SEMESTER : III
CORE COURSE : V

Inst Hours : 5
Credit : 5
Code : [illegible]

OPERATIONS RESEARCH

UNIT 1:
Linear programming problem - Mathematical formulation – Illustrations on Mathematical formulation on Linear Programming Problems – Graphical solution method - some exceptional cases - Canonical and standard forms of Linear Programming Problem - Simplex method.
(Chapter 2: Sec 2.1 to 2.4, Chapter 3: Sec 3.1 to 3.5, Chapter 4: Sec 4.1 , 4.3)

UNIT 2:
Use of Artificial Variables (Big M method - Two phase method) – Duality in Linear Programming - General primal-dual pair - Formulating a Dual problem - Primal-dual pair in matrix form -Dual simplex method.
(Chapter 4: Sec 4.4, Chapter 5: Sec 5.1 to 5.4, 5.9)

UNIT 3:
Transportation problem - LP formulation of the TP - Solution of a TP - Finding an initial basic feasible solution (NWCM - LCM -VAM) – Degeneracy in TP – Transportation Algorithm (MODI Method) – Assignment problem - Solution methods of assignment problem – special cases in assignment problem.
(Chapter 10: Sec 10.1, 10.2, 10.8, 10.9, 10.12, 10.13, Chapter 11: Sec 11.1 to 11.4)

UNIT 4:
Queuing theory - Queuing system - Classification of Queuing models – Poisson Queuing sys
Model I (M/M/1):(∞/FIFO) only - Games and Strategies – Two person zero sum Games - Som
basic terms - the maximin-minimax principle -Games without saddle points-Mixed strategies
graphic solution 2xn and mx2 games.
(Chapter 21: Sec 21.1, 21.2, 21.7 to 21.9, Chapter 17: Sec 17.1 to 17.6)

UNIT 5:
PERT and CPM – Network: Basic components – logical sequencing - Rules of network construction- Critical path analysis - Probability considerations in PERT.
(Chapter 25: Sec 25.1 to 25.4, 25.6, 25.7)

Text Book

Kanti Swarup, P.K. Gupta and ManMohan, Operations Research, 13th Edition, Sultan Chand and Sons, 2007.

Books for Reference

1. Sundaresan V, Ganapathy Subramanian. K.S. and Ganesan K, Resource Management Techniques, A.R. Publications, 2002.
2. Taha H.A., Operations Research: An introduction, 7th edition, Pearson Prentice Hall.

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

UNIT - I

Linear Programming Problem :-

Linear Programming can be used in a Variety of situations. In most of the business or economical situations, resource will be limited. The problem there will be to make use of the available resources in such a way that to Maximise the production or to minimize the expenditure. These datas can be formulated as linear Programming Models.

The objective of the linear Programming Problem is to Maximize the profit and Minimize the total cost.

The LPP is to determine the values of the decision variables such that all the constraints are satisfied and gives the Maximum or Minimum value for the objective function. The Maximum or Minimum value of the objective function is called an optimum value.

Formulation of LPP :-

The Formulation of any situation to a LPP is based on the following guidelines.

(1) Identification of Decision variables

(2) Formation of objective function which is to be either Maximize or Minimize

(3) The various constraints involved due to the limited availability of resources.

Mathematical Formulation of LPP :-

The general form of LPP is as follows

$$\text{Max or Min } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Subject to,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_2$$

$$\dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \left(\begin{smallmatrix}\le\\ \ge\\ =\end{smallmatrix}\right) b_m$$

$$x_i \ge 0 \ \forall i$$

Hence the Variable x_i are called decision Variables. The Function z is called the objective function and $c_1, c_2 \dots c_n$ are called cost Variables.

The linear constraints of the general LPP involved one of the relations, "= (or) ≤ (or) ≥" The addition of some non-negative variables on the left hand side of the inequations may convert inequations into equations. For that purpose we use the Variables called slack Variables or Surplus variables.

<u>Slack Variable :-</u>

The variable which we add to constraint equation is called slack variable.

<u>Surplus Variable :-</u>

The Variable which we subtract to constraint equation is called surplus Variable.

<u>Feasible solution :-</u>

An n tuple $(x_1, x_2 \dots x_n)$ of real numbers which satisfies the linear constraint and the non negative restriction of the given LPP is called feasible solution.

<u>Optimum Feasible Solution :-</u>

An n tuple $(x_1, x_2 \dots x_n)$ of real numbers which satisfies the linear constraints and the non negative restriction of the given LPP and optimize the objective function is called optimum feasible solution.

Degenerate :-

A basic solution to the system $AX = B$ is called degenerate if one or more of the basic variables are zero.

Problem :-

① A Firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two Machines M_1 and M_2. Type A requires 1 minute of processing time on M_1 and 2 minutes on M_2. Type B requires 1 Minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hrs 40 min while Machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP so as to Maximize the profit.

Soln :- Let the firm decide to produce x_1 units of product A and x_2 units of product B to Maximize its profit.

To produce these units of type A and type B products, it requires

$x_1 + x_2$ processing minutes on M_1

$2x_1 + x_2$ processing minutes of M_2.

Since machine M_1 is available for not more than 6 hrs and 40 min.

Machine M_2 is available for 10 hours.

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
and $x_1, x_2 \geq 0$

Since the profit from type A is Rs. 2 and profit from type B is Rs. 3, the total profit is $2x_1 + 3x_2$

$\therefore$ Max $Z = 2x_1 + 3x_2$

Subj to constraints

$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
and $x_1, x_2 \geq 0$.

(3) A firm produces an alloy having the following items
(i) specific gravity ≤ 0.98
(ii) chromium $\geq 8\%$
(iii) Melting point $\geq 450°c$

Raw Materials A, B & C having the properties shown in the table can be used

Property	Raw Material A	B	C
Specific Gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440°c	490°c	480°c

Cost of the various raw material per unit are Rs. 90 for A, Rs. 280 for B and Rs. 40 for C. Find the proportions in which A, B and C be used to obtain an alloy of desired properties while the cost of raw material is minimum.

Soln :-

Let x_1, x_2 and x_3 be the tons of raw materials A, B and C to be used for make the alloy.

Since the cost of various raw materials per unit ton are Rs 90 for A, Rs 280 for B and Rs. 40 for C.

the total cost is $90x_1 + 280x_2 + 40x_3$.

$\therefore$ The complete Formulation of the LPP is

$$\text{Minimize } z = 90x_1 + 280x_2 + 40x_3$$

Subject to

$$0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$$

$$7x_1 + 13x_2 + 16x_3 \geq 8$$

$$440x_1 + 490x_2 + 480x_3 \geq 450$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Graphical Method :-

If the objective function z is a function of two variable then the problem can be solved by graphical method. The procedure is as follow

Step 1: First of all we consider the constraints as equalities or equations.

Step 2: Then we draw the lines in the plane corresponding to each equation obtained in step 1 and non-negative restriction.

Step 3: Then we find the permissible region for the values of the variable which is the region bounded by the lines drawn in step 2.

Step 4: Finally we find a point in the permissible region which gives the optimum value of the objective function.

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① Solve the following LPP by graphical method

$$\max Z = 3x_1 + 2x_2$$

Subject to,

$$-2x_1 + x_2 \le 1$$

$$x_1 \le 2$$

$$x_1 + x_2 \le 3$$

$$\text{and } x_1, x_2, x_3 \geq 0.$$

Soln :-

$-2x_1 + x_2 = 1 \to ①$; $x_1 = 2 \to ②$; $x_1 + x_2 = 3 \to ③$

$x_1 = 0 \to ④$; $x_2 = 0 \to ⑤$

put $x_1 = 0$ in ① $\Rightarrow x_2 = 1 \Rightarrow (0,1)$

$x_2 = 0$ in ① $\Rightarrow \cancel{2}x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5, 0)$

$\therefore$ $(0,1)$ and $(-0.5, 0)$ are the points

put $x_2 = 0$ in ② $\Rightarrow x_1 = 2 \Rightarrow (2, 0)$

$\therefore$ $(2, 0)$ are the points.

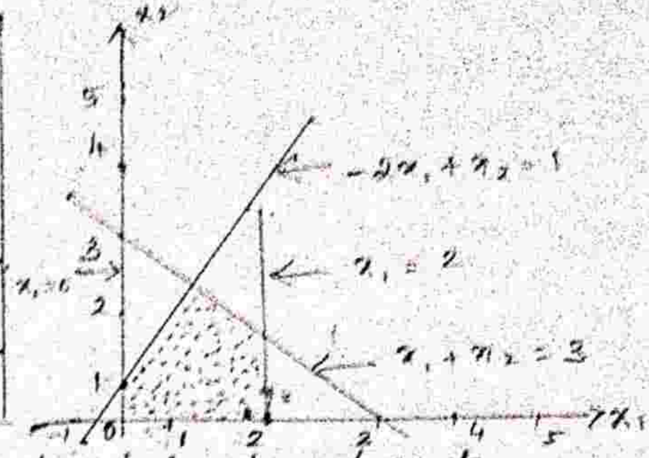
put $x_1 = 0$ in ③ $\Rightarrow x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0$ in ③ $\Rightarrow x_1 = 3 \Rightarrow (3, 0)$

$\therefore$ The vertices of the solution space are $O(0,0)$, $A(2,0)$, $B(2,1)$ $C(2/3, 7/3)$ and $D(0,1)$.

The values of z at these vertices are given by

Vertex	Value of z ($z = 3x_1 + 2x_2$)
O(0, 0)	0
A(2, 0)	6
B(2, 1)	8
C(2/3, 7/3)	20/3
D(0, 1)	2



Since the problem is of Maximization type the optimum solution to the LPP is

$$\text{Max } z = 8,\ x_1 = 2,\ x_2 = 1.$$

(2) Find the Maximum value of z to the following LPP using graphical method, Max $z = 5x_1 + 7x_2$.
Subject to $x_1 + x_2 \le 4$; $3x_1 + 8x_2 \le 24$;
$10x_1 + 7x_2 \le 35$ and $x_i \geq 0,\ \forall\ i = 1, 2$.

Soln :-
The given problem contains only two variables x_1 and x_2. Consider the equations

$x_1 + x_2 = 4 \rightarrow (1)$
$3x_1 + 8x_2 = 24 \rightarrow (2)$
$10x_1 + 7x_2 = 35 \rightarrow (3)$

Now to find the points,

$x_1 + x_2 = 4 \Rightarrow$ put $x_1 = 0 \Rightarrow x_2 = 4$; put $x_2 = 0 \Rightarrow x_1 = 4$

$\therefore$ (0, 4) and (4, 0) are the points.

$3x_1 + 8x_2 = 24 \Rightarrow$ put $x_1 = 0 \Rightarrow x_2 = 3$; put $x_2 = 0 \Rightarrow x_1 = 8$

$\therefore$ (0, 3) and (8, 0) are the points.

$10x_1 + 7x_2 = 35 \Rightarrow$ put $x_1 = 0 \Rightarrow x_2 = 5$; put $x_2 = 0 \Rightarrow x_1 = 3.5$

$\therefore$ (0, 5) and (3.5, 0) are the points.

$\therefore$ Solution is $x_1 = 1.6$, $x_2 = 2.4$ and Max $z = 24.8$.

Simplex Method :-

Step 1 : The objective function of the given LPP is to be Maximised or convert it into a Maximization Problem.

Step 2 : All the b_i's should be non-negative. If some b_i is negative multiply the corresponding equation by -1

Step 3 : The inequations of the constraints must be converted into equations by introducing the slack variable or surplus variable in the constraints. The cost of these variables are taken to be zero.

Step 4 : obtain the initial basic feasible solution $X_B = B^{-1}b$ and form the starting simplex table

Step 5 : Compute the net evaluation $z_j - c_j$ $(j = 1, 2 \ldots n)$, where $z_j = \sum_{j=1}^{n} c_{Bi} y_{ij}$ $(i = 1, 2 \ldots m)$. If all $z_j - c_j \geq 0$, then the initial basic feasible soln is a optimum basic feasible solution.

Step 6 : Choose the most negative $z_j - c_j$. Let it be $z_r - c_r$. If all $y_{ir} < 0$, then there is an indication of unbounded solution.

Step 7 : Compute the ratio $\frac{x_{Bi}}{y_{ir}}$. The $(k, r)^{th}$ element is called leading element or pivot element.

Step 8 : Divide each element of the k^{th} row by the pivot element and make all other elements in the r^{th} column to zero by elementary row operation.

Step 9 : Repeat step 5 to 8 Until an optimum Solution is obtained.

Problem :-

① Use simplex method to solve the LPP

Maximize $z = 4x_1 + 10x_2$

Subj to constraints,

$2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$ & $x_1, x_2 \geq 0$.

Soln :-

By introducing the slack variable S_1, S_2 & S_3

Maximize $Z = 4x_1 + 10x_2 + 0\cdot S_1 + 0\cdot S_2 + 0\cdot S_3$

Subject to

$2x_1 + x_2 + 1\cdot S_1 + 0\cdot S_2 + 0\cdot S_3 = 50$

$2x_1 + 5x_2 + 0\cdot S_1 + 1\cdot S_2 + 0\cdot S_3 = 100$

$2x_1 + 3x_2 + 0\cdot S_1 + 0\cdot S_2 + 1\cdot S_3 = 90$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

Since, 3 equation with 5 variables.

$5 - 3 = 2$

$x_1 = 0$ & $x_2 = 0$ (Non basic Variable)

$S_1 = 50$, $S_2 = 100$, $S_3 = 90$ (Basic Variable)

C_j	4	10	0	0	0	X_B	B	C_B	$\theta = Min\left(\frac{X_B}{P_c}\right)$
X_j	x_1	x_2	S_1	S_2	S_3				
	2	1	1	0	0	50	S_1	0	$\frac{50}{1} = 50$ ← PR
	2	5	0	1	0	100	S_2	0	$\frac{100}{5} = 20$
	2	3	0	0	1	90	S_3	0	$90/3 = 30$
Z_j	0	0	0	0	0				
$Z_j - C_j$	-4	-10	0	0	0				

↑ PC

Since $Z_j - C_j < 0$

The current solution is not optimal

Most Negative $Z_2 - C_2 = -10$

To find Ratio.

Ratio $\theta = Min\left(\frac{X_B}{P_c}\right)$

$$= \text{Min}\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\}$$

$$= \text{Min}\{50, 20, 30\}$$

C_j	4	10	0	0	0				
X_j	x_1	x_2	S_1	S_2	S_3	X_B	B	C_B	$\theta = \text{Min}\left(\frac{X_B}{PC}\right)$
	$-8/5$	0	1	$-1/5$	0	30	S_1	0	
	$2/5$	1	0	$1/5$	0	20	x_2	10	
	$4/5$	0	0	$-3/5$	1	30	S_3	0	
Z_j	4	10	0	2	0	200			
$Z_j - C_j$	0	0	0	2	0	200			

All $Z_j - C_j \geq 0$

The current basic feasible solution is optimal.

$\therefore$ The optimal solution is

$\text{Max } z = 200.$

Since $x_1 = 0, \; x_2 = 10.$

$$\text{Max } z = 4x_1 + 10x_2$$
$$= 4 \times 0 + 10 \times 20$$
$$= 200.$$

2. Solve the following LPP by simplex method

Minimize $z = 8x_1 - 2x_2$

Subject to,

$$-4x_1 + 2x_2 \leq 1$$
$$5x_1 - 4x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

Soln :-

The general objective function is of Minimization type we shall convert to Maximization type.

$$\text{Maximize } (-z) = \text{Maximize } (z^*) = -8x_1 + 2x_2$$

Subject to,

$$-4x_1 + 2x_2 \le 1$$

$$5x_1 - 4x_2 \le 3$$

$$x_1, x_2 \geq 0$$

By introducing a slack variables s_1, s_2

$$\text{Maximize } (z^*) = -8x_1 + 2x_2 + 0s_1 + 0s_2$$

Subject to constraints,

$$-4x_1 + 2x_2 + 1 \cdot s_1 + 0 \cdot s_2 = 1$$

$$5x_1 - \cancel{3}\,4x_2 + 0 \cdot s_1 + 1 \cdot s_2 = 3$$

and $x_1, x_2, s_1, s_2 \geq 0$

$$x_1 = x_2 = 0$$

$$s_1 = 1, \quad s_2 = 3.$$

c_j	-8	2	0	0	x_B	B	c_B	$\theta = \text{Min}\left(\frac{x_B}{P_C}\right)$
x_j	x_1	x_2	s_1	s_2				
	-4	2	1	0	1	s_1	0	$1/2$ ← PR
	5	-4	0	1	3	s_2	0	
z_j	0	0	0	0	0			
$z_j - c_j$	8	-2	0	0				
		↑ PC						

Since $z_2 - c_2 = -2 < 0$

The current solution is Not optimal.

c_j	-8	2	0	0				
x_j	x_1	x_2	s_1	s_2	x_B	B	c_B	$\theta = \min\left(\frac{x_B}{p_c}\right)$
	-2	1	1/2	0	1/2	x_2	2	
	-3	0	2	1	5	s_2	0	
z_j	-4	2	1	0	1			
$z_j - c_j$	4	0	1	0	1			

for all $z_j - c_j \geq 0$

The current BFS is optimal

∴ The optimal solution is given by.

Maximize $z^* = 1$.

Since. $x_1 = 0$, $x_2 = 1/2$.

$$\text{Maximize } z^* = -8x_1 + 2x_2$$
$$= -8 \times 0 + 2 \times (1/2)$$
$$= 1$$

Minimize z = - Maximize (z^*)

∴ Maximize $z = -1$, $x_1 = 0$ & $x_2 = 1/2$.

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(3) Find the non negative values are x_1, x_2, x_3 which maximize $z = 3x_1 + 2x_2 + 5x_3$

Subject to.

$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + x_2 + x_3 \leq 430.$$

Soln :-

By introducing the slack variables s_1, s_2, s_3

Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to

$x_1 + 4x_2 + 1S_1 + 0 \cdot S_2 + 0 \cdot S_3 = 420$

$3x_1 + 2x_3 + 0 \cdot S_1 + 1 \cdot S_2 + 0 \cdot S_3 = 460$

$x_1 + 2x_2 + x_3 + 0 \cdot S_1 + 0 S_2 + 1 S_3 = 430$

and $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Since 3 equations with 6 variables

$6 - 3 = 3$ (Variables are zero)

$x_1 = 0, \ x_2 = 0, \ x_3 = 0$ (Non Basic Variable)

$S_1 = 420, \ S_2 = 460, \ S_3 = 430$ (Basic Variable)

c_j	3	2	5	0	0	0				
x_j	x_1	x_2	x_3	S_1	S_2	S_3	x_B	B	C_B	$\theta = \text{Min}\left(\frac{x_B}{PC}\right)$
	1	4	0	1	0	0	420	S_1	0	0
	3	0	2	0	1	0	460	S_2	0	230 ←PR
	1	2	1	0	0	1	430	S_3	0	430
Z_j	0	0	0	0	0	0				
$Z_j - C_j$	-3	-2	-5	0	0	0				

↑ PC

Since $Z_j - C_j < 0$

The current solution is not optimal

Most Negative $Z_2 - C_2 = -5$.

To find ratio

ratio $\theta = \text{Min}\left\{\frac{x_B}{PC}\right\}$

$= \text{Min}\left\{ \frac{420}{0}, \frac{460}{2}, \frac{430}{1} \right\}$

$= \text{Min}\{0, 230, 430\}$

C_j	3	2	5	0	0	0	X_B	B	C_B	$\theta = \text{Min}\left(\frac{X_B}{P_C}\right)$
x_j	x_1	x_2	x_3	S_1	S_2	S_3				
	1	4	0	1	0	0	420	S_1	0	105
	3/2	0	1	0	1/2	0	230	x_3	5	0
	−1/2	2	0	0	−1/2	1	200	S_3	0	100 ← PR
Z_j	15/2	0	5	0	5/2	0	1150			
$Z_j - C_j$	9/2	−2	0	0	5/2	0				
		↑ PC								

Since $Z_j - C_j < 0$

The current solution is not optimal.

Most Negative $Z_2 - C_2 = -2$.

ratio θ $\theta = \text{Min}\{420/4, 230/0, 200/2\} = \text{min}\{105, 0, 100\}$

C_j	3	2	5	0	0	0	X_B	B	C_B	$\theta = \text{Min}\left(\frac{X_B}{P_C}\right)$
x_j	x_1	x_2	x_3	S_1	S_2	S_3				
	2	0	0	1	1	−2	20	S_1	0	
	3/2	0	1	0	1/2	0	230	x_3	5	
	−1/4	1	0	0	−1/4	1/2	100	x_2	2	
Z_j	−1	2	5	0	2	1				
$Z_j - C_j$	4	0	0	0	2	1				

All $z_j - c_j \geq 0$

The current Basic feasible solution is optimal

$\therefore$ The optimal solution is Max $z = 1350$

Since, $x_1 = 0$, $x_2 = 100$, $x_3 = 230$

Max $z = 3 \times 0 + 2 \times 100 + 5 \times 230$

$= 1350$.

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(4) Solve the following LPP by simplex method

Minimize $z = x_2 - 3x_3 + 2x_5$.

Subject to

$3x_2 - x_3 + 2x_5 \leq 7$

$-2x_2 + 4x_3 \leq 12$

$-4x_2 + 3x_3 + 8x_5 \leq 10$.

Soln :-

Since the given objective function is of Maximization type. we shall convert it into a Maximization type as follow .

Maximize $(-z) = -x_2 + 3x_3 - 2x_5$

$z = -x_2 + 3x_3 - 2x_5 + 0s_1 + 0s_2 + 0s_3$

Subject to,

$3x_2 - x_3 + 2x_5 + 1 \cdot s_1 + 0 \cdot s_2 + 0s_3 = 17$

$-2x_2 + 4x_3 + 0s_1 + 1 \cdot s_2 + 0 \cdot s_3 = 12$

$-4x_2 + 3x_3 + 8x_5 + 0s_1 + 0s_2 + 1 \cdot s_3 = 10$

and $x_2, x_3, x_5, s_1, s_2, s_3 \leq 0$

Since 3 equations with 5 variables

$x_2 = 0$, $x_3 = 0$, $x_5 = 0$ (non basic Variables)

$s_1 = 17$, $s_2 = 12$, $s_3 = 10$ (basic Variable).

c_j	-1	3	-2	0	0	0	X_B	B	C_B	$\theta = \min\left(\frac{X_B}{PC}\right)$	
x_j	x_2	x_3	x_5	S_1	S_2	S_3					
	3	-1	2	1	0	0	17	S_1	0	-7	
	-2	4	0	0	1	0	12	S_2	0	3	PR
	-4	3	8	0	0	1	10	S_3	0	$10/3$	
Z_j	0	0	0	0	0	0					
$Z_j - C_j$	1	-3	2	0	0	0					
		↑PC									

Since $Z_j - C_j \leq 0$

The current solution is not optimal Most negative $Z_2 - C_2 = -3$

To find ratio $\theta = \min\left(\frac{X_B}{PC}\right)$

$= \min\{-7/1, +12/4, 10/3\}$

$= \min\{-7, 3, 10/3\}$.

c_j	-1	3	-2	0	0	0	X_B	B	C_B	$\theta = \min\left(\frac{X_B}{PC}\right)$	
X_j	x_2	x_3	x_5	S_1	S_2	S_3					
	$5/2$	0	2	1	$1/4$	0	10	S_1	0	4	PR
	$-1/2$	1	0	0	$1/4$	0	3	x_3	3	-6	
	$-5/2$	0	8	0	$-3/4$	1	1	S_3	0	$-2/5$	
Z_j	$-3/2$	3	0	0	$3/4$	0	9				
$Z_j - C_j$	$-1/2$	0	2	0	$3/4$	0	9				
	PC										

Since $z_j - c_j < 0$

The current solution is not optimal

Most negative $z_1 - c_1 = -1/2$

To find ratio

$$\theta = \min\{4, -6, -2/5\}.$$

c_j	-1	3	-2	0	0	0	x_B	B	C_B	$\theta = \min\left(\frac{x_B}{P_r}\right)$
x_j	x_2	x_3	x_5	s_1	s_2	s_3				
	1	0	4/5	2/5	1/10	0	4	x_2	-1	
	0	1	2/5	1/5	1/5	0	5	x_3	3	
	0	0	10	1	-1/2	1	-10	s_1	0	
z_j	-1	3	2/5	1/5	1/2	0	11			
$z_j - c_j$	0	0	12/5	1/5	1/2	0				

Since all $z_j - c_j \geq 0$

The current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

Maximize $z^* = 11$,

Since, $x_2 = 4$, $x_3 = 5$, $x_5 = 0$.

Maximize $z^* = -x_2 + 3x_3 - 2x_5$

$= -4 + 3(15) - 2 \times 0$

$= 11$.

$\therefore$ Minimize $= -(\text{Maximize } z^*) = -11$

$z = -11$, $x_2 = 4$, $x_3 = 5$, $x_5 = 0$.

UNIT - II

Simplex Method for $\geq$ Constraints :-

(1) Solve the following LPP by simplex method

Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$; $3x_1 + 4x_2 \geq 12$ & $x_1, x_2 \geq 0$.

Soln :-

By introducing the Non - negative slack Variable s_1 and surplus variable s_2

The standard form of the LPP

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to,

$2x_1 + x_2 + 1 \cdot s_1 + 0 \cdot s_2 = 2$

$3x_1 + 4x_2 + 0s_1 + 1 \cdot s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$.

Add the artificial variable R_1 to the LHS of the constraint.

which does not possess the slack variable and assign $-M$ to the artificial variable in the objective function.

Max $z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MR_1$

subject to

$2x_1 + x_2 + s_1 + 0s_2 = 2$

$3x_1 + 4x_2 + 0s_1 - s_2 + R_1 = 12$

$x_1, x_2, s_1, s_2, R_1 \geq 0$

The IBFS is given by

$x_1 = x_2 = s_2 = 0$ (Non Basic)

$s_1 = 2$, $R_1 = 12$ (Basic)

C_j	3	2	0	0	$-M$	x_B	B	C_B	θ
x_j	x_1	x_2	s_1	s_2	R_1				
	2	1	1	0	0	2	s_1	0	$2/1 = 2$ ← PR
	3	4	0	-1	1	12	R_1	$-M$	$12/4 = 3$
Z_j	$-3M$	$-4M$	0	M	$-M$				
$Z_j - C_j$	$-3m-3$	$-4M-2$	0	M	0				

↑ pc

Since there are $z_j - c_j < 0$

The Current BFS is Not optimal.

C_j	3	2	0	0	$-M$	x_B	B	C_B	θ
x_j	x_1	x_2	s_1	s_2	R_1				
	2	1	1	0	0	2	x_2	2	
	-5	0	-4	-1	1	4	R_1	$-M$	
Z_j	$4+5M$	2	$2+4m$	M	$-M$	$4-4m$			
$Z_j - C_j$	$5m+1$	0	$4m+2$	M	0	$-4m+4$			

for all $z_j - c_j \geq 0$ and an Artificial Variable R_1 appear in the basis at non zero level. The given LPP does not possess any FS. But the LPP possess a pseudo optimal solution.

Big-M Method [Method of penalty]

In the Big-M Method huge negative cost $(-M)$ is assigned to the artificial value, while zero cost to the slack and surplus variable in the objective function. The usual

Simplex Method is then followed. At any iteration there can arise any one of the following two cases.

Case i :-

There is atleast one vector corresponding to some artificial variable in the basis at zero level and all the net evaluation $z_j - c_j \geq 0$. In this case the given LPP doesnot possess an optimum basic feasible solution.

Case ii :-

There is atleast one vector corresponding to some artificial variable in the basis not at zero level and all the net evaluation $z_j - c_j \geq 0$. In this case the given LPP does not posses an optimum basic feasible solution.

Whenever the artificial variable leaves from the basis we omit all the entries corresponding to that variable from the simplex method table in the next iteration. A draw back of this method is the possible computational error that could result from assigning a large value to M.

Problem :-

① Use Big-M method to solve minimize

$z = 4x_1 + 3x_2$

Subject to, $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$ and $x_1, x_2 \geq 0$.

Soln :-

That is Max $z = -4x_1 - 3x_2$

Subject to,

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

By introducing the non negative slack, surplus and artificial variables the standard form of LPP becomes,

Max $z = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MR_1 - MR_2$

Subject to,

$2x_1 + x_2 - S_1 + 0 \cdot S_2 + 0 \cdot S_3 + R_1 = 10$

$-3x_1 + 2x_2 + 0S_1 + S_2 + 0 \cdot S_3 = 6$

$x_1 + x_2 + 0S_1 + 0S_2 - S_3 + R_2 = 6$

and $x_1, x_2, x_3, S_1, S_2, S_3, R_1, R_2 \geq 0$

The IBFS is given by

$R_1 = 10$, $S_2 = 6$, $R_3 = 6$ (Basic Variable)

$x_1 = x_2 = S_1 = S_3 = 0$ (Non Basic Variable).

C_j	-4	-3	0	0	0	-M	-M	X_B	B	C_B	θ
x_j	x_1	x_2	S_1	S_2	S_3	R_1	R_2				
	2	1	-1	0	0	1	0	10	R_1	-M	5 ←PR
	-3	2	0	1	0	0	0	6	S_2	0	-2
	1	1	0	0	-1	0	1	6	R_2	-M	6
Z_j	-3m	-2m	M	0	M	-M	-m				
$Z_j - C_j$	-3m+4	-2m+3	M	0	M	0	0				

↑PC

Since, $Z_j - C_j < 0$.

The current basic feasible solution is not optimal.

C_j	-4	-3	0	0	0	$-M$	X_B	B	C_B	θ
x_j	x_1	x_2	S_1	S_2	S_3	R_2				
	1	$1/2$	$-1/2$	0	0	0	5	x_1	-4	$5/2 = 2.5$
	0	$7/2$	$-3/2$	1	0	0	21	S_2	0	$21/7 = 3$
	0	$1/2$	$1/2$	0	-1	1	1	R_2	$-M$	2 ← PR
Z_j	-4	$-2-M/2$	$2-M/2$	0	M	$-M$	$-20+4M$			
$Z_j - C_j$	0	$-\frac{M}{2}+1$	$-\frac{M}{2}+2$	0	M	0	$4M-20$			
		↑PC								

All $Z_j - C_j < 0$, the current BFS is not optimal.

C_j	-4	-3	0	0	0	X_B	B	C_B	θ
x_j	x_1	x_2	S_1	S_2	S_3				
	1	0	-1	0	1	4	x_1	-4	
	0	0	-5	1	7	14	S_2	0	
	0	1	1	0	-2	2	x_2	-3	
Z_j	-4	-3	$4+3$	0	$-4+6$	-22			
$Z_j - C_j$	0	0	7	0	2	-22			

Since all $Z_j - C_j \geq 0$

The current BFS is optimal

$$\begin{aligned} \text{But } \operatorname{Min} Z &= -\operatorname{Max}(-Z) \\ &= -\operatorname{Max} Z \\ &= -(-22) \\ &= 22. \end{aligned}$$

$\therefore$ The current solution is

$\min z = 22, x_1 = 4, x_2 = 2$.

Two phase - simplex method :-

Two phase method consist of two different phase.

phase i :-

In phase (i), we consider the objective function $\min z^* =$ sum of the artificial variables and cost of the artificial variable are taken to be -1 in the objective function. Here the cost of the given variables, slack variables and surplus variables are taken to be zero. Solve the LPP as usual simplex method.

If $\min z^* = 0$ then we proceed to phase (ii)

phase ii :-

Final table of phase (i) is initial table of phase (ii) but we consider the objective function (i.e) replace the cost by original cost.

Problem :-

(1) Use two phase simplex method to solve

Maximize $z = 5x_1 + 8x_2$

Subject to constraints,

$$3x_1 + 2x_2 \geq 3$$
$$x_1 + 4x_2 \geq 4$$
$$x_1 + x_2 \leq 5.$$
$$x_1, x_2 \geq 0.$$

Soln :-

By introducing the non negative slack and surplus and Artificial variable

The standard form of LPP becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0S_1 + 0S_2 + 0S_3$$

Subject to,

$$3x_1 + 2x_2 - S_1 + 0S_2 + 0S_3 + R_1 = 3$$

$$x_1 + 4x_2 + 0S_1 - S_2 + 0S_3 + R_2 = 4$$

$$x_1 + x_2 + 0S_1 + 0S_2 + S_3 = 5$$

$$x_1, x_2, S_1, S_2, S_3, R_1, R_2 \geq 0$$

S_1, S_2 are Surplus Variable ; S_3 are slack Variable and R_1, R_2 are Artificial Variable.

The IBFS is given by,

$R_1 = 3$, $R_2 = 4$, $S_3 = 5$ (basic Variable)

$x_1 = x_2 = S_1 = S_2 = 0$ (Non basic Variable).

Phase - I :-

Assigning a cost -1 to the Artificial Variable and cost 0 to the all other Variables the objective function of the auxillary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

Subject to,

$$3x_1 + 2x_2 - S_1 + 0S_2 + 0S_3 + R_1 = 3$$

$$x_1 + 4x_2 + 0S_1 - S_2 + 0S_3 + R_2 = 4$$

$$x_1 + x_2 + 0S_1 + 0S_2 + S_3 = 5$$

$$x_1, x_2, S_1, S_2, S_3, R_1, R_2 \geq 0.$$

c_j	0	0	0	0	0	-1	-1				
x_j	x_1	x_2	S_1	S_2	S_3	R_1	R_2	x_B	B	C_B	θ
	3	2	-1	0	0	1	0	3	R_1	-1	$3/2 = 1.5$
	1	4	0	-1	0	0	1	4	R_2	-1	$4/4 = 1$ ← PR
	1	1	0	0	1	0	0	5	S_3	0	$5/1 = 5$
Z_j	-4	-6	1	1	0	-1	-1				
$Z_j - C_j$	-4	-6	1	1	0	0	0				
		↑ PC									

Since $Z_j - C_j < 0$, the current basic feasible solution is not optimal.

c_j	0	0	0	0	0	-1	-1				
x_j	x_1	x_2	S_1	S_2	S_3	R_1	R_2	x_B	B	C_B	θ
	$5/2$	0	-1	$1/2$	0	1	$-1/2$	1	R_1	-1	$2/5$ ← PR
	$1/4$	1	0	$-1/4$	0	0	$1/4$	1	x_2	0	4
	$3/4$	0	0	$1/4$	1	0	$-1/4$	4	S_3	0	$16/3$
Z_j	$-5/2$	0	1	$-1/2$	0	-1	$1/2$	-1			
$Z_j - C_j$	$-5/2$	0	1	$-1/2$	0	0	$3/2$	-1			

Since $Z_j - C_j < 0$

the current Basic feasible solution is not optimal.

C_j	0	0	0	0	0	-1	-1	X_B	B	C_B	θ
x_j	x_1	x_2	S_1	S_2	S_3	R_1	R_2				
	1	0	-2/5	1/5	0	2/5	-1/5	2/5	x_1	0	
	0	1	1/10	-3/10	0	-1/10	3/10	9/10	x_2	0	
	0	0	3/10	1/10	1	-3/10	-1/10	27/10	S_3	0	
z_j	0	0	0	0	0	0	0				
$z_j - c_j$	0	0	0	0	0	1	1				

Since all $z_j - c_j \geq 0$

The current BFS is optimal.

We proceed to phase - II

Phase - II :-

We consider the actual cost associated with the original variables. The new objective functions then become

$$\text{Max } z = 5x_1 + 8x_2 + 0S_1 + 0S_2 + 0S_3$$

The IBFS for this phase is the one obtained at the end of phase I.

C_j	5	8	0	0	0	X_B	B	C_B	θ
x_j	x_1	x_2	S_1	S_2	S_3				
	1	0	-2/5	1/5	0	2/5	x_1	5	2 ← PR
	0	1	1/10	-3/10	0	9/10	x_2	8	-3
	0	0	3/10	1/10	1	37/10	S_3	0	37
z_j	5	8	-6/5	-7/5	0	46/5			
$z_j - c_j$	0	0	-6/5	-7/5	0	46/5			
				↑ PC					

c_j	5	8	0	0	0	x_B	B	C_B	θ
x_j	x_1	x_2	s_1	s_2	s_3				
	5	0	-2	1	0	2	s_2	0	-
	3/2	1	-1/2	0	0	3/2	x_2	8	-
	-1/2	0	1/2	0	1	7/2	s_3	0	7 ← PR
z_j	12	8	-4	0	0	12			
$z_j - c_j$	7	0	-4	0	0	12			
			↑ PC						

$z_j - c_j < 0$.

The current BFS is not optimal.

c_j	5	8	0	0	0	x_B	B	C_B	θ
x_j	x_1	x_2	s_1	s_2	s_3				
	3	0	0	1	4	16	s_2	0	
	1	1	0	0	1	5	x_2	8	
	-1	0	1	0	2	7	s_1	0	
z_j	8	8	0	0	8	40			
$z_j - c_j$	3	0	0	0	8	40			

Since all $z_j - c_j \geq 0$

The current BFS is optimal

$\therefore$ Max $z = 40$

$x_1 = 0$

$x_2 = 5$.

Duality :-

Every Lpp has associated with another Lpp is called the dual of the problem. The given problem is called primal problem. If the optimal solution of one problem is known then the optimal solution of other is known.

Formulation of Dual Lpp :

Consider the following Lpp in canonical form

$$\text{Max } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Subject to,

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \;;\; x_i \geq 0, \forall i$$

The above problem is called the primal problem and the variables are called primal variables and the constraints are called primal constraints.

The dual of the given primal problem is defined as,

$$\text{Min } z^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

Subject to,

$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \geq c_2$$

$$\vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n, \; y_i \geq 0, \forall i$$

Here the variables are called dual variables and the constraints are called dual constraints.

(1) Find the dual of the LPP

$Max\ z = 3x_1 - x_2 + x_3$

Subject to $4x_1 - x_2 \le 8$; $8x_1 + x_2 + 3x_3 \ge 12$;

$5x_1 - 6x_3 \le 13$; $x_1, x_2, x_3 \ge 0$

Soln :-

The dual is

$Min\ z^* = 8y_1 - 12y_2 + 13y_3$

Subject to $4y_1 + 8y_2 + 5y_3 \ge 3$; $y_1 + y_2 \le 1$;

$-3y_2 - 6y_3 \ge 1$ and $y_1, y_2, y_3 \ge 0$.

(2) Construct the dual of the LPP

$Min\ z = 4x_1 + 6x_2 + 18x_3$

Subject to, $x_1 + 3x_2 \ge 3$; $x_2 + 2x_3 \ge 5$ & $x_1, x_2, x_3 \ge 0$.

soln :-

The dual is,

$Max\ z^* = 3y_1 + 5y_2$

Subject to, $y_1 \le 4$; $3y_1 + y_2 \le 6$; $2y_2 \le 18$; &

$y_1, y_2 \ge 0$

(3) Write the dual of the LPP

$Min\ z = x_2 + 3x_3$

Subject to, $2x_1 + x_2 \le 3$; $x_1 + 2x_2 + 6x_3 \ge 5$;

$-x_1 + x_2 + x_3 = 2$ and $x_1, x_2, x_3 \ge 0$.

Soln :-

The dual LPP is

$Max\ z^* = -3y_1 + 5y_2 + 2y_3$

Subject to, $-2y_1 + y_2 - y_3 \le 0$; $-y_1 + 2y_2 + y_3 \le 1$

$6y_2 + y_3 \le 2$ and $y_1, y_2 \ge 0$, y_3 is

Unrestricted

(4) write the dual of the following primal LPP.

$\text{Min } z = 4x_1 + 5x_2 - 3x_3$

Subject to, $x_1 + x_2 + x_3 = 22$; $3x_1 + 5x_2 - 2x_3 \leq 65$;

$x_1 + 7x_2 + 4x_3 \geq 120$; $x_1 \geq 0$, $x_2 \geq 0$ & x_3 unrestricted.

Soln :-

$\text{Max } z^* = 22y_1 - 65y_2 + 120y_3$

Subject to,

$y_1 - 3y_2 + y_3 \leq 4$; $y_1 - 5y_2 + 7y_3 \leq 5$;

$y_1 + 2y_2 + 4y_3 = -3$; and $y_2, y_3 \geq 0$, y_1 is unrestricted.

---

(5) write the dual of the primal

$\text{Max } z = 6x_1 + 6x_2 + x_3 + 7x_4 + 5x_5$

Subject to, $3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$;

$2x_1 + x_2 + 3x_4 + 9x_5 = 6$

$x_1, x_2, x_3, x_4 \geq 0$ and x_5 is unrestricted.

Soln :-

The dual LPP is

$\text{Min } z^* = 2y_1 + 6y_2$

Subject to, $3y_1 + 2y_2 \geq 6$; $7y_1 + y_2 \geq 6$

$8y_1 \geq 1$; $5y_1 + 3y_2 \geq 7$; $y_1 + 9y_2 = 5$

and y_1, y_2 are unrestricted.

Dual Simplex Method :-

The dual simplex method is moreover same as the standard simplex method. Here we do not require any artificial variable. Hence a lot of computational work is minimized in this method. The procedure is as follow.

Step 1 : convert the minimization LPP into a maximization LPP.

Step 2 : Convert all $\geq$ inequalities into $\leq$ inequalities by multiplying the corresponding equation by -1.

Step 3 : Introduce any slack variable and obtain the starting equation.

Step 4 : After finding out the initial basic feasible solution construct the starting simplex table. In this table construct the net evaluation $z_j - c_j$ and test the nature of $z_j - c_j$. Here the following subcases arise.

Case i :- If all $z_j - c_j \geq 0$ and the x_{Bi} column vectors are ≥ 0 optimum basic feasible soln. is obtained.

Case ii :- All $z_j - c_j \geq 0$ & one of $x_{Bi} < 0$ we go the next step.

Case iii :- If one $z_j - c_j < 0$ then the dual simplex method fails.

Step 5 : Determining the leaving variable by selecting the most negative x_{Bi}. Then calculate the replacement ratio Max$\left(\frac{z_j - c_j}{y_{kj}}, y_{kj} < 0\right)$ and determine the entering variable.

Step 6 : The leaving row and entering column intersect at a point and that element is called pivot element.

Step 7 :- Now test for optimality and repeat the process until the optimum basic feasible solution is obtained or there is an indication of no feasible solution.

1. Using dual simplex method solve the LPP

$$\text{Min } z = 2x_1 + x_2$$

Subject to $3x_1 + x_2 \geq 3$; $4x_1 + 3x_2 \geq 6$;

$x_1 + 2x_2 \geq 3$ and $x_1, x_2 \geq 0$.

Soln :-

$$\text{Max } z^* = -2x_1 - x_2$$

Subject to $-3x_1 - x_2 \leq 3$; $-4x_1 - 3x_2 \leq -6$;

$-x_1 - 2x_2 \leq -3$; and $x_1, x_2 \geq 0$.

By introducing the non-negative slack variable S_1, S_2 and S_3, the LPP becomes.

$$\text{Max } z^* = -2x_1 - x_2 + 0S_1 + 0S_2 + 0S_3$$

Subject to $-3x_1 - x_2 + S_1 = -3$

$-4x_1 - 3x_2 + S_2 = -6$

$-x_1 - 2x_2 + S_3 = -3$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

The initial basic solution is given by,

$S_1 = -3$, $S_2 = -6$, $S_3 = -3$ (basic variable)

$x_1 = x_2 = 0$ (Non basic variable).

		C_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-3	-3	-1	1	0	0
0	S_2	-6	-4	(-3)	0	1	0
0	S_3	-3	-1	-2	0	0	1
$Z_j^*-C_j$		0	2	1	0	0	0

Since all $(z_j^* - c_j) \geq 0$, the current solution is not an optimum basic feasible solution

		c_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-1	$(-5/3)$	0	1	$(-1/3)$	0
-1	x_2	2	$4/3$	1	0	$(-1/3)$	0
0	S_3	1	$5/3$	0	0	$(-2/3)$	1
$(Z_j^* - c_j)$		-2	$2/3$	0	0	$1/3$	0

all $z_j^* - c_j \geq 0$, the current solution is not optimum basic feasible solution.

		c_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
-2	x_1	$3/5$	1	0	$-3/5$	$1/5$	0
-1	x_2	$6/5$	0	1	$4/5$	$-3/5$	0
0	S_3	0	0	0	1	-1	1
$(Z_j^* - c_j)$		$-12/5$	0	0	$2/5$	$1/5$	0

The optimum soln/- is $\text{Max } z^* = -12/5$

$x_1 = 3/5, \quad x_2 = 6/5$

But $\text{Min } z = -\text{Max } z^* = -(-12/5) = 12/5$

$\therefore \text{Min } z = 12/5 \ ; \ x_1 = 3/5 \ ; \ x_2 = 6/5$

Transportation Problem :-

The Transportation problem deals with a special class of linear programming problem in which the objective is to transport a product manufactured at several plants to a number of different destination at a minimum cost. our objective is to determine the total minimum shipping cost.

Mathematical Formulation of a Transportation Problem:

Let $a_i \rightarrow$ supply at source i, $b_j \rightarrow$ demand at destination j. $c_{ij} \rightarrow$ unit transportation cost and $x_{ij} \rightarrow$ number of units shifted from source i to destination j.

$$\text{Minimize } z = \sum_{i=1}^{M} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{m} x_{ij} = a_i \quad \text{and} \quad \sum_{i=1}^{n} x_{ij} = b_j$$

$$(i = 1, 2 \ldots m) \qquad (j = 1, 2 \ldots n)$$

and $x_{ij} \geq 0$, for all i & j.

Basic Feasible Solution :-

A solution of $m \times n$ transportation problem is said to be basic feasible solution if the total number of allocations is equal to $m+n-1$.

Optimal solution :-

A Basic feasible solution is said to be optimal when the total transportation cost is minimum.

Methods For Finding Basic Feasible solution :-

(1) North-west Corner Method (NWCM)

(i) First we check the problem is balance (or) unbalance. ~~If~~ If supply equal demands then the problem is balance otherwise unbalance.

(ii) Select the top left corner cell (box) of the transportation problem and allocate as many as unit as possible equal to the minimum between available supply and demand.

(iii) Adjust the supply and demand number in the respective rows and columns.

(iv) If the demand for the first cell is satisfied then move horizontally to the next cell in the second column.

(v) If the supply for the first cell is satisfied then move down in the second row.

(vi) If demand is satisfied then delete (x) the column.

(vii) If supply is satisfied the delete (x) the Row.

(viii) continue the process. untill all supply & demand are satisfied.

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① Determine Basic Feasible Solution to the following transportation problem using North west corner Rule (NWCR)

Sink

		A	B	C	D	E	Supply
Origin	P	2	11	10	3	7	4
	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6	

Soln :-

Since $a_i = b_j = 21$, the given Problem is Balanced.

∴ There exist a Feasible solution to the transportation problem.

2 [3]	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

11 L1	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

4 L2	7	2	1	8
9	4	8	12	9
2	4	5	6	

7 L4	2	1	6
4	8	12	9
4	5	6	

2 L2	1	2
8	12	9
5	6	

8 L2	12	9
3	6	

12 L6	6
6	

Finally the initial basic feasible solution is as shown in the following Table.

2 L3	11 L1	10	3	7
1	4 L2	7 L4	2 L2	1
3	9	4	8 L3	12 L6

$m + n - 1 = 3 + 5 - 1 = 7$

This ensures that the solution is non degenerate basic Feasible.

$\therefore$ The Initial Transportation cost $\Big\} = Rs.\ 2 \times 3 + 11 \times 1 + 4 \times 2 + 7 \times 4 + 2 \times 2 + 8 \times 3 + 12 \times 6$

$= Rs.\ 153.$

Least Cost Method :-

(i) First of all we check the problem is balance or Unbalanced.

(ii) Identify the box having Minimum Unit transportation cost (c_{ij}).

(iii) If the Minimum cost is not Unique then choose the top left corner box for allocation.

(iv) choose the value of the corresponding x_{ij} as Much as possible subject to the supply and demand constraints.

(v) Repeat the above steps Until all restrictions are defined.

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(1) Find the initial basic feasible solution for the following TP by LCM.

From

				Supply
1	2	1	4	30
3	3	2	1	50
4	2	5	9	20
Demand 20	40	30	10	

Soln :-

$\Sigma a_i = \Sigma b_j = 10$. The given TPP is balanced

$\therefore$ A feasible solution to Transportation problem (TP)

2	1 ⌊10	4	10
3	2	1	50
2	5	9	20
40	30	10	

Step (i)

3	2	1 ⌊10	50
2	5	9	20
40	20	10	

Step (ii)

3	$2^{\lfloor 20}$	20
2	5	20
40	20	

Step (iii)

$3^{\lfloor 20}$	20
20	20
40	

Step (iv)

$20^{\lfloor 20}$	20
20	

Step (v)

Finally the initial basic feasible solution is as shown in the following Table

$1^{\lfloor 20}$	2	$1^{\lfloor 10}$	4
3	$3^{\lfloor 20}$	$2^{\lfloor 20}$	$1^{\lfloor 10}$
4	$2^{\lfloor 20}$	5	9

$$m + n - 1 = 3 + 4 - 1 = 6$$

This ensure that the solution is non degenerate basic Feasible.

$\therefore$ The initial Transportation cost $\Big\} = Rs.\ 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$

$= Rs.\ 20 + 10 + 60 + 40 + 10 + 40$

$= Rs.\ 180.$

Vogel's Approximation Method :-

(i) First of all, we check our problem is balance (or) unbalanced.

(ii) Identify the boxes having minimum and next to minimum transportation cost in each

(ii) Identify the row (or) column with the largest difference. let the greatest difference correspond to P_i and c_{ij} be the smallest cost in the i^{th} row.

(iii) Allocate $x_{ij} = \min(a_i, b_j)$ in that cell. If $x_{ij} = a_i$ cross off i^{th} row of the transportation table and decrease b_j by a_i.

(iv) If $x_{ij} = b_j$ cross of j^{th} column of the transportation table and decrease a_i by b_j.
If $x_{ij} = a_i = b_j$, cross of either the first row or first column (but not both).

(v) Repeat step 2, 3 and 4 for the resulting Table. Reduce transportation table untill all requirements are satisfied.

① Find the initial basic Feasible solution for the following transportation problem by VAM.

Distribution Centres

	D_1	D_2	D_3	D_4	Availability
S_1	11	13	17	14	250
Origin S_2	16	18	14	10	300
S_3	21	24	13	10	400
Requirements	200	225	275	250	

Soln :-

$\Sigma a_i = \Sigma b_j = 950$ the given problem is balanced.

∴ There exist a feasible solution to the Problem.

11 [200]	13	17	14	250	(2)
16	18	14	10	300	(4)
21	24	13	10	400	(3)
200	225	275	250		
(5)	(5)	(1)	(0)		

Step (i)

~~13~~ [50]	~~17~~	~~14~~	50	(1)
18	14	10	300	(4)
24	13	10	400	(3)
225	275	250		
(5)	(1)	(0)		

18 [175]	14	10	300	(4)
24	13	10	400	(3)
175	275	250		
(6)	(1)	(0)		

Step (ii)

14	10 ∟125	125	(4)
13	10	400	(3)
275	250		
(1)	(0)		

Step (iv)

13	10 ∟125	400
275	125	

Step (v)

13 ∟275	275
275	

Step (vi)

Finally the initial basic Feasible solution is as shown in the following table.

11 ∟200	13 ∟50	17	14
16	18 ∟175	14	10 ∟125
21	24	13 ∟275	10 ∟125

$$m + n - 1 = 3 + 4 - 1 = 6$$

This ensure that the solution is non degenerate basic feasible.

∴ The initial Transportation cost $\Big\}$ = Rs. $17 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$.

= Rs. 12,075/-.

Transportation problem (or) MODI Method :-

(1) Solve the transportation problem

	1	2	3	4	Supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
Demand	6	10	12	15	

Soln :-

Since $\Sigma a_i = \Sigma b_j = 43$, The given TP is balanced

$\therefore$ There exist a basic feasible solution to this problem.

By vogel's approximation method, the initial solution is

21	16	25	13 (11)
17 (6)	18 (3)	14	23 (4)
32	27 (7)	18 (12)	41

$\therefore m+n-1 = 3+4-1 = 6$

Hence the solution is non degenerate basic feasible.

$\therefore$ The initial transportation cost $\Big\}$ = Rs. $13\times11 + 17\times6 + 18\times3 + 23\times4 + 27\times7 + 18\times12$

= Rs. 796.

<u>To find the optimal solution :-</u>

We apply MODI method.,

We determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij} = u_i + v_j$. 2nd row contain maximum number of allocations, we choose $u_2 = 0$.

$\therefore c_{21} = u_2 + v_1 = 17 \Rightarrow 0 + v_1 \Rightarrow v_1 = 17$

$c_{22} = u_2 + v_2 = 18 \Rightarrow 0 + v_2 \Rightarrow v_2 = 18$

$c_{24} = u_2 + v_4 \Rightarrow 23 \Rightarrow 0 + v_4 \Rightarrow v_4 = 23$

$c_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = -10$

$c_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 9$

$c_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$

21	16	25	13 (11)	$u_1 = -10$
17 (6)	18 (3)	14	23 (4)	$u_2 = 0$
32	27 (7)	18 (12)	41	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

We find the cell evaluation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

We get the following table.

21 7 14	16 8 8	25 -1 26	13 11	$u_1 = -10$
17 6	18 3	14 9 5	23 4	$u_2 = 0$
32 26 6	27 7	18 12	41 32 9	$u_3 = 9$
$v_1 = 17$	$v_2 = 18$	$v_3 = 9$	$v_4 = 23$	

Since all $d_{ij} > 0$, the solution under the test is optimal and unique.

The optimum allocations schedule is given by

$\therefore$ The optimum allocations schedule is given by $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$

$$\text{optimum transportation cost} = Rs.\ 13 \times 11 + 17 \times 6 + 18 \times 3 + 23 \times 4 + 27 \times 7 + 18 \times 12$$

$$= Rs.\ 796.$$

Unbalanced Transportation problem

If the given Transportation problem is unbalanced one. (i.e.) $\Sigma a_i \neq \Sigma b_j$ then convert this into a balanced one by introducing a dummy source or dummy destination with zero cost vectors and then solve by usual method.

Assignment Problem.

Mathematical Formulation of AP :-

Let there be n Machines and n jobs
A job i when processed by Machine j at a cost c_{ij}
$x_{ij} = \begin{cases} 1, & \text{if the } i^{th} \text{ job is assigned to } j^{th} \text{ machine} \\ 0, & \text{otherwise} \end{cases}$

The Total assignment cost is $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Hence, the AP can be stated Mathematically as follows

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \; (i = 1, 2 .. n) \; ; \; \sum_{i=1}^{n} x_{ij} = b_j \; (j = 1, 2 .. n) \; ; \; \sum a_i = \sum b_j \text{ and}$$

$$x_{ij} \geq 0$$

Assignment Algorithm (or) Hungarian Method :-

Step 1 : Subtract the Minimum cost of each row of the cost Matrix from all the elements of the respective row and get the Resultant Matrix.

Step 2 : Subtract the Minimum cost of each Column of the cost Matrix from all the elements of the respective column and get the Resultant matrix. This gives a new cost Matrix.

Step 3 : In the new cost matrix draw minimum possible number of horizontal and vertical lines to cover all the zeros. Here two cases arises.

Case i :- If the number of lines = order of cost matrix optimum assignment has been obtained. Find out the optimum solution.

Case ii :- If the number of lines < order of cost Matrix, we go to step 4.

Step 4: Determine the smallest cost in the new cost matrix not covered by the lines. Subtract this cost from all the surviving elements of the new cost matrix and add the same to all those elements of the new cost matrix that are lying at the intersection of horizontal and vertical lines.

Step 5: Repeat steps 3 and 4 until we get $p = n$.

Step 6: For determining the optimum assignment of the final table. Consider only the zero elements of the matrix. Examine successively the rows (or) columns of the matrix. To find out one with exactly one zero and encircled this zero and mark a cross in the remaining zeros of the row or column.

Step 7: The assignment schedule corresponding to these zeros is the optimum assignment schedule and find out the minimum assignment.

---

(1) Consider the problem of assigning five jobs to five persons. The assignment cost are given as follows.

Jobs

$$\text{person}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix} \end{array}$$

Soln :-

The cost matrix of the given AP is

$$\begin{pmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{pmatrix} \qquad \begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

Step (i) Step (ii)

$$\begin{pmatrix} 7 & 8 & 0 & 6 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{pmatrix} \qquad \begin{pmatrix} 7 & 8 & 0 & 6 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & 0 & 3 \\ 4 & (0) & 2 & 4 & 0 \end{pmatrix}$$

Step (iii) Step (iv)

The optimum assignment schedule is given by

A→5, B→1, C→4, D→3, E→2

∴ The optimum (minimum) assignment cost $\Big\}$ = 1 + 0 + 2 + 1 + 5

= 9 Units of cost.

Unbalanced Assignment Models :-

If the number of rows is not equal to the number of columns in the cost matrix of the given assignment problem, then the given assignment problem is said to be unbalanced.

---

① A batch of 4 jobs can be assigned to 5 different machines. The set up time (in hours) for each job on various machines is given below.

Machine

job	1	2	3	4	5
1	10	11	4	2	8
2	7	11	10	14	12
3	5	6	9	12	14
4	13	15	11	10	7

Soln :-

No. of rows is less than the number of column in the cost matrix, the given AP is unbalanced.

Add dummy job with zero cost elements.

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step (i)

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Step (ii) Step (iii)

$$\begin{array}{c} \begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \checkmark \\ \checkmark \\ \\ \\ \end{array} \\ \checkmark \end{array} \qquad \begin{pmatrix} 9 & 9 & 2 & 0 & 6 \\ 0 & 3 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step (iv) Step (v)

$$\begin{pmatrix} 9 & 9 & 2 & (0) & 6 \\ (0) & 3 & 2 & 6 & 4 \\ \cancel{0} & (0) & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & (0) \\ 1 & \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Step (vi)

The optimum assignment schedule is given by job

job 1 → M/c 4, job 2 → M/c 1, job 3 → M/c 2,

job 4 → M/c 5 (or) M/c 3

∴ The optimum (Minimum) total set up time

$= 2 + 7 + 6 + 7$ hours

$= 22$ hours.

Maximization case in Assignment problem :-

Max $z = -\text{Min}(-z)$, Multiply all the cost elements c_{ij} of the cost Matrix by -1.

(1) Solve the assignment problem for Maximization given the profit Matrix (profit in Rupees)

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{Machines}} \\ & & P & Q & R & S \\ \hline & A & 51 & 53 & 54 & 60 \\ \text{Job} & B & 47 & 50 & 48 & 50 \\ & C & 49 & 50 & 60 & 61 \\ & D & 63 & 64 & 60 & 60 \end{array}$$

Soln :-

$$\begin{pmatrix} 51 & 53 & 54 & 60 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix} \qquad \begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Step (i) Step (ii)

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Step iii step iv

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & (0) & 2 & \cancel{0} \\ 11 & 11 & 1 & (0) \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

∴ The optimum assignment schedule is given by

$A \to R$, $B \to Q$, $C \to S$, $D \to P$

$$\left.\begin{array}{l}\text{The optimum (Maximum)} \\ \text{profit}\end{array}\right\} = \text{Rs } 54 + 50 + 61 + 63$$

$$= \text{Rs. } 228.$$