SEMESTER : II
CORE COURSE : IV

Inst Hour	: 4
Credit	: 4
Code	: 18K2M04

VECTOR ANALYSIS AND FOURIER SERIES

UNIT 1:

Vector differentiation – velocity & acceleration –Vector & scalar fields – Gradient of a vector– Directional derivative -Divergence & curl of vector solenoidal & irrotational vectors – Laplacian double operator – simple problems
(Chapter I and Chapter II of Text book 1)

UNIT 2:

Vector Integration – Line Integral – Conservative field – scalar potential– Work done by a force – Surface integral – Volume integral – simple problems.
(Chapter III of Text book 1)

UNIT 3:

Gauss Divergence Theorem –Simple problems &Verification of the theorem
(Chapter IV- 4.1- 4.2.3 of Text book 1)

UNIT 4:

Stoke's Theorem– Green's Theorem – Simple problems & Verification of the theorems
(Chapter IV- 4.3- 4.5 of Text book 1)

UNIT 5:

Fourier series – definition –Finding Fourier Series expansion of periodic functions with Period 2π and with period 2a – Use of odd & even functions in Fourier Series. Half range Fourier series – definition– Development in Cosine series & in Sine series
(Chapter 6 Sections 1- 5 of Text book 2)

Text Book(s)

[1] K.Viswanatham & S.Selvaraj, Vector Analysis, Emerald Publishers Reprint 1999

[2] T.K.M Pillai & others, Calculus Volume III, S.V Publications 2014

Books for Reference

[1] M.L.Khanna, Vector Calculus

[2] M.D.Raisinghania, Vector Calculus

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

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UNIT - I

Unit - I

Vector Differentiation :

Pb: 1 If $\bar{A} = t^2\bar{i} - t\bar{j} + (2t+1)\bar{k}$ and $\bar{B} = (2t-3)\bar{i} + \bar{j} - t\bar{k}$, find

i, $\frac{d}{dt}(\bar{A}+\bar{B})$ ii, $\frac{d}{dt}(\bar{A}\times\bar{B})$

Soln:

i, $\bar{A} = t^2\bar{i} - t\bar{j} + (2t+1)\bar{k}$

$\Rightarrow \frac{d\bar{A}}{dt} = 2t\bar{i} - \bar{j} + 2\bar{k}$

& $\bar{B} = (2t-3)\bar{i} + \bar{j} - t\bar{k}$

$\frac{d\bar{B}}{dt} = 2\bar{i} - \bar{k}$

i, $\frac{d}{dt}(\bar{A}+\bar{B}) = \frac{d\bar{A}}{dt} + \frac{d\bar{B}}{dt}$

$= (2t\bar{i} - \bar{j} + 2\bar{k}) + (2\bar{i} - \bar{k})$

$= 2(t+1)\bar{i} - \bar{j} + \bar{k}$

$= 4\bar{i} - \bar{j} + \bar{k}$, at $t=1$.

ii, $\frac{d}{dt}(\bar{A}\times\bar{B}) = \bar{A}\times\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\times\bar{B}$

$= [t^2\bar{i} - t\bar{j} + (2t+1)\bar{k}] \times (2\bar{i} - \bar{k})$

$+ (2t\bar{i} - \bar{j} + 2\bar{k}) \times [(2t-3)\bar{i} + \bar{j} - t\bar{k}]$

$= (\bar{i} - \bar{j} + 3\bar{k}) \times (2\bar{i} - \bar{k}) + (2\bar{i} - \bar{j} + 2\bar{k}) \times$

$(-\bar{i} + \bar{j} - \bar{k})$, at $t=1$

$= 7\bar{j} + 3\bar{k}$.

Pb: 2 Find grad r^n, where $r^2 = x^2 + y^2 + z^2$.

Soln:

WKT $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$.

grad $r^n = \nabla r^n$

$= \bar{i} \frac{\partial r^n}{\partial x} + \bar{j} \frac{\partial r^n}{\partial y} + \bar{k} \frac{\partial r^n}{\partial z}$

$= \sum \bar{i}\, n r^{n-1} \frac{\partial r}{\partial x}$

$= \sum \bar{i}\, n r^{n-1} \frac{x}{r}$

$= \sum \bar{i}\, n r^{n-2} x$

$= n r^{n-2} (x\bar{i} + y\bar{j} + z\bar{k})$

$= n r^{n-2} \bar{r}$.

Pb: 3 Find the directional derivative of $\phi = xy + yz + zx$ at $(1, 2, 0)$ in the direction $\bar{i} + 2\bar{j} + 2\bar{k}$. Find also its max. value.

Soln:

$$\phi = xy + yz + zx$$

$$\therefore \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

$$= 2\bar{i} + \bar{j} + 3\bar{k}, \quad \text{at } (1,2,0)$$

Unit vector in the direction $\bar{i} + 2\bar{j} + 2\bar{k}$ is

$$\hat{a} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k}).$$

Then $\nabla\phi \cdot \bar{a} = (2\bar{i} + \bar{j} + 3\bar{k}) \cdot \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$

$$= \frac{2}{3} + \frac{2}{3} + 2 = \frac{10}{3}$$

Also, at $(1,2,0)$, max. directional derivative of $\phi = |\text{grad}\,\phi| = |\nabla\phi| = |2\bar{i} + \bar{j} + 3\bar{k}|$

$$= \sqrt{14}.$$

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Pb: 4 Show that

(a) $\bar{A} = 3y^4z^2\,\bar{i} + 4x^3z^2\,\bar{j} - 3x^2y^2\,\bar{k}$ is solenoidal and

(b) $\bar{B} = (6xy + z^3)\,\bar{i} + (3x^2 - z)\,\bar{j} + (3xz^2 - y)\,\bar{k}$ is irrotational.

Soln: (a) $\bar{A}$ is Solenoidal, if $\operatorname{div}\bar{A} = 0$

$$\operatorname{div}\bar{A} = \nabla \cdot \bar{A}$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0.$$

$\therefore \bar{A}$ - solenoidal.

(b) $\bar{B}$ - irrotational, if $\nabla \times \bar{B} = 0$ (ie, curl $\bar{B} = 0$)

$$\operatorname{curl}\bar{B} = \nabla \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy+z^3) & (3x^2-z) & (3xz^2-y) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right]\bar{i} +$$

$$+ \left[\frac{\partial}{\partial z}(6xy + z^3) - \frac{\partial}{\partial x}(3xz^2 - y)\right]\bar{j}$$

$$+ \left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]\bar{k}$$

$$= (-1+1)\bar{i} + (3z^2 - 3z^2)\bar{j} + (6x - 6x)\bar{k}$$

$$= 0$$

$\therefore \bar{B}$ is irrotational.

Pb: 5 Find $\nabla^2 r^n$ and deduce $\nabla^2\left(\frac{1}{r}\right)$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Soln:

$$\nabla^2 r^n = \nabla \cdot (\nabla r^n)$$

$$= \nabla \cdot (n r^{n-2} \bar{r})$$

$$= n \nabla \cdot (r^{n-2} \bar{r})$$

$$= n \left[r^{n-2} \nabla \cdot \bar{r} + \bar{r} \cdot (\nabla r^{n-2}) \right]$$

$$= n \left[3 r^{n-2} + \bar{r} \cdot (n-2) r^{n-4} \bar{r} \right] \quad (\because \nabla \cdot \bar{r} = 3)$$

$$= n \left[3 r^{n-2} + (n-2) r^{n-4} r^2 \right]$$

$$= n r^{n-2} \left[3 + n - 2 \right]$$

$$= n(n+1) r^{n-2}$$

$$\therefore \nabla^2 (r^n) = n(n+1) r^{n-2}.$$

Also, $\nabla^2\left(\frac{1}{r}\right) = \nabla^2 (r^{-1}) = 0, \quad (\because n = -1)$

Pb: 6 Find the angle between the normals to the surface $xy = z^2$ at $(1, 4, -2)$ and $(-3, -3, 3)$ on it.

Soln.

$$\phi = xy - z^2 = 0$$

$$\Rightarrow \nabla\phi = \nabla(xy - z^2)$$

$$= \bar{i}\frac{\partial(xy - z^2)}{\partial x} + \bar{j}\frac{\partial}{\partial y}(xy - z^2) + \bar{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\bar{i} + x\bar{j} - 2z\bar{k}$$

Normal to the surface at $(1, 4, -2)$ is

$$\nabla_1\phi = 4\bar{i} + \bar{j} + 4\bar{k}$$

Normal to the surface at $(-3, -3, 3)$ is

$$\nabla_2\phi = -3\bar{i} - 3\bar{j} - 6\bar{k}$$

Let θ be the angle between the normals.

Then $\cos\theta = \dfrac{\nabla_1\phi}{|\nabla_1\phi|} \cdot \dfrac{\nabla_2\phi}{|\nabla_2\phi|}$

$$= \frac{4\bar{i} + \bar{j} + 4\bar{k}}{\sqrt{16 + 1 + 16}} \cdot \frac{-3\bar{i} - 3\bar{j} - 6\bar{k}}{\sqrt{9 + 9 + 36}}$$

$$= \frac{-39}{\sqrt{33}\sqrt{54}} = \frac{-13}{3\sqrt{22}}$$

$\therefore$ Acute angle between normals is $\cos^{-1}\left(\frac{13}{3\sqrt{22}}\right)$.

Pb: 7 Prove that $\nabla^2(r^n \bar{r}) = n(n+3)r^{n-2}\bar{r}$.

soln:

$$\nabla^2(r^n \bar{r}) = \nabla[\nabla \cdot r^n \bar{r}]$$

$$= \nabla\left[(\nabla r^n)\cdot \bar{r} + r^n(\nabla \cdot \bar{r})\right]$$

$$= \nabla\left[n r^{n-2}\bar{r})\cdot \bar{r} + 3r^n\right]$$

$$= \nabla\left[n r^n + 3r^n\right] \quad [\because \bar{r}\cdot\bar{r} = r^2]$$

$$= \nabla\left[(n+3) r^n\right]$$

$$= (n+3)\nabla r^n$$

$$= (n+3)\left[n r^{n-2}\bar{r}\right]$$

$$= n(n+3)r^{n-2}\bar{r}.$$

Vector Identities:

1. $\operatorname{div}(\phi \bar{F}) = \phi \operatorname{div}\bar{F} + \bar{F}\cdot \operatorname{grad}\phi$
2. $\operatorname{curl}(\phi \bar{F}) = \phi \operatorname{curl}\bar{F} + (\operatorname{grad}\phi)\times \bar{F}$
3. $\operatorname{div}(\bar{A}\times\bar{B}) = \bar{B}\cdot(\operatorname{curl}\bar{A}) - \bar{A}\cdot(\operatorname{curl}\bar{B})$
4. $\operatorname{curl}(\bar{A}\times\bar{B}) = \bar{A}\operatorname{div}\bar{B} - \bar{B}\operatorname{div}\bar{A} + (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B}$
5. $\operatorname{curl}(\operatorname{grad}\phi) = 0$.
6. $\operatorname{div}(\operatorname{curl}\bar{A}) = 0$
7. $\operatorname{div}(\operatorname{grad}\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$.

Exercise Pbs:

1. If $\phi(x, y, z) = 3x^2y - y^3z^3$, find grad ϕ at the point $(1, -2, -1)$.

2. If $\bar{A} = 2yz\,\bar{i} - x^2y\,\bar{j} + xz^2\,\bar{k}$ and $\phi = 2x^2yz^3$ find i, grad ϕ ii, $\bar{A} \cdot$ grad ϕ iii, $(\bar{A} \cdot \nabla)\phi$

3. Show that grad $\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$.

4. Find the unit vector normal to the surface $x^2 + 2y^2 + z^2 = 7$ at $(1, -1, 2)$.

5. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\bar{i} - \bar{j} - 2\bar{k}$.

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Unit - II

Unit - II

Integration of vectors

I. Line Integrals :

Pb: 1 If $\bar{A} = (3x^2 + 6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, evaluate $\int_C \bar{A}\cdot d\bar{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path C given by $x = t$; $y = t^2$; $z = t^3$.

Soln:

$$\bar{A} = (3x^2 + 6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$$

The path C is given by the parametric eqns.

$$x = t, \quad y = t^2, \quad z = t^3$$

$$\Rightarrow dx = dt, \; dy = 2t\,dt, \; dz = 3t^2 dt$$

& $t = 0$ to $t = 1$

$$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_C (3x^2 + 6y)dx - (14yz)dy + (20xz^2)dz$$

$$= \int_C \left[(3x^2 + 6y)\frac{dx}{dt} - (14yz)\frac{dy}{dt} + (20xz^2)\frac{dz}{dt}\right]dt$$

$$= \int_0^1 \left[(3t^2 + 6t^2)1 - (14t^5)2t + (20t^7)3t^2\right]dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 5$$

Pb: 2 Evaluate $\int_C \bar{F}\cdot d\bar{r}$, when $\bar{F} = yz\bar{i} + 2y\bar{j} - x^2\bar{k}$, where C consists of straight line segments from O(0, 0, 0) to A(0, 0, 1); from A to B(0, -3, 1) and from B to P(2, -3, 1).

Soln:

$$\bar{F} = yz\,\bar{i} + 2y\,\bar{j} - x^2\,\bar{k},$$

& $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BP} \bar{F}\cdot d\bar{r} \quad \text{--- (1)}$$

Along the line OA, joining (0, 0, 0) and (0, 0, 1)

$x = y = 0$, $dx = dy = 0$.

$\therefore \bar{F} = 0$, and $d\bar{r} = dz\,\bar{k}$

$$\therefore \int_{OA} \bar{F}\cdot d\bar{r} = 0.$$

Along the line AB, joining (0, 0, 1) and (0, -3, 1)

$x = 0$, $z = 1$; $dx = 0 = dz$

$\therefore \bar{F} = y\bar{i} + 2y\bar{j}$ and $d\bar{r} = dy\,\bar{j}$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_0^{-3} (2y)\,dy.$$

$$= \left[y^2\right]_0^{-3} = 9$$

Along the line BP, joining (0, -3, 1) and (2, -3, 1)

$y = -3$, $z = 1$, $dy = 0 = dz$, x varies from 0 to 2.

$\bar{F} = -3\bar{i} - 6\bar{j} - x^2\bar{k}$ and $d\bar{r} = dx\,\bar{i}$

$$\int_{BP} \bar{F} \cdot d\bar{r} = \int_0^2 (-3)\,dx = \left[-3x\right]_0^2 = -6.$$

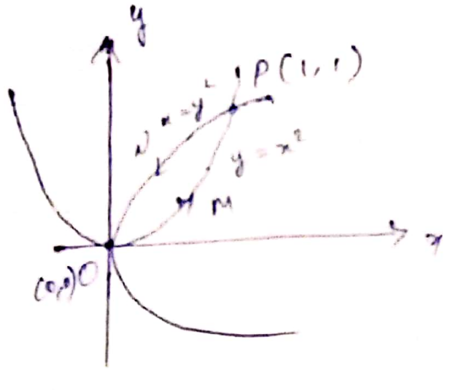
Hence $\int_C \bar{F} \cdot d\bar{r} = 0 + 9 - 6 = 3$

Pb: 3 If $\bar{A} = (x-y)\bar{i} + (x+y)\bar{j}$, evaluate $\int_C \bar{A} \cdot d\bar{r}$ around the closed curve C which is given by $y = x^2$ and $x = y^2$.

Soln:

$y = x^2$ & $y^2 = x$ intersect at the origin O(0, 0) and P(1,1) and form the closed curve. C.

$$\oint_C \bar{A} \cdot d\bar{r} = \int_{OMP} \bar{A} \cdot d\bar{r} + \int_{PNO} \bar{A}\, d\bar{r}$$



Along OMP,

$y = x^2$, $dy = 2x\,dx$, limit : $x=0$ to $x=1$

$\therefore \bar{A} = (x \cdot y)\bar{i} + (x+y)\bar{j}$

$= (x - x^2)\bar{i} + (x + x^2)\bar{j}$

& $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$= dx\,\bar{i} + 2x\,dx\,\bar{j}$, along OMP.

$$\int_{OMP} \bar{A} \cdot d\bar{r} = \int_{x=0}^{x=1} [(x - x^2)dx + (x + x^2)\,2x\,dx]$$

$$= \int_0^1 (x + x^2 + 2x^3)\,dx$$

$$= \frac{4}{3}.$$

Along PNO

$x = y^2$, $dx = 2y\,dy$, limit: $y=1$ to $y=0$

$\therefore \bar{A} = (y^2 - y)\bar{i} + (y^2 + y)\bar{j}$. 4 .

$d\bar{r} = 2y\,dy\,\bar{i} + dy\,\bar{j}$, along PNO.

$$\int_{PNO} \bar{A} \cdot d\bar{r} = \int_{y=1}^{y=0} [(y^2 - y)2y\,dy + (y^2 + y)\,dy]$$

$$= \int_1^0 (y - y^2 + 2y^3)\,dy = -\frac{2}{3}$$

Hence $\oint_C \bar{A} \cdot d\bar{r} = \frac{4}{3} + \left(-\frac{2}{3}\right) = \frac{2}{3}$.

II. Surface Integral :

Pb: 1 Evaluate $\iint_S \bar{A} \cdot \hat{n}\, ds$, if $\bar{A} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Soln:

Given surface is $2x+y+2z=6$.

Then $\nabla(2x+y+2z) = 2\bar{i}+\bar{j}+2\bar{k}$

i.e, $\bar{n} = 2\bar{i}+\bar{j}+2\bar{k}$

$$\Rightarrow \hat{n} = \frac{\bar{n}}{|\bar{n}|} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$$

Consider the orthogonal projection of S on xy-plane.

Unit vector normal to xy plane is $\bar{k}$.

$$\hat{n} \cdot \bar{k} = \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k}) \cdot \bar{k} = \frac{2}{3}$$

$$\bar{A} \cdot \hat{n} = \left[(x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}\right] \cdot \frac{1}{3}(2\bar{i}+\bar{j}+2\bar{k})$$

$$= \frac{2}{3}\left[(x+y^2) - x + 2yz\right]$$

$$= \frac{2}{3}\left[(x+y^2) - x + y(6-2x-y)\right].$$

$\because 2z = 6-2x-y$

$$= \frac{1}{3}(3-x)y$$

$$\therefore \iint_S \bar{A}\cdot\hat{n}\, ds = \iint_R \bar{A}\cdot\hat{n} \frac{dx\,dy}{|\hat{n}\cdot\bar{k}|}$$

$$= \iint_R \frac{1}{3}(3-x)y \frac{dx\,dy}{(2/3)}$$

$$= 2\iint_R (3-x)y \, dx\,dy$$

$$= 2\int_{x=0}^{3}\int_{y=0}^{6-2x} (3-x)y \, dx\,dy$$

$$= 2\int_{x=0}^{3} (3-x)\left[\frac{y^2}{2}\right]_{y=0}^{6-2x} dx$$

$$= 2\int_{0}^{3} (3-x)\frac{(6-2x)^2}{2} dx$$

$$= 4\int_{0}^{3} (3-x)^3 dx$$

$$= -4\left[\frac{(3-x)^4}{4}\right]_0^3$$

$$= 81$$

III. Volume Integral :

Pb: 1 Evaluate $\int_V \bar{F}\, dv$, where $\bar{F} = xy\bar{i} - zx\bar{j} + \bar{k}$, and V is the octant of the sphere $x^2+y^2+z^2 = 4$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

Soln:

$$\int_V (xy\bar{i} - zx\bar{j} + \bar{k})\, dv$$

$$= \bar{i}\int_V xy\, dv - \bar{j}\int_V zx\, dv + \bar{k}\int_V dv \quad \text{—— (1)}$$

Consider

$$\int_V xy\, dv = \iiint_V xy\, dx\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{\sqrt{4-z^2}}\int_{x=0}^{\sqrt{4-y^2-z^2}} xy\, dx\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{\sqrt{4-z^2}} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y^2-z^2}} y\, dy\, dz$$

$$= \frac{1}{2}\int_{z=0}^{2} dz \int_{y=0}^{\sqrt{4-z^2}} y(4-y^2-z^2)\, dy$$

$$= \frac{1}{2}\int_{z=0}^{2}\left[-\frac{(4-y^2-z^2)^2}{4}\right]_{y=0}^{\sqrt{4-z^2}} dz$$

$$= -\frac{1}{8}\int_{0}^{2}(4-z^2)^2\, dz = \frac{32}{15}.$$

By Symmetry,

$$\int_V zx\, dV = \frac{32}{15}$$

Now,

$\int_V dV = V$, where V - volume of the sphere $x^2+y^2+z^2=4$ in $x \geq 0,\ y \geq 0,\ z \geq 0.$

$$= \frac{1}{8} \times V$$

$$= \frac{1}{8} \times \frac{4}{3}\pi(2^3), \qquad (\because r = 2$$

$$= \frac{4}{3}\pi$$

$\therefore$ From ①,

$$\int_V \bar{F}\, dV = \frac{32}{15}\bar{i} - \frac{32}{15}\bar{j} + \frac{4}{3}\pi\bar{k}$$

Pb: 2 Evaluate $\int_V \operatorname{div} \bar{A}\, dv$, where $\bar{A} = 2x^2 y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ and V is the region in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and $x = 2$.

Soln:

$$\operatorname{div} \bar{A} = \nabla \cdot \bar{A}$$

$$= \nabla \cdot (2x^2 y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k})$$

$$= \frac{\partial}{\partial x}(2x^2 y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8xz$$

$$\int_V \operatorname{div} \bar{A}\, dv = \iiint_V (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{z=0}^{3} \int_{y=0}^{\sqrt{9-z^2}} (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{z=0}^{3} dx\, dz \left[2xy^2 - y^2 + 8xzy \right]_{y=0}^{\sqrt{9-z^2}}$$

$$= \int_{x=0}^{2} dx \int_{z=0}^{3} \left[(2x-1)(9-z^2) + 8xz\sqrt{9-z^2} \right] dz$$

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$$= \int_{x=0}^{2} \left[(2x-1)\left(9z - \frac{z^3}{3}\right) - \frac{8x}{3}\left(9-z^2\right)^{3/2} \right]_{z=0}^{3} dx$$

$$= \int_{0}^{2} 18(6x-1)\,dx$$

$$= 180.$$

Ex. Probs :

1. Evaluate $\int_{0}^{\pi/2} (3\sin x\,\bar{i} + 2\cos x\,\bar{j})\,dx$

2. Show that $\int_{2}^{4} \bar{F}(t)\,dt = 50\bar{i} - 32\bar{j} - 24\bar{k}$.

   if $\bar{F}(t) = (3t^2 - t)\bar{i} + (2-6t)\bar{j} - 4t\,\bar{k}$.

3. Show that if $\bar{A} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$,

   $\iint_S \bar{A}\cdot\hat{n}\,ds = \frac{12}{5}\pi R^5$, where S - surface of the sphere of radius R, with centre at origin

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Unit - III

Unit - III

Integral Theorems

1. **Gauss's Divergence Theorem**

It relates the surface integral of a vector fn to the volume integral of the divergence of the vector fn.

2. **Stoke's Theorem:**

It relates the line integral of a vector fn to the surface integral of the curl of the vector fn.

3. **Green's Theorem:**

It is a particular case of Stoke's theorem in which the surface is planar. It relates a line integral to the double integral taken over the region bounded by the closed curve.

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1. Gauss's Divergence Formula

$$\iint_S F \cdot d\bar{s} = \iiint_V \nabla \cdot F \, dv, \quad d\bar{s} = \hat{n}\, ds.$$

(or) $$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy) =$$

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz.$$

Pb: 1. Use Divergence Theorem to show that

$$\iint_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \hat{n}\, ds = \frac{4}{3}\pi (a+b+c).$$

where S is the surface of the sphere

$x^2 + y^2 + z^2 = 1$.

Soln: Let $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$

$\text{div } \bar{F} = a + b + c$

$$\iint_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \hat{n}\, ds = \iint_S \bar{F} \cdot \hat{n}\, ds$$

$$= \iiint_V \text{div } \bar{F}\, dv$$

$$= \iiint_V (a+b+c)\, dv$$

$= (a+b+c)V$, where V is the volume enclosed by S.

$= \frac{4}{3}\pi\,(a+b+c)$.

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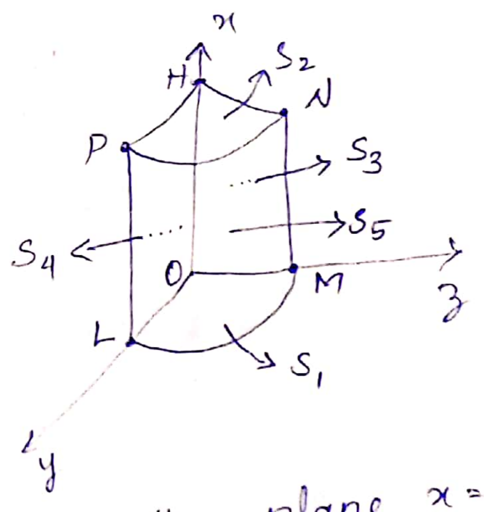
Pb: 2 State Gauss's Divergence theorem and verify it for $\bar{A} = 2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ taken over the region bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $x=2$, $y=0$, $z=0$, lying in the first octant.

Soln:



1. Plane face OLM (quadrant of the circle $y^2+z^2=9$) in the plane $x=0$, denoted by S_1.

2. Plane face HPN (quadrant of the circle) in the plane $x=2$, denoted by S_2.

3. Plane face OMNH in the plane $y=0$, denoted by S_3.

4. Plane HPLO in the plane $z=0$, denoted by S_4. and

(5). Curved surface PLMN in the first octant, denoted by the surface integral

$$\iint_S \bar{A}\cdot\hat{n}\, ds = \iint_{S_1} \bar{A}\cdot\hat{n}_1\, ds_1 + \iint_{S_2} \bar{A}\cdot\hat{n}_2\, ds_2$$

$$+ \iint_{S_3} \bar{A}\cdot\hat{n}_3\, ds_3 + \iint_{S_4} \bar{A}\cdot\hat{n}_4\, ds_4 + \iint_{S_5} \bar{A}\cdot\hat{n}_5\, ds_5 \quad \text{——} (1)$$

$$\bar{A} = 2x^2 y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$$

For S_1, $x=0$ so that $\bar{A} = -y^2\bar{j}$ and $\hat{n}_1 = -\bar{i}$

$$\therefore \iint_{S_1} \bar{A}\cdot\hat{n}_1\, ds_1 = \iint_{S_1} (-y^2\bar{j})\cdot(-\bar{i})\, ds_1 = 0$$

For S_2, $x=2$ so that $\bar{A} = 8y\bar{i} - y^2\bar{j} + 8z^2\bar{k}$ and $\hat{n}_2 = \bar{i}$

$$\therefore \iint_{S_2} \bar{A}\cdot\hat{n}_2\, ds_2 = \iint_{S_2} (8y\bar{i} - y^2\bar{j} + 8z^2\bar{k})\cdot\bar{i}\, ds_2$$

$$= \iint_{S_2} 8y\, ds_2 = \iint_{S_2} 8y\, dy\, dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} 8y\, dy\, dz$$

$$= \int_{y=0}^{3} 8y\sqrt{9-y^2}\, dy$$

$$= -4\int_{y=0}^{3} \sqrt{9-y^2}\, d(9-y^2)$$

$$= -\frac{8}{3}\left[(9-y^2)^{3/2}\right]_0^3$$

$$= 72$$

For S_3, $y=0$ so that $\bar{A} = 4xz^2\bar{k}$ and $\hat{n}_3 = -\bar{j}$

$$\therefore \iint_{S_3} \bar{A}\cdot\hat{n}_3\, dS_3 = \iint_{S_3} (4xz^2\bar{k})\cdot(-\bar{j})\, dS_3 = 0$$

For S_4, $z=0$ so that $\bar{A} = 2x^2y\bar{i} - y^2\bar{j}$ and

$$\hat{n}_4 = -\bar{k}$$

$$\therefore \iint_{S_4} \bar{A}\cdot\hat{n}_4\, dS_4 = \iint_{S_4} (2x^2y\bar{i} - y^2\bar{j})\cdot(-\bar{k})\, dS_4 = 0$$

For the curved surface S_5, $\hat{n}_5 = \frac{\nabla\phi}{|\nabla\phi|}$, where ϕ is the surface $y^2+z^2=9$

$$\hat{n}_5 = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4y^2+4z^2}} = \frac{2y\bar{j}+2z\bar{k}}{\sqrt{36}}$$

$$= \frac{1}{3}(y\bar{j} + z\bar{k}), \quad \because y^2+z^2=9$$

$$\therefore \iint_{S_5} \bar{A}\cdot\hat{n}_5\, dS_5 = \iint_{S_5} (2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k})\cdot\frac{1}{3}(y\bar{j}+z\bar{k})\, dS_5$$

$$= \iint_{S_5} \frac{1}{3}(4xz^3 - y^3)\, dS_5$$

Take $y = 3\cos\theta$, $z = 3\sin\theta$ so that $ds_5 = 3\,d\theta\,dx$. As θ varies from 0 to $\frac{\pi}{2}$ and x varies from 0 to 2, the whole curved surface S_5 is covered.

$$\therefore \iint_{S_5} \bar{A}\cdot\hat{n}_5\, ds_5 = \int_{x=0}^{2}\int_{\theta=0}^{\pi/2} \frac{1}{3}\left[4x(3\sin\theta)^3 - (3\cos\theta)^3\right] 3\,dx\,d\theta$$

$$= \int_{x=0}^{2} 4x\,dx \int_{\theta=0}^{\pi/2} 27\sin^3\theta\, d\theta - \int_{x=0}^{2} dx \int_{\theta=0}^{\pi/2} 27\cos^3\theta\, d\theta$$

$$= 2\times 4\times 27 \times \frac{2}{3} - 2\times 27\times \frac{2}{3}$$

$$= 108.$$

Thus, $\iint_S \bar{A}\cdot\hat{n}\, dS = 0 + 72 + 0 + 0 + 108$

$$= 180$$

Now, the volume integral,

$$\iiint_V \operatorname{div}\bar{A}\, dv = \iiint_V \nabla\cdot(2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k})\,dV$$

$$= \iiint_V (4xy - 2y + 8xz)\,dx\,dy\,dz$$

The volume V is covered as x varies from 0 to 2, y varies from 0 to 3 and z varies from 0 to $\sqrt{9-y^2}$.

$\therefore$ The volume integral

$$= \int_{x=0}^{2} \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left[4xyz - 2yz + 4xz^2 \right]_{z=0}^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left[2y(2x-1)\sqrt{9-y^2} + 4x(9-y^2) \right] dx\, dy$$

$$= \int_{x=0}^{2} \left[-\frac{2}{3}(2x-1)(9-y^2)^{3/2} + 4x\left(9y - \frac{y^3}{3}\right) \right]_{y=0}^{3} dx$$

$$= \int_{x=0}^{2} \left[18(2x-1) + 72x \right] dx$$

$$= \int_{0}^{2} (108x - 18)\, dx$$

$$= \left[108\frac{x^2}{2} - 18x \right]_{0}^{2}$$

$$= \left[108 \times 2 - 18 \times 2 \right]$$

$$= 180$$

II Green's Theorem Formula:

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \text{ where}$$

C - Simple closed curve & described in the +ve direction.

R - region of the given plane.

Pb:1 Evaluate $\oint_C y(2xy-1)\,dx + x(2xy+1)\,dy$, where C is the circle $x^2+y^2=1$, using Green's Theorem.

Soln:

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here $P = y(2xy-1) \Rightarrow \frac{\partial P}{\partial y} = 4xy - 1$

& $Q = x(2xy+1) \Rightarrow \frac{\partial Q}{\partial x} = 4xy + 1$

$$\therefore \oint_C y(2xy-1)\,dx + x(2xy+1)\,dy$$

$$= \iint_R [(4xy+1) - (4xy-1)]\,dx\,dy$$

$$= \iint_R 2\,dx\,dy = 2\iint_R dx\,dy$$

$= 2$ (Area of the region R, which is the circle $x^2+y^2=1$)

$(\because \pi r^2 = \pi)$

$= 2\pi$

III. Stoke's Theorem Formula:

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, ds$$

Pb: 1 Verify Stoke's theorem for the vector fn $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Soln: For the boundary circle C, is $x^2+y^2=1$, $z=0$.

Now, $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$.

$$\bar{F} \cdot d\bar{r} = (y\bar{i} + z\bar{j} + x\bar{k}) \cdot (dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= y\,dx + z\,dy + x\,dz$$

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C y\,dx + z\,dy + x\,dz$$

$$= \oint_C y\,dx, \quad \because z=0 \ \& \ dz=0 \text{ on } C.$$

$$= \int_0^{2\pi} \sin\theta\, d(\cos\theta)$$

$$= -\int_0^{2\pi} \sin^2\theta\, d\theta = -\frac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)\,d\theta$$

$$= -\pi$$

$$\text{curl } \bar{F} = \nabla \times (y\bar{i} + z\bar{j} + x\bar{k})$$

$$= -\bar{i} - \bar{j} - \bar{k}$$

$$\therefore \iint_S (\text{curl}\,\bar{F}) \cdot \hat{n}\, ds = \iint_{S_1} (\text{curl}\,\bar{F}) \cdot \bar{k}\, ds$$

$$= \iint_{S_1} (-\bar{i} - \bar{j} - \bar{k}) \cdot \bar{k}\, ds$$

$$= \iint_{S_1} (-1)\, ds = -S_1$$

where S_1 is the area of the region bounded by circle C on $z=0$.

But the area of the circle $x^2+y^2=1$ is π.

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = \iint_S (\text{curl}\,\bar{F}) \cdot \hat{n}\, ds$$

Hence Stoke's Theorem is verified.