MAJOR

CORE COURSE - III

PROBABILITY THEORY AND RANDOM VARIABLES

UNIT - I

Random experiment - Sample space - Types of events, Definition classical approach to Probability - Mathematical and statistical definition - axiomatic approach to Probability - Addition theorem and Multiplication theorem on Probability. Conditional Probability, Baye's theorem and Boole's inequality theorem - simple problems.

UNIT - II

Random variables - Definition - Discrete random variables - Probability mass function - Distribution functions Properties - simple problems.

UNIT - III

Continuous random Variable - Definition - Distribution function of continuous random variables - properties - Probability density function - simple problems.

UNIT - IV

Mathematical expectation - Definition - properties of Expectation, Addition and Multiplication theorems, Variance, covariance and its properties - simple problems.

UNIT-V

Bivariance probability distribution - joint probability mass function and joint probability density function, joint probability distribution function, Marginal probability density functions, conditional probability density functions, conditional expectation, conditional variance, stochastic independence - Definition and simple problems.

MAJOR

UNIT-I

PROBABILITY THEORY AND RANDOM VARIABLES

RANDOM EXPERIMENT

If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be anyone of the possible outcomes, then such an experiment is called a random experiment.

SAMPLE SPACE

All the trials conducted under the same set of a conditions form a random experiment. The result of a trial in a random experiment is called an outcome, an elementary event or a sample point. The totality of all possible outcomes (ie, sample point) of a random experiment constitutes the sample space.

TYPES OF EVENTS

Any particular performances of a random experiment is called a trial and outcome or combhination of outcome are termed as events.

- Exhaustive event
- Favourable event
- Mutually exclusive event
- Equally likely event
- Independent event

CLASSICAL APPROACH TO PROBABILITY:

If a random experiment or a trial results in 'n' exhaustive, mutually exclusive and equally likely outcomes (or cases), out of which m are favourable to the occurance of an event E, then the probability 'p' of occurance of E usually denoted by P(E).

$$\boxed{P = P(E) = \frac{\text{No. of Favourable case}}{\text{Total no. of exhaustive case}} = \frac{m}{n}}$$

If $P(E) = 1$, E is called a certain event and

If $P(E) = 0$, E is called an impossible event.

If P = success, q = failure then $P + q = 1$.

MATHEMATICAL AND STATISTICAL DEFINITION:

If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the Probability of happening of the event, it being assumed that the limit is finite and unique.

$$\boxed{P(E) = \lim_{N \to \infty} \frac{M}{N}}$$

AXIOMATIC APPROACH TO PROBABILITY:

It includes both the classical and the statistical definitions as particular cases and overcomes the deficiencies of each of them, The axiomatic development of mathematical theory of probability relies entirely upon the logic of deduction.

ADDITIONAL THEOREM & MULTIPLICATION THEOREM

If A and B are any two events (subsets of sample space S) and are not disjoint, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

For two events A and B.

$$P(A \cap B) = P(A) \cdot P(B/A),$$
$$= P(B) \cdot (P(A/B))$$

CONDITIONAL PROBABILITY:

The probability $P(A)$ of an event A represents the likelihood that a random experiment will result in an outcome in the set A relative to the sample space S of the random experiment.

SOME THEOREMS ON PROBABILITY:

- Probability of the impossible event is zero

  ie, $P(\phi) = 0$.

- Probability of the complementary event $\bar{A}$ of A is given by

  $$P(\bar{A}) = 1 - P(A)$$

- For any two events A and B, we have

  $$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
  $$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

- If $B \subset A$, then

(i) $P(A \cap \bar{B}) = P(A) - P(B)$

(ii) $P(B) \leq P(A)$

BOOLE'S INEQUALITY:

For n events $A_1, A_2, \ldots A_n$, we have

a) $$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

b) $$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

STATEMENT	MEANINGS
1) At least one of the events A or B occurs	$\omega \in A \cup B$
2) Both the events A and B occur.	$\omega \in A \cap B$
3) Neither A nor B occurs	$\omega \in \bar{A} \cap \bar{B}$
4) Event A occurs and B doesnot occur.	$\omega \in A \cap \bar{B}$
5) Exactly one of the events A or B occurs	$\omega \in A \Delta B$

6) Not more than one of the events A or B occurs.	$\omega \in (A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})$
7) If event A occurs, so does B.	$A \subset B$
8) Events A and B are mutually exclusive.	$A \cap B = \phi$
9) Complementary event of A.	$\bar{A}$
10) Sample space	universal set S.

EXAMPLE ON CONDITIONAL PROBABILITY:

Data on the readership of a certain magazine show that the proportion of male under 35 is 0.40 and over 35 is 0.20. If the proportion of readers under 35 is 0.70, find the proportion of subscribers that are females over 35 years. Also calculate the probability that a randomly selected male subscriber is under 35 years of age.

Solution:-

let us define the following events:

A: Reader of the magazine is a male

B: Reader of the magazine is over 35.

Then in usual notations, we are given:

$P(A \cap B) = 0.20$, $P(A \cap \bar{B}) = 0.40$ & $P(\bar{B}) = 0.70$

$P(\bar{B}) = 0.30$.

i) females over 35 years is

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= 0.30 - 0.20$$

$$P(\bar{A} \cap B) = 0.10$$

iii) Male under 35 years:

$$P(\bar{B}/A) = \frac{P(A \cap \bar{B})}{P(A)}$$

$$= \frac{0.40}{0.60}$$

$$= \frac{2}{3}$$

$$\left[\because P(A) = P(A \cap B) + P(A \cap \bar{B})\right]$$

UNIT - II

RANDOM VARIABLES

DEFINITION

A Random variable is a function $x(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the event $[\omega : x(\omega) \leq a] \in B$.

DISCRETE RANDOM VARIABLES

A variable which can assume only a countable number of real numbers, values and for which these value which the variable takes depends on chance, is called a discrete random variables.

In other words, a real valued functions defined on a discrete sample space is called a discrete random variables.

PROBABILITY MASS FUNCTION:

If x is a one-dimensional discrete random variable taking at mass a countably infinite number of $x_1, x_2, \ldots$ then its probability behaviour at each real

Point is described by a function which is defined below,

If x is a discrete random variable with distinct values then the function $p(x)$ defined as

$$P_x(x) = \begin{cases} P(x = x_i) = p_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \ ; i = 1, 2, 3. \end{cases}$$

is called the probability mass function of r.v. x.

DISTRIBUTION FUNCTION:

Let x be a random variable. The function is defined for all real x by

$$F(x) = p(x \leq x_i) = p\ x(\omega) \leq x \}, -\infty < x < \infty$$

is called the distribution Function.

PROPERTIES OF DISTRIBUTION FUNCTION:

We now proceed to desire a number of Properties common to all distribution functions.

If k is the d.f. of the r.v. x and if a $a < b$, the $p(a < x \leq b) = p(b) - F(a)$

PROPERTIES OF PROBABILITY MASS - FUNCTION

(i) The numbers $p(x_i)$, $i = 1, 2, \ldots$ must statisfied the following conditions,

$$P(x_i) \geq 0 \ \forall i$$

(ii) $\sum_{i=1}^{\infty} P(x_i) = 1.$

PROBLEM:

If $P(x) = \begin{cases} x/15 & ; x = 1, 2, 3, \ldots \\ 0 & , \text{elsewhere} \end{cases}$

find (i) $P(x = 1 \text{ (or) } 2)$ and

(ii) $P(1/2 < x \ 5/2,\ x > 1)$

Soln :

(i) $P(x = 1 \text{ (or) } 2) = P(x=1) + P(x=2)$

$$= 1/15 + 2/15$$

$$= 1/5$$

(ii) $$P(1/2 < x < 5/2 \mid x > 1) = \frac{P(1/2 < x < 5/2) \cap (x > 1)}{P(x > 1)}$$

$$= \frac{P\{(x=1 \text{ or } 2) \cap (x>h)\}}{P(x>1)}$$

$$= \frac{P(x=2)}{1-P(x=1)}$$

$$= \frac{2/15}{1-(1/15)} = \frac{2/15}{14/15}$$

$$= \frac{1}{7}$$

PROBLEM :

A random variable x has the following Probability mass function.

X :	-2	-1	0	1	2	3
P(x) :	0.1	k	0.2	2k	0.3	k

i) find the value of k and Calculate Mean & Variance

ii) find the cummulative Distributive function.

Soln

Since $\sum_{i=-2} p(x_i) = 1$

(i) $= 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$= 0.6 + 4k = 1$

$4k = 1 - 0.6$

$4k = 0.4$

$k = 0.4/4$

$\boxed{k = 0.1}$

Mean

$\sum_{i=-2}^{3} x\,(p(x))$ $\qquad \because k = 0.1$

$x_i = (-2)(0.1) + (-1)(0.1) + 0(0.2) + 1(2)(0.1) +$

$(2(0.3)) + 3(0.1)$

$= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$

$\boxed{\text{Mean} = 0.8}$

VARIANCE

$V(x) = \Sigma(x^2) - [\Sigma(x_i)]^2$

$(x^2) = \sum_{i=2} x_i^2\, p(x_i)$

$= (-2)^2(0.1) + (-1)^2(0.1) + (0)^2(0.2) +$

$(1)^2(0.2) + 2^2(0.3) + (3^2)(0.1)$

$= 0.4 + 0.1 + 0 + 0.2 + 0.2 + 0.9$

$V = 2.8 - (0.8)^2$

$= 2.8 - (0.64)$

$\boxed{V = 2.16}$

(ii)

x	$f(x)$	c.f.
-2	0.1	0.1
-1	0.2	$0.1+k$
0	0.4	$0.2+0.2$
1	0.6	$0.4+2k$
2	0.9	$0.6+0.3$
3	1.0	$0.9+k$

PROBLEM

The following the distribution function of discrete random variables.

x :	-3	-1	0	1	2	3	5	8
$P(x) =$	0.10	0.30	0.45	0.65	0.75	0.90	0.95	1

i) Find the probability distribution of X.

(ii) Find the X is even.

(iii) $P(1 \leq x \leq 8)$

(iv) $P(x-3/x<0)$ and $p(x \geq 3/x>0)$

soln

i) X : $-3 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8$

$p(x)$: $0.10 \quad 0.20 \quad 0.15 \quad 0.20 \quad 0.10 \quad 0.15 \quad 0.05 \quad 0.05$

(ii) $P(x \text{ is even}) = P(x=2) + P(x=8)$

$= 0.10 + 0.05$

$= 0.15$

(iii) $P(1 \leq x \leq 8) = P(x=1) + P(x=2) + P(x=3) + P(x=5) + P(x=8)$

$= 0.20 + 0.10 + 0.15 + 0.05 + 0.05$

$= 0.45$

(iv) $P(x = -3 / x < 0) = \dfrac{P\{(x=-3) \cap (x<0)\}}{P(x<0)}$

$= \dfrac{P(x=-3)}{P(x=-3,-1)}$

$= \dfrac{0.10}{0.30}$

$= 0.3$

$P(x \geq 3 / x > 0) = \dfrac{P[(x \geq 3) \cap (x>0)]}{P(x>0)}$

$= \dfrac{P(x=3,5,8)}{P(x=1,2,3,5,8)} = \dfrac{0.25}{0.55}$

$= \frac{0.5}{0.11}$

$= 0.45$

PROBLEM

The probability mass function of a random variable also except at the points $x = 0, 1, 2$ at these points it has the values $p(0) = 3c^3$, $P(1) = AC - 10c^2$, $P(2) = 5c - 1$ $\forall c > 0$.

i) Determine the value of c.

(ii) Compute $p(x) < 2$ and $P(1) < x \leq 2$

(iii) Determine Describe the distribution function

(iv) Find the smallest x such that $f(x) \geq 3$.

Soln :-

X :	0	1	2
P(x) :	$3c^3$	$4c - 10c^2$	$5c - 1$

Since $\sum_{i=0}^{2} p(x_i) = 1$

$\Rightarrow 3c^3 + 4c - 10c^2 + 5c - 1 = 1$

$\Rightarrow 3c^3 - 10c^2 + 9c - 1 = 1$

$\Rightarrow 3c^3 - 10c^2\ 9c - 1 - 1 = 0$

$\Rightarrow 3c^3 - 10c^2 + 9c - 2 = 0$

$\Rightarrow c - 1 = 0, \quad 3c^2 - 7c + 2 = 0$

$c = 1,$

$(3c-1)(c-2) = 0$

$c = 1, \quad 3c - 1 = 0, \quad c - 2 = 0$

$c = 1, \quad 3c = 1, \quad c = 2$

$c = 1/3$

$c = 1/3$ where $c = 1, 2$ $\sum_{i=0}^{1} (p(x_i) \neq 1$, where

$c = 1/3 \rightarrow \sum_{i=0}^{1} p(x_i) = 1$

X	0	1	2
P(x)	1/9	2/9	2/3

(ii) $P(x < 2)$

$P(x<2) = P(x=0) + P(x=1)$

$= 1/9 + 2/9 = 3/9 = 1/3$

$P(1 < x \leq 2) = P(x=2) = 2/3$

(iii)

X	:	0	1	2
$P(x)$	:	$1/9$	$2/9$	$2/3$
$F(x)$	:	$1/9$	$1/9 + 2/9 = 3/9$ $= 1/3$	$3/9 + 2/3 = 9/9$ $= 1$

UNIT- III

CONTINUOUS RANDOM VARIABLES

DEFINITION

A random variable x is said to be continuous if it can take all possible values (integral as well as fractional) between certain limits.

A random variable is said to be continuous when its different values cannot be put in 1-1 correspondance with a set of positive integers.

It can measure to any desired degree of accuracy.

PROBLEM:

A random variable x is distributed as random between the values 0 and 1 so that its probability density function $f(x) = kx^2(1-x^3)$ where k is a constant. Find the value of k also find its Mean & Variance.

$$f(x) = k\, x^2 (1-x^3)$$

$$k \int_0^1 x^2 (1-x^3)\, dx = 1$$

$$k \int_0^1 (x^2 - x^5)\, dx = 1$$

$$k \left| \frac{x^3}{3} - \frac{x^6}{6} \right|_0^1 = 1$$

$$k \left[\left(\frac{1}{3} - \frac{1}{6} \right) - 0 \right] = 1$$

$$k \left(\frac{6-3}{18} \right) = 1$$

$$k \left(\frac{3}{18} \right) = 1$$

$$\boxed{k = 6}$$

$$f(x) = \begin{cases} 6x^2 (1-x^3) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Men $\mu_1' = \int_0^1 x\, f(x)\, dx$

$$= \int_0^1 x \left(6x^2 (1-x^3) \right) dx$$

$$= \int_0^1 x (6x^2 - 6x^5)\, dx$$

$$= \int_0^1 (6x^3 - 6x^6)\, dx$$

$$= 6\left|\frac{x^4}{4} - \frac{x^7}{7}\right|_0^1 \Rightarrow 6\left|\frac{1}{4} - \frac{1}{7}\right|$$

$$= 6\left(\frac{7-4}{28}\right)$$

$$= 6\left(\frac{3}{28}\right)$$

$$= \frac{18}{28} = \frac{9}{14} = 0.64$$

$$\mu_2' = \int_0^1 x^2 f(x)\, dx$$

$$= \int_0^1 x^2\, 6x^2 (1-x^3)\, dx$$

$$= \int_0^1 x^2 (6x^2 - 6x^5)\, dx$$

$$= \int_0^1 6x^4 - 6x^7\, dx$$

$$= 6\left|\frac{x^5}{5} - \frac{x^8}{8}\right|_0^1$$

$$= 6\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$= 6\left(\frac{8-5}{40}\right) \Rightarrow 6\left(\frac{3}{40}\right) \Rightarrow \frac{18}{40} = 0.45$$

Variance $= \mu_2' - (\mu_1')^2$

$$= 0.45 - (0.64)^2$$

$$= 0.45 - 0.4096$$

$$= \frac{18}{40} - \left(\frac{9}{14}\right)^2 \Rightarrow \frac{18}{40} - \frac{81}{196} = \frac{288}{7840}$$

CONTINUOUS DISTRIBUTION FUNCTION:

If x is a continuous random Variable with the p.d. of $f(x)$, then the function.

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \ -\infty < x < \infty$$

is called distribution function (or) Random variable.

PROPERTIES OF DISTRIBUTION FUNCTION

(1) $0 \le f(x) \le 1, \ -\infty < x < \infty$

(2) Riemann integral

$$F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0$$

(3) $F(x)$ is non-decreasing function of x

$$F(-\infty) \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$$

$$F(\infty) \lim_{x \to \infty} F(x) = \lim_{x \to \infty} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$0 \le F(x) \le 1$$

(4) $F(x)$ is a continuous function of x on the right.

(5) The discontinous of $f(x)$ are at the mostly Countable.

(6) It may be noted that,

$$P(a \le x \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx)$$

$$= P(x \le b) - P(x \le a) = F(b) - F(a)$$

Ily,

$$P(a < x < b) = P(a < x \le b) = P(a \le x < b)$$

$$= \int_a^b f(t)\,dt.$$

(7) $F'(x) = \frac{d}{dx} F(x) = f(x) \Rightarrow dF(x) = f(x)\,dx$

$dF(x)$ is known as probability differential of x.

Verify that the following is a distribution function.

$$F(x) = \begin{cases} 0 & , \quad x < -a \\ \frac{1}{2}\left(\frac{x}{a} + 1\right) & , \quad -a \le x \le a \\ 1 & , \quad x > a \end{cases}$$

<u>Soln:</u>

obviously the properties 1, 2, 3 and 4 are statisfied. Also we observe that $F(x)$ is continuous at $x = a$ and $x = -a$ as well

Now $\frac{d}{dx} F(x) = \left\{ \begin{array}{ll} 1/2a, & -a \leq x \leq a \\ 0, & \text{otherwise} \end{array} \right\} = f(x)$ say

In order that $F(x)$ is a distribution function, $f(x)$ must be a p.d.f. Thus we have to show that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Now $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-a}^{a} f(x)\,dx = \frac{1}{2a} \int_{-a}^{a} 1.dx = 1$$

Hence $F(x)$ is a distribution function.

PROBABILITY DENSITY FUNCTION:

P.d.f. $f(x)$ of the random variable X is defined as,

$$f_{(x)}(x) = \lim_{x \to 0} \frac{P(x \leq X \leq x + \delta x)}{\delta x}$$

PROPERTIES OF PROBABILITY DENSITY FUNCTION:

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x) = dx = 1$

(3) The probability $P(E)$ given by

$$P(E) = \int_E f(x)\, dx.$$ is well defined for any event E.

Let be (x) or $f(x)$ be the probability density function of a random variable x, where x is defined from a to b. Then

ARITHMETIC MEAN:

$$\int_a^b x\, f(x)\, dx$$

HARMONIC MEAN:

$$\frac{1}{H} = \int_a^b \frac{1}{x}\, f(x)\, dx$$

GEOMETRIC MEAN

$$\log G = \int_a^b \log x\, f(x)\, dx.$$

$$\mu'_r \text{ (about origin)} = \left[\int_a^b x^{\partial} . f(x)\, dx \right]$$

$$\mu_r \text{ (about mean)} = \left[\int_a^b (m - \text{mean})^{\partial} f(x) . dx \right]$$

Hence,

$$\mu_2 = \mu_2' - \mu_1'^2 = \int_a^b x^2 f(x)\,dx - \left(\int_a^b x f(x)\,dx\right)^2$$

MEDIAN:

Median is the point which divides the total into equal parts. Thus

MODE:

It is the value of x for which $f(x)$ is maximum. $\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = 1/2$

Thus solving,

$$\int_M^a f(x)\,dx = 1/2.$$

for M, we get the values of Median.