SEMESTER : III
NON MAJOR ELECTIVE COURSE: I – Mathematics

Inst Hour	**: 2**
Credit	**: 2**
Code	**: 18K3MEL01**

OPERATIONS RESEARCH – I

UNIT – 1:
Introduction to OR .
(Section : 1.1 – 1.8)

UNIT – 2:
Formulation of LPP.
(Section : 2.1 – 2.3)

UNIT – 3:
Graphical solution
(Section : 2.5)

UNIT – 4:
Transportations Model – Mathematical formulation of a TP – methods for finding initial basic feasible solution – North West corner rule and Least cost method.
(Section : 7.1)

UNIT – 5:
Assignment problem (Balanced).
(Section : 8.1 – 8.5)
(In all the units applications of concepts only. No book work)

Text Book:

[1] Prof. V. Sundaresan, Prof. K.S. Ganapathy Subramanian, Prof. K. Ganesan, Resource Management and Techniques, A.R. Publications, Fourth Edition, 2007.

Books for Reference:

[1] Operations Research by Kanti Swarup, Gupta.P.K & Manmohan. (8th edition)
[2] Problems in Operational Research by Gupta.P.K. &Manmohan.
[3] Operational Research by Hamdy A. Taha (Third Edition).

Question Pattern (Both in English & Tamil Version)

Section A : 5 x 5 = 25 Marks, (Any 5 out of 8, No Unit should be omitted)
Section B : 5 x 10 = 50 Marks, (Any 5 out of 8, No Unit should be omitted)

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HOD Department of Mathem...
K. N. GOVERNMENT ARTS COLLEGE
THANJAVUR-613 007

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UNIT – III
GRAPHICAL SOLUTION.

INTRODUCTION:

LPP involving only two variables can be effectively solved by a graphical method. This method is simple to understand and easy to use. Graphical method has multiple Solutions, unbounded solutions and infeasible solutions.

WORKING PROCEDURE FOR GRAPHICAL METHOD:

Given a L.P.P, optimize $z = f(x_i)$ subject to the constraints $g_j(x_j) \leq, =, \geq b_j$ and the non-negativity restrictions $x_i \geq 0$, $i = 1, 2$; $j = 1, 2, 3, \ldots m$.

STEP 1: Consider the inequality constraints as equalities. Draw the Straight lines in the XOY plane corresponding to each equality and non-negativity restrictions.

STEP 2: Find the permissible region for the values of the variables which is the region bounded by the lines drawn in step 1.

STEP 3: Find the points of intersection of the bounded lines by solving the equations of the corresponding lines.

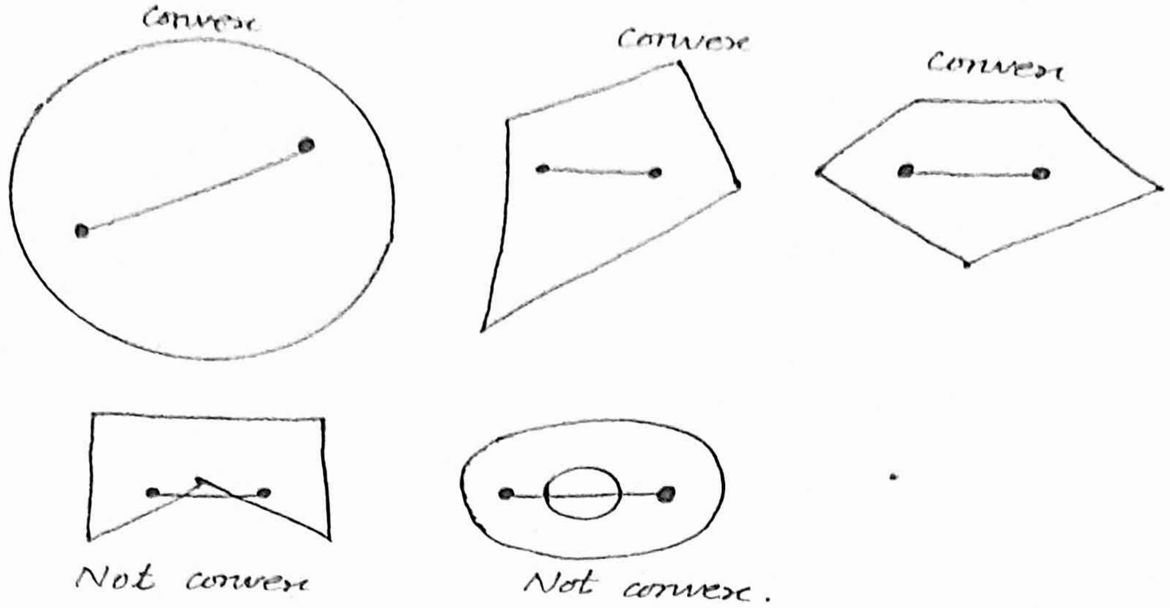
STEP 4: Find the values of z at all vertices of the permissible region.

STEP 5: (i) For maximization problem, choose the vertex for which z is maximum.

(ii) For minimization problem, choose vertex for which z is minimum.

NOTE: A region or a set of points is said to be convex if the line joining any two of its points lies completely within the region.

Example:



Problems:-

1. Solve the following L.P.P. by the graphical method

Max $Z = 3x_1 + 2x_2$.

Subject to,

$$-2x_1 + x_2 \le 1$$
$$x_1 \le 2$$
$$x_1 + x_2 \le 3 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution

Consider the inequalities as equalities.

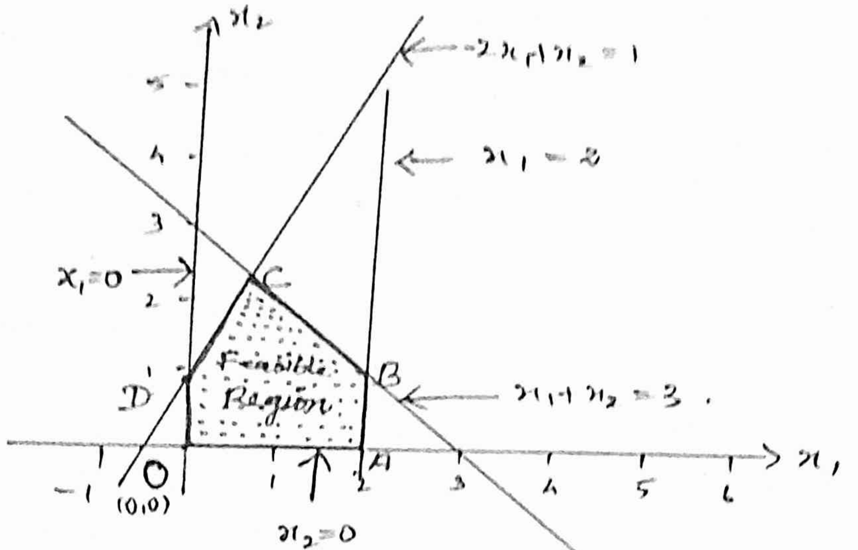
$$-2x_1 + x_2 = 1 \longrightarrow (1)$$
$$x_1 = 2 \longrightarrow (2)$$
$$x_1 + x_2 = 3 \longrightarrow (3)$$
$$x_1 = 0 \longrightarrow (4)$$
$$x_2 = 0 \longrightarrow (5)$$

$(1) \Rightarrow -2x_1 + x_2 = 1,$

put $x_1 = 0 \Rightarrow x_2 = 1 \Rightarrow (0,1)$

put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5, 0)$

So the line (1) passes through the points (0,1) & (−0.5,0). The points on this line satisfy the equation $-2x_1+x_2=1$. Now origin (0,0), on substitution, gives $-0+0=0<1$: hence it also satisfies the inequality $-2x_1+x_2 \le 1$.



Thus all points on the origin side and on this line satisfy the inequality $-2x_1+x_2 \le 1$. Similarly interpreting the other constraints we get the feasible region OABCD. The feasible region is also known as solution space of the L.P.P. Every point within this area satisfies all the constraints.

To find vertices:

B is the point of intersection of $x_1=2$ and $x_1+x_2=3$. Solving these two equations, we have $x_1=2$, $x_2=1$. ∴ we have the vertex B(2,1). Similarly, C is the intersection of $-2x_1+x_2=1$ and $x_1+x_2=3$. Solving these, we have $C(2/3, 7/3)$.

∴ The vertices of the solution space are O(0,0), A(2,0), B(2,1), C(2/3, 7/3) and D(0,1).

The values of z at these vertices are given by

vertex	value of z
O (0,0)	0
A (2,0)	6
B (2,1)	8
C $(2/3, 7/3)$	$20/3$
D (0,1)	2

($\because$ $z = 3x_1 + 2x_2$) since the problem is maximization type. The optimum solution to the L.P.P is

Maximum $z = 8$, $x_1 = 2$, $x_2 = 1$.

2. A Pineapple firm produces two products canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

	Canned Juice	Canned Pineapple	Available resources
Labour (Man hours)	3	2.0	12.0
Equipment (M/c hours)	1	2.3	6.9
Material (Unit)	1	1.4	4.9

Assuming one unit of canned juice and canned pineapple has profit margins Rs. 2 and Rs. 1 respectively. Formulate this as a L.P.P. and solve it graphically also.

Solution:

Let x_1 be the number of units of canned juice and x_2 be the number of units of canned pineapple to be produced.

The constraints or restrictions in this problem are the labour equipment and material.

For labour, $3x_1 + 2x_2 \leq 12$

For Equipment, $x_1 + 2.3x_2 \leq 6.9$

For material, $x_1 + 1.4x_2 \leq 4.9$ and

$x_1, x_2 \geq 0$.

Here our objective is to maximize the profit.

$\therefore$ The objective function is maximize $z = 2x_1 + x_2$.

$\therefore$ The complete formulation of the L.P.P is maximize z

maximize $z = 2x_1 + x_2$.

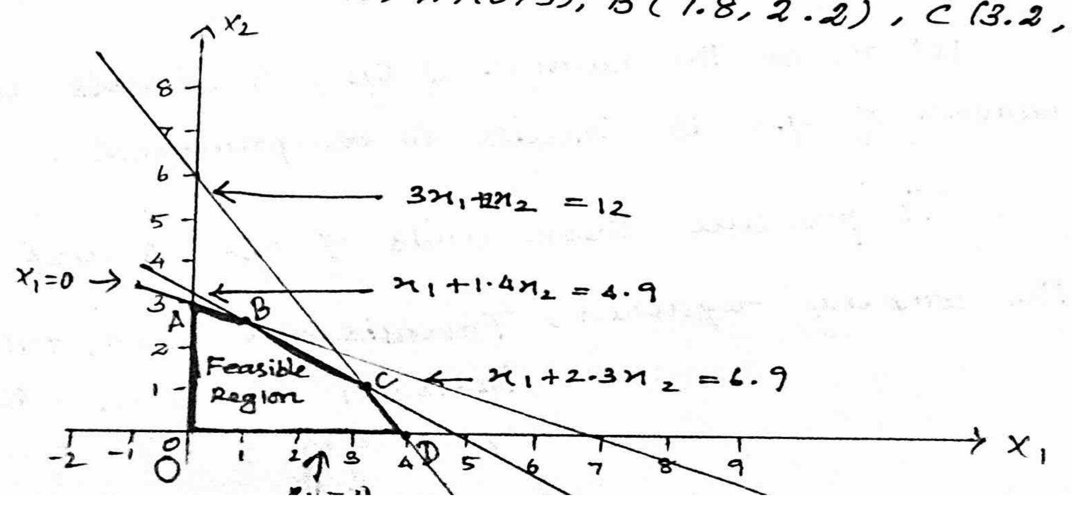
Subject to the constraints,

$3x_1 + 2x_2 \leq 12$

$x_1 + 2.3x_2 \leq 6.9$

$x_1 + 1.4x_2 \leq 4.9$ & $x_1, x_2 \geq 0$.

The solution space is given below with the shaded area with vertices O (0,0), A(0,3), B(1.8, 2.2), C (3.2, 1.2) & D (4,0).



The values of z at these vertices are given by.

vertex	value of z
O (0, 0)	0
A (0, 3)	3
B (1.8, 2.2)	5.8
C (3.2, 1.2)	7.6
D (4, 0)	8

$\because z = 2x_1 + 2x_2$. Since the problem is of maximiza-tion type, the optimum solution is Maximum $z = 8$, $x_1 = 4$, $x_2 = 0$.

3. A company manufactures 2 types of printed circuits. The require-ments of transistors, resistors and capacitors for each type of printed circuits along with other data are given below:

	Circuit A	Circuit B	Stock available
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs. 5	Rs. 8	

How many circuits of each type should the company produce from the stock to earn maximum profit.

Solution:

Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

To produce these units of type A and type B circuits, the company requires, Transistors $= 15x_1 + 10x_2$

Resistors $= 10x_1 + 20x_2$

capacitors $= 15x_1 + 20x_2$.

(1)

since the availability of these transistors, resistors and capacitors are 180, 200 and 210 respectively, the constraints are,

$15x_1 + 10x_2 \le 180$

$10x_1 + 20x_2 \le 200$

$15x_1 + 20x_2 \le 210$ and $x_1 \ge 0, x_2 \ge 0$.

Since the profit from type A is Rs.5 and from type B is Rs. 8 per units, the total profit is $5x_1 + 8x_2$.

$\therefore$ The complete formulation of the L.P.P is

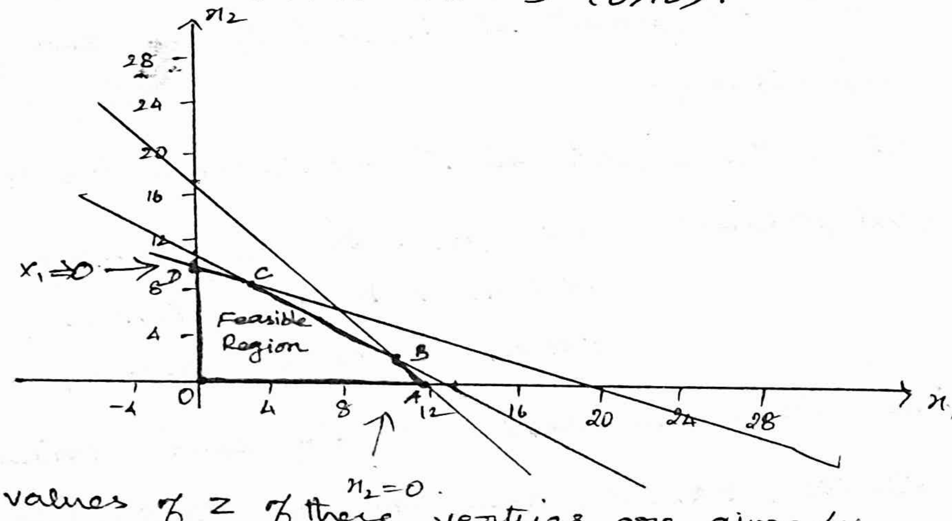
Maximize $Z = 5x_1 + 8x_2$

subject to $15x_1 + 10x_2 \le 180$

$10x_1 + 20x_2 \le 200$

$15x_1 + 20x_2 \le 210$ and $x_1, x_2 \ge 0$.

by using graphical method, the solution space is given below with shaded area OABCD with vertices O(0,0), A(12,0), B(10,3), C(2,9) and D(0,10).



The values of Z of these vertices are given by,

Vertex	values of Z
O (0,0)	0
A (12,0)	60
B (10,3)	74
C (2,9)	82
D (0,10)	80

$\because Z = 5x_1 + 8x_2$

$\therefore$ Max $Z = 82$, $x_1 = 2$, $x_2 = 9$

4. A Company making cold drinks has two bottling plants located at towns T_1 and T_2. Each plant produces three drinks A, B and C and their production capacity perday is given below:

Cold drinks	Plant at	
	T_1	T_2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are Rs 6000 and 4000 res. Find graphically, the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Solution:

Let the plant at T_1, and T_2 be run for x_1 and x_2 days respectively.

Since the plants at T_1 and T_2 run for x_1 and x_2 days, They will produce, $6000x_1 + 2000x_2$ bottles of A

$1000x_1 + 2500x_2$ bottles of B

$3000x_1 + 3000x_2$ bottles of C

Since the demand for the cold drinks A, B and C are 80,000, 22,000, and 40,000 respectively and the production is always greater than or equal to the demand, the constraints are

$6000x_1 + 2000x_2 \geq 80000 \Rightarrow 6x_1 + 2x_2 \geq 80$

$\Rightarrow 3x_1 + x_2 \geq 40$

$1000x_1 + 2500x_2 \geq 22000 \Rightarrow x_1 + 2.5x_2 \geq 22$

$3000x_1 + 3000x_2 \geq 40000 \Rightarrow 3x_1 + 3x_2 \geq 40$ & $x_1, x_2 \geq 0$.

Since the operating costs per day at T_1 is Rs 6000 and at T_2 is Rs 4000 and T_1, T_2 run for x_1 and x_2 days, the total operating cost is Rs. $6000x_1 + 4000x_2$.

$\therefore$ The objective function is minimize $Z = 6000x_1 + 4000x_2$

$\therefore$ The complete formulation of the L.P.P is

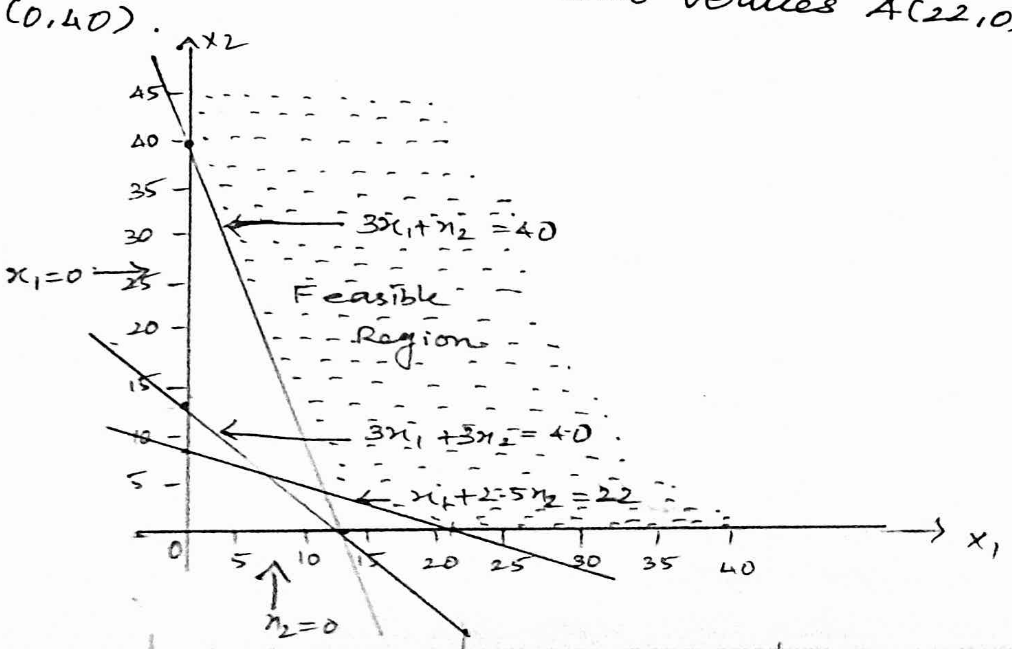
minimize $Z = 6000x_1 + 4000x_2$

Subject to $3x_1 + x_2 \geq 40$

$x_1 + 2.5x_2 \geq 22$

$3x_1 + 3x_2 \geq 40$ & $x_1, x_2 \geq 0$

By using graphical method, the feasible region is given below with shaded area with vertices A(22,0), B(12,4), C(0,40).



From the figure, the constraints $8x_1 + 8x_2 \geq 40$ does not affect the solution space. So $8x_1 + 8x_2 \geq 40$ is a redundant constraint. Also from the direction of the arrows, we see that the solution space is unbounded above.

The values of Z at these vertices $A(22,0)$, $B(12,4)$ and $C(0,40)$ are given by,

Vertex	Values of Z
A (22, 0)	1,32,000
B (12, 4)	88,000
C (0, 40)	1,60,000

$\because Z = 6000x_1 + 4000x_2$. Since the problem is of minimization type, the optimum solution is,

Minimum $Z =$ Rs. 88,000, $x_1 = 12$ days, $x_2 = 4$ days.

NOTE:

From the above examples, for problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. That is, "if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the feasible region".

TRANSPORTATION MODEL:

INTRODUCTION:-

Transportation deals with the transportation of a commodity from 'm' sources to 'n' destinations. It is assumed that (i) Level of supply at each source and the amount of demand at each destination and. (ii) The unit transportation cost of commodity from each source to each destination are known. (given)

It is also assumed that the cost of transportation is linear. The objective is to determine the amount to be shifted from each source to each destination such that the total transportation cost is minimum.

1. Mathematical Formulation of a Transportation Problem:-

Let us assume that there are m sources and n destinations.

Let a_i be the supply at source i, b_j be the demand at destination j, c_{ij} be the unit transportation cost from source i to destination j and x_{ij} be the number of units shifted from source i to destination j.

Then the transportation problem can be expressed mathematically as

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to constraints.

$$\sum_{j=1}^{n} x_{ij} = a_i \ , \ i = 1, 2, 3 \ldots m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ j = 1, 2, 3 \ldots n.$$

and $x_{ij} \geq 0 \ \forall \ i \ \& \ j$.

DEFINITION 1: A set of non-negative values x_{ij}, $i = 1, 2, \ldots m$; $j = 1, 2, \ldots n$. that satisfies the constraints is called a <u>feasible solution</u> to the transportation problem.

NOTE: A balanced transportation problem will always have a feasible solution.

DEFINITION 2: A feasible solution to a $(m \times n)$ transportation problem that contains no more than $m+n-1$ non-negative allocations is called a basic feasible solution (BFS) to the transportation problem.

DEFINITION 3:- A basic feasible solution to a $(m \times n)$ transportation problem is said to be a <u>non-degenerate basic feasible solution</u> if it contains exactly $m+n-1$ non-negative allocations in independent positions.

DEFINITION 4: A basic feasible solution that contains less than $m+n-1$ non-negative allocations is said to be a <u>degenerate basic</u> feasible solution.

DEFINITION 5 : A feasible solution is said to be an optimal solution if it minimizes the total transportation cost.

II METHODS FOR FINDING INITIAL BASIC FEASIBLE SOLUTION.

The transportation problem has a solution if and only if the problem is balanced. Therefore before starting to find the initial basic feasible solution, check whether the given transportation problem is balanced.

METHOD 1 : NORTH WEST CORNER RULE

STEP 1 : The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is $x_{11} = \min\{a_1, b_1\}$.

Case (i) : If $\min\{a_1, b_1\} = a_1$, then $x_{11} = a_1$, decrease b_1 by a_1 and move vertically to the 2^{nd} row (i.e) to the cell (2,1) cross out the first row.

Case (ii) : If $\min\{a_1, b_1\} = b_1$, then put $x_{11} = b_1$, and decrease a_1 by b_1 and move horizontally right (to the cell (1,2) cross out the first column.

Case (iii) :- If $\min\{a_1, b_1\} = a_1 = b_1$ then put $x_{11} = a_1 = b_1$ and move diagonally to the cell (2,2) cross out the first row and the first column.

Step 2 :- Repeat the procedure until all the requirements are satisfied.

METHOD 2 : LEAST COST METHOD (OR) MATRIX MINIMA METHOD (OR) LOWEST COST ENTRY METHOD

Step 1 : Identify the cell with smallest cost and allocate $x_{ij} = \text{Min}(a_i, b_j)$

Case (i) :- If $\min\{a_i, b_j\} = a_i$, then put $x_{ij} = a_i$, cross out the i^{th} row and decrease b_j by a_i, Go to step (2).

Case (ii) :- If $\min\{a_i, b_j\} = b_j$ then put $x_{ij} = b_j$ cross out the j^{th} column and decrease a_i by b_j Go to step (2).

Case (iii) :- If $\min\{a_i, b_j\} = a_i = b_j$, then put $x_{ij} = a_i = b_j$. Crossout either i^{th} row or j^{th} column but not both. Go to step (2).

Step (2) :- Repeat step (1) for the resulting reduced transportation table until all the requirements are satisfied.

METHOD 3 : VOGEL'S APPROXIMATION METHOD (VAM) (or) UNIT COST PENALTY METHOD :

STEP (1) :- Find the difference between the smallest & next smallest costs in each row (column) and write them in brackets against the corresponding row (column).

Step (2) : Identify the row (or) column with largest penalty. If a tie occurs, break the tie arbitrarily.

choose the cell with smallest cost in that selected row or column and allocate as much as possible to that cell and cross out the satisfied row or column and go to step (3).

Step (5): Again compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the rim requirements are satisfied.

Examples

1. Determine basic feasible solution to the following transportation problem using North West Corner Rule.

		A	B	C	D	E	Supply
	P	2	11	10	3	7	4
Origin	Q	1	4	7	2	1	8
	R	3	9	4	8	12	9
Demand		3	3	4	5	6.	

Solution :-

Since $\sum a_i = \sum b_j = 21$. The given problem is balanced.

∴ There exists a feasible solution to the transportation problem

2 3	11	10	3	7	4
1	4	7	2	1	8
3	9	4	8	12	9
3	3	4	5	6	

Following North West Corner Rule, The first allocation is made in the cell (1,1).

Here $x_{11} = \min\{a_1, b_1\} = \min\{4,3\} = 3$.

$\therefore$ Allocate 3 to the cell (1,1) and decrease 4 by 3 ie $4-3=1$.

As the first column is satisfied, we cross out the first column and the resulting reduced Transportation table is

11 1	10	3	7	1
4	7	2	1	8
9	4	8	12	9
3	4	5	6	

Here The North West Corner cell is (1,2)

So allocate $x_{12} = \min\{1,3\} = 1$ to the cell (1,2) and move vertically. to The cell (2,2). The resulting reduced transportation table is.

4 2	7	2	1	8
9	4	8	12	9
2	4	5	6	

Allocate $x_{22} = \min\{8,2\} = 2$ to The cell {2,2} and move horizontally to the cell (2,3). The resulting Transportation table is

7 4	2	1	6
4	8	12	9
4	5	6	

Allocate x_{23} = min (6,4) = 4 and move horizontally to the cell (2,4). The resulting reduced transportation table is

2 2	1	2
8	12	9
5	6	

Allocate x_{24} = (2,5) = 2 and move vertically to the cell (3,4). The resulting reduced transportation table is

8 3	12	9
3	6	

Allocate x_{34} = min (9,3) = 3 and move horizontally to the cell (3,5), which is

12 6	6
6	

Allocate x_{35} = min (6,6) = 6

Finally the initial basic feasible solution is as shown in the following table.

2 3	11 1	10	3	7
1	4 2	7 4	2 2	1
3	9	4	8 3	12 6

From this table, the no/ of +ve independent allocat is equal to m+n−1 = 3+5−1 = 7 This ensures that the soln/ is non degenerate basic feasi

∴ The initial transportation cost

= Rs(2 × 3)+(1 × 1)+(4 × 2)+(7 × 4)+(2 × 2)+(8 × 3) + (12 × 6)

= Rs 153/–.

2. Find the starting solution of the following transportation model.

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

using (i) North West Corner Rule
(ii) Least Cost Method
(iii) Vogel's approximation method.

Solution :-

Since $\Sigma a_i = \Sigma b_j = 30$, the given transportation problem is balanced. Hence there exists a basic feasible solution to this problem.

(i) North West Corner Rule!
(ii) Least cost method
(iii) Vogel's approximation method.

(i) North West Corner Rule:

(i)

1 (7)	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(ii)

0 (3)	4	2	12
3	1	5	11
3	10	10	

(iii)

4 (8)	2	9
1	5	11
10	10	

(iv)

1 (1)	5	11
1	10	

(v)

5 (10)	10
10	

. The starting solution is as shown in the following table.

1_{7}	2	6
0_{3}	4_{9}	2
3	1_{1}	5_{10}

∴ The initial transportation cost

$= Rs (1\times7)+(0\times3)+(4\times9)+(1\times1)+(5\times10)$

$= Rs\ 94/-$

(ii) Least Cost Method :- Using this method, the allocation are as shown in the table below.

(i)

1	2	6_{7}	7
0_{10}	4	2	12
3	1	5	11
10	10	10	

(ii)

2	6	7
4	2	2
1_{10}	5	11
10	10	

(iii)

6	7
2_{2}	2
5	1
10	

(iv)

6	7
5_{1}	1
8	

(v)

6_{7}	7
7	

The starting solution is as shown in the following table:

1	2	6_{7}
0_{10}	4	2_{2}
3	1_{10}	5

∴ The initial transportation cost $= Rs. (6\times7)+(0\times10)+(2\times2)+(1\times10)+(5\times1)$

$= Rs\ 61/-$

(iii) VOGEL'S APPROXIMATION METHOD: Using this method, the allocation are shown in the table given below:

(i)

1	2	6	7	(1)
0	4	2_{10}	12	(2)
3	1	5	11	(2)
10 (1)	10 (1)	10 (3)		

(ii)

1	2	7	(1)
0_{2}	4	2	(4)
3	1	11	(2)
10 (1)	10 (1)		

(iii)

1	2	7	(1)
3	1_{10}	10	(2)
8	10		
(2)	(1)		

(iv)

1_7	7
3	1
8	

(v)

3_1	1
1	

The starting solution is as shown in the foll/. table

1_7	2	6
0	4	2_{10}
3_1	1_{10}	5

∴ The initial transportation

= Rs. (1 × 7) + (0 × 2) + (2 × 10) + (3 × 1) + (1 × 10)

= Rs 40/-

NOTE: For the above problem, the no/. of positive allocations in independent positions is $(m+n-1)$. i.e.,

$m+n-1 = 3+3-1 = 5$.

UNIT – V

ASSIGNMENT PROBLEMS

8.1 INTRODUCTION:

The assignment problem is a particular case of transpor-tations problem in which the objective is to assign a no. of tasks to an equal no. of facilities at a minimum cost.

The assignment problem can be stated in the form of $m \times n$ matrix (c_{ij}) called Cost matrix (or) Effectiveness matrix where c_{ij} is the cost of assigning i^{th} machine to the j^{th} job.

$$
\begin{array}{cc|cccccc}
 & & & & \text{Jobs} & & & \\
 & & 1 & 2 & 3 & \cdots & & n \\
\text{Machines} & 1 & c_{11} & c_{12} & c_{13} & \cdots & & c_{1n} \\
 & 2 & c_{21} & c_{22} & c_{23} & \cdots & & c_{2n} \\
 & 3 & c_{31} & c_{32} & c_{33} & \cdots & & c_{3n} \\
 & \vdots & \cdots & \cdots & \cdots & \cdots & & c_{4n} \\
 & \vdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
 & m & c_{m1} & c_{m2} & c_{m3} & \cdots & & c_{mn}
\end{array}
$$

8.2 Mathematical formulation of an assignment problem:

Consider an assignment problem of assigning n jobs to n machines job to one machine. Let c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and

$$\text{Let } x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine.} \end{cases}$$

The assignment model is then given by the following LPP.

minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1, j = 1,2,\dots n$,

$\sum_{j=1}^{n} x_{ij} = 1, i = 1,2,\dots n$ & $x_{ij} = 0$ (or) 1.

8.3 Comparison with Transportation Model :-

The assignment problem may be considered as a special case of the transportation problem. Consider a transportation problem with 'n' sources and 'n' destinations.

Destination

		1	2	3	......	n	supply (a_i)
	1	c_{11}	c_{12}	c_{13}	......	c_{1n}	a_1
	2	c_{21}	c_{22}	c_{23}	......	c_{2n}	a_2
Source	3	c_{31}	c_{32}	c_{33}	......	c_{3n}	a_3
	⋮	⋮	⋮	⋮	......	c_{4n}	...
	⋮	⋮	⋮	⋮	......	c_{5n}	...
	n	c_{n1}	c_{n2}	c_{n3}	......	c_{mn}	a_n
Demand (b_j)		b_1	b_2	b_3	......	...	...

we have to find x_{ij} $(i, j = 1,2,3\dots n)$ for which the total transportation cost

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \text{ is minimized.}$$

Subject to the constraints $\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,\dots n$

$\sum_{i=1}^{n} x_{ij} = b_j, \ j = 1,2,\dots n, \ \sum a_i = \sum b_j, \ i,j = 1,2,\dots n$

& $x_{ij} \geq 0, \ i,j = 1,2,\dots n$.

Here the 'sources' represent 'facilities' or 'machines' and 'destinations' represent 'jobs'.

Suppose that the supply available at each source is 1. ie $a_i = 1$ and the demand required at each destination is 1 ie $b_j = 1$.

Let c_{ij} be the unit transportation cost from the ith source to the jth destination. Here it means the cost of assigning the ith machine to the jth job.

Let x_{ij} be the amount to be shipped from ith source to the jth destination. Here it means the assignment of the ith machine to the jth job. We can restrict the value of x_{ij} to be either 0 (or) 1. $x_{ij} = 0$ means that the ith machine does not get the jth job and $x_{ij} = 1$ means that the ith machine gets the jth job.

Since each machine should be assigned to only one job and each job requires only one machine, the total assignment value of the ith machine is 1, (ie.,) $\sum x_{ij} = 1$ and the total assignment value of the jth job. is 1. (ie.) $\sum x_{ij} = 1$.

Hence the assignment problem can be expressed as

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$. where c_{ij} is the cost of assigning ith machine to the jth job subject to the constraints.

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th machine is assigned to the } j\text{th job} \\ 0, & \text{if } i\text{th machine is not assigned to the job.} \end{cases}$$

$\sum_{j=1}^{n} x_{ij} = 1$, $i = 1, 2, \ldots n$ & $\sum_{i=1}^{n} x_{ij} = 1$, $j = 1, 2, \ldots n$.

From this we see that assignment problem represent a transportation problem with all demands and supplies equal to 1.

The units available at each source and units demanded at each destination are equal to 1. It means exactly that there is only one occupied cell in each row and each column of the transportation table. i.e., only 'n' occupied cells in place of the required $n+n-1 = 2n-1$

Hence an assignment problem is always a degenerate form of a Transportation problem.

But the Transportation technique (or) simplex method can not be used to solve the assignment problem because of degeneracy. In fact a very convenient iterative procedure is available for solving an assignment problem.

The technique used for solving assignment problem makes use of the following two theorems.

Theorem 1 :- The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment cost matrix.

Theorem 2 :- If for an assignment problem all $c_{ij} > 0$. then an assignment schedule (x_{ij}) which satisfies $\Sigma c_{ij} x_{ij} = 0$, must be optimal.

8.4 : Difference between the transportation problem and the assignment Problem :-

Transportation Problem	Assignment Problem
(a) Supply at any source may be any positive quantity a_i	(a) Supply at any source (machine) will be 1. ie $a_i = 1$.
(b) Demand at any destination may be any positive quantity b_j	(b) Demand at any destination (job) will be 1 ie, $b_j = 1$.
(c) One or more source to any number of destinations	(c) One source (machine) to only one destination (job).

8.5 Assignment Algorithm (or) Hungarian Method :-

First check whether the number of rows is equal to the no. of columns, if it is so, the assignment problem is said to be balanced. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm.

Step (i) :- Substract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

Step (ii) :- Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1.

Step (iii) :- (Assigning the zeros)

(a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this encircle zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(b). Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

Step 4 : (Apply optimal TEST)

(a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

(b) If atleast one row/column is without as assignment (ie., if there is atleast one row/column is without one encircled zero), then the current assignment is not optimal. Go to step 5.

Step 5 : Cover all the zeros by drawing a minimum no. of straight lines as follows.

(a) Mark (x) the rows that do not have assignment

(b) Mark (x) the columns (not already marked) that have zeros in marked rows.

(c) Mark (x) the rows (not already marked) that have assignments in marked columns.

(d) Repeat (b) and (c) until no more marking is required.

(e) Draw lines through all unmarked rows and marked columns. If the no. of these lines is equal to the order of the matrix then it is optimum solution otherwise not.

Step 6:- Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

Step 7:- Repeat steps (1) to (6) until an optimum assignment is attained.

Note: (i) In case some rows columns contain more than one zero, encircle any unmarked zero arbitrarily and cross all other zeros in its column or row. Proceed in this way until no zeros is left unmarked or encircled.

Examples:

1. Consider the problem of assigning five jobs to five persons. The assignment are given as follows:

$$
\text{person}\quad
\begin{array}{c}
A\\ B\\ C\\ D\\ E\\ F
\end{array}
\overset{\text{Jobs}}{\begin{pmatrix}
1 & 2 & 3 & 4 & 5\\
8 & 4 & 2 & 6 & 1\\
0 & 9 & 5 & 5 & 4\\
3 & 8 & 9 & 2 & 6\\
4 & 3 & 1 & 0 & 3\\
9 & 5 & 8 & 9 & 5
\end{pmatrix}}
$$

Determine the optimum assignment schedule.

Solution: The cost matrix of the given assignment problem is

$$
\begin{pmatrix}
8 & 4 & 2 & 6 & 1\\
0 & 9 & 5 & 5 & 4\\
3 & 8 & 9 & 2 & 6\\
4 & 3 & 1 & 0 & 3\\
9 & 5 & 8 & 9 & 5
\end{pmatrix}
$$

since the no. of rows is equal to the no. of columns in the cost matrix, the given assignment problem is balanced.

Step(i) :- Select the smallest cost element in ~~each~~ each row & substract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{pmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{pmatrix}$$

Step(ii) :-

$$\begin{bmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{bmatrix}$$

Select the smallest cost element in each column and substract this from all the elements of the correspon-ding column, we get the reduced matrix.

Since each row and each column contains atleast one zero, we shall make assignments in the reduced matrix.

Step(iii)

$$\begin{bmatrix} 7 & 3 & \cancel{0} & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & \cancel{0} & 3 \\ 4 & (0) & 2 & 4 & \cancel{0} \end{bmatrix}$$

Examine the rows successively until a row with exactly one unmarked zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zeros of its column. The 3rd row ~~column~~ contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 4th row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. The 1st row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Finally the last row contains exactly one unmarked zero, so encircle this zero and cross all other zeros in its column. Like wise examine the columns successively. The assignments in rows and colums in the reduced matrix is given by

$$\begin{pmatrix} 7 & 3 & \cancel{0} & 5 & (0) \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & \cancel{(0)} & 3 \\ 4 & (0) & 2 & 4 & \cancel{0} \end{pmatrix}$$

Step 4: Since each row and each column contains exactly one assignment (i.e., exactly one encircled zero) the current assignment is optimal.

∴ The optimum assignment schedule is given by A → 5, B → 1, C → 4, D → 3, E → 2.

The optimum (minimum) assignment cost = (1+0+2+1+5) cost units = 9 units of cost.

3. The assignment cost of assigning any one operator to any one machine is given in the following table.

Operators

		I	II	III	IV
Machine	A	10	5	13	15
	B	3	9	18	3
	C	10	7	3	2
	D	5	11	9	7

Find the optimal assignment by Hungarian method.

Solution:-

The cost matrix of the given assignment problem is

$$\begin{bmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{bmatrix}$$

Since the number of rows is equal to the no. of columns in the cost matrix, the given assignment problem is balanced.

Select the smallest cost element in each row and substract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{bmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$$

Select the smallest cost element in each column and substract this from all the elements of the corresponding column, we get the reduced matrix.

$$\begin{bmatrix} 5 & 0 & 7 & 10 \\ 0 & 6 & 14 & 0 \\ 8 & 5 & 0 & 0 \\ 0 & 6 & 3 & 2 \end{bmatrix}$$

Since each row and each column contain atleast one zero, we shall make the assignment in rows and columns of this reduced cost matrix.

$$\begin{bmatrix} 5 & (0) & 7 & 10 \\ \cancel{0} & 6 & 14 & (0) \\ 8 & 5 & (0) & \cancel{0} \\ (0) & 6 & 3 & 2 \end{bmatrix}$$

Since each row and column contain exactly one assignment (i.e., exactly one encircled zero). The current assignment is optimal.

The optimum assignment schedule is

A → II, B → IV, C → III, D → I and the optimum (minimum) assignment cost = Rs. (5+3+3+5) = Rs. 16/-.

3. The processing time in hours for the jobs when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

Machines

Jobs	M_1	M_2	M_3	M_4	M_5
J_1	9	22	58	11	19
J_2	43	78	72	50	63
J_3	41	28	91	37	45
J_4	74	42	27	49	39
J_5	36	11	57	22	25

Solution :- The cost matrix of the given problem is

$$\begin{bmatrix} 9 & 22 & 58 & 11 & 19 \\ 43 & 78 & 72 & 50 & 63 \\ 41 & 28 & 91 & 37 & 45 \\ 74 & 42 & 27 & 49 & 39 \\ 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Since the no. of rows is equal to the no. of columns in the cost matrix, the given assignment problem is balanced.

STEP (i) :-

Select the smallest cost element in each row and substract this from all the elements of the corresponding row, we get the reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Step 2 :-

Select the smallest cost element in each column and substract this from all the elements of the corresponding column, we get the following reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{bmatrix}$$

Step 3 :- Now we shall examine the rows successively. 2nd row contains a single unmarked zero, encircle this zero and cross all other zeros in its column all other zeros. 3rd row contains a single unmarked zero encircle this this zero and cross all other zeros in its column. 4th row contains a single unmarked zero, encircle this zero and cross all other zeros in its column. After this no row is with exactly one unmarked zero. So go for columns.

Examine the columns successively. Fourth column contains exactly one unmarked zero, encircle this zero and cross all other zeros in its row. After examining all the rows and columns, we get

$$\begin{bmatrix} \cancel{0} & 13 & 49 & (0) & \cancel{0} \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

Step 4 :- step 4, since the 5th row and 6th column do not have any assignment the current assignment is not optimal.

Step 5 :

$$\begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & (4) \end{bmatrix}$$

Step 6 :-

$$\begin{bmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{0} & 17 & 49 & (0) & \cancel{0} \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & (0) \end{bmatrix}$$

∴ The optimum assignment schedule is $J_1 \to M_4$, $J_2 \to M_1$, $J_3 \to M_2$, $J_4 \to M_3$, $J_5 \to M_5$ and the optimum (minimum) processing time.

$= 11 + 43 + 28 + 27 + 25$ hours

$= 134$ hours.