UNIT - 3

PROBABILITY

1) Probability :

"Probability is the science of decision-making with calculated risks in the face of uncertainty"

- Ya Lin chou

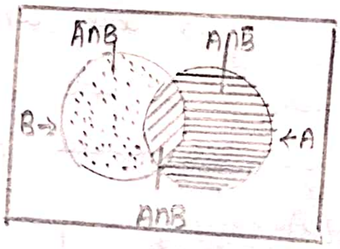
2) Addition Theorem of probability:

If A and B are any two events (subsets of sample space S) and are not disjoint, then

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Proof:

From the venn diagram, we have



$A\cup B = A\cup(\bar{A}\cap B)$,

where A and $\bar{A}\cap B$ are mutually disjoint

$\therefore P(A\cup B) = P[A\cup(\bar{A}\cap B)]$

$= P(A) + P(\bar{A}\cap B)$ [By Axiom 3]

$= P(A) + P(B) - P(A\cap B)$

OR from (*) onwards [From theorem 3.3 (ii)]

$P(A\cup B) = P(A) + [P(\bar{A}\cap B) + P(A\cap B)] - P(A\cap B)$

$= P(A) + P[(\bar{A}\cap B)\cup(A\cap B)] - P(A\cap B)$

[$\because$ $(\bar{A}\cap B)$ and $(A\cap B)$ are disjoint]

$\Rightarrow P(A\cup B) = P(A) + P(B) - P(A\cap B)$

3) Multiplication Theoram of probability

For two events A and B

$$\left.\begin{aligned} P(A\cap B) &= P(A) \cdot P(B/A), \quad P(A)>0 \\ &= P(B) \cdot P(A/B), \quad P(B)>0 \end{aligned}\right\}$$

where $P(B/A)$ represents conditional probability of occurence of B when the A event has already happened and $P(A/B)$ is conditional probability of happened of A, given that B has already happened.

Proof :

In the usual relations, we have

$$P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)} \quad \text{or} \quad P(A\cap B) = \frac{n(A\cap B)}{n(S)}$$

For the conditional event A/B, the favourable Outcomes must be one of the sample points of B i.e., for the event A/B, the sample space is B and out of the n(B) sample point, $n(A\cap B)$ per tain to the occurence of the event A. Hence

$$P(A/B) = \frac{n(A\cap B)}{n(B)}$$

Rewritting, we get

$$P(A\cap B) = \frac{n(B)}{n(S)} \times \frac{n(A\cap B)}{n(B)} = P(B) \cdot P(A/B)$$

similarly

$$P(A\cap B) = \frac{n(A)}{n(S)} \times \frac{n(A\cap B)}{n(A)} = P(A) \cdot P(B/A).$$

4) Continuous Distribution Function :

If x is a continuous random variable with the p.d.f. $f(x)$, then the function

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty \quad \ldots\ldots (5.12)$$

is called the distribution function (d.f) or sometimes the cummulative distribution function (c.d.f) of the random variables X.

5) Properties of Distribution function:

1. $0 \leq F(x) \leq 1$, $-\infty < x < \infty$
2. From analysis (Riemon integral), we know that

$$F'(x) = \frac{d}{dx} F(x) = f(x) \geq 0$$

$\Rightarrow F(x)$ is non-decreasing function of x.

3. $F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 1$

$$\Rightarrow 0 \leq F(x) \leq 1$$

4. $F(x)$ is a continuous function of x on the right.
5. The discontinuities of $F(x)$ are at the most countable.
6. It may be noted that

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= P(X \leq b) - P(X \leq a)$$

$$= F(b) - F(a) \quad \ldots\ldots (1)$$

similarly,

$$P(a < x < b) = P(a < x \leq b) = P(a \leq x < b) = \int_{a}^{b} f(t)\,dt \quad \ldots (2)$$

7) $F'(x) = \frac{d}{dx} F(x) = f(x) \Rightarrow dF(x) = f(x)\, dx.$

$dF(x)$ is known as probability differential of X.

6) Write Axioms of probability?

The axioms thus provide a set of rules which define relationships between abstract entities. These rule can be used to deduce between and the theorems can be brought together to deduce more complex theorams. These theorams have no emprical meaning althrough they can be given an interpretation in terms of emprical phenomenon. The important point, however, is that the mathematical development of probability theorem is, in no way, conditional upon the interpretation given to the theory.

Definition:

The set of all possible outcomes of a given random experiment is called the sample space associated with the experiment. Each possible outcome or element in a sample space is called a sample point or an elementary event.

Random Variable:

A random variable (r.v) is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the event

$\{\omega : X(\omega) \leq a\} \in B.$

Properties of Distributive Function :

If F is a d.f of the r.v x and of $a < b$, then $P(a < x \leq b) = F(b) - F(a)$.

Proof: The events '$a < x \leq b$' and '$x \leq a$' are disjoint and their union is the event '$x \leq b$'. Hence by addition theorem of probability :

$$P(a < x \leq b) + P(x \leq a) = P(x \leq b)$$

$$\Rightarrow P(a < x \leq b) = P(x \leq b) - P(x \leq a)$$

$$= F(b) - F(a)$$

If F is d.f of one dimensional r.v. X, then

(i) $0 \leq F(x) \leq 1$, (ii) $F(x) \leq F(y)$ if $x < y$

In other words, all distribution functions are monotomically non-decreasing and lie between 0 and 1.

If F is d.f of one-dimensional r.V x, then

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0 \text{ and } F(\infty) = \lim_{x \to \infty} F(x) = 1$$

Let the express which the sample space S as a countable values of written as follows.

$$P(S) = \left\{ \sum_{n=1}^{\infty} P(-n < x \leq x \leq -n+1) + \sum_{n=0}^{\infty} P(n < x \leq n-1) \right.$$

$$1 = \lim_{a \to \infty} \sum_{n=1}^{a} [F(-n+1) - F(-n) + \lim_{b \to \infty} \sum_{n=0}^{b} [(n+1) - F(n)]$$

$$= \lim_{a \to \infty} \{F(0) - F(-a)\} + \lim_{b \to \infty} [F(b+1) - F(0)\}$$

$$= \{F(0) - F(-\infty)\} + \{F(\infty) - F(0)\}$$

$$1 = F(\infty) - F(-\infty) \quad \text{———} (*)$$

Since $-\infty < \infty$, $F(-\infty) \leq F(\infty)$. Also $F(-\infty) \geq 0$ and $F(\infty) \leq 1$

$$\therefore 0 \leq F(-\infty) \leq F(\infty) \leq 1 \quad \text{———} (**)$$

From $(*)$ and $(**)$, we get $F(-\infty) = 0$ and $F(\infty) = 1$

A random variable x has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the constant 'a' (ii) compute $P(x<3)$, $P(x \geq 3)$

Soln

(i) $$a+3a+5a+7a+9a+11a+13a+15a+17a = 1$$

$$81a = 1$$

$$\boxed{a = \frac{1}{81}}$$

(ii) $$P(X<3) = P(X=0)+P(X=1)+P(X=2)$$

$$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81}$$

$$= \frac{9}{81}$$

$$P(X<3) = \frac{1}{9}$$

$$P(X \geq 3) = 1 - P(X<3)$$

$$= 1 - \frac{1}{9}$$

$$P(X \geq 3) = 8/9$$

X	$F_X(x) = P(X \leq x)$
0	$a = 1/81$
1	$3a = 3/81$
2	$5a = 5/81$
3	$7a = 7/81$
4	$9a = 9/81$
5	$11a = 11/81$
6	$13a = 13/81$
7	$15a = 15/81$
8	$17a = 17/81$

What is the chance that a leap year selected at random will contain 53 sundays?

Soln

Leap year = 366 days

= 52 weeks and 2 days

The following are the possible combinations for these two 'over' days :

i) {Sun, Mon} {Mon, Tue} {Tue, wed}, {Wed, Thu}

{Thu, Fri} {Fri, sat} {sat, sun}

Required probability = $\frac{n(S)}{n(A)}$

n(S) = {Sun, Mon} {sat, sun}

n(A) = {sun} {Mon} {Tue} {wed} {Thu} {Fri} {sat}

$P(S) = \frac{2}{7}$

Required probability $= \frac{2}{7}$

UNIT - 4

MATHEMATICAL EXPECTATION

Mathematical Expectation of Random

The probability distribution for a random variable, we often want to compute the mean or expected value of the random variable. The expected value of a random variable is a weights are the probabilities associated with the corresponding values. The mathematical expression for computing the expected value of a discrete random variable X with probability mass function (p.m.f) f(x) is given below:

$$E(X) = \sum_x x f(x), \text{ (for discrete r.v)} \quad \text{— (6.1)}$$

The Mathematical expression for computing the expected value of a continuous random variable X with probability density function (p.d.f) f(x) is, however, as follows:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx, \text{ (for continuous r.v.)} \quad \text{— (6.2a)}$$

Provided the right hand integral in (2) or in (1) is absolutely covergent i.e., provided

$$\int_{-\infty}^{\infty} |x f(x)|\, dx = \int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty \quad \text{— (6.2)}$$

$$\sum_x |x f(x)| = \sum_x |x|\, f(x) < \infty \quad \text{— (6.2a)}$$

Addition Theorem of Expectation :

If x and y are random variables, then $E(x+y) = E(x) + E(y)$, provided all the expectations exist.

Proof: Let x and y be continuous r.v's with joint p.d.f. $f_{xy}(x,y)$ and marginal p.d.f's $f_X(x)$ and $f_Y(y)$ respectively. Then by def.,

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad \text{——— (1)}$$

and

$$E(y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy \qquad \text{——— (2)}$$

$$E(x+y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{xy}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{xy}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{xy}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{xy}(x,y)\,dy\right] dx + \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\right] dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= E(x) + E(y)$$

The result can be extended by n variable is given below.

Multiplication Theorem of Expectation

If x and y are independent random variable, then $E(xy) = E(x) \cdot E(y)$.

Proof: proceeding as in property I, we have

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_x(x)\, f_y(y)\, dx\, dy$$

[since x and y are Independent]

$$= \int_{-\infty}^{\infty} x f_x(x)\, dx \int_{-\infty}^{\infty} y f_y(y)\, dy$$

[using Addition Theorem of expectation ① and ②]

$= E(x)\, E(y)$, provided x and y are independent

Properties of Expectation:

Property 1: Addition Theorem of Expectation

If x and y are random variable, then $E(x+y) = E(x) + E(y)$ provided all the expectation exists.

Proof: Let x and y be continuous r.v's with joint p.d.f $f_{xy}(x,y)$ and marginal p.d.f's $f_x(x)$ and $f_y(y)$ respectively. Then by defn,

$$E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx \quad \text{———} \quad (1)$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \quad \text{——} (2)$$

$$E(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\right] dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E(X) + E(Y)$$

The result can be extendend to n variables as given below.

Property 2: Multiplication Theorem of Expectation

If X and Y are independent random variable, then $E(XY) = E(X) \cdot E(Y)$.

Proof: proceeding as in property 1, we have

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

[since X and Y are Independent]

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$= E(X)\,E(Y)$, provided X and Y are Independent.

property 3:

If x is a random variable and 'a' is constant, then

(i) $E[a\psi(x)] = aE[\psi(x)]$ ——— (1)

(ii) $E(\psi(x)+a] = E[\psi(x)]+a.$ ——— (2)

where $\psi(x)$, a function of X, is a r.v and all expectations exist.

The last integral does not depend on t. If it tends to zero as $h\to 0$ then $\phi_x(t)$ is

Now $|e^{ihx}-1| \le |e^{ihx}| + 1 \le 1+1 = 2$

$$\therefore \int_{-\infty}^{\infty} |e^{ihx}-1|\, dF(x) \le 2\int_{-\infty}^{\infty} dF(x) = 2$$

Hence by Dominated covergence theorem (D.C.T), taking the limit inside the integral sign in (*), we get.

$$\lim_{h\to 0} |\phi_x(t+h) - \phi_x(t)| \le \int_{-\infty}^{\infty} \lim_{h\to 0} |e^{ihx}-1|\, dF(x) = 0$$

$$\Rightarrow \lim_{h\to 0} \phi_x(t+h) = \phi_x(t), \forall t.$$

Hence $\phi_x(t)$ is uniformly continuous in 't'.

Property 3: $\phi_x(-t)$ and $\phi_x(t)$ are conjugate function.

i.e., $\phi_x(-t) = \overline{\phi_x(t)}$ where it is the complete conjugate of a

Proof: $\phi_x(t) = E(e^{ax}) = E[(\cos tx + i\sin tx)]$

$$\therefore \overline{\phi_x(t)} = E[(\cos tx - i\sin tx)] = E[\cos(-t)x + i\sin(-t)x]$$
$$= E(e^{-ax}) = \phi_x(-t)$$

Property : 4

If the distribution function of a r.v. X is symmetrically about zero, i.e., if $1 - F(x) = F(-x) \Rightarrow f(-x) = f(x)$, ——— (1)

then $\phi_x(t)$ is real valued and even function of t.

Proof: By definition, we have

$$\therefore \phi_x(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \int_{-\infty}^{\infty} e^{-ity} f(-y)\,dy \qquad (x = -y)$$

$$= \int_{-\infty}^{\infty} e^{ity} f(y)\,dy \qquad [\because f(-y) = f(y)]$$

$$= \phi_x(t) \qquad \text{——— (2)}$$

$\Rightarrow \phi_x(t)$ is an even function of t.

From (1) and (2) we get : $\phi_x(t) = \phi_x(-t) = \overline{\phi_x(t)}$

Hence $\phi_x(t)$ is a real valued and even function of t.

Property : 5

If X is some r.v with characteristic function $\phi_x(t)$, and if $\mu_r' = E(x^r)$ exists,

then $$\mu_r' = (-i)^r \left[\frac{d^r}{dt^r}\phi(t)\right]_{t=0} \qquad \text{——— (1)}$$

Proof: $$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

Differentiating (under the Integral sign) 'r' times w.r.to t, we get

$$\frac{d^r}{dt^r}\phi(t) = \int_{-\infty}^{\infty} (ix)^r e^{itx} f(x)\,dx = (i)^r \int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx$$

$$\therefore \left[\frac{d^r}{dt^r}\phi(t)\right]_{t=0} = (i)^r \left|\int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx\right|_{t=0}$$

$$= (i)^r \int_{-\infty}^{\infty} x^r f(x)\,dx = i^r E(x^r) = i^r \mu_r'.$$

Hence, $$\mu_r' = \left(\frac{1}{i}\right)^r \left|\frac{d^r}{dt^r}\phi(t)\right|_{t=0} = (-i)^r \left|\frac{d^r}{dt^r}\phi(t)\right|_{t=0}$$

Property 6:

$\phi_{cx}(t) = \phi_x(ct)$, c being a constant.

Property 7:

If X_1 and X_2 are Independent random variables, from

$$\phi_{x_1+x_2}(t) = \phi_{x_1}(t)\, \phi_{x_2}(t) \quad \text{--- (1)}$$

More generally, for Independent random variables X_i, $i = 1, 2, \dots, n$, we have

$$\phi_{x_1+x_2+\dots+x_n}(t) = \phi_{x_1}(t)\, \phi_{x_2}(t) \dots \phi_{x_n}(t) \quad \text{--- (2)}$$

Conditional Expectation:

If X and Y are two random variables, then covariance between item is defined as

$$\begin{aligned} cov(X,Y) &= E[(X-E(X))][Y-E(Y)] \\ &= E[XY - XE(Y) - YE(X) + E(X)E(Y)] \\ &= E(XY) - E(Y)E(X) - E(X)E(Y) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

If X and Y are Independent then $E(XY) = E(X)E(Y)$ and hence in this case

$$cov(X,Y) = E(X)E(Y) - E(X)E(Y) = 0$$

Expected Value of a Random Variable :-

The Mathematical expression for computing the expected value of a discrete random variable X with probability mass function (p.m.f) f(x) is given below

$$E(X) = \sum_x x f(x), \text{ for discrete r.v)}$$

UNIT-5

BINOMIAL DISTRIBUTION

Binomial Distribution :

A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(X=x) = P(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & ; x=0,1,2,\dots,n \ ; q=1-p \\ 0 & ; \text{otherwise} \end{cases}$$

Moments of Binomial distribution :

The first four moments about origin of Binomial distribution are obtained as follows:

$$\mu_1' = E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$$= np(q+p)^{n-1} = np$$

$$\because \binom{n}{x} = \frac{n}{x}\binom{n-1}{x-1} = \frac{n}{x} \cdot \frac{n-1}{x-1} \cdot \binom{n-2}{x-2} = \frac{n}{x} \cdot \frac{n-1}{x-1} \cdot \frac{n-2}{x-2}\binom{n-3}{x-3},$$

and so on.

Thus the mean of the binomial distribution is np.

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \frac{n(n-1)}{x(x-1)} \cdot \binom{n-2}{x-2} p^x q^{n-x}$$

$$= n(n-1)p^2 \left\{ \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} \right\} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{n} x^3 p(x) = \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\} \binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x}$$

$$+ 3n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)(n-2)p^3(q+p)^{n-3} + 3n(n-1)p^2(q+p)^{n-2} + np$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

similarly

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

Let $x^4 = Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + x$

(By giving to x the values 1, 2 and 3, we find the values of arbitrary constants A, B and C)

$$\therefore \mu_4' = E(x^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

[On simplification].

Central Moment of Binomial distribution:

$$\mu_2 = \mu_2' - \mu_1'^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} - 3\{n(n-1)p^2 + np\}np + 2(np)^3$$

$$= np(-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq)$$

$$= np\{3np(1-p) + 2p^2 - 3p + 1 - 3npq\}$$

$$= np(2p^2 - 3p + 1) = np(2p^2 - 2p + q) = npq(1-2p)$$

$$= npq\{q + p - 2p\} = npq(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2' - 3\mu_1'^4 = npq\{1 + 3(n-2)pq\}$$

{On simplification}

Hence $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \dfrac{(q-p)^2}{npq} = \dfrac{(1-2p)^2}{npq}$

$\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{npq\{1+3(n-2)pq\}}{n^2p^2q^2} = \dfrac{1+3(n-2)pq}{npq}$

$= 3 + \dfrac{1-6pq}{npq}$

$\gamma_1 = \sqrt{\beta_1} = \dfrac{q-p}{\sqrt{npq}} = \dfrac{1-2p}{\sqrt{npq}}$.

$\gamma_2 = \beta_2 - 3 = \dfrac{1-6pq}{npq}$

Poisson Distribution :

A random variable X is said to be follow a poisson distribution if it assumes only one-negative values and its probability mass function is given by :

$$P(x,\lambda) = P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & ; x = 0, 1, 2, \ldots ; \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Here λ is know as the parameter of the distribution if it assumes, We shall use the notation $X \sim P(\lambda)$ to denote that X is a poisson variate with parameter λ

Moments of poisson Distribution :

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\,P(x,\lambda) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda}\left\{\sum_{x=0}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}\right\}$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Hence the mean of the poisson distribution is λ.

$$\mu_2' = E(x^2) = \sum_{x=0}^{\infty} x^2 p(x,\lambda) = \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda} \left[\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right] + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$\mu_3' = E(x^3) = \sum_{x=0}^{\infty} x^3 p(x,\lambda) = \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^3 \left\{ \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 3e^{-\lambda}\lambda^2 \left\{ \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right\} + \lambda$$

$$= e^{-\lambda}\lambda^3 e^{\lambda} + 3e^{-\lambda}\lambda^2 e^{\lambda} + \lambda = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(x^4) = \sum_{x=0}^{\infty} x^4 . P(x,\lambda)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^4 \left\{ \sum_{n=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} \right\} + 6e^{-\lambda}\lambda^3 \left\{ \sum_{n=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 7e^{-\lambda}\lambda^2 \left\{ \sum_{n=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right\} + \lambda$$

$$= \lambda^4 (e^{-\lambda}e^{\lambda}) + 6\lambda^3 (e^{-\lambda}e^{\lambda}) + 7\lambda^2 (e^{-\lambda}e^{\lambda}) + \lambda$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are now obtained as follows:

$$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Thus the mean and the variance of the poisson distribution are each equal to λ

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4$$
$$= 3\lambda^2 + \lambda$$

Co-efficients of skewness and kurtosis are given by:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \quad \text{and} \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda} \quad \text{and} \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$

Hence the poisson distribution is always a skewed distribution

Proceeding to the limit as $\lambda \to \infty$, $\beta_1 = 0$ and $\beta_2 = 3$.

Recurrence Relation for the moments of Binomial Distribution:

By def., $\mu_r = E\{X - E(X)\}^r = \sum_{x=0}^{n} (x-np)^r \binom{n}{x} p^x q^{n-x}$

Differentiating w.r.to p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x} \Big[-nr(x-np)^{r-1} p^x q^{n-x} + (x-np)^r \{x p^{x-1} q^{n-x} - (n-x) p^x q^{n-x-1}\} \Big]$$

$$= -nr \sum_{x=0}^{n} \binom{n}{x} (x-np)^{r-1} p^x q^{n-x} + \sum_{x=0}^{n} \binom{n}{x} (x-np)^r p^x q^{n-x} \left(\frac{x}{p} - \frac{n-x}{q}\right)$$

$$= -nr \sum_{x=0}^{n} (x-np)^{r-1} p(x) + \sum_{x=0}^{n} (x-np)^r p(x) \frac{(x-np)}{pq}$$

$$= -nr \sum_{x=0}^{n} (x-np)^{r-1} P(x) + \frac{1}{pq} \sum_{x=0}^{n} (x-np)^{r+1} P(x)$$

$$= -nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right) \text{ (Renovsky Formula)}$$

putting $r=1, 2$ and 3 successively in, we get

$$\mu_2 = pq\left(n\mu_0 + \frac{d\mu_1}{dp}\right) = npq$$

$$[\because \mu_0 = 1 \text{ and } \mu_1 = 0]$$

$$\mu_3 = pq\left[2n\mu_1 + \frac{d\mu_2}{dp}\right] = pq \cdot \frac{d(npq)}{dp} = npq\frac{d}{dp}\{p(1-p)\}$$

$$= npq\frac{d}{dp}(p-p^2) = npq(1-2p) = npq(q-p)$$

and

$$\mu_4 = pq\left[3n\mu_2 + \frac{d\mu_3}{dp}\right] = pq\left[3n.npq + \frac{d}{dp}\{npq(q-p)\}\right]$$

$$= pq\left[3n^2pq + n\frac{d}{dp}\{p(1-p)(1-2p)\}\right]$$

$$= pq\left[3n^2pq + n\frac{d}{dp}(p-3p^2+2p^3)\right]$$

$$= pq\left[3n^2pq + n(1-6p+6p^2)\right]$$

$$= pq\left[3n^2pq + n(1-6pq)\right]$$

$$= npq\left[3npq + 1 - 6pq\right]$$

$$= npq\left[1 + 3pq(n-2)\right].$$