Discrete Distribution UNIT-3

Code: 18K3S04

1. Poisson distribution limiting case of the binomial distribution under the following conditions:-

i) n, the number of trials is indefinitely large i.e. $n \to \infty$

ii) P, the constants probability of success for each trial is indefinitely small $P \to 0$.

iii) $np = \lambda$, (say) is finite.

Thus, $P = \lambda/n$, $q = 1 - \lambda/n$, where λ is a positive real number. The probability of x successes in a series of n independent trial is,

$$b(x:n,p) = \binom{n}{x} p^x q^{n-x} \; ; x = 0,1,2\ldots n \longrightarrow (1)$$

under the above conditions. We want the limiting from of (1) under the above conditions. Hence,

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

Using Stirling's approximation for $n!$ as $n \to \infty$ viz.

$$\lim_{n\to\infty} n! \approx \sqrt{2\pi}\, e^{-n} n^{n+(1/2)}, \text{ we get.}$$

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty}\left\{\frac{\sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}}{x!\sqrt{2\pi}\, e^{-(n-x)}.(n-x)^{n-x+(1/2)}}\right\}\left(\frac{\lambda}{n}\right)^x \left(1-\lambda/n\right)^{n-x}$$

$$= \frac{\lambda^x}{e^x x!} \lim_{n\to\infty} \frac{(1-\lambda/n)^{n-x}}{(1-x/n)^{n-x+(1/2)}} \Rightarrow \frac{\lambda^x}{e^x x!} \cdot \frac{\lim_{n\to\infty}(1-\lambda/n)^n \lim_{n\to\infty}(1-\lambda/n)^{-x}}{\lim_{n\to\infty}(1-x/n)^n \lim_{n\to\infty}(1-x/n)^{-x+(1/2)}}$$

But we know that.

$$\lim_{n\to\infty}(1-\lambda/n)^n = e^{-\lambda}, \quad \lim_{n\to\infty}(1-\lambda/n)^{\infty} = 1, \; \infty \text{ is not a function of } n.$$

$$\therefore \lim_{n\to\infty} b(x;n,p) = \frac{e^{-\lambda}\lambda^x}{x!} \; ; x = 0,1,\ldots,\infty$$

which is the required probability function of the poisson distribution. 'λ' is known the parameter of Poisson distribution.

Definition:

A random variable X is said to follow a poisson distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(x,\lambda) = P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} \end{cases} ; \; x = 0,1,2\ldots ; \; \lambda > 0$$

Here λ is known as the parameter of the distribution. we shall use the notation $X \sim P(\lambda)$, to denote that X is a poisson variate with parameter λ.

Moments of the Poisson ~~Distributi~~ Distribution!

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, p(x,\lambda) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda} \left\{ \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right\}$$

$$= \lambda e^{-\lambda} (1 + \lambda + \lambda^2/2! + \lambda^3/3! + \cdots) = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Hence the mean of the Poisson distribution is λ.

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x,\lambda) = \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \lambda^x/x! + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} 3x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^4 \left\{ \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} \right\} + 6 e^{-\lambda}\lambda^3 \left\{ \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 7 e^{-\lambda}\lambda^2 \left\{ \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right\} + \lambda$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are now obtained as follows:

$$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Thus ~~the~~ the mean and variance of the poisson distribution are each equal to λ.

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3+3\lambda^2+\lambda) - 3\lambda(\lambda^2+\lambda) + 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= (\lambda^4+6\lambda^3+7\lambda^2+\lambda) - 4\lambda(\lambda^3+3\lambda^2+\lambda) + 6\lambda^2(\lambda^2+\lambda) - 3\lambda^4 = 3\lambda^2+\lambda$$

Coefficients of Skewness and Kurtosis are given by:

$$\beta_1 = \mu_3^2/\mu_2^3 = \lambda^2/\lambda^3 = 1/\lambda, \text{ and } \gamma_1 = \sqrt{\beta_1} = 1/\sqrt{\lambda}$$

Also $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3 + 1/\lambda$ and $\gamma_2 = \beta_2 - 3 = 1/\lambda$,

Hence the poisson distribution is always a skewed distribution. proceeding to the limit as $\lambda \to \infty$, $\beta_1 = 0$ and $\beta_2 = 3$.

Mode of the poisson Distribution:

$$\frac{P(x)}{P(x-1)} = \frac{\dfrac{e^{-\lambda}\lambda^x}{x!}}{\dfrac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \lambda/x$$

We discuss the following cases:

Case:1 When λ is not an integer. Let us suppose that s is the integral part of λ,

so that,

$$\lambda = s+f, \quad 0 < f < 1.$$

$$\frac{P(x)}{P(x-1)} = \frac{s+f}{x} \begin{cases} >1, & \text{if } x = 0,1,\dots,s \\ <1, & \text{if } x = s+1, s+2, \dots \end{cases}$$

$$\frac{P(1)}{P(0)} > 1, \frac{P(2)}{P(1)} > 1, \dots \frac{P(s-1)}{P(s-2)} > 1, \frac{P(s)}{P(s-1)} > 1,$$

and $$\frac{P(s+1)}{P(s)} < 1, \frac{P(s+2)}{P(s+1)} < 1, \dots$$

combining the above expressions into a single expression, we get.

$$P(0) < P(1) < P(2) < \dots < P(s-2) < P(s-1) < P(s) > P(s+1) > P(s+2) > \dots$$

which shows that $P(s)$ is the maximum value. Hence in this case, the unimodal and the integral part of λ is the unique model value.

Case : II When $\lambda = k$ (say) is an integer. Here, as in case I. we have,

$$\frac{P(1)}{P(0)} > 1, \quad \frac{P(2)}{P(1)} > 1, \ldots \frac{P(k-1)}{P(k-2)} > 1$$

and $$\frac{P(k)}{P(k-1)} = 1, \quad \frac{P(k+1)}{pk} < 1, \quad \frac{P(k+2)}{P(k+1)} < 1, \ldots$$

$\therefore P(0) < P(1) < P(2) < \ldots < P(k-2) < P(k-1) = P(k) > P(k+1) > P(k+2) \ldots$

In this case we have two maximum value, viz, $P(k-1)$ and $P(k)$ and thus the distribution is bimodal and two modes are at $(k-1)$ and k, i.e., at $(\lambda-1)$ and λ,

Recurrence Relation for Moments of the Poisson Distribution:

$$\mu_r = E(X-E(x))^r = \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

differentiating w. r. to λ, we get,

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\{x\lambda^{x-1}e^{-\lambda} - \lambda^x e^{-\lambda}\}$$

$$= -r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\{\lambda^{x-1}e^{-\lambda}(x-\lambda)\}$$

$$= -r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} + 1/\lambda\sum_{x=0}^{\infty}(x-\lambda)^{r+1}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= -r\mu_{r-1} + 1/\lambda\,\mu_{r+1}$$

$\cancel{\mu_{r+1} = r\lambda \frac{d\mu_r}{}}$

$$\Rightarrow \mu_{r+1} = r\lambda\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}$$

Putting $r = 1, 2$ and 3 successively, we get

$$\mu_2 = \lambda\mu_0 + \lambda\frac{d\mu_1}{d\lambda} = \lambda, \quad \mu_3 = 2\lambda\mu_1 + \lambda\frac{d\mu_2}{d\lambda} = \lambda$$

$$\mu_4 = 3\lambda\mu_2 + \lambda\frac{d\mu_3}{d\lambda} = 3\lambda^2 + \lambda$$

Moment Generating function of the poisson Distribution:

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1+\lambda e^t + \frac{(\lambda e^t)^2}{2!}+\cdots\right\} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

— x —

Characteristic function of the Poisson Distribution:

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

— x —

Cumulants of the Poisson Distribution:

$$k_X(t) = \log M_X(t) = \log\left[e^{\lambda(e^t-1)}\right] = \lambda(e^t-1)$$

$$= \lambda\left[(1+t+t^2/2!+t^3/3!+\cdots+t^r/r!+\cdots)-1\right]$$

$$= \lambda\left[t+t^2/2!+t^3/3!+\cdots+t^r/r!+\cdots\right]$$

$K_r = r^{th}$ cumulant $=$ Coefficient of $t^r/r!$ in $K_x(t) = \lambda$

$\Rightarrow k_r = \lambda;\ r = 1,2,3\ldots$

Hence, all cumulants of the Poisson distribution are equal, each being equal to λ. In particular, we have,

$$\text{Mean} = k_1 = \lambda,\ \mu_2 = \lambda,\ \mu_3 = \lambda \text{ and } \mu_4 = k_4 + 3k_2^2 = \lambda + 3\lambda^2$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \lambda^2/\lambda^3 = 1/\lambda \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\lambda+3\lambda^2}{\lambda^2} = 1/\lambda + 3$$

— x —

Additive Property:

Sum of independent Poisson variates is also a Poisson variate. More elaborately, if $x_i\ (i=1,2,\ldots,n)$ are independent Poisson variates with parameters $\lambda_i;\ i=1,2,\ldots n$, respectively, then $\sum_{i=1}^{n} x_i$ is also a poisson variate with parameter $\sum_{i=1}^{n}\lambda_i$.

Proof:

$$M_{x_i}(t) = e^{\lambda_i(e^t-1)} \quad ; i = 1,2,\ldots,n$$

$$M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t)\,M_{x_2}(t)\ldots M_{x_n}(t),$$

$$= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} \cdots e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)(e^t-1)}$$

Which is the m.g.f of a Poisson Variate with parameter $\lambda_1+\lambda_2+\cdots+\lambda_n$, Hence, by uniqueness theorem of m.g.f's, $\sum_{i=1}^{n} x_i$ is also a Poisson Variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Probability Generating Function of poission Distribution:

$$\text{P.G.F of } X = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot s^k = \sum_{k=0}^{\infty} e^{-\lambda} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

UNIT - 4

Discrete Uniform Distribution:

Definition:

A r.v X is said to have a discrete uniform distribution over the range $[1, n]$ if its p.m.f is expressed as follows:

$$P(X=x) = \begin{cases} 1/n & \text{for } x = 1,2,\ldots,n) \\ 0, & \text{otherwise.} \end{cases}$$

Here n is known as the parameter of the distribution and lies in the set of all positive integers.

Moments:

$$E(X) = 1/n \sum_{i=1}^{n} i = \frac{n+1}{2}, \quad E(X^2) = 1/n \sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)\cancel{(n+1)}(n-1)}{12}$$

The m.g.f of x is:

$$M_x(t) = E[e^{tx}] = 1/n \sum_{x=1}^{n} e^{tx} = \frac{e^t(1-e^{nt})}{n(1-e^t)}$$

Negative Binomial Distribution:

Definition:

A r.v X is said to follow a negative binomial distribution with Parameter r and P if its P.m.f is given by.

$$P(X=x)=p(x)=\begin{cases}\binom{x+r-1}{r-1}p^r q^x ; & x=0,1,2,\ldots \\ 0, & \text{otherwise.}\end{cases}$$

Derivation of Negative binomial distribution:

Suppose we have a succession of n bernoulli trials. We assume that i) the trials ~~are~~ are indepented ii) the probability of success 'p' in a trial remains constant from ~~trait to trait~~ trial to trial.

Let $f(x; r, p)$ denote the probability that there are x failures preceding the r^{th} success in $x+r$ trials. Now, the last trial must be a success, whose probability is p. In the remaining $(x+r-1)$ trials we must have $(r-1)$ successes whose probability is given by binomial probability law by the expression:

$$\binom{x+r-1}{r-1}p^{r-1}q^x.$$

$\therefore$ by Compound probability theorem, $f(x; r, p)$ is given by the product these two probabilityies ie,

$$f(x; r, p)=\binom{x+r-1}{r-1}p^r q^x$$

Also, $\binom{x+r-1}{r-1}=\binom{x+r-1}{x}=\dfrac{(x+r-1)(x+r-2)\ldots(r+1)r}{x!}=(-1)^x\binom{-r}{x}$

$$P(x)=\binom{-r}{x}p^x(-q)^x ; \quad x=0,1,2,\ldots$$

which is the $(x+1)^{th}$ ~~the~~ term in the expansion of $p^r(1-q)^{-r}$, a binomial expansion with a negative index. Hence the distribution is known ~~as~~ as NBD.

Also,
$$\sum_{x=0}^{\infty}p(x)=p^r\sum_{x=0}^{\infty}\binom{-r}{x}(-q)^x=p^r(1-q)^{-r}=1$$

Therefore, $p(x)$ represent the probability function and the ~~pdf~~ probability function is discrete Variable which follows this ~~pdf~~ probability function is called the ~~pb~~ negative binomial variate.

If $P=1/Q$ and $q=P/Q$ so that $Q-P=1$ $\because p+q=1$,

then,
$$P(x)=\binom{-r}{x}Q^{-r}\left(-P/Q\right)^x ; x=0,1,2\ldots$$

This is the general term in the negative binomial expression $(Q-P)^{-r}$

MGF of Negative Binomial Distribution:

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} \binom{-r}{x} Q^{-r} \left(-\frac{Pe^t}{Q}\right)^x = (Q - Pe^t)^{-r}$$

$$\mu_1' = \left[\frac{d}{dt} M(t)\right]_{t=0} = \left[-r(-Pe^t)(Q-Pe^t)^{-r-1}\right]_{t=0} = rP$$

$\therefore$ Mean of the negative binomial distribution is rP.

$$\mu_2' = \left[d^2/dt^2\, M(t)\right]_{t=0} = \left[rPe^t (Q-Pe^t)^{-r-1} + (-r-1)rPe^t (Q-Pe^t)^{-r-2}(-Pe^t)\right]_{t=0}$$

$$= rP + r(r+1)P^2$$

$\therefore \mu_2 = \mu_2' - \mu_1'^2 = r(r+1)P^2 + rP - r^2P^2 = rPQ$

As $Q > 1$, $rP < rPQ$, ie Mean < Variance, which is a distinguishing feature of the negative binomial distribution.

~~Cumulants of~~

Cumulants of NBD:

$$K_X(t) = \log M_X(t) = -r\log(Q - Pe^t) = -r\log\left[Q - P\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots\right)\right]$$

$$= -r\log\left[1 - P\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right]$$

Mean $= k_1 = rP$, $\mu_2 = rP(1+P) = rPQ$,

$\mu_3 = k_3 = rP(1 + 3P + 2P^2) = rP(1+P)(1+2P) = rPQ(Q+P)$,

$k_4 = rP(1+P)(1 + 6P + 6P^2) = rPQ(1 + 6PQ)$

$\therefore \mu_4 = k_4 + 3k_2^2 = rPQ[1 + 3PQ(r+2)]$

Since $Q = 1/p$, $P = qQ = P/q$, we have in terms of p and q,

Mean $= \dfrac{rq}{p}$, Variance $= \mu_2 = \dfrac{rq}{p^2}$, $\mu_3 = \dfrac{rq(1+q)}{p^3}$,

$\mu_4 = \dfrac{rq(p^2 + 3q(r+2))}{p^4}$

$$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(1+q)^2}{rq}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{p^2 + 3q(r+2)}{rq}$$

$$\therefore \gamma_1 = \sqrt{\beta_1} = (1+q)/\sqrt{rq} \text{ and } \gamma_2 = \beta_2 - 3 = (p^2 + 6q)/rq$$

— x —

Poisson Distribution as a limiting case of the NBD:

Negative binomial distribution tends to Poisson distribution as $P \to 0$, $r \to \infty$ such that $rp = \lambda$ (finite).

Proceeding to the limits, we get.

$$\operatorname{Lim} p(x) = \lim \binom{x+r-1}{r-1} p^r q^x = \lim \binom{x+r-1}{x} Q^{-r} \left(\frac{P}{Q}\right)^x$$

$$= \lim_{r \to \infty} \frac{(x+r-1)(x+r-2)\ldots(r+1)r}{x!} (1+p)^{-r} \left(\frac{P}{1+P}\right)^x$$

$$= \lim_{r \to \infty} \left\{ 1/x! \left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\ldots(1+1/r) \cdot 1 \cdot r^x (1+p)^{-r} \left(\frac{P}{1+P}\right)^x \right\}$$

$$= 1/x! \lim_{r \to \infty} \left\{ (1+p)^{-r} \left(\frac{rp}{1+p}\right)^x \right\}^{*}$$

$\because rp = \lambda$

$$= \frac{\lambda^x}{x!} \lim_{r \to \infty} [(1+\lambda/r)]^{-r} \lim_{r \to \infty} (1+\lambda/r)^{-x}$$

$$= \frac{\lambda^x}{x!} \cdot e^{-\lambda} \cdot 1 = \frac{e^{-\lambda} \lambda^x}{x!} \quad //$$

which is the probability function of the poisson distribution with parameter 'λ'.

Probability Generating function of Negative binomial distribution:

$$P_X(s) = E(s^X) = \sum_{x=0}^{\infty} s^x p(x) = \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-qs)^x$$

$$= p^r (1-qs)^{-r}$$

$$= [P/(1-qs)]^r$$

UNIT – 5

Geometric Distribution:

A random Variable X is said to have a geometric distribution if it assumes only non-negative Values and its Probability mass function is given by.

$$P(X=x)=\begin{cases} q^x p & ; x=0,1,2\ldots ; 0<p\leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Lack of Memory:

The geometric distribution is said to ~~lack~~ lack of memory in a Certain Sense. suppose an event E can occur at one of the times $t=0,1,2,\ldots$ and the occurrence time X has a geometric distribution with parameter P.

Thus $P(X=t)=q^t p$; $t=0,1,2,\ldots$

Suppose we know that the event E has not ~~occ~~ occurred before k, ie, $X\geq k$. Let $Y=X-k$. Thus, Y is the amount of additional time needed for E to occur. We can show that.

$$P(Y=t/X\geq k)=P(X=t)=pq^t$$

which implies that the additional time to wait has the same distribution as initial time to wait.

Since the distribution ~~does~~ does not depend upon k, it, in a sense, 'Lack memory' of how much we shifted the time Origin. If 'B' were waiting for the event E and is relieved by 'c' immediately before time k, then the waiting time distribution of 'c' is the same as that of 'B'.

Proof:

We have.

$$P(X \geq r) = \sum_{S=r}^{\infty} pq^{S} = p(q^{r} + q^{r+1} + q^{r+2} + \cdots) = \frac{pq^{r}}{(1-q)} = q^{r}$$

$$P(Y \geq t \mid X \geq k) = \frac{P(Y \geq t \cap X \geq k)}{P(X \geq k)} = \frac{P(X-k \geq t \cap X \geq k)}{P(X \geq k)}$$

$$= \frac{P(X \geq k+t)}{P(X \geq k)} = \frac{q^{t+k}}{q^{k}} = q^{t}$$

$$\therefore P(Y=t/X \geq k) = P(Y \geq t / X \geq k) - P(Y \geq t+1 / X \geq k)$$

$$= q^{t} - q^{t+1}$$

$$= q^{t}(1-q)$$

$$= pq^{t} = P(X=t)$$

Moments of Geometric Distribution:

$$\text{Mean} = \mu_1' = \sum_{x=0}^{\infty} x\, P(X=x) = \sum_{x=1}^{\infty} x pq^{x}$$

$$= pq \sum_{x=1}^{\infty} x q^{x-1} = pq(1-q)^{-2} = q/p$$

$$V(X) = E(X^2) - [E(X)]^2 = E[X(X-1)] + E(X) - [E(X)]^2$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)pq^{x} = \sum_{x=2}^{\infty} x(x-1)pq^{x}$$

$$= 2pq^{2} \sum_{x=2}^{\infty} \left[\frac{x(x-1)}{2\times1} q^{x-2}\right]$$

$$= 2pq^{2}(1-q)^{-3} = \frac{2q^{2}}{p^{2}}$$

$$V(X) = \mu_2 = 2q^{2}/p^{2} + q/p - q^{2}/p^{2} = q^{2}/p^{2} + q/p = q/p^{2}$$

Moment Generating Function of Geometric Distribution:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} q^x p = p \sum_{x=0}^{\infty} (e^t q)^x$$

$$= p(1-qe^t)^{-1} = \frac{p}{(1-qe^t)}$$

$$\mu_1' = \left[\frac{d}{dt} M(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} p(1-qe^t)^{-1}\right]_{t=0} = pq(1-q)^{-2} = \frac{q}{p}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M(t)\right]_{t=0} = \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2} = \frac{q^2 + pq}{p^2} = \frac{q}{p^2}$$

Hence the mean and Variance of the geometric distribution are q/p and q/p^2 respectively.

Hyper-Geometric Distribution:

Definition:

A discrete r.v X is said to follow the hypergeometric distribution with parameter N, M and n if it assumes only non-negative values and its P.m.f is given by.

$$P(X=k) = h(k; N, M, n) = \begin{cases} \dfrac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} & ; k = 0, 1, 2 \ldots \min(n, M) \\ 0, & \text{otherwise} \end{cases}$$

Where N is a positive Integer, M is a positive Integer not exceeding N and n is a positive Integer that is at most N.

Mean and Variance of the Hypergeometric Distribution!

$$E(x) = \sum_{k=0}^{n} k\,p(x=k) = \sum_{k=0}^{n} k \left\{ \binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n} \right\}$$

$$= \frac{M}{\binom{N}{n}} \sum_{k=1}^{n} \left\{ \binom{M-1}{k-1}\binom{N-M}{n-k} \right\} = \frac{M}{\binom{N}{n}} \sum_{x=0}^{m} \binom{A}{x}\binom{N-A-1}{m-x}$$

where $x = k-1,\ m = n-1,\ M-1 = A$

$$= \frac{M}{\binom{N}{n}} \binom{N-1}{m} = \frac{M}{\binom{N}{n}} \binom{N-1}{n-1} = \left(\frac{nM}{N}\right)$$

$$E\{x(x-1)\} = \sum_{k=0}^{n} k(k-1) \left\{ \binom{M}{k}\binom{N-M}{n-k} \div \binom{N}{n} \right\}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \sum_{k=2}^{n} \left\{ \binom{M-2}{k-2}\binom{N-M}{n-k} \right\}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \binom{N-2}{n-2} = \frac{M(M-1)\,n(n-1)}{N(N-1)}$$

$$\therefore E(x^2) = E[x(x-1)] + E(x)$$

$$= \frac{M(M-1)\,n(n-1)}{N(N-1)} + \frac{nM}{N}$$

Hence

$$V(x) = \frac{M(M-1)\,n(n-1)}{N(N-1)} + \frac{nM}{N} - \left(\frac{nM}{N}\right)^2$$

$$= \frac{nM(N-M)(N-n)}{N^2(N-1)}$$