Unit - 4

QUEUING THEORY

21.1 : Introduction :-

In every day life it is seen that a number of people arrive at a cinema ticket window. If the people arrive too frequently they will have to wait for getting tickets or sometimes do without it. Such a problem arise in Railways, Airlines etc., under such circumstances the only alternatives is to form a Queue called the Waiting Line in order to get the service more effectively.

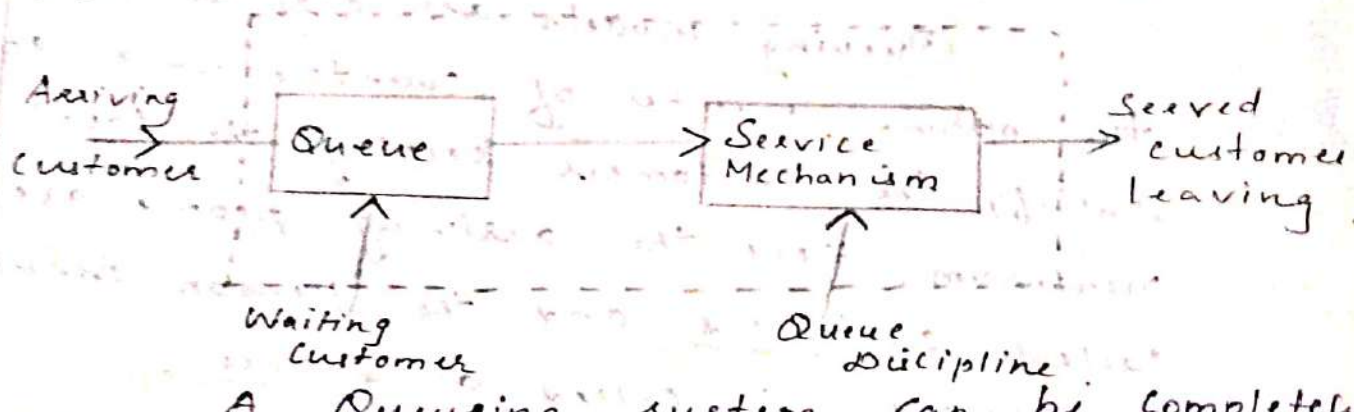
Queuing models are aids to determine the optimal number of counters so as to satisfy the customers keeping the total cost minimum. Here the arriving people are called customers and the person issuing the tickets is called a server.

Servers may be in parallel or in series. service time may be constant or variable and customers may be served singly or in batches.

2.2 : Queueing System :-

The Mechanism of Queueing system is very simple. Customers arrive at a service counter and are attended to by one or more of the servers. As soon as a customer is served, it depart from the system. Thus a Queuing System can be described as consisting of customers arriving for service waiting for service if it is not immediate and leaving the system after being served.

The general frame work of a Queuing system is shown in the following figure



A Queueing system can be completely described by:

(a). the input (or arrival pattern)

(b) the service mechanism (or service pattern)

(c) the Queue discipline

(d) Customer's behaviour

(a) The input (or arrival pattern)

The input describes the way in which the customer arrive and join the system. Thus the arrival pattern can best be described in terms of probabilities and consequently the probability distribution for inter arrival times (the time between two successive arrivals) must be defined. We deal with those Queuing system in which the customer arrive in 'poisson' fashion. Mean arrival rate is denoted by λ.

(b) The Service Mechanism (or Service pattern)

The Service pattern is specified when it is known how many customers can be served at a time, what the statistical distribution of the service time is and when the service is available. Service time may be constant or a random variable. Distribution of service time which is important in practice is the negative exponential distribution. The Mean service rate is denoted by μ.

(c) The Queue discipline :-

The Queue discipline is the rule determining the formation of the Que the Manner of the customer's behaviour while waiting, and the manner in which they are chosen for service.

This simplest discipline is "First come, First served" according to which the customers are served in the order of their arrival.

Such a type of Queue discipline is observed at a ration shop. If the order is reversed, we have 'Last come first served' discipline, as in the case of a big godown the items which come last are taken out first.

Some of the Queue disciplines are

1. FIFO - First in, First out
   (OR)
   FCFS - First come, First served.
2. LIFO - Last in, First out
   (OR).
   LCFS - Last come, First served
3. SIRO - Service in Random order.

(d) <u>Customer's behaviour</u> :.

The customers generally behave in 4 ways.

(i) Balking :

A customer may leave the Queue, if there is no waiting space

(ii) Reneging :-

This occurs when the waiting customer leaves the Queue duie to impatience

(iii) incapa

(iii) Priorities :-

In certain applications some customers are served before other regardless of their order of arrival.

(iv) Jockeying :-

Customers may jump from one waiting line to another.

21.7 : Classification of Queueing models :-

Generally Queueing model may be completely specified in the following symbolic form

$$(a/b/c) : (d/e)$$

The first and second symbols denote the type of distributions of inter-arrival times and of inter-service time, respectively. Third symbol specifies the number of servers, whereas fourth symbol stands for the capacity of the system and the last symbol denotes the Queue discipline.

If we specify the following letter as,

M ≡ poisson arrival or departure distribution

E_k ≡ Erlangian (or) Gamma inter-arrival for service time distribution

GI ≡ General Input distribution

G ≡ General Service time distribution

then (M/E_k/1) : (∞/FIFO) defines a Queueing system in which the arrivals follow poisson distribution, service time are Erlangian single server, infinite capacity and 'first in, first out' Queue discipline.

21.8 : Definition of Transitent and Steady States:-

Transitent state :-

A Queueing system is said to be in transitent state when its operating characteristics (like input, output, mean Queue length, etc) are dependent upon time.

If the Characteristic of the Queueing System becomes independent of time, then the steady state condition is said to prevail

Steady state :-

A System is said to be in steady state when the behaviour of

the system is independent of time.
Let $P_n(t)$ denote the probability that there are 'n' units in the system at time t. Then in steady state

$$\Rightarrow \lim_{t \to \infty} P_n'(t) = 0$$

21.9 : Poisson Queueing Systems :-

Queues that follow the poisson arrivals (exponential inter arrival time) and poisson service (exponential service time) are called poisson Queues.

Model I {(M/M/1) : (∞/FIFO)}

This model deals with a Queueing system having single service channel. poisson input, Exponential service and there is no limit on the system capacity While the customers are served on a "first in, first out" basis.

With usual notation, show that probability distribution of Queue length is given by $\rho^n(1-\rho)$ where $\rho = \lambda/\mu < 1$ and $n > 0$

Soln :-

The probability that there will be n units in the system time $(t+\Delta t)$ may be expressed as the sum of four independent Compound probabilities, by using the fundamental properties of probability, poisson arrival and of exponential service times.

$$(i.e)\ P_n(t+\Delta t) = P_n(t)[1-(\lambda+\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + P_n(t)(\lambda\mu)(\Delta t)^2 \quad \rightarrow (1)$$

For there are following 4 cases.

time t No of units	Arrival.	service	Time (t+Δt) No. of units.
n	0	0	n
n-1	1	0	n
n+1	0	1	n
n	1	1	n

From (1)

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

In the steady state $P_n'(t) \to 0$

$P_n(t) \to P_n$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \quad \rightarrow (2)$$

Probability that there will no unit at time $(t+\Delta t)$ will be the sum of the following independent probabilities

(i) Prob { that there is no unit in the system at time t and no arrival in time Δt }

$$= P_0(t)(1-\lambda\Delta t)$$

{ Question of any service in Δt will not arise as there are no units at time t }

(ii) Prob { that there is one unit in the system at time t, one unit serviced in Δt, and no arrival in Δt }

$$= P_1(t)\,\mu\Delta t\,(1-\lambda\Delta t)$$

adding these probabilities,

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\,\mu\Delta t\,(1-\lambda\Delta t)$$

$\Rightarrow$

as $\Delta t \to 0$ $$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t)$$

for $n=0$

In steady state

$$\Rightarrow 0 = -\lambda P_0 + \mu P_1$$

$$\mu P_1 = \lambda P_0 .$$

$$P_1 = \frac{\lambda}{\mu} P_0 .$$

Use $n=1$ in (2)

then, $$P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad , \text{ for } n \geq 0$$

Since $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \cdots + = 1$$

$$P_0 \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots \right] = 1$$

(or) $$P_0 \left[\frac{1}{1 - \lambda/\mu} \right] = 1$$

for $\lambda/\mu < 1$ and the infinite series in LHS is infinite Geometric series whose I term $= 1$ and common Ratio $= \lambda/\mu$.

Measure of Model I :-

(1) To find the average (expected) number of units in the system, L_s.

Soln :-

By definition of Expected value

$$L_s = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \left(\lambda/\mu\right)^n \left(1 - \lambda/\mu\right)$$

$$= \left(1 - \lambda/\mu\right)\left(\lambda/\mu\right) \sum_{n=1}^{\infty} \left(\lambda/\mu\right)^{n-1}$$

$$= \left(1 - \lambda/\mu\right)\left(\lambda/\mu\right) \left[1 + 2\frac{\lambda}{\mu} + 3\left(\frac{\lambda}{\mu}\right)^2 + \cdots \right]$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{-2}$$ (using Binomial Series)

$$= \frac{\lambda/\mu}{1 - \lambda/\mu}$$

$$L_s = \frac{\rho}{1-\rho}$$, where $\rho = \lambda/\mu < 1$

② To find the average Length of Queue, L_q

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= \frac{\rho}{1-\rho} - \frac{\lambda}{\mu}$$

$$= \frac{\rho}{1-\rho} - \rho.$$

$$= \frac{\rho - \rho(1-\rho)}{1-\rho}$$

$$= \frac{\rho - \rho + \rho^2}{1-\rho}$$

$$L_q = \frac{\rho^2}{1-\rho}$$

③ Expected waiting time in the System :-

$$W_s = \frac{L_s}{\lambda}$$

$$= \frac{\rho}{1-\rho} \times \frac{1}{\lambda}$$

$$= \frac{\lambda/\mu}{1-\lambda/\mu} \times \frac{1}{\lambda}$$ (where $\rho = \lambda/\mu$)

$$= \frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}} \times \frac{1}{\lambda}$$

$$= \frac{\lambda}{\mu-\lambda} \times \frac{1}{\lambda}.$$

$$W_s = \frac{1}{\mu - \lambda}$$

(4) Waiting time in the Queue :-

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\rho^2}{1-\rho} \times \frac{1}{\lambda}$$

$$= \frac{(\lambda/\mu)^2}{1-(\lambda/\mu)} \times \frac{1}{\lambda} \quad (\text{where } \rho = \lambda/\mu)$$

$$= \frac{\lambda^2/\mu^2}{\frac{\mu-\lambda}{\mu}} \times \frac{1}{\lambda}$$

$$= \frac{\lambda^2/\mu}{\mu-\lambda} \times \frac{1}{\lambda}$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

(5) Expected waiting time of a customer who has to wait $(w \mid w > 0)$

$$= \frac{1}{\mu - \lambda}$$

(6) Expected length of the non-empty Queue, $(L \mid L > 0)$

$$= \frac{\mu}{\mu - \lambda}$$

(7) probability of Queue size $\geq N$ is ρ^N

(8) Probability [waiting time in the system $\geq t$]

$$= \int_t^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)w}\, dw$$

(9) Probability [waiting time in the Queue > t]

$$= \int_t^\infty \rho(\mu-\lambda)e^{-(\mu-\lambda)w}\,dw$$

(10) Traffic Intensity = λ/μ

Examples

(1) In a railway Marshalling yard, goods train arrive at a rate of 30 Trains per day. Assuming that inter arrival time follows an exponential distribution and the service time distribution is also exponential, with an average of 36 minutes. Calculate the following

(i) the mean Queue size (line length)

(ii) the probability that queue size exceeds 10

(iii) If the input of the Train increases to an average 33 per day, what will be the changes in (i), (ii)?

Soln :-

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$$

$$\mu = \frac{1}{36} \text{ trains per minute}$$

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

(i) $$L_s = \frac{\rho}{1-\rho}$$

(i) $L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3$ trains

(ii) $P(\geq 10) = (0.75)^{10} = 0.056$

(iii) when the input increases to 33 trains per day.

We have $\lambda = \frac{33}{60 \times 24} = \frac{1}{480}$

$\mu = \frac{1}{36}$ trains per minute

Now, $L_s = \frac{\rho}{1-\rho}$ = where $\rho = \frac{\lambda}{\mu}$

$\rho = 0.825$

$L_s = \frac{0.825}{1-0.825} = 5$ trains (appr.)

Also, $P(\geq 10) = \rho^{10} = (0.825)^{10}$
$= 0.1460$

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② Customer arrive at a one window drive - in bank according to poisson distribution with mean 10 per hour. The space in front of the window including that for the serviced car can accomadate a maximum of 3 cars. Other can wait outside thei space.

(i) What is the probability that an arriving customer can drive directly to the space in front of the window?

(ii) What is the probability that an arriving customer will have to wait outside the indicated space?

(iii) How long the arriving customer is expected to wait before starting service?

Soln :-

W.K.T P_n denotes the probability of n units in the system and

$$P_n = (\lambda/\mu)^n (1-\lambda/\mu)$$

⇒ (i) the probability that an arriving customer can drive directly to the space in front of the window

$$= P_0 + P_1 + P_2$$

$$P_0 = 1-\lambda/\mu, \quad P_1 = (1-\lambda/\mu)$$

$$P_2 = (\lambda/\mu)^2 (1-\lambda/\mu).$$

$$\Rightarrow P_0 + P_1 + P_2 = (1-\lambda/\mu)\left[1+\lambda/\mu+(\lambda/\mu)^2\right]$$

Here $\lambda = 10$ per hr.

$$\mu = 1/5 \times 60 = 12 \text{ per hr}.$$

$$\lambda/\mu = 10/12.$$

$$\therefore P_0 + P_1 + P_2 = (1-10/12)\left[1+10/12+\left(\frac{100}{144}\right)\right]$$

$$= 0.42.$$

(ii) Probability that an arriving customer will have to wait outside the indicated space $= 1-0.42 = 0.58$

(iii) Average Waiting time of a customer in the Queue $= \lambda/\mu \cdot \frac{1}{\mu - \lambda}$

$= 10/12 \times \frac{1}{12 - 10}$

$= 5/12$

$= 0.417$

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(5) In a super Market, the average arrival rate of customer is 10 in every 30 minutes following poisson processes. The average time taken by the cashier to list and Calculate the customer's purchases in 2.5 minutes, following exponential distribution. What is the probability that the Queue length exceeds 6?

What is the expected time spent by a customer in the system?

Soln :-

Here the Mean arrival rate

$\lambda = 10/30$ per minute

Mean service rate $= \frac{1}{2.5}$ per minute

$\Rightarrow \rho = \lambda/\mu = \frac{1/3}{1/2.5} = 0.8333$

(i) The probability of Queue size $> n = \rho^n$

when $n = 6 \Rightarrow (0.8333)^6 = 0.3348$

(ii) $W_s = \frac{L_s}{\lambda} = \frac{(\rho/1-\rho)}{\lambda}$, $\rho = \lambda/\mu$

$= \frac{0.8333}{1 - 0.833} \times 3 = \frac{2.499}{0.167}$

(4) In a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there is just one phone, find (i) Expected number of callers in the booth at any time (ii) the proportion of the time the booth is expected to be Idle?

Soln :-

Mean arrival rate $\lambda = 15$ per hour

Mean service rate $\mu = \frac{1}{3} \times 60 = 20$ per hour.

∴ (i) Expected length of the non empty

$$\text{Queue} = \frac{\mu}{\mu - \lambda} = \frac{20}{20-15} = 4$$

(ii) The service is busy means $= \lambda/\mu$

$$= 15/20 = 3/4.$$

∴ the booth expected to Idle for

$$1 - 3/4 = 1/4 \text{ hrs}$$
$$= 15 \text{ mins}$$

(5) On an average 96 patients per 24 hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs.100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs.10 per

(5)

patient treated, how much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patient to $\frac{1}{2}$ patient?

Soln :-

Here $\lambda = 96/24 = 4$ patient/hr

$\mu = 1/10 \times 60 = 6$ patient/hr.

Average number of patients in the queue

$$L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\lambda - \mu} = \frac{4}{6} \cdot \frac{4}{6-4} = 1\frac{1}{3}$$

This number is reduced from $1\frac{1}{3}$ to $\frac{1}{2}$

$$L_q' = \lambda/\mu' \; \frac{\lambda}{\mu' - \lambda}$$

(or) $$\frac{1}{2} = \frac{4}{\mu'} \cdot \frac{4}{\mu' - 4}$$

(or) $$\mu'^2 - 4\mu' - 32 = 0$$

$$(\mu' - 8)(\mu' + 4) = 0$$

$\mu' = 8$ patients/hr as $\mu' = 4$ illogical.

Average time required by each patient

$= 1/8$ hrs $= 15/2$ minutes.

⇒ Decrease in the time required by each patient $= 10 - 15/2 = 5/2$ minutes.

∴ The budget required for each patient

$= 100 + (5/2 \times 10)$

$=$ Rs. 125

⇒ To decrease the size of the queue, the budget per patient should be increased from Rs. 100 to Rs. 125.

(6) A T.V repairman finds that the time spent on his job has an exponential distribution with Mean 30 Minutes. If he repairs sets in the order in which they came in and if the arrival of sets is poisson with an average rate of 10 per 8 hr day, what is the expected idle time day? How many jobs are ahead of the average set just brought in?

Soln :-

Mean service rate $\mu = 1/30$ per min

$= 1/30 \times 60 = 2$ set per hr.

Mean arrival rate $= 10/8$ per hr

$\rho = \lambda/\mu$, $\mu = 2$ p/hr.

$\lambda = 5/4$ p/hr.

The utilisation factor λ/μ is $\frac{5}{4 \times 2} = 5/8$

$\Rightarrow$ for 8 hr day, Repairman's busy time $= 8 \times 5/8$

$= 5$ hrs.

Idle time of repairman $= 8 - 5$ hrs $= 3$ hrs.

The number of jobs ahead = No. of units in the system.

$$= \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = \frac{5/8}{3/8} = 5/3$$

$=$ 2 app, Tv. sets.

(7) Cars arrive at a petrol pump, having one petrol unit, in poisson fashion with an average of 10 cars per hour. The service time is distributed exponentially with a mean of 3 minutes. Find (i) average number of cars in the system (ii) average waiting time in the queue (iii) average queue length (iv) the probability that the number of cars in the system is 2.

Soln :-

Mean arrival rate, $\lambda = 10$ per hour

Mean service rate, $\mu = \frac{1}{3} \times 60 = 20$ per hour

$$\rho = \frac{\lambda}{\mu} = \frac{10}{20} = \frac{1}{2}.$$

(i) Average no. of cars in the system,

$$L_s = \frac{\rho}{1-\rho}$$

$$= \frac{1/2}{1-1/2} = 1 \text{ car}.$$

(ii) Average waiting time in the queue

$$= L_q/\lambda = \frac{0.5}{10} = 0.05 \text{ hr}$$

$$= 3 \text{ minutes}.$$

(iii) Average queue length

$$L_q = \frac{\rho^2}{1-\rho} = \frac{1/4}{1-1/2} = 0.5 \text{ car}$$

(iv) Probability of n units in the system $P_n = \rho^n (1-\rho)$

When $n = 2$, $P_2 = (1/2)^2 (1/2) = 1/8$.

$(L > 0) = \frac{\mu}{\mu - \lambda} = \frac{0.025}{0.25 - 0.085} = 1.5.$

(9) Arrivals at a telephone booth are considered to be poisson with an average time of 10 min bet one arrival & next. The duration of phone call is assumed to be exponentially distributed with mean 3 minutes.

(a) What is the pbty. that a person arriving at the booth will have to wait?

(b) The Telephone dept will Install a second booth when convinced that an arrival would expect waiting for atleast 3 minute for phone. By how much should the flow of arrival increase in order to justify the second booth?

(c) Find the avg. no. units in the system

(d) Estimate the fraction of the day that the phone will be in use.

(e) What is the pbty. that it will take more than 10 min altogether to wait for phone & complete the call?

Soln :-

$$\lambda = 1/10 = 0.10 \; ; \; \mu = 1/3 = 0.33$$

(a) $P(w > 0) = 1 - P_0$

$$= 1 - (1 - \lambda/\mu) = \lambda/\mu$$

$$= 0.10/0.33 = 0.3$$

(b) $$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$3 = \frac{\lambda'}{0.33(0.33 - \lambda)}$$

$W_q = 3$ & $\lambda = \lambda'$ for second booth

$\lambda' = 0.16$

Arrival rate should become 0.16 person per minute to justify the second booth.

Increase in the arrival rate is $0.16 - 0.10 = 0.06$ arrivals per minute.

(c) Avg. number of units in the system is given by $L_s = \frac{\rho}{1-\rho} = \frac{0.3}{1-0.3} = 0.43$ Customer

(d) The fraction of a day that the phone will be busy = traffic Intensity.

$= \rho = \lambda/\mu = 0.3$

(e) $P(W \geq 10) = \int_{10}^{\infty} (\mu - \lambda) e^{-(\mu-\lambda)t} dt$

$= 0.10$.

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(10) People arrive at a Theatre ticket booth in poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate

(a) The mean number in the waiting line

(b) The mean waiting time

(c) The utilization factor.

Soln :- $\lambda = 25$ per hr

$\mu = 1/2 \times 60 = 30$ per hr

$\therefore \rho = \lambda/\mu = 25/30 = 5/6 = 0.833$

(i) Length of the Queue

$L_q = \frac{\rho^2}{1-\rho^2} = \frac{(0.833)^2}{1-0.833}$

$$= \frac{0.693889}{0.167}$$

$= 4$ (appl.)

(ii) Mean waiting time $= \frac{L_q}{\lambda}$

$= 4/25$

$= 9.6$ minutes

(iii) Utilization factor $\rho = \lambda/\mu = 0.833$

GAMES AND STRATEGIES

17.1 Introduction :-

Many practical problems require decision - making in a competitive situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent.

A competitive situation will be called a 'Game', if it has the following properties:

(i) There are a finite number of competitors (participants) called players.

(ii) Each player has a finite number of strategies (alternatives) available to him.

(iii) A play of the game take place when each player employs his strategy.

(iv) Every game results in an outcome eg. loss (or) gain (or) a draw, usually called payoff, to some player.

17.2 : Two person zero sum Games :-

When there are two competitors playing a game, it is called a 'two - person game'. In case the number of competitors exceeds two, say n, then the game is termed as 'n - person game'.

If the algebraic sum of gains and losses of all the players is zero in a game then such a game is called Zero - Sum Game otherwise the game is called a non-zero sum game.

Zero sum game with two players are called two person zero sum games. In this case the loss (gain) of one player is exactly equal to gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero sum character or simply a non zero sum game.

17.3 : Some Basic Terms :-

① Player :

The competitors in the game are known as players. A player may be individual or group of individuals or an organisation.

② Strategy :-

A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt. A strategy of may be of two types:

(a) pure strategy

(b) mixed strategy

(a) pure strategy :-

If the players select the same strategy each time, then it is referred to as pure strategy.

(b) Mixed strategy :-

When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occassion then this is known as mixed strategy (i.e) Maximin ≠ Minimax

③ optimum strategy :-

A course of action or play which puts the player in the most preferred position irrespective compe of the strategy of his competitors is called an optimum strategy

④ Value of the Game :

It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of game is zero and unfair if it is non-zero

⑤ Pay off Matrix :-

When the players select their particular strategies, the pay off (gains or losses) can be represented in the form of a matrix called the pay off matrix.

Since, the game is zero-sum, therefore gain of one player is equal to the loss of the other and vice-versa

17.4 : The Maximin - Minimax Principle :-

Consider the following game

player B

player A $\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$

We find Row - Minima, Column Maxima

	player B		
Player A	B_1	B_2	Row Minima
A_1	1	1	1
A_2	4	-3	-3
Column Maxima	4	1	

We Write Minimum of Maximum by Minimax and Maximum of Minimum by Maximin.

This selection of strategies by A and B was based upon maximin, minimax principles.

Saddle point :-

A saddle point of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima.

The payoff at the saddle point is called the value of the game denoted by V.

The saddle point need not be unique. We shall denote the maximin value of the game by $\underline{v}$ and the minimax value of the game by $\bar{v}$.

A game is said to be fair if $\underline{v} = 0 = \bar{v}$

A game is said to be strictly determinable if $\underline{v} = v = \bar{v}$

Examples

① Solve the game whose pay off matrix is given by.

Player B

		B_1	B_2	B_3
Player A	A_1	1	3	1
	A_2	0	-4	-3
	A_3	1	5	-1

Soln :-

	B_1	B_2	B_3	Row Minima
A_1	1	3	1	1
A_2	0	-4	-3	-4
A_3	1	5	-1	-1
Column Maxima	1	5	1	

Minimax = 1, Maximin = 1 ⇒ Saddle point (

$S_0 = (A_1, B_1)$ (or) (A_1, B_3).

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② Solve the following game whose pay-off matrix is given below:

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

Solution :

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \quad \begin{matrix} \text{Row minima} \\ 0 \\ 4 \\ 2 \\ 1 \end{matrix}$$

Column Maxima: 9 6 4 8 8

Max of Row Minima = 4

Min of Column Maxima = 4

S_0 = Row 2 and column 3

Value of the game $V = 4$

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(2) For what value of λ, the game with the following matrix is strictly determinable.

Player B

	B_1	B_2	B_3
Player A A_1	λ	6	2
A_2	-1	λ	-7
A_3	-2	4	λ

Ignoring the value of λ

Solution :-

	B_1	B_2	B_3	Row minima
A_1	λ	6	2	2
A_2	-1	λ	-7	-7
A_3	-2	4	λ	-2
Column maxima	-1	6	λ	

We know, if

$\underline{v} = v = \bar{v}$, then the game is

strictly determinable

Here, $\underline{v} = 2$ and $\bar{v} = 1$

$\Rightarrow -1 \le \lambda \le 2$.

---

(7) For the game with payoff matrix

player B

$$\text{player A} \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

Determine the best strategies for the players A and B and also the value of the game. Is this game (i) Fair (ii) strictly determinable?

Solution :-

	player B			Row minima
player A	-1	2	-2	-2
	6	4	-6	-6
Column maxima	6	4	-2	

$\therefore$ Max of Row minima $= -2$

Min of column maxima $= -2$

S_0 = Row 1 and column 3

Value of the game $= -2$

The game is Not Fair but strictly determinable

---

(5) Determine the range of value of p and q that will make the payoff element a_{22} a saddle point for the game whose payoff matrix (a_{ij}) is given below

Player B

$$\text{Player A} \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 8 \end{bmatrix}$$

Soln :-

Ignoring p and q will determine the maximin and minimax values of the pay of matrix.

	B_1	B_2	B_3	Row minima
A_1	2	4	5	2
A_2	10	7	q	7
A_3	4	p	8	4
Column maxima	10	7	9	

Maximin value = 7 $(= \underline{v})$

Minimax value = 7 $(= \bar{v})$

Thus there exists a saddle point at position (2, 2)

Which implies the condition on p as $p \le 7$ and q as $q \geq 7$

$\therefore$ Range required $p \le 7$, $q \geq 7$.

17.5 : Games without Saddle points - Mixed Strategies

As determining the Minimum of Column Maxima and Maximum of row minima are two different operations, there is no reason to expect that they should alway lead to Unique pay off position - the saddle point

Solution of 2 x 2 Games without Saddle point.

METHOD I :-

Let the payoff matrix be as follows

The payoff Matrix (for the player A) is given by.

Player B

$$\text{Player A} \begin{array}{c} H \\ T \end{array} \overset{\begin{array}{cc} H & T \end{array}}{\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}}$$

Determine the strategy for A, B, value of the Game.

Solution :-

Player B

		H	T	Row Min
player A	H	2	-1	-1
	T	-1	0	-1
	Column maxi	2	0	

Hence $\underline{V} = -1 \neq 0 = \overline{V}$

$\Rightarrow$ The Matrix without saddle point

Let A play H with Prob/- x

A play T with prob/- $1-x$

If player B plays H all the times

Then A's expected gain is

$E(A, H) = x \cdot 2 + (1-x)(-1)$

$\therefore E(A, H) = 3x - 1 \rightarrow$ ①

III^ly B plays T all the time

Then, A's expected gain is

$E(A, T) = x \cdot (-1) + (1-x)(0)$

$\therefore E(A, T) = -x \rightarrow$ ②

from ① & ②

$E(A, H) = E(A, T) = E(A)$.

We determine the best strategy for A

$\Rightarrow 3x - 1 = -x$

$(i.e)\ x = 1/4$

$\Rightarrow 1 - x \Rightarrow 1 - \frac{1}{4} = 3/4$.

$\therefore$ best strategy for player A is to play H and T with prob/- 1/4 & 3/4.

We Apply the same procedure for B. Let the pbty/- of the choice of H be denoted by y and that of T be (1-y).

$E(B, H) = E(B, T) = E(B)$

$y \cdot 2 + (1-y)(-1) = y(-1) + (1-y)0$

$4y = 1$

$y = 1/4$

$1 - y \Rightarrow 1 - 1/4 = 3/4$

Now using x, y in E(A), E(B)

E(A) = -1/4 E(B) = -1/4

A's strategy = (1/4 3/4)

B's strategy = (1/4 3/4)

Value of Game = $2 \times \frac{1}{4} - 1 \times \frac{3}{4}$

$= -1/4$

(OR)

METHOD II :-

For any 2×2 two person zero sum game without any saddle point having the payoff matrix for player A is

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

Where, $P_1 = \dfrac{a_{22} - a_{21}}{\lambda}$ and $P_2 = 1 - P_1$

$q_1 = \dfrac{a_{22} - a_{12}}{\lambda}$ and $q_2 = 1 - q_1$

Where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$

$\therefore V = \dfrac{a_{11} a_{22} - a_{12} a_{21}}{\lambda}$

Examples

(1) Solve the following 2 × 2 game

$$A \overset{B}{\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}}$$

Soln :-

Let B play 5 with pbty/. x and 1 with pbty/. $(1-x)$

$E(A,5) = 5x + 1\cdot(1-x)$

III^{ly} $E(A,3) = 3x + 4\cdot(1-x)$

$\therefore E(A) = E(A,5) = E(A,3)$

$\Rightarrow 5x + (1-x) = 3x + 4 - 4x$

$4x + 1 = -x + 4$

$5x = 3$

$x = 3/5$

$\Rightarrow 1 - x = 1 - 3/5 = 2/5$

III^{ly} For A with pbty/. y and $1-y$

$\rightarrow 5y + 3(1-y) = 1\cdot y + 4(1-y)$

$5y + 3 - 3y = y + 4 - 4y$

$2y + 3 = 4 - 3y$

$5y = 1$

$y = 1/5$

$\Rightarrow 1 - y = 1 - 1/5 = 4/5$.

Now, Strategy for B = $(3/5, 2/5)$

Strategy for A = $(1/5, 4/5)$

Value of the Game $= 5 \times \frac{1}{5} + 3\left(\frac{4}{5}\right)$

$= 1 + 12/5 = 17/5$

(7) Solve the following 2×2 Game

$$A \overset{B}{\begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}}$$

Soln :-

	B		Row minima
A	2	5	2
	7	3	3
Column maxima	7	5	

Min of column maxima = 5

Max of Row minima = 3

$$\bar{v} \neq \underline{v}$$

$\Rightarrow$ No saddle point

(i.e) Let B choose 2, 5 with probability $x, 1-x$ and 7, 3 with pbty $x, 1-x$.

$$\Rightarrow 2x + 5(1-x) = 7x + 3(1-x)$$

$$2x + 5 - 5x = 7x + 3 - 3x$$

$$5 - 3x = 4x + 3$$

$$-7x = -2$$

$$x = 2/7$$

$$\therefore 1 - x = 5/7$$

Strategy for B $= \left(\frac{2}{7}, \frac{5}{7}\right)$

Strategy for A :

If A choses 2, 7 with pbty/- $y, 1-y$ and 5, 3 with pbty/- $y, 1-y$

$$2y + 7(1-y) = 5y + 3(1-y)$$

$$2y + 7 - 7y = 5y + 3 - 3y$$

$$7 - 5y = 2y + 3$$

$$4 = 7y$$

$$\Rightarrow y = 4/7$$

$$\Rightarrow 1 - y = 1 - 4/7 = 3/7$$

Value of the game $= \left(2 \times \frac{4}{7}\right) + 7\left(\frac{3}{7}\right)$

$$= \frac{8}{7} + \frac{3}{1}$$

$$= 29/7$$

Strategy for A $= \left(\frac{2}{7}, \frac{5}{7}\right)$

Strategy for B $= \left(\frac{4}{7}, \frac{3}{7}\right)$

Value of the Game $= 29/7$

---

8. In a game of matching coins with two players, suppose A wins one unit value when there are two heads, wins nothing when there are two tails and looses 1/2 unit value when there are one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game.

Solution :-

Formulation

$$A \begin{array}{c} \\ H \\ T \end{array} \overset{B}{\begin{array}{cc} H & T \\ 1 & -1/2 \\ -1/2 & 0 \end{array}}$$

Let this be

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

Now, $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$

$= 1 + 0 - (-1/2 - 1/2)$

$= 2$

$$P_1 = \frac{a_{22} - a_{21}}{\lambda}$$

$$= \frac{0 - (-1/2)}{2}$$

$= 1/4$

$\Rightarrow P_2 = 1 - P_1 = 1 - 1/4 = 3/4$.

$$q_1 = \frac{a_{22} - a_{12}}{\lambda}$$

$$= \frac{0 - (-1/2)}{2}$$

$= 1/4$

$\Rightarrow q_2 = 1 - q_1 = 1 - 1/4 = 3/4$

Value of the game $v = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

$= \dfrac{1(0) - (-1/2)(-1/2)}{2}$

$= -1/8$

---

9 Solve the following game without saddle point

$$A \overset{B}{\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}}$$

Soln :-

Let the given pay off - matrix be

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \overset{\begin{array}{cc} B_1 & B_2 \end{array}}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

The optimum mixed strategies

Strategy for A : $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$

Strategy for B : $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$

Now, $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$

$= 2 + 1 - (5 + 4)$

$= -6$

$P_1 = \dfrac{a_{22} - a_{21}}{\lambda} = \dfrac{1-4}{-6} = \dfrac{1}{2}$

$P_2 = 1/2$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda}$$

$$= \frac{1-5}{-6} = \frac{2}{3} \quad \frac{1}{3} = 1 - 1/3 = 2/3$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ 2/3 & 1/3 \end{bmatrix}$$

Value of the game $= \dfrac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

$$= \frac{2-20}{6}$$

$$= 3.$$

17.6 Graphic Solution of 2×n and m×2 Games :-

Dominance property :-

General Rule :-

(i) If all the elements of a row, say k^{th} are less than or equal to the corresponding elements of any other row, say r^{th} then k^{th} row is dominated by r^{th} row.

(ii) If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by r^{th} column.

(iii) omit dominated rows or columns

(iv) If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar arguments follow for column

Examples

① Solve the following game using dominance property.

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7
	III	6	1	6

Soln :-

Row III is dominated by Row II and Column III is dominated by column I.

∴ We omit Row III and column III

		B	
		I	II
A	I	1	7
	II	6	2

Using strategy for B as $x, 1-x$ Probty

$$1 \cdot x + 7(1-x) = 6 \cdot x + 2(1-x)$$

$$x + 7 - 7x = 6x + 2 - 2x$$

$$7 - 6x = 4x + 2$$

$$5 = 10x$$

$$x = 1/2$$

$$\Rightarrow 1 - x = 1 - 1/2 = 1/2$$

∴ Strategy for B : $\{1/2, 1/2, 0\}$

We have,

$$1 \cdot y + 6(1-y) = 7y + 2(1-y)$$

$y + 6 - 6y = 7y + 5 - 2y$

$6 - 5y = 5y + 2$

$4 = 10y$

$y = 2/5$

$\Rightarrow 1 - y = 1 - 2/5 = 3/5$

$\therefore$ Strategy for A : $\{2/5, 3/5, 0\}$

Value of the game $= 1 \cdot x + 7(1-x)$

$= 1 \cdot 1/2 + 7 \cdot 1/2$

$= 1/2 + 7/2 = 4$

---

(2) Using Dominance property solve

B

		I	II	III	IV
A	1	-5	3	1	20
	2	5	5	4	6
	3	-4	-2	0	-5

Soln :-

Row III is dominated by Row II and Column II is dominated by column I

$\Rightarrow$ We omit Row III, column II

B

		I	III	IV
A	1	-5	1	20
	2	5	4	6

Column IV is dominated by column I

We omit colum IV.

B

		I	III	Row minima
A	1	-5	1	-5
	2	5	4	4
Column Maxima		5	4	

Min of column Maxima = 4

Max of Row minima = 4

Saddle point = (2, III)

Value of the game = 4

---

(8) Solve the rectangular game whose payoff Matrix for player A is $\begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & -12 \end{bmatrix}$

Soln :-

Row III is dominated by Row II,

Column I is dominated by Column II.

Omit Row III, Column I

$$\begin{bmatrix} -2 & 8 \\ 5 & -1 \end{bmatrix}$$

If pbty/- for A are $x, 1-x$

$$-2x + 5(1-x) = 8x + (-1(1-x))$$

$$-2x + 5 - 5x = 8x - 1 + x$$

$$5 - 7x = 9x - 1$$

$$6 = 16x$$

$$x = 3/8.$$

$$\Rightarrow 1 - x = 1 - 3/8 = 5/8$$

$$\therefore S_A = (3/8, 5/8, 0)$$

Similarly for B

$$-2y + 8(1-y) = 5y - 1(1-y)$$

$$-2y + 8 - 8y = 5y - 1 + y$$

$$8 - 10y = 6y - 1$$

$$-16y = -9$$

$$y = 9/16$$

$$\Rightarrow 1-y = 1 - 9/16 = 7/16$$

$\therefore S_B = (0, 9/16, 7/16)$

Value of the Game $= -2x + 5(1-x)$

$= -6/8 + 5/8$

$V = -1/8$

---

(4) Solve the following 2 x 4 game graphically.

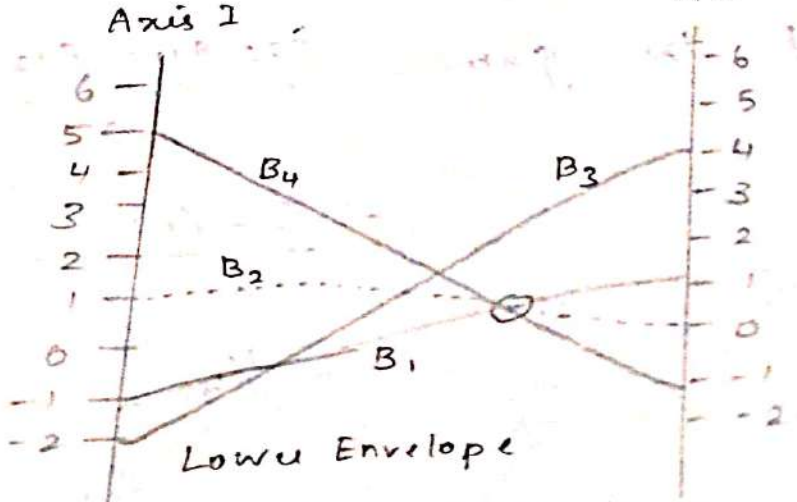
player B

Player A $\begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}$

soln :-

Consider two axes say axis I, axis II unit distance apart.

Vertically at



Maximin Lines B_1, B_4

Maximin lines B_1, B_4

$$\begin{array}{c} \quad B \\ A \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix} \end{array}$$

Now, If the Matrix be $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\lambda = a_{11} + a_{22} - (a_{21} + a_{12})$$

$$= 6 - (-2)$$

$$\lambda = 8$$

$$P_1 = \frac{a_{22} - a_{12}}{\lambda}$$

$$P_1 = \frac{5+1}{8} = \frac{3}{4}$$

$$P_2 = 1/4$$

$$S_A : (3/4 \quad 1/4)$$

$$q_1 = \frac{a_{22} - a_{21}}{\lambda}$$

$$q_1 = 6/8 = 3/4$$

$$\Rightarrow q_2 = 1/4$$

$$S_B : (3/4 \quad 0 \quad 0 \quad 1/4)$$

Value of the game $= \dfrac{a_{22}\,a_{11} - a_{12}\,a_{21}}{\lambda}$

$$= \frac{5 - (1)}{8}$$

$$= 1/4$$

(5) Solve the rectangular game whose payoff matrix for player A is $\begin{pmatrix} 2 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}$

Soln :-

$$A \overset{B}{\begin{pmatrix} 2 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}}$$

Column I is dominated by column II

Omit column I

$$A \overset{B}{\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}}$$

For A : $2 \cdot x + 3(1-x) = 3 \cdot x + 2(1-x)$

$$2x + 3 - 3x = 3x + 2 - 2x$$

$$3 - x = 2 + x$$

$$1 = 2x$$

$$x = 1/2.$$

$$\Rightarrow 1 - x = 1 - 1/2 = 1/2$$

$$S_A : (1/2, 1/2)$$

For B : $2y + 3(1-y) = 3y + 2(1-y)$

$$2y + 3 - 3y = 3y + 2 - 2y$$

$$3 - y = y + 2$$

$$y = 1/2$$

$$\therefore 1 - y = 1 - 1/2 = 1/2$$

$$S_B : (0, 1/2, 1/2)$$

Value of the Game = $2x + 3(1-x)$

$= 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2}$

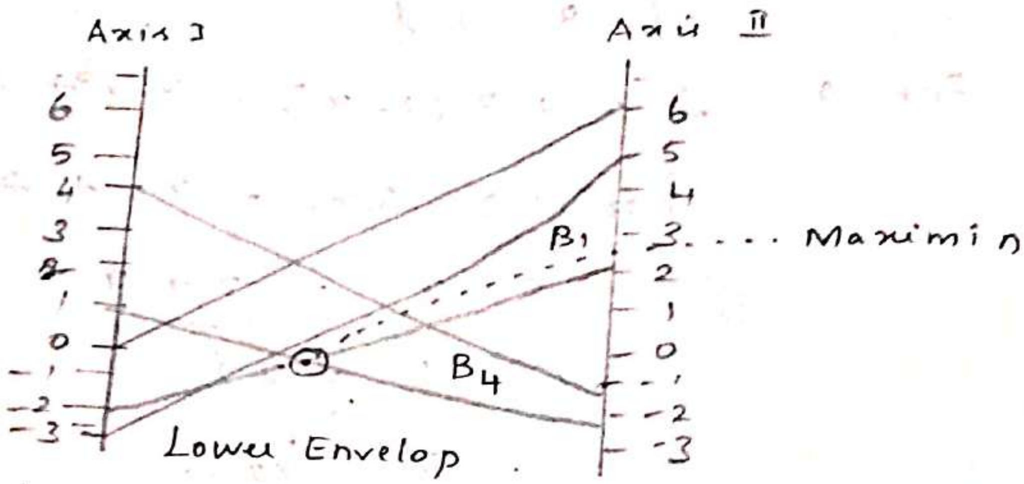
$= 5/2$.

---

(6) Using Graphical method, solve the rectangular game whose payoff matrix for player

A is $\begin{pmatrix} 2 & -1 & 5 & -2 & 6 \\ -2 & 4 & -3 & 1 & 0 \end{pmatrix}$

Soln :-



Reduced payoff Matrix for A is $\begin{pmatrix} 2 & -2 \\ 2 & 1 \end{pmatrix}$

Solving using probabilities

$2x + (1-x)(-2) = -2x + 1(1-x)$

$4x - 2 = -3x + 1$

$7x = 3$

$x = 3/7$

$\Rightarrow 1-x = 1 - 3/7 = 4/7$.

$\therefore S_A : (3/7, 4/7)$.

For player B

$2y + (1-y)(-2) = -2y + 1(1-y)$

$2y - 2 + 2y = -2y + 1 - y$

$4y - 2 = 1 - 3y$

$7y = 3$

$y = 3/7$

$\Rightarrow 1 - y = 1 - 3/7 = 4/7$

$\therefore S_B : (3/7, 0, 0, 4/7, 0)$

$\therefore$ Value of the Game $= 2 \cdot 3/7 + (-2)(4/7)$

$= 6/7 - 8/7$

$\therefore = -2/7$.

---

(7) Is the following two person zero sum game stable? solve the game.

		I	II	III	IV
Player A	1	5	-10	9	0
	2	6	7	8	1
	3	8	7	15	1
	4	3	4	-1	4

Soln :-

The game has no saddle point and hence it is not stable.

Row III dominates Row I and Row II

$\therefore$ we omit Row I and Row II

$$\begin{array}{c c} & B \\ A \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 8 & 7 & 15 & 1 \\ 3 & 4 & -1 & 4 \end{pmatrix} \end{array}$$

Column II is dominated by column IV

omit Column III

$$A \begin{pmatrix} I & III & IV \\ 8 & 15 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

$\frac{15+1}{2}$, $\frac{-1+4}{2}$ = Average of Column III and Column IV

8 , 1.5 = Avg. of Column III & IV

∴ As Column I is dominated by 8, 1.5

omit column I

$$\begin{array}{c c} & \begin{array}{cc} III & IV \end{array} \\ A \begin{array}{c} 3 \\ 4 \end{array} & \begin{pmatrix} 15 & 1 \\ -1 & 4 \end{pmatrix} \end{array}$$

$$\therefore S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 5/19 & 14/19 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 3/19 & 16/9 \end{bmatrix}$$

$$V = 61/19$$

Exercise

(1) Determine which of the following two person zero sum games are strictly determinable and fair. Give optimum strategy for each player in the case of strictly determinable games :

(a) Player A
Player B
$$\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) Player A
Player B
$$\begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$$

(2) Consider a "modified" form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Rs. 1.00 if the coins turn both tails. The non matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non matching player, which one would you choose and what would be your strategy?

(3) Solve the following 2 × 2 game graphically?

Player B

Player A	B_1	B_2	B_3	B_4
A_1	2	1	0	-2
A_2	1	0	3	2

(4) Obtain the optimal strategies for both persons and the value of the game for

Zero sum two person game whose payoff matrix is as follows

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

(5) Solve the following problem graphically

Player B

$$\text{Player A} \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}$$

(6) Using graphical method in solving the following game

Player A

$$\text{Player B} \begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix}$$

UNIT-5

PERT and CPM - Network

25.1 :- Introduction :-

Network Scheduling is a technique used for planning and scheduling large projects in the fields of Construction, Maintenance, fabrication, Purchasing, research and development designs etc.

There are two basic planning and control techniques that utilize a network to complete a pre-determined project or schedule. These are

(i) Program Evaluation and Review Technique (PERT)

(ii) Critical Path Method (CPM).

25.2 :- Network : Basic Components :-

A Network is a graphic Representation of a project's operations and is composed of activities and events that must be completed to reach the end objective of a Project.

Activity :-

An activity is a task, or item of work to be done, that consumes time, effort, Money or other resources.

It lies between two events called the "Preceding" and "Succeeding" ones.

An activity is represented by an arrow with its head indicating the sequence in which the events are to occur.

Event :-

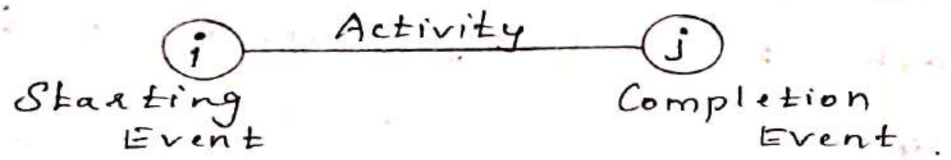
An Event represent the start or completion of some activity and such it consumes no time. It has no time duration and does not consume any resource. An event is nothing but a Node.

Node is generally represented on the network by a circle, rectangle, hexagon or some other Geometric shape.

An Event is not complete untill all the activities flowing into it are completed.

NOTE :-

Activities are identified by the number of their starting (tail or initial) event and ending (head or terminal) event. An arrow (i, j) extended between two events; the tail event i represents the start of the activity and head event j represents the completion of the activity as shown below.



The activities can be further classified into the following three categories.

① **Predecessor Activity :-**

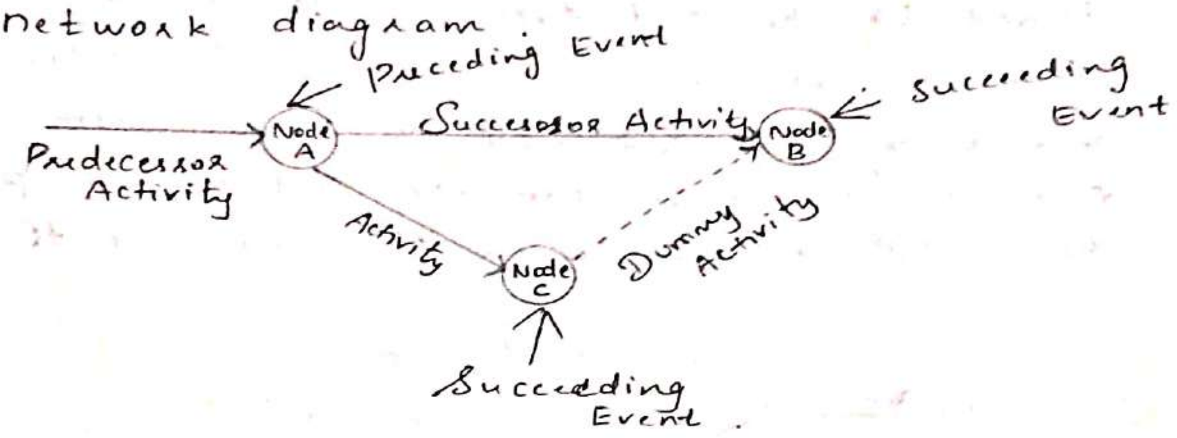
An activity which must be completed before one or ~~another~~ more other activities start is known as Predecessor activity.

② Successor Activity :-

An activity which started immediately after one or more of other activities are completed is known as successor activity.

③ Dummy Activity :-

An activity which does not consume either any resource and time is known as Dummy activity. A dummy activity is depicted by dotted line in the network diagram.



NOTE :-

A Dummy activity is depicted by a dotted line in the network diagram.

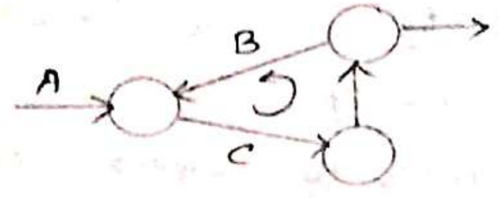
25.3 : Logical Sequencing :-

In logical sequencing, following three types of errors are most common while drawing a network diagram.

① Looping
② Dangling
③ Dependency Relationship.

① **Looping :-**

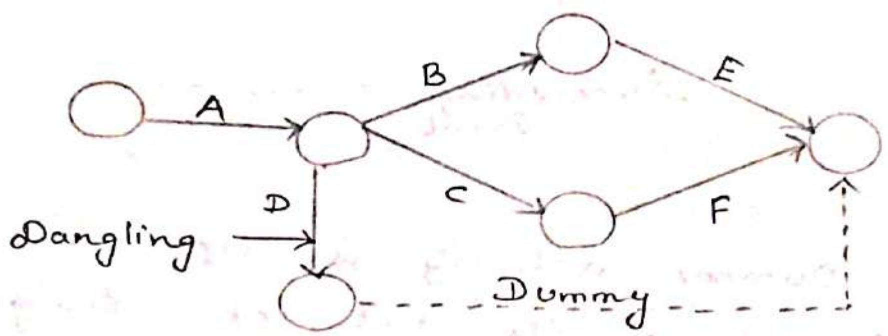
If an activity were represented as going back in time, a closed loop would occur as shown in following figure.



Closed Loop

② **Dangling :-**

No activity should end without being joined to the end event. If it is not so, a dummy activity is introduced in order to maintain the continuity of the system. Such end-events other than the end of the project as a whole are called dangling events.



③ **Dependency Relationships :-**

When two chains of activities have a common event, wholly or partly independent of each other, a dummy activity is used to establish proper logical relationships.

2.6.4 Rules of Network Construction :-

For the construction of a network, generally, the following rules are followed

(1) Each activity is represented by one and only one arrow

(2) Each activity must be identified by its starting and end node which implies that

(i) Two activities should not be identified by the same completion events and

(ii) Activities must be represented either by their symbols or by the corresponding ordered pair of starting - completion events.

(3) Nodes are numbered to identify an activity uniquely. Tail node should be lower than the head node of an activity.

(4) Between any pair of nodes, there should be one and only one activity

(5) Arrow should be kept straight and not curved or bent.

(6) The logical sequence between activities must follow the following rules.

(i) An event cannot occur until all the incoming activities into it have been completed

(ii) An activity cannot start unless all the preceding activities on which it depends, have been completed.

(iii) Dummy activity should only be introduced, if absolutely necessary.

Numbering the Events:- (Ford and Fulkerson's Rule)

After the network is drawn in logical sequence, every event is assigned a number. The number sequence must be such so as to reflect the flow of the network.

In event numbering, the following rules should be observed.

(a) Event Numbers should be Unique

(b) Event Numbering should be carried out on a sequential basis from left to right.

(c) The initial Event which has all outgoing arrows with no incoming arrow is numbered 0 or 1.

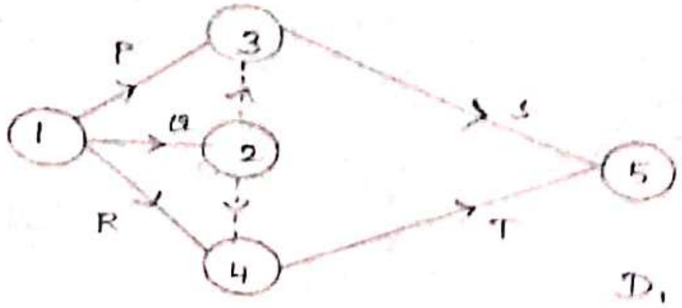
(d) The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.

(e) Gaps should be left in the sequence of event numbering to accomodate subsequent inclusion of activities.

Sample Problems

① If there are Five activities P, Q, R, S and T such that P, Q, R have no immediate predecessors but S and T have immediate predecessor P, Q and Q, R respectively. Represent this situation by a network.

Solution :-

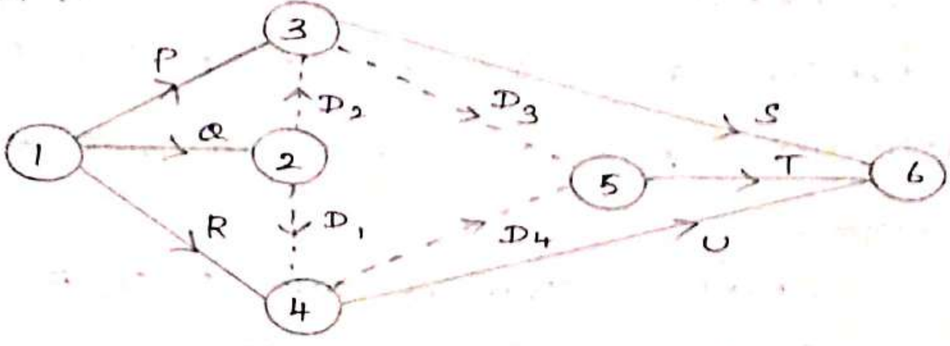


D_1 and D_2 are Dummy Activity

(2) Draw the network for the project whose activities and their precedence relationships are given below.

Activity :	P	Q	R	S	T	U
Predecessor :	–	–	–	P, Q	P, R	Q, R

Solution :-



D_1, D_2, D_3 and D_4 are dummy activities.

(3) Draw the network for the project whose activities with their predecessor relationship are given below.

A, C, D can start simultaneously.

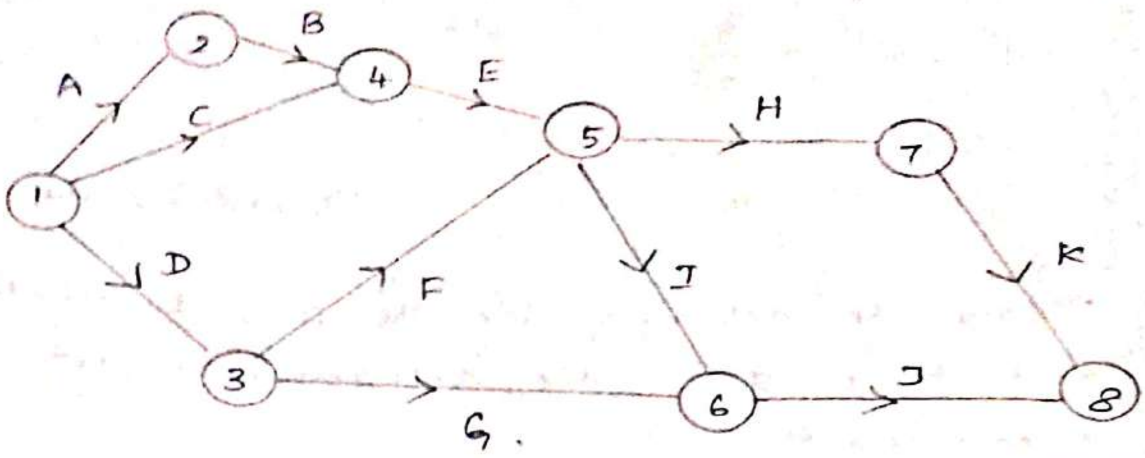
E > B, C ; F, G > D ; H, I > E, F ;

J > I, G ; K > H ; B > A .

Solution :-

Identify the start activities, (i.e) activities which have no predecessors.

They are A, C, D as given.



(4) Construct the network for the project whose activities and their relationship are as given below.

Activities : A, D, E can start simultaneously

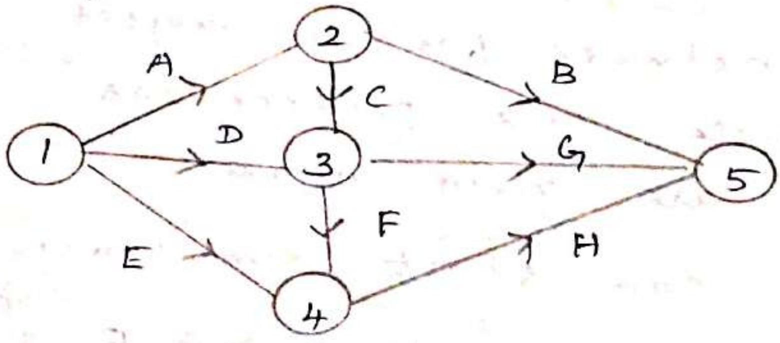
Activities : B, C > A ; G, F > D, C ; H > E, F

Solution :-

Start activities are A, D, E

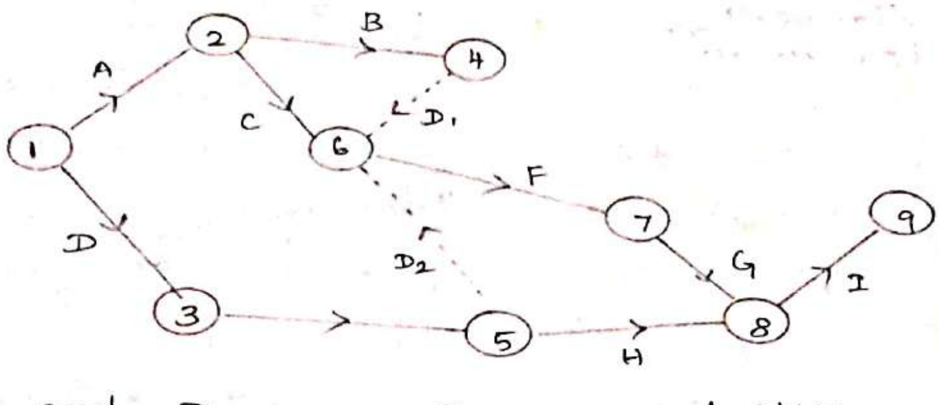
End activities are H, G, B

The required network is



(5) Draw the network for the project whose activities and their precedence relationship are as given below:

Activities :	A	B	C	D	E	F	G	H	I
Immediate Predecessors :	-	A	A	-	D	B, C, E	F	E	G, H



D_1 and D_2 are Dummy activities.

(6) Construct the network for the project whose precedence relationship are as given below.

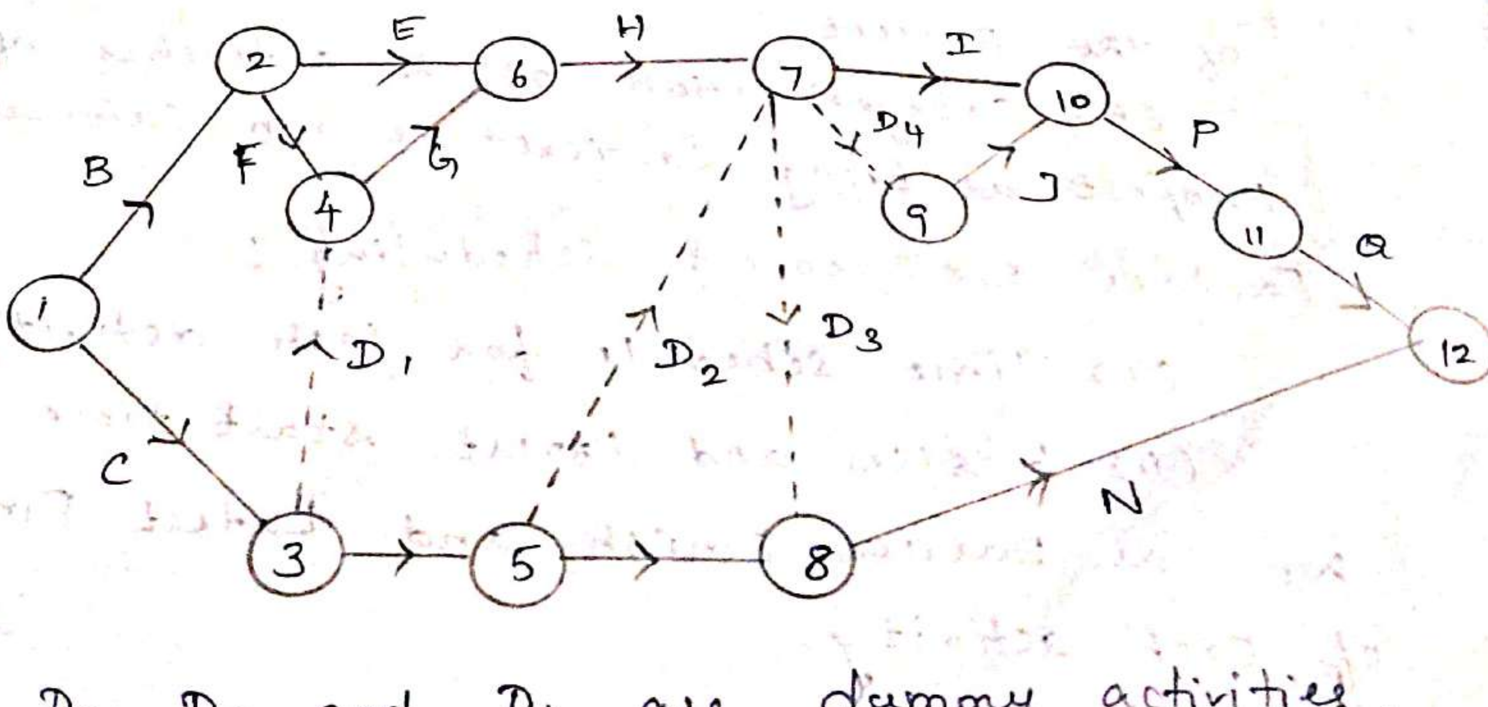
B < E, F ; C < G, L ; E, G < H ; L, H < I ;

L < M ; H, M < N ; I < J ; I, J < P ; P < Q .

Solution :

Start activities : B, C

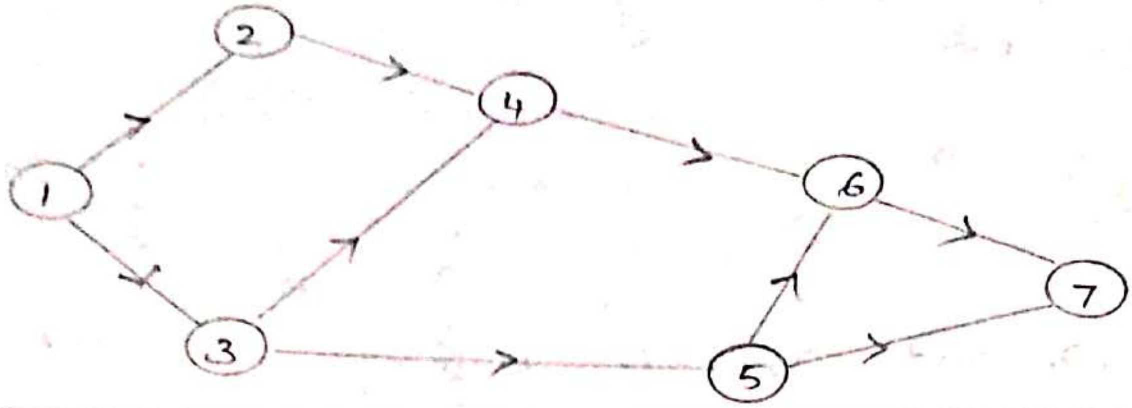
End activities : N, Q



D_1, D_2, D_3 and D_4 are dummy activities.

(7) Draw the event oriented network for the following data

Event No :	1	2	3	4	5	6	7
Immediate Predecessors :	-	1	1	2, 3	3	4, 5	5, 6



25.6 : <u>Critical path Analysis</u> :-

The purpose of analysis is to find the critical path. i.e, the sequence of activities with the longest duration and to find the float associated with each non critical activity

To achieve this objective, we carry out

(a) Total duration needed for the Completion of the project.

(b) Categorization of the activities of the project as being Critical or non Critical.

Factors For project scheduling :-

(i) Time schedule for Each activity

(ii) Earlier and latest start time as well as Earlier Finish and Latest Finish of Each activity.

(iii) Float for each activity

(iv) Critical activities and critical path for the Network.

Calculating various times of Events and activities, the following terms shall be used in critical path calculations

E_i = Earliest occurence time of event i

L_j = Latest occurence time of event j

t_{ij} = Duration of activity (i, j)

The critical path calculations are done in the following ways.

(a) Forward pass Calculation

(b) Backward pass Calculation

(a) <u>Forward pass calculation :-</u>

Step 1 : set $E_1 = 0$; i = 1 (initial node)

Step 2 : Set the Earliest Start time for each activity that beigns at node i as

$ES_{ij} = E_i$; for all activities (i, j) that start at node i

Step 3 : Compute the Earliest Finish (EF) time of each activity that begins at node i by adding the Earliest start (ES) time of the activity to the duration of the activity. Thus

$$EF_{ij} = ES_{ij} + t_{ij}$$
$$= E_i + t_{ij}.$$

Step 4 : Move on to next node, say node j (j > i) and compute the earliest occurence for node j, using

$$E_j = \max_i \{EF_{ij}\} = \max_i \{E_i + t_{ij}\}$$

for all immediate predecessor activities.

Step 5 : If j = n (final node), then the earliest finish time for the project is given by

$$E_n = \max \{EF_{ij}\} = \max \{E_{n-1} + t_{ij}\}.$$

Backward pass calculation :-

Step 1 : $L_n = E_n$; for $j = n$

Step 2 : Set the latest finish time of each activity that ends at node j as

$$LF_{ij} = L_j.$$

Step 3 : Compute the latest occurence times of all activities ending at j by subtracting the duration of each activity from the latest finish time of the activity. Thus,

$$LS_{ij} = LF_{ij} - t_{ij} = L_j - t_{ij}$$

Step 4 : Proceed backward to the node in the sequence, that decrease j by 1. Also compute the latest occurence time of node i (i < j) using

$$L_i = \min_j \{LS_{ij}\} = \min_j \{L_j - t_{ij}\}$$

for all immediate successor activities.

Step 5 : If $j = 1$ (initial node), then

NOTE : $L_1 = \text{Min}\{LS_{ij}\} = \text{Min}\{L_j - t_{ij}\}$.

Based on the above calculations, an activity (i, j) will be critical if it satisfies the following condition

(i) $E_i = L_i$ and $E_j = L_j$

(ii) $E_j - E_i = L_j - L_i = t_{ij}$

An activity that does not satisfy the above condition is termed as non-critical.

Critical path :-

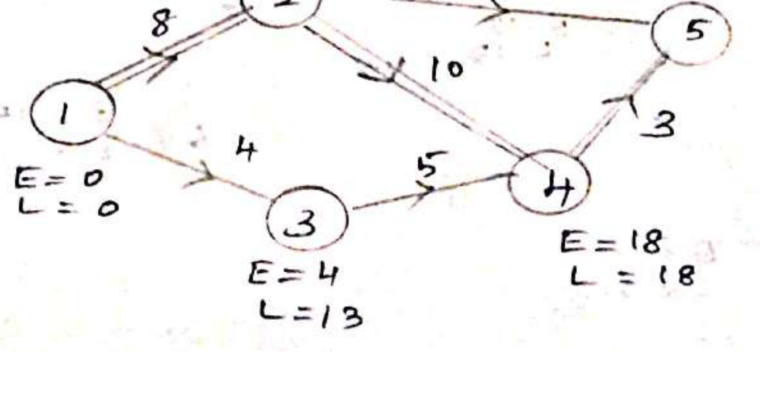
Path, connecting the first initial node to the very last terminal node, of longest duration in any project network is called the critical path.

Example

① Compute the Earliest Start, Earliest Finish, Latest Start and Latest Finish of each activity of the project given below.

Activity :	1-2	1-3	2-4	2-5	3-4	4-5
Duration (in days) :	8	4	10	2	5	3

Soln :-



Activity	Duration (days) t_{ij}	Earliest Start ES	Earliest Finish $EF = ES + t_{ij}$	Latest Start $LS = LF - t_{ij}$	Latest Finish LF
1-2	8	0	8	0	8
1-3	4	0	4	9	13
2-4	10	8	18	8	18
2-5	2	8	10	19	21
3-4	5	4	9	13	18
4-5	3	18	21	18	21

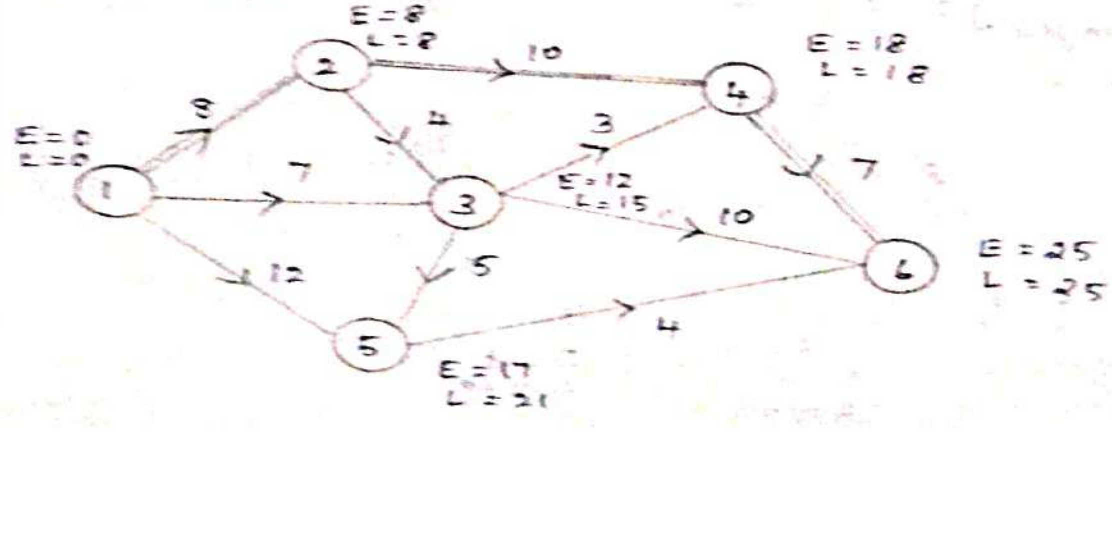
Critical path : 1 - 2 - 4 - 5

(2) Calculate the Earliest start, Earliest Finish, Latest start and Latest Finish of each activity of the Project given below and determine the critical path of the project.

Activity :	1-2	1-3	1-5	2-3	2-4	3-4
Duration (in weeks) :	8	7	12	4	10	3

Activity :	3-5	3-6	4-6	5-6
Duration (in weeks) :	5 ~~3~~	~~5~~ 10	~~10~~ 7	4

Solution :-



Activity	Duration (in weeks)	Earliest		Latest	
		Start	Finish	Start	Finish
	(t_{ij})	ES	$ES + t_{ij}$	$LF - t_{ij}$	LF
1 - 2	8	0	8	0	8
1 - 3	7	0	7	8	15
1 - 5	12	0	12	9	21
2 - 3	4	8	12	11	15
2 - 4	10	8	18	8	18
3 - 4	3	12	15	15	18
3 - 5	5	12	17	16	21
3 - 6	10	12	22	15	25
4 - 6	7	18	25	18	25
5 - 6	4	17	21	21	25

Critical path : 1 - 2 - 4 - 6

<u>Float</u> (or slack) of an Activity and Event :-

The float of an activity is the amount of time by which it is possible to delay its completion time without affecting the total project completion time.

① <u>Event Float</u> :-

The float of an event is the difference between its latest time (L_i) and its Earliest time (E_i). That is,

Event Float $= L_i - E_i$

② **Activity Float :-**

It is the float in the activity time estimates.

Total Float :

Total float of an activity (TF) is defined as the difference between the latest finish and the earliest finish of the activity (OR) the difference between the latest start and the earliest start of the activity.

$$\left.\begin{array}{l}\text{Total float of}\\ \text{activity } i-j\end{array}\right\} = (LF)_{ij} - (EF)_{ij}$$

(OR)

$$= (LS)_{ij} - (ES)_{ij}.$$

Free Float :-

Free Float (FF) of an activity is that portion of the total float which can be used for rescheduling that activity without affecting the succeeding activity.

$$\left.\begin{array}{l}\text{Free Float of}\\ \text{an activity } i-j\end{array}\right\} = \text{Total float } i-j - (L-E) \text{ of the event } j$$

$$= \text{Total float of } i-j - \text{slack of the head event}$$

Where L = Latest occurrence, E = Earliest occurrence

Note : Free Float ≤ Total float for any activity

Independent Float :-

Independent Float (I.F) of an activity is the amount of time by which the activity

Can be rescheduled without affecting the preceding or succeeding activities of that activity.

Independent float of an activity i-j } = Free float of i-j − (L − E) of event i

= Free Float of i-j − slack of the tail event i

Note :-

Independent Float ≤ Free Float for any activity.

I.F ≤ F.F ≤ T.F

<u>Interfering Float</u> (or) <u>Interference Float</u> :-

Interference float of an activity i-j is nothing but the slack of the head event j.

obviously,

Interfering Float of i-j = Total float of i-j − Free Float of i-j.

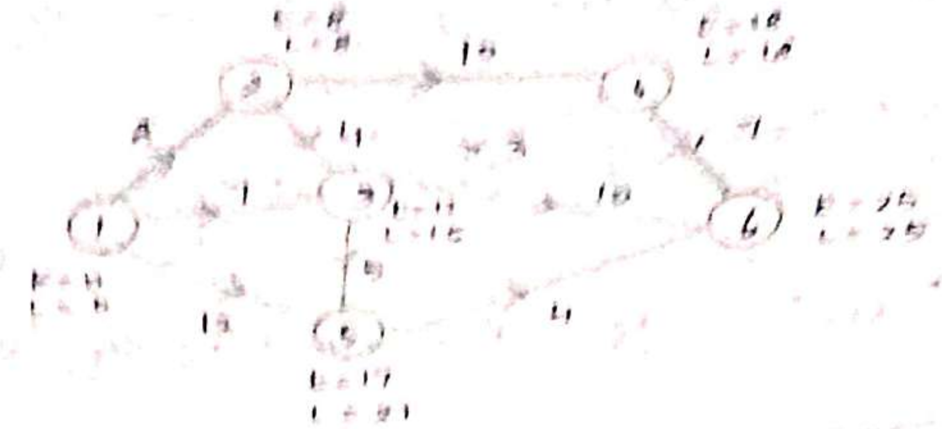
Example.

① Calculate the Total float, Free Float and independent Float for the project whose activities are given below.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6
Duration (in Weeks)	8	7	12	4	10	3	5	10	7

Activity :	5-6
Duration :	4

Solution :



Activity	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Floats TF	Floats FF	Floats IF
1-2	8	0	8	0	8	0	0	0
1-3	7	0	7	8	15	8	5	5
1-5	13	0	13	9	21	9	5	5
2-3	4	8	12	11	15	3	0	0
2-4	10	8	18	8	18	0	0	0
3-4	3	12	15	15	18	3	3	0
3-5	5	12	17	16	21	4	0	-3
3-6	10	12	22	15	25	3	3	0
4-6	7	18	25	18	25	0	0	0
5-6	4	17	21	21	25	4	4	0

Path Activity	Duration
1 - 2 - 4 - 6	25
1 - 2 - 3 - 4 - 6	22
1 - 2 - 3 - 6	22
1 - 3 - 4	10
1 - 3 - 6	17
1 - 3 - 5 - 6	16
1 - 5 - 6	16

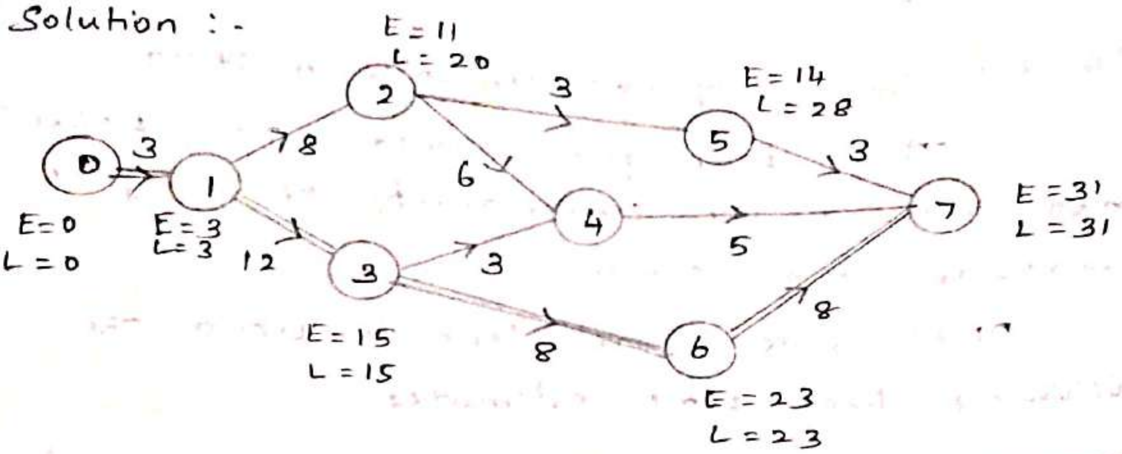
∴ The Critical path is 1 - 2 - 4 - 6.

Project duration is 25 weeks.

(4) Construct the network for the project whose activities are given below, and compute the Total, Free and independent Float of each activity and hence determine the critical path and the Project duration.

Activity:	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration (in Weeks):	3	8	12	6	3	3	8	5	3	8

Solution :-



Activity	Duration (in Weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Floats TF	FF	IF
0-1	3	0	3	0	3	0	0	0
1-2	8	3	11	12	20	9	0	0
1-3	12	3	15	3	15	0	0	0
2-4	6	11	17	20	26	9	1	-8
2-5	3	11	14	25	28	14	0	-9
3-4	3	15	18	23	26	8	0	0
3-6	8	15	23	15	23	0	0	0
4-7	5	18	23	26	31	8	8	0
5-7	3	14	17	28	31	14	14	0
6-7	8	23	31	23	31	0	0	0

Path Activity	Duration
0 - 1 - 2 - 5 - 7	17
0 - 1 - 2 - 4 - 7	22
0 - 1 - 3 - 4 - 7	23
0 - 1 - 3 - 6 - 7	31

∴ The critical path is 0 - 1 - 3 - 6 - 7

Project Duration = 31 weeks.

25.7 : Probability Considerations in PERT

This technique unlike CPM, take into account the uncertainity of project durations into account.

PERT calculations depends upon the following three time estimates.

Optimistic (least) time estimate : (t_o or a)

is the shortest possible time to complete the activity if all goes well.

Pessimistic (Greatest) time estimate : (t_p or b)

is the Longest time that an activity could take if everything goes wrong.

Most Likely time Estimate : (t_m or m)

is the estimate of the normal time an activity would take. If only one time were available, this would be it.

Otherwise, it is the mode of the Probability distribution.

PERT Procedure :-

(1) Draw the project network

(2) Compute the expected duration of each activity $t_e = \frac{t_0 + 4t_m + t_p}{6}$

(3) Compute the expected Variance $\sigma^2 = \left(\frac{t_p - t_0}{6}\right)^2$ of each activity

(4) Compute the Earliest Start, Earliest Finish, Latest Start and Latest Finish, Total float of each activity.

(5) Determine the critical path and identify critical activities.

(6) Compute the expected Variance of the Project length (also called the Variance of the Critical path) σ_c^2 which is the sum of the Variances of all the critical activities.

(7) Compute the expected standard deviation of the Project length σ_c and calculate the standard normal derivative $\frac{T_S - T_E}{\sigma_c}$ where

T_S = Specified (or) scheduled time to complete the Project

T_E = Normal Expected project Duration

σ_c = Expected standard deviation of the Project length.

(8) Using (7) one can estimate the Probability of completing the Project within a specified time, using the normal curve Tables

Examples :

(1) Construct the network for the project whose activities and the three time estimates of these activities (in weeks) are given below. Compute

(a) Expected duration of each activity

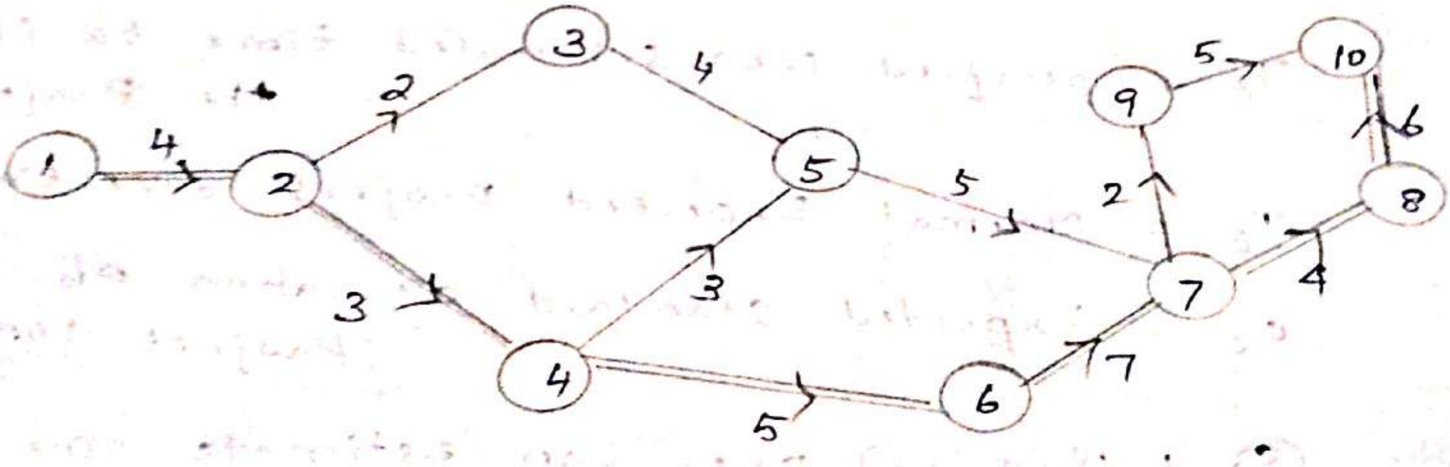
(b) Expected variance of each activity

(c) Expected variance of project length

Activity	1-2	2-3	2-4	3-5	4-5	4-6	5-7
t_o	3	1	2	3	1	3	4
t_m	4	2	3	4	3	5	5
t_p	5	3	4	5	5	7	6

Activity	6-7	7-8	7-9	8-10	9-10
t_o	6	2	1	4	3
t_m	7	4	2	6	5
t_p	8	6	3	8	7

Solution :



Activity	t_o	t_m	t_p	Expected duration $t_e = \frac{t_o + 4t_m + t_p}{6}$	Expected Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1-2	3	4	5	4	1/9 = 0.11
2-3	1	2	3	2	1/9 = 0.11
2-4	2	3	4	3	1/9 = 0.11
3-5	3	4	5	4	1/9 = 0.11
4-5	1	3	5	3	4/9 = 0.44
4-6	3	5	7	5	4/9 = 0.44
5-7	4	5	6	5	1/9 = 0.11
6-7	6	7	8	7	1/9 = 0.11
7-8	2	4	6	4	4/9 = 0.44
7-9	1	2	3	2	1/9 = 0.11
8-10	4	6	8	6	4/9 = 0.44
9-10	3	5	7	5	4/9 = 0.44

Critical path is 1 - 2 - 4 - 6 - 7 - 8 - 10

Expected Project duration = 29 weeks.

(c) Expected Variance of the Project length } = Sum of the expected Variance of all the Critical activities

$= 1/9 + 1/9 + 4/9 + 4/9 + 4/9 + 4/9$

$= 15/9 = 5/3$

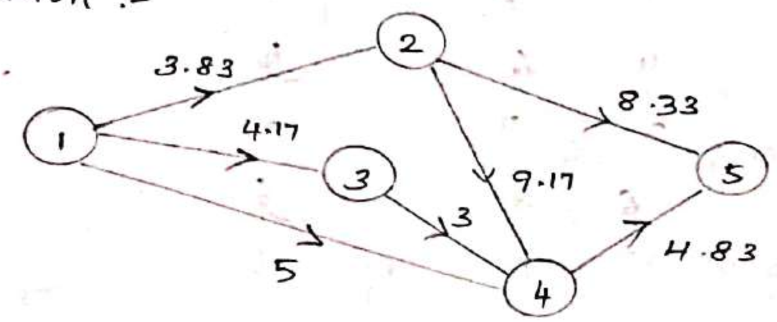
$= 1.67$.

② The following table indicates the details of a Project. The duration are in days. 'a' refers to optimistic time, 'm' refers to most likely time and 'b' refers to pessimistic time duration.

Activity	1-2	1-3	1-4	2-4	2-5	3-5	4-5
a	2	3	4	8	6	2	2
m	4	4	5	9	8	3	5
b	5	6	6	11	12	4	7

(a) Draw the network (b) Find the critical path (c) Determine the expected standard deviation of the completion time.

Solution :-



Activity	a	m	b	Expected duration t_e	Expected variance σ^2
1-2	2	4	5	3.83	1/4
1-3	3	4	6	4.17	1/4
1-4	4	5	6	5	1/9
2-4	8	9	11	9.17	1/4
2-5	6	8	12	8.33	1
3-4	2	3	4	3	1/9
4-5	2	5	7	4.83	25/36

Critical path 1 - 2 - 4 - 5

Expected project duration = 17.83 days

Expected Variance of the Completion Time $\Big\} = \frac{1}{4} + \frac{1}{4} + \frac{25}{36} = \frac{43}{36}$

Expected standard deviation of Completion time $\Big\} = \sqrt{\frac{43}{36}} = 1.09$ (nearly)

---

③. A project consist of the following activities and time estimates

Activity	1-2	1-3	1-4	2-5	2-6	3-6	4-7	5-7	6-7
least time (days)	3	2	6	2	5	3	3	1	2
Greatest time (days)	15	14	30	8	17	15	27	7	8
Most likely time (days)	6	5	12	5	11	6	9	4	5

(a) Draw the network.

(b) What is the probability that the project will be completed in 27 days?

Soln :-

Greatest time = Pessimistic time = t_p

Least time = optimistic time = t_o

Most likely time = t_m.

Activity	t_o	t_m	t_p	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 - 2	3	15	6	7	4
1 - 3	2	14	5	6	4
1 - 4	6	30	12	14	16
2 - 5	2	8	5	5	1
2 - 6	5	17	11	11	4
3 - 6	3	15	6	7	4
4 - 7	3	27	9	11	16
5 - 7	1	7	4	4	1
6 - 7	2	8	5	5	1

Critical path 1 - 4 - 7

Expected project duration = 25 days

Expected Variance of the project length = Sum of the expected variances of all the critical activities

= 16 + 16

= 32.

σ_c = Standard deviation of the project length

$= \sqrt{32} = 4\sqrt{2} = 5.656$

$Z = \frac{T_S - T_E}{\sigma_c} = \frac{27 - 25}{5.656} = \frac{2}{5.656} = 0.35$

Probability that the Project will be completed in 27 days $\Big\} = P(T_S \le 27)$

$= P(Z \le 0.35)$

$= 0.1365 + 0.5$

$= 0.6368$

$= 63.7\%$

---

(4) Three Time estimates (in months) of all activities of a project are as given below.

Time in months

Activity	a	m	b
1 - 2	0.8	1.0	1.2
2 - 3	3.7	5.6	9.9
2 - 4	6.2	6.6	15.4
3 - 4	2.1	2.7	5.1
4 - 5	0.8	3.4	3.6
5 - 6	0.9	1.0	1.1

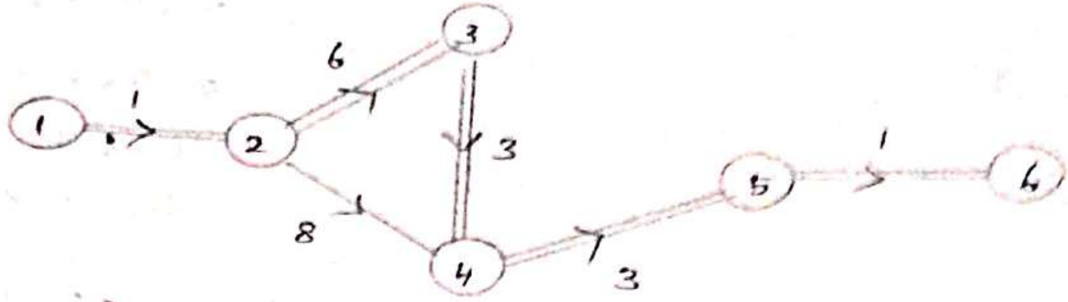
(a) Find the expected duration and standard deviation of each activity

(b) Construct the project network

(c) Determine the critical path, expected project length and expected variance of the project length.

(d) What is the probability that the Project will be completed (i) two months later than expected (ii) Not more than 3 months earlier than expected (iii) What due date has about 90 % chance of being met?

Solution. :-



Activity	a	m	b	t_e	σ
1 - 2	0.8	1.0	1.2	1	0.067
2 - 3	3.7	5.6	9.9	6	1.03
2 - 4	6.2	6.6	15.4	8	1.53
3 - 4	2.1	2.7	5.1	3	0.5
4 - 5	0.8	3.4	3.6	3	0.47
5 - 6	0.9	1.0	1.1	1	0.033

Critical path : 1 - 2 - 3 - 4 - 5 - 6 .

Expected project length = 14 Months

Expected Variance $= (0.067)^2 + (1.03)^2 + (0.5)$
$+ (0.47)^2 + (0.033)$

$= 1.5374$

$\sigma_c = \sqrt{1.5374} = 1.2399$

(d) (i) $T_S = 16$, $T_E = 14$, $\sigma_c = 1.2399$

$$Z = \frac{16 - 14}{\sigma_c} = \frac{2}{1.2399}$$

$$= 1.61$$

$p(T_S \leq 16) = 0.9463$

$p(T_S \leq 16) = 94.63\%$

(ii) $T_S = 11$, $T_E = 14$, $\sigma_c = 1.2399$

$$Z = \frac{T_S - T_E}{\sigma_c} = \frac{11 - 14}{1.2399}$$

$$= -\frac{3}{1.2399}$$

$$= -2.42$$

$p(T_S \leq 11) = 0.5 - 0.4922 = 0.0078$

Required Probability $= 0.78\%$

(iii) Z value for 90% area in the table $= 1.28$ (nearly)

Let T_S be the required due date

Then $Z = \dfrac{T_S - T_E}{\sigma_c}$

(i.e) $1.28 = \dfrac{T_S - 14}{1.6023}$

$T_S = 14 + 1.28 \times 1.2399$

$= 15.59$ Months nearly.

Exercise.

(1) (a) Construct the Network for each of the projects whose activities and their precedence relationship are given below.

(i) Activity : A B C D E F G H I J K

Immediate Predecessor : – – – A B B C D E H, I F, G

(ii) A < C, D, I ; B < G, F ; D < G, F ; F < H, K ;
G, H < J ; I, J, K < E

(iii) A, B, C can start simultaneously
A < F, E ; B < D ; C, E, D < G.

(iv) Activity : A B C D E

Immediate Predecessor : – A A A B, C, D

(v) A < C, D ; B < C, D ; C < E ; D, E < F

(vi) Activity A B C D E F G H I

Immediate Predecessor – – A, B B B A, B F, D F, D C, G.

(vii) A < C, D ; B < E ; C, E < F, G ; D < H ; G < I ;
H, I < J.

(viii) Draw the network for the following data and number the events.

A < C, B ; B < D, E ; C < F ; E < G ; F < I, J ;
J < K ; G < L ; K, L < M.

project schedule has the following characteristics.

Activity	1-2	1-3	2-4	3-4	3-5	4-9
Time	4	1	1	1	6	5

Activity	5-6	5-7	6-8	7-8	8-10	9-10
Time	4	8	1	2	5	7

Construct network and find the critical path.

(3) Draw the network and determine the critical path for the given data

Jobs	1-2	1-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration	6	5	10	3	4	6	2	9

Find the TF, FF and IF of each activity.

(4) Construct the network for the project

A < D, E ; B, D < F ; C < G ; B, G < H ; F, G < I

Find also the minimum time of completion of the project, when the time (in days) of completion of each task is as follows.

Task :	A	B	C	D	E	F	G	H	I
Time :	23	8	20	16	24	18	19	4	10

(5) A small project consist of seven activities for which the relevant data are given below.

Activity	A	B	C	D	E	F	G
Preceding Activities	–	–	–	A, B	A, B	C, D, E	C, D, E
Activity Duration (Days)	4	7	6	5	7	6	5

(i) Draw the network and find the project completion time.

(ii) Calculate Total float for each activities and highlight the critical path

(iii) Draw the Time

(E) The following table list the jobs of a network along with their estimates with time

jobs	1-2	1-3	2-4	3-4	4-5	3-5
Optimistic time	2	9	5	2	6	8
Most likely time	5	12	14	5	6	17
Pessimistic time	14	15	17	8	12	20

(a) Draw the Network

(b) Calculate the length and variance of the critical path

(c) Find the probability that the project will be completed within 30 days?

(7) A project consist of eight activities with the following relevant information.

Activity	A	B	C	D	E	F	G	H
Immediate Predecessor	-	-	-	A	B	C	D, E	F, G
optimistic	1	1	2	1	2	2	3	1
Most likely	1	4	2	1	5	5	6	2
Pessimistic	7	7	8	1	14	8	15	3

(i) Draw the PERT network and find out the expected project completion time

(ii) What duration will have 95 % confidence for project completion

(iii) If the average duration for activity F increases to 14 days, what will be its effect on the expected projection completion time which will have 95 % confidence?.

(For standard normal Z = 1.645, area under the standard normal curve from 0 to Z is 0.45).

(8) A small project is composed of seven activities whose time estimates are listed in the table

Activity		Estimated duration (weeks)		
i	j	Optimistic	Most likely	Pessimistic
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

(a) Draw the network

(b) Find the expected duration and variance of each activity. What is expected project length?

(c) Calculate the variance and standard deviation of project length. What is the probability that the project will be completed.

(i) at least 4 weeks earlier than expected?

(ii) no more than 4 weeks later than expected?

(d) If the project due date is 19 weeks, what is the probability of meeting the due date?

Given

Z	0.50	0.67	1.00	1.33	2.00
P	0.3085	0.2514	0.1587	0.0918	0.0228