between these eqns.

We get $h = \frac{-p}{4}$

$$\therefore 4\left(\frac{-p}{4}\right)^3 + 3p\left(\frac{-p}{4}\right)^2 + 2q\left(\frac{-p}{4}\right) + r = 0$$

$$4\left(\frac{-p^3}{64}\right) + 3p\left(\frac{p^2}{16}\right) - \frac{2pq}{4} + r = 0$$

$$\frac{-4p^3}{64} + \frac{3p^3}{16} - \frac{2pq}{4} + \frac{r}{1} = 0$$

$$\frac{-p^3 + 3p^3 - 8pq + 16r}{16} = 0.$$

(64, 16,

2 (16, 16,

8 (8, 8,

1,

11/02/19

DESCARTE'S RULE OF SIGNS :

Show that the equation $x^{10}+10x^3+x-4=0$ has eight imaginary roots

Soln

Let $f(x) = x^{10}+10x^3+x-4$

The signs of the terms in $f(x)$: + + + -

Therefore the no. of changes of sign in $f(x)$: one

Hence there cannot be more than one positive roots → (1).

Now, subs $f(x) = f(-x)$

$$f(-x) = (-x)^{10} + 10(-x)^3 + (-x) - 4$$

$$= x^{10} - 10x^3 - x - 4$$

The signs of the terms in $f(-x) = +\ -\ -\ -$

∴ the no. of changes in $f(-x)$ = one

Hence there cannot be more than one negative roots → (2)

From (1) and (2) we see that the eqn has got at the most two roots

Since $f(x) = 0$, is of degree 10, it has at~~least~~ eight imaginary roots.

Show that the equation $x^6 + 3x^2 - 5x + 1 = 0$ has atleast 4 imaginary roots

Let $f(x) = x^6 + 3x^2 - 5x + 1$

The signs of terms in $f(x)$: $+\ +\ -\ +$

~~There~~ ∴ No. of changes of sign in $f(x)$: 2.

Hence there cannot be more than 2 +ve roots

Substitute $f(x) = f(-x)$

$$f(-x) = \cancel{(-x)^6 + 3(-x)^3}, (-x)^6 + 3(-x^2) - 5(-x) + 1$$

$$= x^6 + 3x^2 + 5x + 1$$

The signs of terms of $f(-x) = +\ +\ +\ +$

No. of changes in $f(-x)$ = 0

Hence there cannot be no negative roots

From (1) & (2) we get atleast 2 real roots

$f(x)=0$ is of degree 6, it has 4 imaginary

HW 11-02-14

1. Find the real roots of $x^7-x^5-x^4-6x^2+7=0$

Let $f(x) = x^7 - x^5 - x^4 - 6x^2 + 7$

The signs of terms in $f(x) = + \; - \; - \; - \; +$

No. of changes of signs in $f(x) = 2$

$f(x) = f(-x)$

$\therefore f(-x) = (-x)^7 - (x^5) - (-x^4) - 6(-x^2) + 7$

$= -x^7 + x^5 - x^4 - 6x^2 + 7$

$$\boxed{f(-x) = -x^7 + x^5 - x^4 - 6x^2 + 7}$$

The signs of terms in $f(-x) = - \; + \; - \; - \; +$

No. of changes of sign in $f(-x) = 3$

Hence it has got atmost 5 real roots

2. $x^5 - 6x^2 - 4x + 5 = 0$

$f(x) = x^5 - 6x^2 - 4x + 5$

The signs of terms in $f(x)$: $+ \; - \; - \; +$

No. of changes of sign in $f(x) = 2$

$f(x) = f(-x)$

$f(-x) = (-x)^5 - 6(-x)^2 - 4(-x) + 5$

$$\boxed{f(-x) = -x^5 - 6x^2 + 4x + 5}$$

12/02/19

5. MATRICES

1. Find the x, y, z and w that satisfy the matrix relationship

$$\begin{bmatrix} x+3 & 2y+5 \\ z+4 & 4x+5 \\ w-2 & 3w+1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -4 & 2x+1 \\ 2w+5 & -20 \end{bmatrix}$$

$$x + 3 = 1 \Rightarrow \boxed{x = -2}$$

$$2y + 5 = -5$$

$$2y = -10$$

$$\boxed{y = -5}$$

$$3w + 1 = -20$$

$$3w = -21$$

$$\boxed{w = -7}$$

$$z + 4 = -4$$

$$\boxed{z = -8}$$

2. solve the eqn for the matrix A

$$3A + \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$$

$$3A = \begin{bmatrix} -6 & 3 \\ 3 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

.02.18

S.T THE MATRIX

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \text{ SATISFIES } A(A-I)(A+2I)=0$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$A(A - I) = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix}$$

$$A(A - I) = \begin{bmatrix} \cancel{2} & \cancel{-6} & \cancel{-5} \\ \cancel{-9} & \cancel{0} & \cancel{0} \\ -5 & 6 & 20 \end{bmatrix}$$

$$A(A - I) = \begin{bmatrix} -12 & -4 & -12 \\ -9 & -3 & -9 \\ 21 & 7 & 21 \end{bmatrix}$$

$$A(A-I)(A+2I) = \begin{bmatrix} -12 & -4 & -12 \\ -9 & -3 & -9 \\ 21 & 7 & 21 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -48-12+60 & 36-12-24 & -12-12+24 \\ -36-9+27 & 27-9-18 & +9+9-18 \\ 84+21-105 & -63+21+42 & 21+21-42 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ show that A satisfies the equation $A^2 - 5A - 2I = 0$ using this result determine A^{-1}

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$5A = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

$$2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7-7 & 10-10 \\ 0 & 22-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A^2 = 5A + 2I$

$A^4 = (5A + 2I)(5A + 2I)$

$= 25A^2 + 10AI + 10AI + 4I^2$

$= 25A^2 + 20AI + 4I^2$

$= 25(5A + 2I) + 20A + 4I$

$= 125A + 50I + 20A + 4I$

$= 125A + 50I + 20A + 4I$

$A^4 = 145A + 54I$

$A^4 \cdot A = 145A^2 + 54AI$

$= 145(5A + 2I) + 54A$

$= 725A + 290I + 54A$

$A^5 = 779A + 290I$

$A^5 = 779A + 290I$

$779A + 290I = 779\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 290\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 779 & 1558 \\ 2337 & 3116 \end{bmatrix} + \begin{bmatrix} 290 & 0 \\ 0 & 290 \end{bmatrix}$

$A^5 = \begin{bmatrix} 1069 & 1558 \\ 2337 & 3406 \end{bmatrix}$

Show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ satisfies $B = A + I$ the eqn $A^2 = -I$, use the result to calculate the 16^{th} power of matrix, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-1 & 0+0 \\ 0+0 & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A^2 = -I}$$

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = A + I$$

$$\Rightarrow B^2 = (A+I)^2$$

$$B^2 = \cancel{(A+I)} A^2 + 2AI + I^2$$

$$\Rightarrow B^2 = A^2 + 2A + I$$

$$B^2 = -I + 2A + I$$

$$B^2 = 2A$$

$$(B^2)^8 = (2A)^8$$

$$B^{16} = 256 \times A^8$$

$$= 256 \, [(A^2)^4$$

$$= 256 \, (-I)^4$$

$$= 256 \, (I)$$

$$= 256 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^{16} = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -1 \\ 0 & 16 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 \to \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\rho(A) = 2$$

FIND THE RANK (pg 98)

1. $\begin{bmatrix} 3 & 4 & -6 \\ 2 & -1 & 7 \\ 1 & -2 & 8 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 4 & -6 \\ 2 & -1 & 7 \\ 1 & -2 & 8 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & 8 \\ 2 & -1 & 7 \\ 3 & 4 & -6 \end{bmatrix} \quad R_1 \longleftrightarrow R_3$

$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & 3 & -9 \\ 3 & 4 & -6 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$

$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & 3 & -9 \\ 0 & 10 & -30 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$

$= \cancel{\begin{bmatrix} 1 & -2 & 8 \\ 0 & 3 & -9 \\ 0 & 1 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/3 \\ R_3 \to R_3/10 \end{matrix}$

$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & 3 & -9 \\ 0 & 0 & -18 \end{bmatrix} \quad R_3 \to 3R_3 - R_2 \qquad = \begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$

$\cancel{\rho(A) = 2.} \quad \cancel{= \begin{bmatrix} 1 & -2 & 8 \\ 0 & 3 & -9 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2}$

$18 + 2(-9)$

$18 - 18$

$R_3 \cdot 2R_2$

2. $\begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & -2 \\ -6 & 7 & 8 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & -2 \\ -6 & 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & 4 \\ 8 & 7 & -6 \end{bmatrix} C_1 \leftrightarrow C_3$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & -9 & -30 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 8R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 3R_2$$

$\rho(A) = 2$

(3)

$$\Rightarrow \begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} 1 & 11 & 3 & 5 \\ -1 & 13 & 5 & 11 \\ 4 & 2 & -2 & -8 \end{bmatrix} C_1 \leftrightarrow C_3$$

$$= \begin{bmatrix} 1 & 11 & 3 & 5 \\ 0 & 24 & 8 & 16 \\ 0 & -42 & -14 & -28 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 11 & 3 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & -6 & -2 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2/8 \\ R_3 \to R_3/7 \end{matrix}$$

$$= \begin{bmatrix} 1 & 11 & 3 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 2R_2 \quad \rho(A) = 2$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} = 0$$

$$= (2-\lambda)\left[(1-\lambda)(-1-\lambda) - 3\right] + 2(-1-\lambda-1) + 3(3-(1-\lambda))\Big]$$

$$\Rightarrow (2-\lambda)(-1-\lambda+\lambda+\lambda^2-3) + 2(-2-\lambda) + 3(3-1+\lambda)$$

$$\Rightarrow (2-\lambda)(\lambda^2-4) - 4 - 2\lambda + 6 + 3\lambda = 0$$

$$2\lambda^2 - 8 - \lambda^3 + 4\lambda - 4 - 2\lambda + 6 + 3\lambda = 0$$

$$-\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$\begin{array}{c|cccc} & 1 & -2 & -5 & 6 \\ 3 & 0 & 3 & 3 & -6 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$\therefore \lambda = 3$

$\lambda^2 + \lambda - 2 = 0$

$(\lambda+2)(\lambda-1) = 0$

$\lambda = -2, 1$

$\lambda = -2, 1, 3$

$\lambda = 1$ in $A - \lambda I$

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} = 0$$

$x_1 - 2x_2 + 3x_3 = 0 \rightarrow ①$

$x_1 + x_3 \quad = 0$

$x_1 + 3x_2 - 2x_3 = 0$

$x_1 - 2x_2 + 3x_3 = 0$

$x_1 + 3x_2 - 2x_3 = 0$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \end{vmatrix}$$

$$\frac{x_1}{4-9} = \frac{-x_2}{-2-3} = \frac{x_3}{5}$$

$$\frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{5}$$

$\times 5$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1} \qquad (x_1, x_2, x_3) = (-1, 1, 1)$$

when $\lambda = -2$ in $A - \lambda I$

$$= \begin{bmatrix} 2+2 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$4x_1 - 2x_2 + 3x_3 = 0 \rightarrow ①$

$x_1 + 3x_2 + x_3 = 0 \rightarrow ②$

$x_1 + 3x_2 + x_3 = 0$

$$A = \begin{vmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$\frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14} \qquad \lambda I - A$$

$\lambda_2 = (-11, -1, 14)$.

λ_3 substitute in $A - \lambda I$

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$$

$$-x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + 3x_2 - 4x_3 = 0$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\times 4 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} \quad (\text{e.g. } x_3 = (1,1,1))$$

$$P = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix} \left[\text{From } \lambda \text{ values}\right] \quad \lambda_1 \quad \lambda_2 \quad \lambda_3$$

$$P^{-1} = \frac{1}{|P|}(\text{adj } P)$$

$$|P| = \begin{vmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{vmatrix}$$

$$= -1(1+14) - 11(1-1) + 1(-14-1)$$

$$= -15 - 15$$

$$\boxed{|P| = -30}$$

$$\text{adj } P = \begin{bmatrix} +(1+14) & -(1-1) & +(-14-1) \\ -(11+14) & +(-1-1) & -(14-11) \\ +(11-1) & -(-1-1) & +(-1-11) \end{bmatrix}^T$$

$$= \begin{bmatrix} 15 & 0 & -15 \\ -25 & -2 & -3 \\ 10 & 2 & -12 \end{bmatrix}^T$$

$$= \begin{bmatrix} 15 & -25 & 10 \\ 0 & -2 & 2 \\ -15 & 13 & \cancel{-12} \end{bmatrix}$$

$D = P^{-1}AP$

$A = P^{-1}PD$

$$P^{-1} = \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 13 & 12 \end{bmatrix}$$

Hence, $D = P^{-1}AP \qquad A = DP^{-1}P$

D by using λ values

$$\begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 13 & 12 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -12 & 1 \end{bmatrix}$$

$$\times \left(= \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 13 & 12 \end{bmatrix}\begin{bmatrix} -1 & 11 & 1 \\ -2 & -2 & -2 \\ 3 & -42 & 3 \end{bmatrix}\right.$$

$$= \frac{1}{30}\begin{bmatrix} -15-50-30 & 165-50+420 & -10-4+24 \\ 0-4-6 & 0-22+84 & 0-4-6 \\ -15-26+36 & 165-26-504 & 15-26+36 \end{bmatrix}$$

$A = DP^{-1}P$

$$P^{-1}P = \frac{1}{30}\begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$A = DP^{-1}P$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

CAYLEY'S HAMILTON THEOREM

STATEMENT :

Every matrix satisfies its characteristic equation

PROOF :

Let A be a matrix of order n

The matrix $[A-\lambda I]$ is given by

$$[A-\lambda I] = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Let $|A-\lambda I|$ be $a_0+a_1\lambda+a_2\lambda^2+\cdots+a_n\lambda^n$

We ~~have~~ know that

$$\boxed{(\text{adj} A)(A) = |A| I}$$

Here $A = [A-\lambda I]$

$$\boxed{\therefore \{\text{adj}[A-\lambda I]\}[A-\lambda I] = |A-\lambda I| I} \to (1)$$

Let $\text{adj}[A-\lambda I] = B_0 - B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$

Here B_0, B_1, B_2 denotes matrices

Substitute the value of $\text{adj}[A-\lambda I] = B_0+B_1\lambda+B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$ in (1)

$$(B_0+B_1\lambda+B_2\lambda^2+\cdots+B_{n-1}\lambda^{n-1})[A-\lambda I] = (a_0+a_1\lambda+a_2\lambda^2+\cdots+a_n\lambda^n) I$$

Equating the like powers of λ on b.s

$B_0 A = a_0 I$

$B_1 A - B_0 I = a_1 I$

$\cdots\cdots$

$B_{n-1} A - B_{n-2} = a_{n-1} I$

$-B_{n-1} = a_n I$

Multiplying these eqns by $I, A, A^2 \dots A^{n-1}, A^n$

and adding $a_n A^n + a_{n-1} A^{n-1} + \dots a_1 A$

we get

$B_0 A + B_1 A^2 - B_0 A + B_2 A^3 - B_1 A^2 = a_0 I + a_1 A + a_2 A^2 \dots a_n A^n$

Hence A satisfies its characteristic equation

IMPORTANT APPLICATION OF CAYLEY'S THEOREM :

An important application of cayley's theorem is to express the inverse of a matrix in terms of powers of A. We have shown that,

$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0$

where $a_0 \neq 0$ and $|A| \neq 0$

$\therefore a_0 I = -a_1 A - a_2 A^2 - \dots - a_n A^n$

Premultiplying by A^{-1} we get

$a_0 A^{-1} I = -a_1 A^{-1} A - a_2 A^2 A^{-1} - \dots - a_n A^n A^{-1}$

$\therefore a_0 A^{-1} = -a_1 I - a_2 A - \dots - a_n A^{n-1}$

$$\therefore A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \dots - \frac{a_n}{a_0} A^{n-1}$$

Another important application is to calculate the higher powers.