Inst Hour	: 4
Credit	: 3
Code	: 18K2MAM3

SEMESTER : II
ALLIED COURSE : III - Mathematics

NUMERICAL METHODS – III
(For B.Sc., Mathematics Major)

UNIT 1:
Finite Differences: First Differences – Operators and their relations - Interpolation – Gregory Newton's Backward and Forward Formula for Interpolation.
(Chapter 5- 5.1, 5.2 and Chapter6 - 6.1,6.2 ,6.3)

UNIT 2:
Interpolation with unequal intervals – Divided differences and their properties – Newton's divided difference formula.
(Chapter 8 -8.1-8.3, 8.5)

UNIT 3:
Finite Differences: Factorial Polynomial - Inverse Operator- Summation of Series – Montmort's Theorem
(Chapter 5 -5.4, 5.6, 5.7, 5.8)

UNIT 4:
Numerical Differentiation – Newton's Forward and Backward Difference Formula – Derivative using Stirling Formula - Maxima and Minima of the function given the Tabular values.
(Chapter 9- 9.1-9.4, 9.6)

UNIT 5:
Numerical Integration – Newton – Cote's Formula - Trapezoidal Rule –Simpson's 1/3 Rul[e]
Simpson's 3/8 Rule - Weddle's Rule
(Chapter 9- 9.7 - 9.11, 9.13 - 9.15)

*****(In all the Units SIMPLE PROBLEMS ONLY)**

Text Book:

1. Kandasamy.P, Thilagavathy. K., Gunavathi.K., Numerical Methods , S.Chand& Company 2015.

Reference Books:

[1] S.S.Sastry, Introductory Methods of Numerical Analysis, Prentice Hall of India, Private Limited, New Delhi – 11, Fourth Edition.

[2] H.C.Saxena, Finite Differences and Numerical Analysis, S.Chand & Company Limited, New Delhi-110055, Ninth Edition

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each U[nit]
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

UNIT - I

Finite Differences

6.1 First difference

Let $y = f(x)$ be a given function of x and let $y_0, y_1, y_2, \ldots y_n$ be the values of y corresponding to $x_0, x_1, x_2 \ldots x_n$ the values of x. The independent variable x is called the argument and the corresponding dependent value y is called, the entry. In general, the difference between any two consective values of x need not be same or equal.

We can write the arguments and entries as below.

x	x_0	x_1	x_2	...	x_{n-1}	x_n
y	y_0	y_1	y_2	...	y_{n-1}	y_n

If we subtract from each value of y (except y_0) the preceding value of y, we get

$$y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2, \ldots y_n - y_{n-1}$$

These results are called the first differences of The first differences of y are denoted by Δy.

That is
$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

Here, the symbol Δ denotes an operation, called fo difference operator.

Higher differences:

The second and higher differences are defined as below:

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta(y_n - y_{n-1}) = \Delta y_n - \Delta y_{n-1}$$

Here, Δ^2 is an operator called, second order forward difference operator. In the same way, the third order forward difference operator Δ^3 is as follows:

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$
$$\cdots\cdots\cdots \text{etc}$$

In general $\quad \Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$

Though the arguments $x_0, x_1, x_2, \ldots$ need not, in general, be equally spaced, for purpose of practical work, we take them equally spaced.

Usually, the arguments are taken as

$$x_0,\ x_0+h,\ x_0+2h,\ x_0+3h,\ \ldots$$

so that $\quad x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = h$

Here, h is called the interval of differencing

Therefore, $\quad \Delta f(x) = f(x+h) - f(x)$

$$\begin{aligned}
\Delta^2 f(x) &= \Delta(\Delta f(x)) \\
&= \Delta(f(x+h) - \Delta f(x)) \\
&= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)] \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}$$

$$\begin{aligned}
\Delta^3 f(x) &= \Delta^2[\Delta f(x)] = \Delta[\Delta^2 f(x)] \\
&= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x) \\
&= f(x+3h) - f(x+2h) - 2[f(x+2h) - f(x+h)] \\
&\quad + f(x+h) - f(x) \\
&= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)
\end{aligned}$$

and so on.

<u>Operators</u>. We have already defined the forward difference operator Δ. We will now see some more operators and the relations connecting them.

Backward difference operator (∇) is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

By definition $\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$ etc.

Hence $\nabla^2 f(x) = \nabla [f(x) - f(x-h)]$

$= \nabla f(x) - \nabla f(x-h)$

$= f(x) - f(x-h) - [f(x-h) - f(x-2h)]$

$= f(x) - 2f(x-h) + f(x-2h)$

Central difference operator (δ):

The central difference operator δ is defined by

$$\delta f(x) = f(x+h/2) - f(x-h/2)$$

or $\delta y_x = y_{x+h/2} - y_{x-h/2}$

Shifting or displacement or translation operator E: We define the Shifting operator E such that.

$$Ef(x) = f(x+h)$$

or $Ey_x = y_{x+h}$

Hence $Ey_1 = y_2$ $\quad E(y_2) = y_3$ etc.

$E^2 y_x = E(y_{x+h}) = y_{x+2h}$

$E^n y_x = y_{x+nh}$ and $E^n f(x) = f(x+nh)$

Inverse operator E^{-1} is such that

$$E^{-1}Ef(x) = f(x)$$

Suppose $E^{-1} f(x) = \varphi(x)$

then $E\,\varphi(x) = f(x)$

Hence $\varphi(x+h) = f(x)$

$\varphi(x) = f(x-h)$

$\therefore E^{-1} f(x) = f(x-h)$

Similarly $E^{-2} f(x) = f(x-2h)$

Averaging operator μ:

The averaging operator μ is defined by

$$\mu y_x = \frac{1}{2}\left[y_{x+h/2} + y_{x-h/2}\right]$$

i.e, $$\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

Differential operator D:

The differential operator D is defined by

$$Df(x) = \frac{d}{dx} f(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) \text{ etc.}$$

UNIT operator 1:

The unit operator 1 is such that

$$1 \cdot f(x) = f(x)$$

Properties of operators

1. The operators $\Delta, \nabla, E, S, \mu$ and D are all linear operators.

<u>Proof</u>: $\Delta[af(x) + b\varphi(x)] = [af(x+h) + b\varphi(x+h)] - [af(x) + b\varphi(x)]$

$$= a[f(x+h) - f(x)] + b[\varphi(x+h) - \varphi(x)]$$

$$= a\Delta f(x) + b\Delta\varphi(x)$$

Hence Δ is a linear operator.

Putting $a=b=1$, $\Delta[f(x)+g(x)] = \Delta f(x) + \Delta g(x)$

and by putting $b=0$, $\Delta(af(x)) = a\Delta f(x)$

2. The operator is distributive over addition.

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x).$$

$$\Delta^m \Delta^n f(x) = (\Delta.\Delta \ldots m \text{ factors})(\Delta.\Delta \ldots n \text{ factors}) f(x)$$

$$= \Delta^{m+n} f(x)$$

All the above operators obey index laws.

3. Also $\Delta[f(x)+g(x)] = \Delta[g(x)+f(x)]$

Relation between E and ∇:

$$\nabla f(x) = f(x) - f(x-h)$$
$$= 1\cdot f(x) - E^{-1} f(x)$$
$$= (1-E^{-1}) f(x)$$
$$\nabla = 1 - E^{-1}$$
$$E^{-1} = 1 - \nabla$$
$$E = (1-\nabla)^{-1} \quad \text{since } (E^{-1})^{-1} = E$$

Relation between E and δ

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$
$$= E^{1/2} f(x) - E^{-1/2} f(x)$$
$$= (E^{1/2} - E^{-1/2}) f(x)$$
$$\therefore \delta = E^{1/2} - E^{-1/2} \quad = E^{-1/2}[E-1] = E^{-1/2}\Delta$$

Also $\delta = E^{1/2}[1-E^{-1}] = E^{1/2}\nabla$

$$\therefore \delta = E^{-1/2}\Delta = E^{1/2}\nabla$$

Relation between E and μ:

$$\mu f(x) = \frac{1}{2}\left[f(x + h/2) + f(x - h/2)\right]$$
$$= \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$
$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$
$$\therefore \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

Relation between D and Δ:

$$Df(x) = \frac{d}{dx} f(x)$$

By Taylor's theorem,

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \dots \text{ to } \infty$$
$$\therefore Ef(x) = f(x) + \frac{h}{1!} Df(x) + \frac{h^2}{2!} D^2 f(x) + \dots \text{ to } \infty$$
$$= \left[1 + \frac{hD}{1!} + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots \text{ to } \infty\right] f(x)$$
$$= e^{hD} f(x) \Rightarrow E = e^{hD}$$

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$\therefore E = 1+\Delta = e^{hD}$

$hD = \log E = \log(1+\Delta)$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]$$

Forward Difference Table.

The finite forward differences of a function are represented below in a tabular form.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Difference Table:

The backward differences are given in the following backward difference table.

x	y	∇	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
		∇y_0					
x_1	y_1		$\nabla^2 y_0$				
		∇y_1		$\nabla^3 y_0$			
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$		
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$	
x_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$		$\nabla^6 y_0$
		∇y_3		$\nabla^3 y_2$		$\nabla^5 y_1$	
x_4	y_4		$\nabla^2 y_3$		$\nabla^4 y_2$		
		∇y_4		$\nabla^3 y_3$			
x_5	y_5		$\nabla^2 y_4$				
		∇y_5					
x_6	y_6						

Ex. 1 Find the 7^{th} term of the sequence 2, 9, 28, 65, 126, 217 and also find general term. (7)

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		7			
1	9		12		
				6	
2	28	19			0
			18		
3	65	37		6	0
		61	24		
4	126			6	
		91	30		
5	217				

7^{th} term $= y_6 = y_0 + 6c_1 \Delta y_0 + 6c_2 \Delta^2 y_0 + 6c_3 \Delta^3 y_0 + 6c_4 \Delta^4 y_0$

$= 2 + 6(7) + 15(12) + 20(6) + 15(0)$

$= 2 + 42 + 180 + 120 = 344$

$y_n = y_0 + nc_1 \Delta y_0 + nc_2 \Delta^2 y_0 + nc_3 \Delta^3 y_0 + nc_4 \Delta^4 y_0 + \cdots$

$= 2 + n(7) + \frac{n(n-1)}{2}(12) + \frac{n(n-1)(n-2)}{6}(6) + 0$

$= 2 + 7n + 6n^2 - 6n + n^3 - 3n^2 + 2n$

$= n^3 + 3n^2 + 3n + 2$

$= (n+1)^3 + 1$

$\therefore y_6 = (6+1)^3 + 1 = 344$

Ex. 2 Find the sixth term of the sequence 8, 12, 19, 29, 42 ...

Soln: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	8			
		4		
1	12		3	
				0
2	19	7		
			3	
3	29	10		0
		13	3	
4	42			

6^{th} term $= y_5 = E^5 y_0 = (1+\Delta^5) y_0$

$= y_0 + 5\Delta y_0 + 10\Delta^2 y_0 + 10\Delta^3 y_0 + \cdots$

$= 8 + 5(4) + 10(3) + 10(0) + \cdots$

$= 58$

5.2 Express any value of y in term of y_n and backward difference of y_n

We know $\nabla y_n = y_n - y_{n-1}$

$\therefore \quad y_{n-1} = y_n - \nabla y_n = (1-\nabla) y_n$

similarly $y_{n-2} = y_{n-1} - \nabla y_{n-1}$

$= (1-\nabla) y_{n-1}$

$= (1-\nabla)(1-\nabla) y_n$

$= (1-\nabla)^2 y_n$ and $y_{n-3} = (1-\nabla)^3 y_n$

Generalising this concept,

$y_{n-k} = (1-\nabla)^k y_n$

$= 1 - k_{C_1} \nabla + k_{C_2} \nabla^2 - \cdots + (-1)^k \nabla^k) y_n$

$y_{n-k} = y_n - k_{C_1} \nabla y_n + k_{C_2} \nabla^2 y_n - \cdots + (-1)^k \nabla^k y_n$

Ex: 1 Find $y(-1)$ if $y(0) = 2$, $y(1) = 9$, $y(2) = 28$, $y(3) = 65$, $y(4) = 126$, $y(5) = 217$

Soln: Forming the difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

$\nabla y_5 = 91,\ \nabla^2 y_5 = 30,\ \nabla^3 y_5 = 6 \quad \nabla^5 y_5 = 0$

$y(-1) = y_{-1} = y_{5-6}$

$= y_5 - 6C_1 \nabla y_5 + 6C_2 \nabla^2 y_5 - 6C_3 \nabla^2 y_5 + 6C_6 \nabla^4 y_5 \cdots$

$= 217 - 6(91) + 15(30) - 20(6) + 0$

$= 217 - 546 + 450 - 120$

$= 667 - 666 = 1$

6.1 Linear interpolation or method of proportional parts

The simplest of all interpolations is the case in which the interpolating polynomial is linear. Let us assume that the set of values of x and y are as given below:

$x: \ x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots x_n$

$y: \ y_0 \quad y_1 \quad y_2 \quad y_3 \quad \cdots, y_n$

Now we require the value of y corresponding to x_k which lies between x_r and x_{r+1}.

We will assume the polynomial to be linear.

The line equation is $\dfrac{y - y_r}{x - x_r} = \dfrac{y_{r+1} - y_r}{x_{r+1} - x_r}$.

$\therefore \ y_k = y_r + \left[\dfrac{y_{r+1} - y_r}{x_{r+1} - x_r}\right](x_k - x_r)$ gives the value of y at $x = x_k$ $\quad x_r < x_k < x_{r+1}$.

This method may be successful if the difference between succeeding pairs of values of the variables are small and regular. But, if the intervals between the two pairs of values are large, and irregular, this method of simple proportion cannot be used without large error.

Ex: 1 Using the method of proportional parts, find y at $x=0.5$, $x=0.75$, give the following table. (6)

x:	0	1	2	5
y:	2	3	12	147.

Soln:
$$y_k = y_n + \left(\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\right)(x_k - x_n)$$

$$y(0.5) = 2 + \frac{(3-2)}{(1-0)}(0.5-0) = 2.5$$

$$y(0.75) = 2 + \frac{(3-2)}{(1-0)}(0.75-0) = 2.75.$$

6.2 Gregory - Newton forward interpolation formula (or)

Newton's forward interpolation formula (for equal interval)

$$P_n(x) = P_n(x_0+uh) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \cdots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0$$

6.3 Gregory - Newton Backward interpolation formula (for equal interval)

$$P_n(x) = P_n(x_n+vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \dfrac{x-x_n}{h}.$

Example 1: Find the values of y at $x=21$ and $x=28$ from the following data.

x:	20	23	26	29
y:	0.3420	0.3907	0.4384	0.4848

Soln: Since $x=21$ is nearer to the beginning of the table, we use Newton's forward formula.

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We form the difference table. Also h = constant = 3 (11)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.0010	
				-0.0003
		0.0477		
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

The topmost diagonal gives the forward differences of y_0 while the lower most diagonal gives the backward differences of y_n.

There are only 4 data given. Hence the collection polynomial will be of degree 3.

By Newton's forward interpolation formula.

$$y(x) = P_3(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \dfrac{x-x_0}{h} = \dfrac{21-20}{3} = 0.3333$

$$y(21) = P_3(21) = 0.3420 + (0.3333)(0.0487) + \frac{(0.3333)(-0.6666)}{2}(-0.001)$$
$$+ \frac{(0.3333)(0.6666)(-1.6666)}{6}(-0.0003)$$

$\therefore y(21) = 0.3583$

Since $x = 28$ is nearer to end value, we use Newton's backward interpolation formula.

$$y(x) = P_3(x) = P_3(x_n + vh)$$
$$= y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n + \cdots$$

$$y(28) = P_3(28) = P_3\left[29 + \left(-\tfrac{1}{3}\right)3\right]$$

where $v = \dfrac{x-x_n}{h} = \dfrac{28-29}{3} = -\tfrac{1}{3}$

$$= 0.4848 + \left(-\tfrac{1}{3}\right)(0.0464) + \frac{\left(-\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)}{2}(-0.0013) \quad 2$$

$$+ \frac{\left(-\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)}{6}(-0.0003) + \cdots$$

$$= 0.4848 - 0.015465 + 0.0001444 + 0.0000185$$

$$y(28) = 0.4695$$

UNIT - II

Interpolation with Unequal Intervals

8.1 Introduction

In the previous chapters on Interpolation we had the intervals of differencing to be a constant h. In other words, we had $x_i - x_{i-1} = h$ constant. for $i = 1, 2 \ldots n$. If the values of x's are given at unequal intervals. Our Newton's forward, backward formulae and central difference interpolation formulae will not hold good. Hence, we introduce a new idea of divided differences. These divided differences take into consideration the changes of the values of the function $f(x)$ and also the changes in the values of the arguments x.

8.2 Divided differences.

Let the function $y = f(x)$ assume the values $f(x_0), f(x_1), \ldots f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots x_n$ respectively where the intervals $x_1 - x_0, x_2 - x_1, \ldots x_n - x_{n-1}$ need not be equal.

Definitions

The first divided difference of $f(x)$ for the arguments x_0, x_1 is defined as $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$. It is denoted by $f(x_0, x_1)$ or $[x_0, x_1]$ (or) $\underset{x_1}{\Delta} f(x_0)$. In other words

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

In the same notation, we have

$$f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and $f(x_{n-1}, x_n) = \underset{x_n}{\Delta} f(x_{n-1}) = \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$, $n = 1, 2, \cdots n$

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

This shows that to find a second divided difference, we require three continuous arguments.

In the same way, we define the third divided differences of $f(x)$ for the four arguments x_0, x_1, x_2, x_3 as

$$\underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

Equation (1), (2), (3) refer to divided differences of order one, two and three respectively.

We will see below the divided difference table.

Arguments x	Entry y	1st divided difference $\Delta f(x)$	2nd divided difference $\Delta^2 f(x)$	3rd divided difference. $\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$	
		$f(x_3, x_4)$		
x_4	$f(x_4)$			

Ex:1 Form the divided difference table for the following data

x	:	-2	0	3	5	7	8
$y=f(x)$	:	-792	108	-72	48	-144	-252

Soln: We form the table below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	-792				
		$\frac{108-(-792)}{0-(-2)}=450$			
0	108		-102		
		$\frac{-72-108}{3-0}=-60$		18	
3	-72		24		-3
		$\frac{48-(-72)}{5-3}=60$		-9	
5	48		-39		2
		$\frac{-144-48}{7-5}=-96$		7	
7	-144		-4		
		$\frac{-252-(-144)}{8-7}=-108$			
8	-252				

Ex:2 Find the divided differences of $f(x)=x^3+x+2$ for the arguments 1, 3, 6, 11

Soln: We form below the divided difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)=1$
1	4			
		$\frac{32-4}{3-1}=14$		
			$\frac{64-14}{6-1}=10$	
3	32			$\frac{20-10}{11-1}=1$
		$\frac{224-32}{6-3}=64$		
			$\frac{224-64}{11-3}=20$	
6	224			
		$\frac{1344-224}{11-6}=224$		
11	1344			

Ex: 3 If $f(x) = \frac{1}{x}$, show that

$$f(x_0, x_1, \cdots x_n) = \frac{(-1)^n}{x_0 x_1 x_2 \cdots x_n}$$, where n is any positive integer

Soln: Step 1:

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0} = -\frac{1}{x_0 x_1} = \frac{(-1)^1}{x_0 x_1} \cdots\cdots \rightarrow (1)$$

Step 2: Let us prove the result by induction.

Let the result be true for $r = n$

Then $$f(x_0, x_1, \cdots x_n) = \frac{(-1)^n}{x_0 x_1 x_2 \cdots x_n} \cdots\cdots \rightarrow (2).$$

Further $$f(x_0, x_1, x_2, \cdots x_n) = \frac{f(x_1, x_2 \cdots x_{n+1}) - f(x_0, x_1, \cdots x_n)}{x_{n+1} - x_0}$$

$$= \frac{\frac{(-1)^n}{x_1 x_2 \cdots x_{n+1}} - \frac{(-1)^n}{x_0 x_1 \cdots x_n}}{x_{n+1} - x_0}$$

$$= \frac{(-1)^n}{x_0 x_1 \cdots x_{n+1}} \cdot \left(\frac{x_0 - x_{n+1}}{x_{n+1} - x_0} \right)$$

$$= \frac{(-1)^{n+1}}{x_0 x_1 x_2 \cdots x_{n+1}} \cdots\cdots \rightarrow (3).$$

Hence, the result is true for $r = n+1$ by equation (3)

In other words, if the result is true for $r = n$, then it is true for $r = n+1$. But by (1), the result is true for $r = 1$. Hence by induction, the result is true for any positive integer n.

Ex: 4 Show that $\underset{bcd}{\Delta^3} \left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Soln:

If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$

$$f(a,b) = \underset{b}{\Delta} \left(\frac{1}{a}\right) = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c - a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c - a} = \frac{1}{abc}\left(\frac{c - a}{c - a}\right) = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a} = \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{1}{abcd}\left(\frac{a-d}{d-a}\right) = -\frac{1}{abcd} \quad (5)$$

Therefore $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

8.3. properties of divided differences.

Property 1: The value of any divided difference is independent of the order of the arguments. That is, the divided differences are symmetrical in all their arguments.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0) \quad \cdots\cdots (1)$$

Again $f(x_0, x_1) = \dfrac{f(x_0)}{x_0 - x_1} - \dfrac{f(x_1)}{x_0 - x_1} = \dfrac{f(x_0)}{x_0 - x_1} + \dfrac{f(x_1)}{x_1 - x_0} \quad \cdots\cdots (2)$

In the same way, $f(x_1, x_0) = \dfrac{f(x_1)}{x_1 - x_0} + \dfrac{f(x_0)}{x_0 - x_1} \quad \cdots\cdots (3)$

From (2) and (3), we have $f(x_0, x_1) = f(x_1, x_0)$

similarly

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\left(\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}\right) - \left(\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}\right)\right]$$

$$= \frac{1}{x_2 - x_0}\left[\left(\frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0}\right)f(x_1) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{x_2 - x_0}{(x_1 - x_2)(x_1 - x_0)} f(x_1) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \quad \cdots\cdots (4)$$

From (4), we find

$$f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_1, x_2, x_0) = \cdots$$

This shows that $f(x_0, x_1, x_2)$ is independent

By mathematical induction, we can prove that

$$f(x_0, x_1, x_2 \dots x_n) = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$
$$+ \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} + \dots + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

This is symmetrical w.r.t any two arguments, therefore the divided differences are symmetrical w.r.t any two arguments.

<u>Property 2</u>: The operator Δ is linear.

Proof: If $f(x)$ and $g(x)$ are two functions and α, β are constants then,

$$\Delta[\alpha f(x) + \beta g(x)] = \frac{[\alpha f(x_1) + \beta g(x_1)] - [\alpha f(x_0) + \beta g(x_0)]}{x_1 - x_0}$$
$$= \alpha \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \beta \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$
$$= \alpha \Delta f(x) + \beta \Delta g(x).$$

<u>Property 3</u>. The nth divided differences of a polynomial of degree n are constants.

Proof: Taking $f(x) = x^n$ where n is a positive integer

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^n - x_0^n}{x_1 - x_0}$$
$$= (x_1^{n-1} + x_0 x_1^{n-2} + x_0^2 x_1^{n-3} + \dots + x_0^{n-1})$$

= a polynomial function of degree $(n-1)$ and symmetrical in x_0, x_1 with leading coefficient 1

Again.

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$
$$= \frac{(x_2^{n-1} + x_1 x_2^{n-2} + \dots + x_1^{n-1}) - (x_0^{n-1} + x_1 x_0^{n-2} + \dots + x_1^{n-1})}{x_2 - x_0}$$

$$= \frac{x_2^{n-1} - x_0^{n-1}}{x_2 - x_0} + \frac{x_1 (x_2^{n-2} - x_0^{n-2})}{x_2 - x_0} + \cdots + \frac{x_1^{n-2}(x_2 - x_0)}{x_2 - x_0}$$ (1)

$$= \left(x_2^{n-2} + x_0 x_2^{n-3} + \cdots + x_0^{n-2}\right) + x_1\left[x_2^{n-3} + x_0 x_2^{n-4} + \cdots + x_0^{n-3} + \cdots + x_1^{n-2}\right.$$

= a polynomial of degree $(n-2)$ and ~~p~~ symmetrical i x_0, x_1, x_2 with leading coefficient 1

Proceeding in this way, the rth divided differences of x^n will be a polynomial of degree $(n-r)$ and symmetrical in $x_0, x_1, x_2, \ldots x_r$ with leading coefficient 1.

Hence nth order divided differences of x^n will be a polynomial of degree $n-n=0$, with leading coefficient 1
That is, its value is 1.

That is $\Delta^n x^n = 1$

$$\Delta^{n+i} x^n = 0, \text{ for } i = 1, 2, \ldots$$

Hence $\Delta^n [a_0 x^n + a_1 x^{n-1} + \cdots + a_n]$

$$= a_0 \Delta^n x^n + a_1 \Delta^n x^{n-1} + \cdots + \Delta^n a_n$$

$$= a_0 \cdot 1 + 0 + 0 + \cdots + 0 = a_0$$

8.5 Theorem: Newton's interpolation formula for unequal intervals (or Newton's divided difference formula)

Let $y = f(x)$ take values $f(x_0), f(x_1) \ldots f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots x_n$

By definition,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\therefore f(x) = f(x_0) + (x - x_0) f(x, x_0) \qquad \cdots (1).$$

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Similarly, $f(x, x_0, x_1) = \dfrac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$ (8)

$\therefore f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1)$

Using this value of $f(x, x_0)$ in (1), we have

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x, x_0, x_1) \quad \cdots\cdots \to (2)$$

Again $f(x, x_0, x_1, x_2) = \dfrac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$

$\therefore f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2)$

Using this value in (2), we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1, x_2) \cdots (3)$$

Continuing in this manner, we get,

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots \\ &+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) f(x_0, x_1, x_2, x_3 \ldots x_n) \\ &+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n) f(x, x_0, x_1, x_2, \ldots x_n) \cdots (4) \end{aligned}$$

If $f(x)$ is a polynomial of degree n, then

$f(x, x_0, x_1 \ldots, x_n) = 0$ ($\because (n+1)^{th}$ difference)

Hence the last equation (4) becomes,

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f(x_0, x_1, \ldots x_n) \cdots (5) \end{aligned}$$

Equation (5) is called Newton's divided difference interpolation for unequal intervals.

Ex:1 Show that the divided difference of second order can be expressed as the quotient of two determinants of third order.

Soln: We have seen already,

$$f(x_0, x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x - x_0)(x - x_1)}$$

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$$= \frac{(x_2-x_1) f(x_0) + (x_0-x_2) f(x_1) + (x_1-x_0) f(x_2)}{(x_0-x_1)(x_1-x_2)(x_2-x_0)}$$ (9)

$$= \frac{(x_1-x_2) f(x_0) + (x_2-x_0) f(x_1) + (x_0-x_1) f(x_2)}{-(x_0-x_1)(x_1-x_2)(x_2-x_0)}$$

$$= \frac{\begin{vmatrix} f(x_0) & f(x_1) & f(x_2) \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_0^2 & x_1^2 & x_2^2 \\ x_0 & x_1 & x_2 \\ 1 & 1 & 1 \end{vmatrix}}$$

Ex: 2 Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ and $f(15)$ given the following table:

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

Soln: We form the divided difference table since the intervals are unequal.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$	
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

By Newton's divided difference Interpolation formula.

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \dots \quad (2)$$

In our problem $x_0 = 4, x_1 = 5, x_2 = 7, x_3 = 10, x_4 = 11, x_5 = 13$ and $f(x_0) = 48$, $f(x_0,x_1) = 52$, $f(x_0,x_1,x_2) = 15$, $f(x_0,x_1,x_2,x_3) = 1$.

Hence using these value in (2), we have.

$$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)1$$
$$f(2) = 48 - 104 + 90 - 30 = 4$$
$$f(8) = 48 + (4)52 + (4)(3)15 + (4)(3)(1)1 = 448$$
$$f(15) = 48 + 11\times52 + 11\times10\times15 + 11\times10\times8 = 3150.$$

Ex:8 From the following table find $f(x)$ and hence $f(6)$ using Newton's Interpolation formula.

x :	1	2	7	8
$f(x)$:	1	5	5	4

soln: Evidently, intervals are not equal. We form the divided difference table below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1} = 4$		
2	5		$\frac{0-4}{7-1} = -\frac{2}{3}$	
		$\frac{5-5}{7-2} = 0$		$\frac{-\frac{1}{6}+\frac{2}{3}}{8-1} = \frac{1}{14}$
7	5		$\frac{-1-0}{8-2} = \frac{-1}{6}$	
		$\frac{4-5}{8-7} = -1$		
8	4			

By Newton's divided difference formula.

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \dots$$
$$= 1 + (x-1)4 + (x-1)(x-2)\left(\frac{-2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$
$$= \frac{1}{42}(3x^3 - 58x^2 + 321x - 224)$$
$$f(6) = \frac{1}{42}[3\times216 - 36\times58 + 1926 - 224]$$
$$= 6.238095 24$$

Ex: 3 Find the function $f(x)$ from the following table (11)
hence evaluate $f(x)$

x	0	1	2	4	5	7
$f(x)$	0	0	-12	0	600	7308

Soln: Since 6 data are given, we assume the polynomial to be of degree 5.

Since $f(0)=0$, $f(1)=0$, and $f(4)=0$, it is a clear $x(x-1)(x-4)$ is a factor of $f(x)$

So, Let $f(x) = x(x-1)(x-4)\varphi(x)$ where $\varphi(x)$ is a quadratic polynomial

Now, $\varphi(x) = \dfrac{f(x)}{x(x-1)(x-4)}$

$$\therefore \quad \varphi(2) = \frac{f(2)}{2(1)(-2)} = \frac{-12}{-4} = 3$$

$$\varphi(5) = \frac{f(5)}{5(4)(1)} = \frac{600}{20} = 30$$

$$\varphi(7) = \frac{f(7)}{7(6)(3)} = \frac{7308}{126} = 58$$

Now we will find $\varphi(x)$ using divided difference formula of Newton.

x	$\varphi(x)$	$\Delta\varphi(x)$	$\Delta^2\varphi(x)$
2	3		
		$\frac{30-3}{5-2} = 9$	
5	30		$\frac{14-9}{7-2} = 1$
		$\frac{58-30}{7-5} = 14$	
7	58		

By Newton's formula,

$$\varphi(x) = \varphi(x_0) + (x-x_0)\,\varphi(x_0,x_1) + (x-x_0)(x-x_1)\,\varphi(x_0,x_1,x_2)$$
$$= 3 + (x-2)\,9 + (x-2)(x-5)\,1$$
$$= x^2 + 2x - 5$$

Hence, $f(x) = x(x-1)(x-4)(x^2+2x-5)$

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Ex: 4 From the following table, obtain $f(x)$ as a polynomial in powers of $(x-5)$.

x :	0	2	3	4	5	6
$f(x)$:	4	26	58	112	466	922

Using Newton's method.

Soln: We will form the divided difference table below.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	4			
		11		
2	26		7	
		32		1
3	58		11	
		54		1
4	112		16	
		118		1
5	466		22	
		228	$\frac{p-228}{-2}=b$	$\frac{b-22}{1}=2$
6	922			
		$\frac{a-922}{-4}=p$		$\frac{d-b}{5-7}=1$
5	a		$\frac{q-p}{5-9}=d$	
5		q		$\frac{k-d}{5-9}=1$
5			k	

Since the third differences are constants $(=1)$, we extend the table by introducing $x=5$ three times and introducing unknowns from the last column.

$$\frac{b-22}{1}=1 \Rightarrow b=23, \quad \frac{p-228}{-2}=23 \Rightarrow p=182$$

$$\frac{a-922}{-4}=182 \Rightarrow a=194$$

$$\frac{d-b}{-2}=1 \Rightarrow d=21; \quad \frac{q-p}{-4}=21 \Rightarrow q=98$$

$$\frac{k-d}{-4}=1 \Rightarrow k=17.$$

Now take 5 as the origin and proceed.

$$f(x)=f(5)+(x-5)f(x_0,x_1)+(x-5)(x-5)f(x_0,x_1,x_2)+(x-5)^3 f(x_0,x_1,x_2,x_3)$$

$$=a+(x-5)q+(x-5)^2k+(x-5)^3\times 1$$

$$=194+98(x-5)+\underline{17}(x-5)^2+(x-5)^3$$

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UNIT - III

Finite Differences

5.4 Factorial polynomial

A factorial polynomial $x^{(n)}$ is defined as

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-\overline{n-1}h)$$

where n is a positive integer.

Thus, $x^{(1)} = x$, $x^{(2)} = x(x-h)$, $x^{(3)} = x(x-h)(x-2h)$, ... etc

Differences of $x^{(n)}$

(i) $\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$

$$= (x+h)(x)(x-h)\cdots[x-(n-2)h] - x(x-h)(x-2h)\cdots[x-(n-1)h]$$

$$= x(x-h)(x-2h)\cdots[x-(n-2)h][(x+h)-(x-\overline{n-1}h)]$$

$$= x^{(n-1)}\cdot nh$$

$$= nh\,x^{(n-1)}$$

Similarly $\Delta^2 x^{(n)} = \Delta[nh\,x^{(n-1)}]$

$$= nh(n-1)h\,x^{(n-2)}$$

$$= n(n-1)h^2 x^{(n-2)}$$

Proceeding like this $\Delta^r x^{(n)} = n(n-1)(n-2)\cdots(n-r+1)h^r x^{(n-r)}$

where r is a positive integer and $r<n$.

Reciprocal factorial

The reciprocal factorial function $x^{(-n)}$ is defined as $x^{(-n)} = \frac{1}{(x+h)(x+2h)\cdots(x+nh)}$ where n is a positive integer.

Difference of a reciprocal factorial function.

(i) $\Delta x^{(-n)} = (x+h)^{(-n)} - x^{(-n)}$

$$= \frac{1}{(x+2h)(x+3h)\cdots[x+(n+1)h]} - \frac{1}{(x+h)(x+2h)\cdots[x+nh]}$$

$$= \frac{1}{(x+h)(x+2h)\cdots[x+(n+1)h]}\left[(x+h) - x + \overline{n+1}h\right]$$

$$= \frac{-nh}{(x+h)(x+2h)\cdots[x+(n+1)h]} = (-n)\,h\,x^{\{-(n+1)\}}$$

(ii) $\Delta^2 x^{(-n)} = \Delta(\Delta x^{(-n)})$

$= \Delta(-nh\, x^{(-(n+1))}) = (-nh)[-(n+1)h]\, x^{(-(n+2))}$

$= (-1)^2 h^2 n(n+1)\, x^{(-(n+2))}$

Similarly $\Delta^r x^{(-n)} = (-1)^r n(n+1)(n+2)\cdots(n+r-1)\, x^{(n+r)} \cdot h^r$.

Polynomial in factorial notation

Any polynomial $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ can be expressed in the factorial polynomial form as

$A_0 x^{(n)} + A_1 x^{(n-1)} + A_2 x^{(n-2)} + \cdots + A_n$

since, $f(x) = A_0x^{(n)} + A_1 x^{(n-1)} + \cdots + A_n$

$= A_0 x(x-h)\cdots(x-\overline{n-1}h) + A_1 x(x-h)\cdots(x-\overline{n-2}h)$

$+ A_2 x(x-h)\cdots(x-\overline{n-3}h) + \cdots + A_{n-1}x + A_n \quad \cdots (1)$

Dividing the R.H.S of (1), by x, the remainder is A_n and dividing the quotient again by $x-h$, the remainder A_{n-1} and then dividing the quotient again by $x-2h$, the remainder is A_{n-2} etc.

Thus dividing $f(x)$ successively by $x, x-h, x-2h, \ldots$ The coefficients $A_n, A_{n-1}, A_{n-2}, \cdots$ are got which are nothing but the remainders of $f(x)$ in that order

---

Ex: 1 Express (i) $x^4 + 3x^3 - 5x^2 + 6x - 7$

(ii) $3x^3 - 2x^2 + 7x - 6$ in factorial polynomial and get their successive forward differences taking h

Soln: (i) First, divide $x^4+3x^3-5x^2+6x-7$ successively by $x, x-1, x-2 \cdots$ by synthetic division method.

0	1	3	-5	6	-7
		0	0	0	0
1	1	3	-5	6	$\boxed{-7}$
		1	4	-1	

$$
\begin{array}{c|cccc}
 & 1 & 4 & -1 & \lfloor 5 \\
2 & & 2 & -12 & \\
\hline
 & 1 & 6 & \lfloor 11 & \\
3 & & 3 & & \\
\hline
 & 1 & \lfloor 9 & &
\end{array}
$$

∴ factorial polynomial is

$f(x) = 1\cdot x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$

$\Delta f(x) = 4x^{(3)} + 27x^{(2)} + 22x^{(1)} + 5$

$\Delta^2 f(x) = 12x^{(2)} + 54x^{(1)} + 22$

$\Delta^3 f(x) = 24x^{(1)} + 54$

$\Delta^4 f(x) = 24$

$\Delta^r f(x) = 0$ if $r > 4$

(ii) Now express $\varphi(x) = 3x^3 - 2x^2 + 7x - 6$ is factorial polynomic
Using synthetic division process.

$$
\begin{array}{c|cccc}
 & & & 7 & -6 \\
0 & 3 & -2 & & 0 \\
 & & 0 & 0 & \\
\hline
1 & 3 & -2 & 7 & \lfloor -6 \\
 & & 3 & 1 & \\
\hline
2 & 3 & 1 & \lfloor 8 & \\
 & & 6 & & \\
\hline
3 & & \lfloor 7 & &
\end{array}
$$

Hence, $\varphi(x) = 3x^{(3)} + 7x^{(2)} + 8x^{(1)} - 6$ (here $h=1$)

$\Delta\varphi(x) = 9x^{(2)} + 14x^{(1)} + 8$

$\Delta^2\varphi(x) = 18x^{(1)} + 14$

$\Delta^3\varphi(x) = 18$

$\Delta^r\varphi(x) = 0$ for $r > 3$.

Ex: 2 Find the forward differences of

(i) $\dfrac{1}{x(x+4)(x+8)}$ (ii) $\dfrac{1}{(3x+1)(3x+4)(3x+7)}$

Soln: (i) $y = \dfrac{1}{x(x+4)(x+8)} = (x-4)^{(-3)}$ where $h=4$

$$\Delta^2 y = (-3)(-4)(4)^2 (x-4)^{(-5)} = \frac{192}{x(x+4)(x+8)(x+12)(x+16)}$$

(ii) $f(x) = \dfrac{1}{(3x+1)(3x+4)(3x+8)}$, if $h=1$

$$= \frac{1}{3^3 (x+1/3)(x+4/3)(x+7/3)} = \frac{1}{27}\left(x - 2/3\right)^{(-3)}$$

$$\Delta f(x) = \frac{1}{27}(-3)\left(x - 2/3\right)^{-4} \text{ since } h=1$$

$$\Delta^2 f(x) = \frac{(-3)(-4)}{27}\left(x - 2/3\right)^{-5}$$

$$= \frac{4}{9} \frac{1}{(x+1/3)(x+4/3)(x+7/3)(x+10/3)(x+\frac{13}{3})}$$

$$= \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}.$$

Ex: 3 Find $\Delta^3 f(x)$ if

$f(x) = (3x+1)(3x+4)(3x+7)\ldots(3x+19)$

Soln: $f(x) = (3x+1)(3x+4)(3x+7)\ldots(3x+19)$ (contains 7 factors)

$$= 3^7 (x+1/3)(x+4/3)\ldots\left(x+\frac{19}{3}\right) = 3^7\left(x+\frac{19}{3}\right)^{(7)}$$

$$\Delta f(x) = 3^7 (7)\left(x+\frac{19}{3}\right)^{(6)}$$

$$\Delta^2 f(x) = 3^7 \times 7 \times 6 \left(x+\frac{19}{3}\right)^{(5)}$$

$$\Delta^3 f(x) = 3^7 \times 7 \times 6 \times 5 \times \left(x+\frac{19}{3}\right)^{(4)}$$

$$= 3^3 \times 7 \times 6 \times 5\ (3x+19)(3x+16)(3x+13)(3x+10)$$

Standard Results

Prove (i) $\Delta[f(x)\,g(x)] = f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

(ii) $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$

Proof: $\Delta[f(x)\,g(x)] = f(x+h)\,g(x+h) - f(x)\,g(x)$

$= f(x+h)\,g(x+h) - f(x+h)\,g(x)$

$+ [f(x+h)\,g(x) - f(x)\,g(x)]$

⑤

$= f(x+h)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)]$

$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

(ii) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{g(x)[f(x+h)-f(x)] - f(x)[g(x+h)-g(x)]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

Ex:4 Evaluate (i) $\Delta^n(e^{ax+b})$ (ii) $\Delta^n[\sin(ax+b)]$ (iii) $\Delta^n[\cos(ax+b)]$ (iv) $\Delta[\log(ax+b)]$ (v) $\Delta^n(a^{bx+c})$ (vi) $\Delta \log f(x)$ (vii) $\Delta(\tan^{-1}x)$

soln: (i) $\Delta(e^{ax+b}) = e^{a(x+h)+b} - e^{ax+b} = e^{ax+b}(e^{ah}-1)$

$\Delta^2(e^{ax+b}) = (e^{ah}-1)\,\Delta(e^{ax+b}) = (e^{ah}-1)^2 e^{ax+b}$

similarly $\Delta^n e^{ax+b} = e^{ax+b}(e^{ah}-1)^n$

(ii) $\Delta[\sin(ax+b)] = \sin[a(x+h)+b] - \sin(ax+b)$

$= 2\cos\left(ax+b+\frac{ah}{2}\right)\sin\frac{ah}{2} = 2\sin\frac{ah}{2}\cdot\sin(ax+b+\frac{\pi+ah}{2})$

$\Delta^2\sin(ax+b) = 2\sin\frac{ah}{2}\,\Delta\left[\sin\left(ax+b+\frac{\pi+ah}{2}\right)\right]$

$= \left(2\sin\frac{ah}{2}\right)^2\cdot\sin\left[ax+b+2\left(\frac{\pi+ah}{2}\right)\right]$

similarly proceeding,

$\Delta^n\sin(ax+b) = \left(2\sin\frac{ah}{2}\right)^n\sin\left(ax+b+\frac{n(\pi+ah)}{2}\right)$

(iii) $\Delta[\cos(ax+b)] = \cos(ax+ah+b) - \cos(ax+b)$

$= -2\sin\left(ax+b+\frac{ah}{2}\right)\sin\frac{ah}{2}$

$= 2\sin\frac{ah}{2}\cos\left(\pi/2+ax+b+\frac{ah}{2}\right)$

$= 2\sin\frac{ah}{2}\cos\left(ax+b+\frac{\pi+ah}{2}\right)$

$\Delta^2[\cos(ax+b)] = \left(2\sin\frac{ah}{2}\right)^2\cdot\cos\left(ax+b+\frac{2(\pi+ah)}{2}\right)$

Proceeding like this

$\Delta^n\cos(ax+b) = \left(2\sin\frac{ah}{2}\right)^n\cos\left[ax+b+\frac{n(\pi+ah)}{2}\right]$

(iv) $\Delta[\log(ax+b)] = \log(ax+ah+b) - \log(ax+b)$

$$= \log\left[\frac{ax+ah+b}{ax+b}\right] = \log\left[1+\frac{ah}{ax+b}\right]$$

$$= \log\left[1+\frac{\Delta(ax+b)}{ax+b}\right]$$

(v) $\Delta(a^{bx+c}) = a^{b(x+h)+c} - a^{bx+c}$

$$= a^{bx+c}(a^{bh}-1)$$

$$\Delta^2(a^{bx+c}) = a^{bx+c}(a^{bh}-1)^2$$

$$\Delta^n(a^{bx+c}) = a^{bx+c}(a^{bh}-1)^n$$

(vi) $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{Ef(x)}{f(x)}\right]$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(vii) $\Delta(\tan^{-1}x) = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\frac{(x+h)-x}{1+x(x+h)} = \tan^{-1}\frac{h}{1+x(x+h)}$$

Ex:5 Evaluate (i) $\Delta(e^{3x}\log 2x)$ (ii) $\Delta(x\sin x)$ (iii) $\Delta(xe^x)$ (iv) $\Delta\left(\frac{2^x}{x!}\right)$

(v) $\Delta\left(\frac{x}{\sin 2x}\right)$

Soln: (i) $\Delta(e^{3x}\cdot\log 2x) = e^{3(x+h)}\Delta\log 2x + \log 2x\,\Delta(e^{3x})$

$$= e^{3(x+h)}\cdot\log\left(1+\frac{\Delta(2x)}{2x}\right) + \log 2x\cdot e^{3x}(e^{3h}-1)$$

$$= e^{3x}\left[e^{3h}\log(1+h/x) + (e^{3h}-1)\log(2x)\right]$$

(ii) $\Delta(x\sin x) = (x+h)\Delta(\sin x) + \sin x\cdot\Delta(x)$

$$= (x+h)\cdot 2\sin h/2\,\sin\left(x+\frac{\pi+h}{2}\right) + h\sin x$$

(iii) $\Delta(xe^x) = (x+h)\Delta e^x + e^x\Delta(x)$

$$= (x+h)e^x(e^h-1) + e^x\cdot h$$

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(iv) $\Delta\left(\frac{2^x}{x!}\right) = \frac{x!\,\Delta(2^x) - 2^x\,\Delta(x!)}{(x+h)!\,x!}$ (7)

$$= \frac{x!\,2^x(2^h-1) - 2^x\cdot[(x+h)!-x!]}{(x+h)!\,x!}$$

If $h=1$, then

$$\Delta\left(\frac{2^x}{x!}\right) = \frac{x!\,2^x - 2^x\cdot x\cdot x!}{(x+1)!\,x!} = \frac{2^x(1+x)}{(x+1)!}$$

(v) $\Delta\left(\frac{x}{\sin 2x}\right) = \frac{\sin 2x\,\Delta(x) - x\,\Delta\sin 2x}{\sin(2x+2h)\sin 2x}$

$$= \frac{h\sin 2x - x(2\sin h)\sin\left(2x+\frac{\pi+2h}{2}\right)}{\sin(2x+2h)\cdot\sin 2x}.$$

Example: 6 Evaluate (i) $\Delta^3(1-x)(1-2x)(1-3x)$ if $h=1$

(ii) $\Delta^{10}(1-x)(1-2x)(1-3x)\ldots(1-10x)$ taking $h=1$

(iii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)$ if $h=2$.

Soln: (i) $\Delta^3(1-x)(1-2x)(1-3x) = \Delta^3[-6x^3 + \text{terms of lesser powers}]$

$$= (-6)\,3!+0 = -36$$

(ii) $\Delta^{10}(1-x)(1-2x)\ldots(1-10x) = \Delta^{10}[10!\,x^{10} + \text{terms involving Lesser degree}]$

$$= 10!\,10! + 0 = (10!)^2$$

(iii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^2)(1-4x^4)]$

$$= \Delta^{10}[24x^{10} + \text{terms of Lesser degree}]$$

$$= 24\,(10!)\,2^{10}+0,\ \text{since } \Delta^n(a_0x^n) = a_0 n!\,h^n$$

$$= 24\,(10!)\,2^{10}$$

Ex: 7 prove $\left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{Ee^x}{\Delta^2 e^x} = e^x$, taking h as the interval of differencing

Soln: $\left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{E(e^x)}{\Delta^2(e^x)} = (E^{-1}\Delta^2)\,e^x\cdot\frac{E(e^x)}{\Delta^2(e^x)}$

$$= (E^{-1})(\Delta^2 e^x)\,\frac{e^{x+h}}{e^x(e^h-1)^2} = E^{-1}[e^x(e^h-1)^2]\,\frac{e^h}{(e^h-1)^2}.$$

$$= e^{x-h}(e^h-1)^2\,\frac{e^h}{(e^h-1)^2} = e^x \;//$$

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Ex: 8 Given $y_1 = 2$, $y_4 = -6$, $y_5 = 8$, $y_6 = 9$ and $y_7 = 17$, calculate $\Delta^4 y_3$

Soln: $\Delta^4 y_3 = (E-1)^4 y_3 = (E^4 - 4E^3 + 6E^2 - 4E + 1) y_3$

$$= E^4 y_3 - 4E^3 y_3 + 6E^2 y_3 - 4E y_3 + y_3$$

$$= y_7 - 4y_6 + 6y_5 - 4y_4 + y_3$$

$$= 17 - 4(9) + 6(8) - 4(-6) + 2 = 55$$

Ex: 9 Estimate the production for 1964 and 1966 from the following data.

year :	1961	1962	1963	1964	1965	1966	1967.
Production:	200	220	260	-	350	-	430.

Soln: Since five values are given, collocation polynomial is of degree four. Hence $\Delta^5 y_k = 0$.

i.e., $(E-1)^5 y_k = 0$

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_k = 0$

$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$, taking $k=0$

$y_5 - 5(350) + 10y_3 - 10(260) + 5(220) - 200 = 0$

i.e., $y_5 + 10y_3 = 3450$ ....→ (1)

Taking $k=1$

$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 = 0$

$5y_5 + 10y_3 = 5010$

solving for y_3, y_5 from (1) and (5), $y_3 = 306$, $y_5 = 390$

Hence missing values are 306 and 390.

5.6 Finite integration (or Inverse operator Δ^{-1})

If $\Delta y_x = u_x$ then $y_x = \Delta^{-1} u_x$. Here Δ^{-1} is called finite integration operator or inverse of operator Δ.

If $C(x)$ is a periodic function of period h which is equal to the interval of differencing i.e. same h, then $\Delta C(x) = 0$

Hence if $\Delta y(x) = u(x)$

then $\Delta(y(x) + C(x)) = \Delta y(x) + \Delta C(x)$

$$= \Delta y(x) + 0 = u(x)$$

$$\therefore \Delta^{-1} u(x) = y(x) + C(x)$$

where $C(x)$ is the periodic function of period h.

The following inverse operator results can be remembered from the corresponding forward operator results.

1. $\Delta^{-1}(e^{ax+b}) = \dfrac{e^{ax+b}}{e^{ah}-1}$

   Hence $\Delta^{-1} e^x = \dfrac{e^x}{e^h - 1}$

2. $\Delta^{-1}(a^x) = \dfrac{a^x}{a^h - 1}, \ a \neq 1$

3. $\Delta^{-1}(u_x + v_x) = \Delta^{-1} u_x + \Delta^{-1} v_x$

4. $\Delta^{-1}(c u_x) = c \Delta^{-1} u_x$

5. $\Delta^{-1}(a+bx)^{(n)} = \dfrac{(a+bx)^{(n+1)}}{(n+1)hb}, \ n \neq -1$

6. $\Delta^{-1} x^{(n)} = \dfrac{x^{n+1}}{n+1}, \ n \neq -1$ and $h = 1$

5.7 Summation of series

An important application of finite calculus is finding the sum of series. Let us find the sum of the series.

$$u_1 + u_2 + u_3 + \cdots + u_n$$

Let the x^{th} term u_x be such that $u_x = \Delta y_x$

$$\therefore u_x = \Delta y_x = y_{x+1} - y_x \qquad (\text{here } h = 1)$$

Hence $\quad u_1 = y_2 - y_1$

$u_2 = y_3 - y_2$

$u_3 = y_4 - y_3$

$\dots\dots$

$\dots\dots\dots$

$u_n = y_{n+1} - y_n$

Adding vertically

$$S_n = u_1 + u_2 + \dots + u_n = y_{n+1} - y_1 = (y_x)_1^{n+1} = \left[\Delta^{-1}(u_x)\right]_1^{n+1}$$

Hence, $\displaystyle\sum_{x=1}^{n} u_n = \left[\Delta^{-1} u_x\right]_1^{n+1}$

Ex: Find $\Delta^{-1} x(x+1)(x+2)$

Soln: Here $(x+2)(x+1)x = (x+2)^{(3)}$, if $h=1$

Hence, $\Delta^{-1}(x+2)(x+1)x = \Delta^{-1}(x+2)^{(3)}$

$$= \frac{(x+2)^{(4)}}{4} + c(x)$$

$$= \frac{(x+2)(x+1)(x)(x-1)}{4} + c(x)$$

where $c(x)$ is a periodic function of period 1.

Ex: Find $\Delta^{-1} \dfrac{1}{x(x+1)(x+2)}$

soln:

$$\frac{1}{x(x+1)(x+2)} = (x-1)^{(-3)}$$

$$\Delta^{-1}\frac{1}{x(x+1)(x+2)} = \Delta^{-1}\left[(x-1)^{-3}\right]$$

$$= \frac{(x-1)^{-2}}{(-2)} + c(x) = -\frac{1}{2} \cdot \frac{1}{x(x+1)}$$

Ex: If $\Delta f(x) = 2x^3 - 6x^2 + 7x + 10$. find $f(x)$ Express $2x^3 - 6x^2 + 7x$ in factorial polynomial.

Soln:

0	2	-6	7	10
		0	0	0
1	2	-6	7	$\boxed{10}$
		2	-4	
2	2	-4	$\boxed{3}$	
		4		
	2	10		

Hence, $2x^3-6x^2+7x+10 = \phi(x) = 2x^{(3)}+3x^{(1)}+10$

$$\therefore \Delta f(x) = 2x^{(3)}+3x^{(1)}+10$$

$$f(x) = \Delta^{-1}(2x^{(3)}+3x^{(1)}+10)$$

$$= 2\frac{x^{(4)}}{4}+3\frac{x^{(2)}}{2}+10x^{(1)}+c(x)$$

$$= \frac{1}{2}x(x-1)(x-2)(x-3)+\frac{3}{2}x(x-1)+10x+c(x)$$

where $c(x)$ is periodic function of period 1.

Ex: Sum the series to n terms of.

$$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$$

Soln: n^{th} term $= u_n = n(n+1)(n+2)$

Sum of Series to n terms $= \sum_{x=1}^{n} u_x$

$$= \left[\Delta^{-1}u_x\right]_1^{n+1} = \left[\Delta^{-1}(x+2)^{(3)}\right]_1^{n+1} = \left[\frac{(x+2)^{(4)}}{4}\right]_1^{n+1}$$

$$= \frac{1}{4}\left[(n+3)^{(4)}-3^{(4)}\right] = \frac{1}{4}\left[(n+3)(n+2)(n+1)n-3\cdot2\cdot1\cdot0\right]$$

$$= \frac{1}{4}(n+3)(n+2)(n+1)n.$$

Ex: sum to n terms of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots$$

Soln: $u_x = \dfrac{1}{x(x+1)(x+2)} = (x-1)^{(-3)}$

Sum to n terms $= \sum_{x=1}^{n} u_x = \left(\Delta^{-1}u_x\right)_1^{n+1} = \left[\Delta^{-1}(x-1)^{(-3)}\right]_1^{n+1}$

$$= \left[\frac{(x-1)^{(-2)}}{-2}\right]_1^{n+1} = -\frac{1}{2}\left[n^{(-2)}-0^{(-2)}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{(n+1)(n+2)}-\frac{1}{1\cdot2}\right] = \frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

5.8. Montmort's theorem.

$$u_0+u_1x+u_2x^2+\cdots \text{ to } \infty = \frac{u_0}{1-x}+\frac{x\Delta u_0}{(1-x)^2}+\frac{x^2\Delta^2u_0}{(1-x)^3}+\cdots \text{ to } \infty$$

$u_0+u_1x+u_2x^2+\cdots$ to ∞

$$= u_0+xEu_0+x^2E^2u_0+\cdots \text{ to } \infty$$

$$= (1+xE+x^2E^2+\cdots \text{ to } \infty)u_0$$

$$= \frac{1}{1-xE} u_0 = \frac{1}{1-x(1+\Delta)} u_0 = \frac{1}{1-x-x\Delta} u_0$$

$$= \frac{1}{(1-x)\left[1-\frac{x\Delta}{1-x}\right]} u_0 = \frac{1}{1-x}\left[1-\frac{x\Delta}{1-x}\right]^{-1} u_0$$

$$= \frac{1}{(1-x)}\left[1+\frac{x\Delta}{1-x}+\frac{x^2\Delta^2}{(1-x)^2}+\cdots \text{ to } \infty\right] u_0$$

$$= \frac{u_0}{1-x} + \frac{x}{1-x}\Delta u_0 + \frac{x^2}{(1-x)^3}\Delta^2 u_0 + \cdots \text{ to } \infty$$

Ex: Using Montmort's theorem sum the series $1\cdot3 + 3\cdot5x + 5\cdot7x^2 + 7\cdot9x^3 + \cdots$ to ∞

Soln: $u_0 = 1\cdot3 = 3$; $u_1 = 3\cdot5 = 15$; $u_2 = 5\cdot7 = 35$; $u_3 = 63$.

We form the difference table for the coefficients

		Δu	$\Delta^2 u$	$\Delta^3 u$
u_0	3			
		12		
u_1	15		8	
		20		0
u_2	35		8	
		28		
u_3	63			

$\therefore u_0 = 3,\ \Delta u_0 = 12,\ \Delta^2 u_0 = 8,\ \Delta^3 u_0 = 0$

Therefore, $1\cdot3 + 3\cdot5x + 5\cdot7x^2 + 7\cdot9x^3 + \cdots$ to ∞

$$= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots \text{ to } \infty$$

$$= \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \cdots$$

$$= \frac{3}{1-x} + \frac{12x}{(1-x)^2} + \frac{8x^2}{(1-x)^3},\quad x \neq 1$$

Ex: Using the method of separation of symbols, show that

$$(u_1-u_0) - x(u_2-u_1) + x^2(u_3-u_2) - \cdots \text{ to } \infty$$
$$= \frac{\Delta u_0}{1+x} - x\frac{\Delta^2 u_0}{(1+x)^2} + \frac{x^2\Delta^3 u_0}{(1+x)^3} + \cdots \text{ to } \infty$$

Soln:

$$L.H.S = \Delta u_0 - x\Delta u_1 + x^2\Delta u_2 - \cdots$$
$$= \Delta u_0 - x\Delta E u_0 + x^2\Delta E^2 u_0 - \cdots$$
$$= \Delta(1 - xE + x^2E^2 - \cdots) u_0$$
$$= \left[\Delta(1+xE)^{-1}\right] u_0.$$

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$$= \left(\frac{\Delta}{1+xE}\right)u_0 = \left[\frac{\Delta}{1+x(1+\Delta)}\right]u_0 = \frac{\Delta}{(1+x)\left[1+\frac{x\Delta}{1+x}\right]}u_0$$

$$= \frac{1}{1+x}\left[1-\frac{x\Delta}{1+x}+\frac{x^2\Delta^2}{(1+x)^2}-\cdots\right]\Delta u_0$$

$$= \frac{\Delta u_0}{1+x} - \frac{x\Delta^2 u_0}{(1+x)^2} + \frac{x^2\Delta^3 u_0}{(1+x)^3} - \cdots = R.H.S$$

Ex: Using the method of separation of symbols, prove

$$y_x = y_{x-1} + \Delta y_{x-2} + \cdots + \Delta^{n-1} y_{x-n} + \Delta^n y_{x-n}$$

Soln: $R.H.S = E^{-1}y_x + \Delta E^{-2}y_x + \cdots + \Delta^{n-1}E^{-n}y_x + \Delta^n E^{-n}y_x$

$$= E^{-1}\left[1+\frac{\Delta}{E}+\frac{\Delta^2}{E^2}+\cdots+\frac{\Delta^{n-1}}{E^{n-1}}\right]y_x + \frac{\Delta^n}{E^n}y_x$$

$$= E^{-1}\left[\frac{\left(\frac{\Delta}{E}\right)^n - 1}{\frac{\Delta}{E}-1}\right]y_x + \frac{\Delta^n}{E^n}y_x$$

$$= E^{-1}\left[\frac{\Delta^n - E^n}{\Delta - E}\cdot\frac{1}{E^{n-1}}\right]y_x + \frac{\Delta^n}{E^n}y_x$$

$$= \left[-\frac{\Delta^n - E^n}{E^n}\right]y_x + \frac{\Delta^n}{E^n}y_x = \left[1-\frac{\Delta^n}{E^n}+\frac{\Delta^n}{E^n}\right]y_x$$

$$= y_x = L.H.S$$

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