OPERATIONS RESEARCH - I

18K3SAS4

UNIT - I

Introduction - origin - nature of OR - structure - characteristics - OR in decision making - models in OR - phase of OR - uses and limitations of OR - LPP - mathematical formulation of LPP - Graphical method.

UNIT - II

LPP - standard form of LPP - Maximization simplex method - artificial variable technique Two phase method - Big M method

UNIT - III

Duality in LPP - Formulation of Dual LPP - Primal - Dual relationship - Solving LPP using Dual concepts - Dual simplex method.

UNIT - IV

Transportation problem - Balanced & unbalanced T.P - Initial basic feasible solution - north west corner Rule - Row minima - column minima - matrix minima (LCM) Vogel's Approximation method - optimality Test - MODI Method.

UNIT - V

Assignment Problem - Introduction - Balanced - unbalanced - maximization - minimization - Hungarian method

UNIT - I

NATURE AND FEATURES OF OR

After tracing the process of establishment and growth of operation Research we can considered it as a source to other new sciences. Literally the word 'operation' may be defined as some action that we applied to some problems ~~or~~ OR hypotheses and the word 'research' is an organised process of seeking out facts about the same.

OR has been variously described as the "science of use", "quantitative common sense", "scientific approach" to decision making problems", etc. But only a few are commonly used and widely accepted, namely,

Definition: (i) "OR is a scientific method of providing executive departments with a quantitative basis for decisions under their control"

— P.M Morse and G.E Kimbal

scientific method in O.R.

The scientific method in operation Research is its most important feature. It consists of the following three phases:

Judgement phase:

This phase includes :

(i) identification of the real life Problems.

(ii) Selection of an appropriate goal and the value of various variables related to the goals.

(iii) appropriate scale of measurement and

(iv) formulations of an appropriate model of the problems. abstracting the essential information so that a solution at the decision - maker's goal can be sought.

Research phase:

This phase is the largest and longest among the other two. However, the remaining two are also equally important as they are also equally important as they provide the basis for a scientific method.

This phase utilizes : (i) observations and data collections for a better understanding of what the Problems it.

(ii) formulations of hypothesis and models

(iii) observations and experimentation to test the hypothesis on the basis of

additional data
(iv) analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness (v) Predictions of various results from the hypothesis and
(vi) Generalization of the results and consideration of alternative methods.

3

Action Phase:

This phase consists of making recommendations for decision process by those who first posed the problem for consideration, or by anyone in a position to make a decision influencing the operation in which the problem occurred.

Modelling in O.R.

A model in O.R is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. Models, in general cannot represent every aspect of reality because of innumerable and changing characteristics of the real life Problems to be represented. Instead, they are limited approximation of reality.

The Objective of a model is to provide a means for analysing the behaviour of the system for the Purpose of improving its Performance.

There are several models in each area of business, or industrial activity. for instance an account model is a typical budget in which business accounts are referred to with the intention to with the Intention of Providing measurements such as rate of expenses, quantity sold, etc. a mathematical equation may be considered to be a mathematical model in which a relationship between constants and variables is represented.

A model which has the possibility of measuring observations may be called a quantitative model; 5 a Product a device or any tangible thing used for experimentation may represent a physical model.

Following are the main characteristics that a good model for operations research study have:

(i) A good model should be capable of taking into account new formulation without having any significant change in its frame.

(ii) Assumptions made in the model should be as small as possible.

(iii) It should be simple and coherent. Number of variables used should be less.

(iv) It should be open to parametric type of treatment.

(v) It should not take much time in its construction for any Problem.

Advantages and Limitations of Models.

6 Models in OR are used as an ~~aid~~ aid for analysing complex problems the main advantages of a model are :

(i) Through a model ~~the~~ the Problem ~~The main advantages of~~ under consideration becomes controllable.

(ii) It provides some logical and systematic approach to the problem.

(iii) It indicates the limitations and scope of an activity.

(iv) It helps in incorporating useful tools that eliminate duplication of methods applied ~~to~~ solve any specific Problems

(v) It helps in finding avenues for new research and improvements in a system.

vi) It provides economic description and explanations of the operations of the system they represent.

However, besides the above advantages a model has the following limitations:

a) Models are only an attempt in understanding operations and should never be considered as absolute in any sense.

b) validity of any model with regard to corresponding operation can only be verified by carried out the experiment and observing relevant data characteristics.

c) Construction of models require experts from various disciplines.

General solutions method for O.R. models.

There are in general three methods for deriving the solution to an O.R. models: the analytical methods, the numerical methods and the Monte-Carlo methods. The first one is detective by nature while the second one is Inductive.

Analytical or Deductive Method

In these method classical optimization techniques such as Calculus Finite Differences, etc. are used for solving an O.R model. The kind of mathematics required depends upon the nature of the model. For example, in the inventory model, in order to calculate the economic order quantity (EOQ), the analytical method requires the first derivative of the mathematical expression.

$$TC = (D/Q)\, C_s + (Q/2)\, C_1,$$

equal to zero, as the first step towards identifying the optimum value of EOQ, viz. $\sqrt{2DC_1/C_s}$. This is because of the concept of maxima and minima for optimality. Here, TC = total annual cost, D = annual demand, Q = size of an order, C_s = set up cost per production run, and C_1 = holding cost for a unit in inventory.

Numerical Methods. Numerical methods are concerned with the iterative or trial and error procedures, through the use of numerical computation at each step. These numerical methods are used when some analytical methods fail to derive the solution.

9 The numerical method starts with a trial (initial) solution and continued with a set of rules for improving it towards optimality. The trial solution is then replaced by the improved one and the process is repeated until either no further improvement is possible or the cost of further computations cannot be justified. Thus the numerical methods are hit and trial methods that end at a certain step after which no further improvement can be made.

Monte Carlo methods.

These involve the use of probability and sampling concepts. The various step associated with a Monte Carlo method are as follows

a) For appropriate model of the system, make sample observations and determine the Probability distribution for the variables of interest.

b) Convert the probability distribution to cumulative distribution.

10

c) Select the sequence of random numbers with the help of random tables.

d) Determine the sequence of values of variables of interest with the sequence of random numbers obtained in the above step.

e) Fit an appropriate standard mathematical function to the values obtained in step (d).

The Monte Carlo method is essentially a simulation technique in which statistical distribution functions are created by generating a series a random ~~rat~~ numbers.

Methodology of operations Research

The systematic methodology developed for an O.R. study deals with problems involving conflicting multiple objectives policies and alternatives.

OR in the final analysis is a scientific methodology which is applied to the study of the operations of large complex organisations or activities with a view to assessing the overall implications of various alternative courses of action, thus
11 providing an improved basis for managerial decisions.

The O.R. approach to problem solving consists of the following seven steps.

1. Formulate the problem.
2. Construct a mathematical model.
3. Acquire the input data.
4. Derive the solution from the model.
5. Validate the model.
6. Establish control over the solution.
7. Implement the final results.

Formulating the problem:

The first step in O.R is to develop a clear and concise statement of the problem, ie to identify the problem correctly. Once identified the problem the problem is rightly identified we can say that fifty per cent of the problem is solved. After identification, formulation of the problem is to be carried out. For this it is necessary to study comprehensively the components of the problems, viz

a) the decision - maker.

12 b) the objective.

c) the ~~relative~~ alternative courses of action, and

d) the environment.

After the identifying the components of the problem, the relationship that exists among the components of the problem should be analysed. Also there must be complete agreement on the above points between person initiating the O.R study and the persons performing the study.

2) Constructing a Mathematical Model.

The next step is to build a suitable mathematical model. Model construction consists of hypothesizing relationships between variables subject to, and not subject to control by decision-maker.

A mathematical model might take should include mainly the following three basic sets of elements:

a) Decision variables and Parameters

b) Constraints or Restriction, and

c) objective function.

A generalised mathematical model might take the form: $E = f(x_1, y_1)$ where f represents a system of mathematical relationships between the measures of effectiveness of the objective sought (E) and the variables. both controllable (x_1) and uncontrollable (y_1)

3) obtaining the Input Data. once the mathematical model of the problem has been formulated. the next step

is to obtain the data to be used in the model as input. Since the quality of data determines the quality of output, the importance of obtaining accurate and complete data cannot be over emphasised.

Obtaining correct and relevant data may, however, be a difficult exercise when relatively large problems are involved. A number of sources. Such as company reports and documents, interviews with the company personnel and so on may by used for collecting data

4) Deriving the solution from the Model.
Having collect the input data the next step is to determine the values of decision variables that optimize the given objective function. This deals with the mathematical calculations for obtaining the solution to the model

It may be noted that the solutions can be categorised as being

(i) feasible or infeasible,

(ii) optimum or non-optimum,

(iii) unique or multiple

* A solution that satisfies all the constraints of the problem including the condition of non-negativity is known as a feasible solution, while the infeasible solution is one that does not satisfy the condition of non-negativity of variables.

* An optimum solution to the problem is one of feasible solutions which also optimises the objective function. The feasible solutions other than the optimum solution are called non-optimum solutions.

* If there exists only one optimum solution to be problem it is said to be an unique optimum solution. On the other hand if two or more optimum solutions to the problem exists then multiple optimum solutions are said to exist.

While algorithms exist for most of the standardised problems, there are also some numerical techniques, which yield solutions that are not necessarily optimum. Heuristics and simulation illustrate those methods. Heuristics are step by step logical rules that yield values of the variables which satisfy all the constraints, but not necessarily provide optimum solution. However these values provide an acceptable value for the objective function. Similarly, the technique of simulation ~~solution~~ is applied where a given system is soug ~~up~~ and experimented with. Solutions using simulation need not be optimal because technique only descriptive in nature.

In addition to the solution of the , it is also sometimes essential to perform sensitivity analysis i.e to determine the behaviour of the system changes in the system's parameters and specification. This is done because the input may not be accurate or stable, and the structural assumptions of the model may not be valid.

5. Validation of the Model. Validating a model requires of determining as if the model can reliably predict the actual system

If during validation, the solution cannot be implemented, one needs to be modified. In such a case one must return to the problem formulation step and carefully make the appropriate modifications to represent more realistic situation (formulation step and carefully make the appropriate)

A model must be applicable for a reasonable time period and should be updated from time to time taking into consideration the past, present and future aspects of the problem.

Establishing control over the solution:

After testing the models and its solution, the next step of the study is to establish control over the solution by proper feedback of the information on ~~of the information on~~ the variables which deviated significantly. As soon as one or more of the controlled variable change significantly the solution goes out of control. In such a situation the model may accordingly be modified.

Implementation of the final results:

Finally tested the result of the model are implemented to work.

This would basically involve a careful explanation of the solution to be adopted and its relationship with the operating realities. This stage of O.R investigation is executed primarily through the co-operation of both the O.R experts and those who are responsible for managing and operating the system.

Operation Research and decision-making.

Every industrial enterprise or business house works on a properly established management information system. such a usually assumes a three-tier position at the level of

(i) strategic planning,
(ii) managerial control, and (iii) operational control

These levels are identified with the top managements, middle management and supervisory management respectively. Many prefer to call these three-tier activities as policy-making decision making and activity oriented

For a competent management, decision making it is a major task.

Decision making is not the headache of only the management, rather all of us make decision. We daily decide about many minor and major issues. The essential characteristics of all decisions are: 1) objective 2) alternatives at the disposal, 3) influencing factors.

once these characteristics are known, one can think of improving the characteristics so as to improve upon the decision itself.

Let us consider a situation where a decision concerns spending summer vacations at a hill resort the problem may be to decide the mode of conveyance from amongest the three alternatives: train, bus and a taxi.

At the first level of decision-making, bus is chosen at the mode of conveyance just by intuition. At the second level of decision-making, the three modes are compared and it is decided qualitively that the bus will be preferred since it is less time consuming compared to the train and cheaper as compared to taxi. At the third level of decision making, the three alternatives are compared and it is suggested that the bus will be chosen as it will be taking only half the time taken by train and shall be 40% less costlier than the taxi.

Although outcome of all these decision is the same, one can easily judge the quality of each decision. We may brand the first decision as 'bad' since it is highly emotional. While we may call the second decision as good since it is scientific though qualitative. The third decision is doubtlessly the best as it is scientific as well as quantitative.

It is this scientific quantification used in O.R. that helps management to make better decision.

Operations Research may be regarded as a tool employed to increase the effectiveness of managerial decisions as an objective supplement to the subjective ~~feel~~ feelings of the decision maker

For instance, in distribution allocation areas, O.R may suggest. The best locations for agencies, warehouse as well as the most economical kind of transportation : in marketing areas it may aid in indicating the most profitable type, use and size of advertising campaigns in regard to available financial limits. O.R may suggest alternative courses of action when a problem is analysed and solution is attempted. However, the study of complex problems by O.R. techniques becomes useful only when a choice between two or more courses of decision is possible.

OR may be regarded as a tool that enables the decision - maker to be objective in choosing an alternative from among many that he can conceive of following are the salient advantages of an operations research study approach in decision making.

(i) Better decisions.

OR models frequently yield actions that do improve on intuitive decisions making. A situation may be so complex that the human mind can never hope to assimilate all the significant factors without the aid of O.R guided computer analysis.

(ii) Better coordination.

Sometimes operations research has been instrumental in bringing order out of chaos. For instance, an O.R - oriented planning mode becomes a vehicle for coordinating marketing decisions within the limitations imposed on manufacturing capabilities

iii) Better control

The management of large organisations recognize that it is extremly costly to provide continuous executive supervisions over routing decisions. An OR - approach, thereby regained new freedom to the executive to devote his attention to more pressing matters. The most frequently adopted application in this category deal with production scheduling and inventory replenishment.

(v) Better systems

often an OR study is initiated to analyse a particular decision problem, such as whether to open a new warehouse, afterwards the approach is further development into a system to be employed repeatedly. Thus the cost of undertaking the first application may produce benefits.)

LINEAR PROGRAMMING PROBLEM

A linear programming problem (LPP) consists of three components namely the (i) decision variables (activities) (ii) the objective (goal) (iii) the constraints (restrictions)

(i) The decision variables refer to the activities that are competing one another for sharing the resources available. These variables are usually enter related in terms of utilisation of resources and need simultaneous solution. All the decisions variables are considered as continuous, controllable and non-negative.

(ii) A linear programming problem must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of profit (sales) maxisation cost (time) minimisation and so on. The relationship among the variables representing objective must be linear.

(iii) There are always certain limitations (or constraints) on the use of resources, such as labour, space, raw material, money etc. that limit the degree to which an objective can be achieved.

Such constrains must be expressed as linear inequalities or equalities in terms of decision variables

Basic assumptions.

The following four basic assumptions are necessary for all linear programming problems.

a) Certainty: In all LPP's it is assumed that all the parameters such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decision variables must be known and fixed. In other words, this assumption means that all the coefficient in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

b) Divisibility (or continuity)

This implies that solution values of the decision variables and resources can take on any non-negative values, including fractional values of the decision variables, for instance it is Possible to Produce 4.35 quintals of wheat or 17.35 thousand kilometers of cloth or 6.52 thousand kilo liters of milk, so these variables are divisible But it is not possible to produce 2.6 refrigerators. Such variables are not divisible and hence are to be assigned integer values. when it is necessary to have integer variables the integer programming problem is considered to attain the desired values.

c) Proportionality :

This requires the contribution of each decision variables is both the objective function and the constraints to be directly proportional to the value of the variables. for example If Production of one unit of a particular product uses 3 hours of a particular resource. Then the production of

6 units of the Product uses
3×6, i.e 18 hours of that resource

d) Additivity.

The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable and the sum of the resources used by each decision variables respectively. for example, the total profit earned by the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumer by A and B must be equal to the sum of resources used for A and B individually.

Mathematical Formulation of the problem.

Procedure

the procedure for mathematical formulation of a linear programming problem consists of the following major steps:

Step 1: study the given situation to find the key decisions to be made.

Step 2: Identify the variables involved and designate them by symbols x_j ($j = 1, 2, \ldots$)

Step 3: State the feasible alternative which generally are: $x_j \geq 0$, For all j.

Step 4: identify the constraints in the problem and express then as linear in equalities or equations. L.H.S of which are linear function of the decision variables

Step 5:

Identify the objective function and express it as a linear function of the decision variables.

OM

Graphical Solution method:

the major steps in the solution of a linear Programming Problem by graphical method are summarised as follows

Step 1: Identify the Problem the decision variables, the objective and the restriction

Step 2: Set up the mathematical formulation of the Problem.

Step 3: Plot a graph representing all the constraints of the Problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the Problem and is restricted to the first quadrant only.

Step 4: the feasible region obtained in step 3 may be bounded or unbounded computed the coordinates of all the corner points of the feasible region

Step 5: Find out the value of the objective function at each corner (solution) point determined in step 4

Step 6: select the corner point that optimises (maximizes or minimizes) the value of the objective function.

It gives the optimum feasible solution

Remarks 1: The above method is known as search Approach method.

2) Another method known as Iso-profit or Iso-cost approach, involves the following steps.

a) first four steps are same as in the search approach. In the fifth step we choose a convenient profit (or cost) and draw iso-profit (iso-cost) line so that it falls within the feasible region.

b) Move this iso-profit (or iso-cost) line parallel to itself farther (closer) from (to) the origin.

c) Identify the optimum solution as the co-ordinates of that point on the feasible region touched by the highest possible iso-profit line (or lower-possible iso-cost line)

d) compute the optimum feasible solution.

PROBLEM:

Find the minimum value of $z = 20x_1 + 10x_2$ subject to the constraint:

$$x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30,$$

$$4x_1 + 3x_2 \geq 60, \text{ and } x_1, x_2 \geq 0$$

CALCULATION:

$$x_1 + 2x_2 = 40 \rightarrow ①$$

$$3x_1 + x_2 = 30 \rightarrow ②$$

$$4x_1 + 3x_2 = 60 \rightarrow ③$$

Put $x_1 = 0$ in ① we get

$$0 + 2x_2 = 40$$

$$x_2 = 20 \; [x_1 = 0, x_2 = 20]$$

0, 20
40, 0

Put $x_2 = 0$, in ① we get

$$x_1 = 40 \quad [x_1 = 40, x_2 = 0]$$

40, 20

Put $x_1 = 0$ in ② we get

$$3(0) + x_2 = 30$$

$$x_2 = 30 \quad [x_1 = 0, x_2 = 30]$$

Put $x_2 = 0$, in ② we get

$$x_1 = \frac{30}{3} = 10 \quad [x_1 = 10, x_2 = 0]$$

Put $x_1 = 0$ in ③ we get

$$3x_2 = 60$$

$$x_2 = \frac{60}{3} = 20 \quad [x_1 = 0, x_2 = 20]$$

Put $x_2 = 0$ in ③ we get

$$4x_1 = 60$$

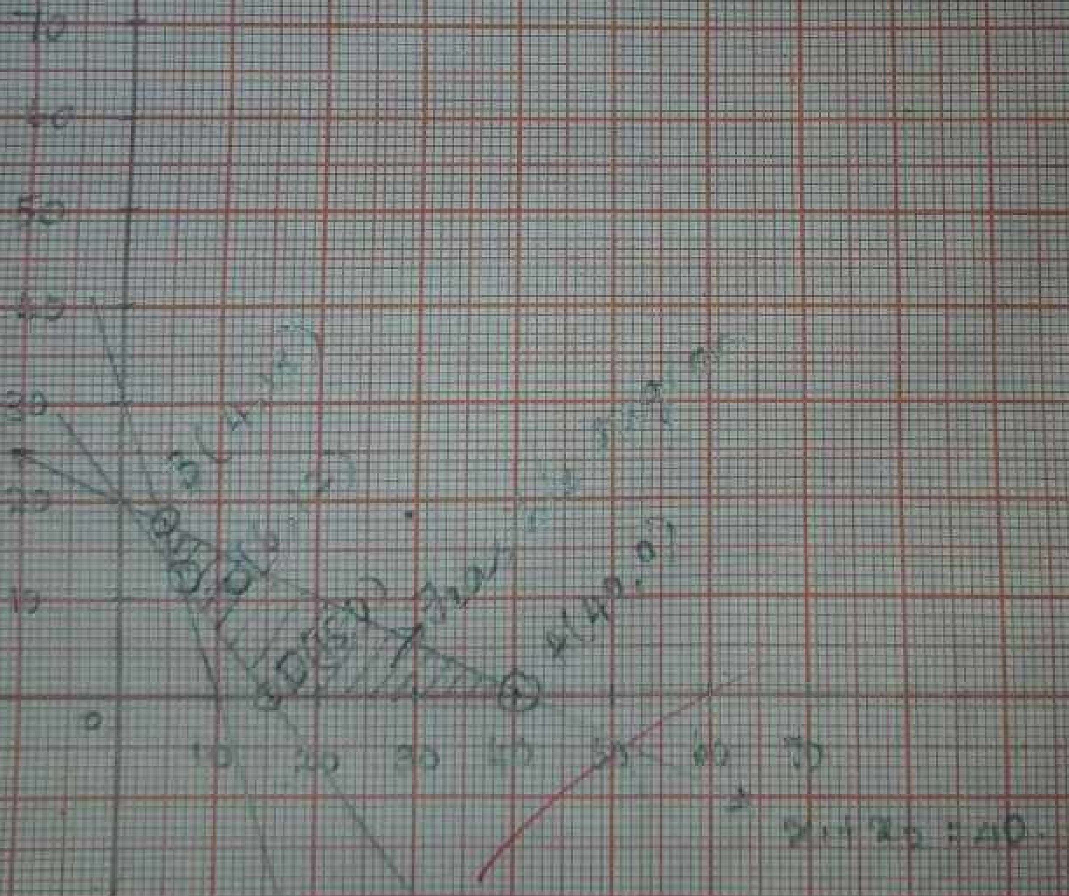
$$x_1 = \frac{60}{4} = 15 \quad [x_1 = 15, x_2 = 0]$$

RESULT:

The feasible region is ABCD

Corner Points	minimize $z = 20x_1 + 10x_2$
A(15, 0)	300
B(40, 0)	800
C(4, 18)	260
D(6, 12)	240 (minimum)

The minimum value of x occurs at D (6, 12). Hence the optimal solution is $x_1 = 6$, $x_2 = 12$.


70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
A(40,0)
$x_1+2x_2=40$

PROBLEM:

Find the maximum value of $z = 7x_1 + 3x_2$ subject to the constraints

$$x_1 + 2x_2 \geq 3$$

$$x_1 + x_2 \leq 4$$

$$0 \leq x_1 \leq 5/2$$

$$0 \leq x_2 \leq 3/2$$

CALCULATION:

$$x_1 + 2x_2 = 3 \rightarrow ①$$

$$x_1 + x_2 = 4 \rightarrow ②$$

$$x_1 = 5/2$$

$$x_2 = 3/2$$

Put $x_1 = 0$ in equation ① we get

$$0 + 2x_2 = 3$$

$$x_2 = 3/2 \quad [x_1 = 0, \; x_2 = 3/2]$$

Put $x_2 = 0$ in equation ① we get

$$x_1 + 2(0) = 3$$

$$x_1 = 3 \quad [x_1 = 3, \; x_2 = 0]$$

Put $x_1 = 0$ in equation ② we get

$$0 + x_2 = 4$$

$$x_2 = 4 \quad [x_1 = 0, \; x_2 = 4]$$

Put $x_2=0$ in equation ② we get,

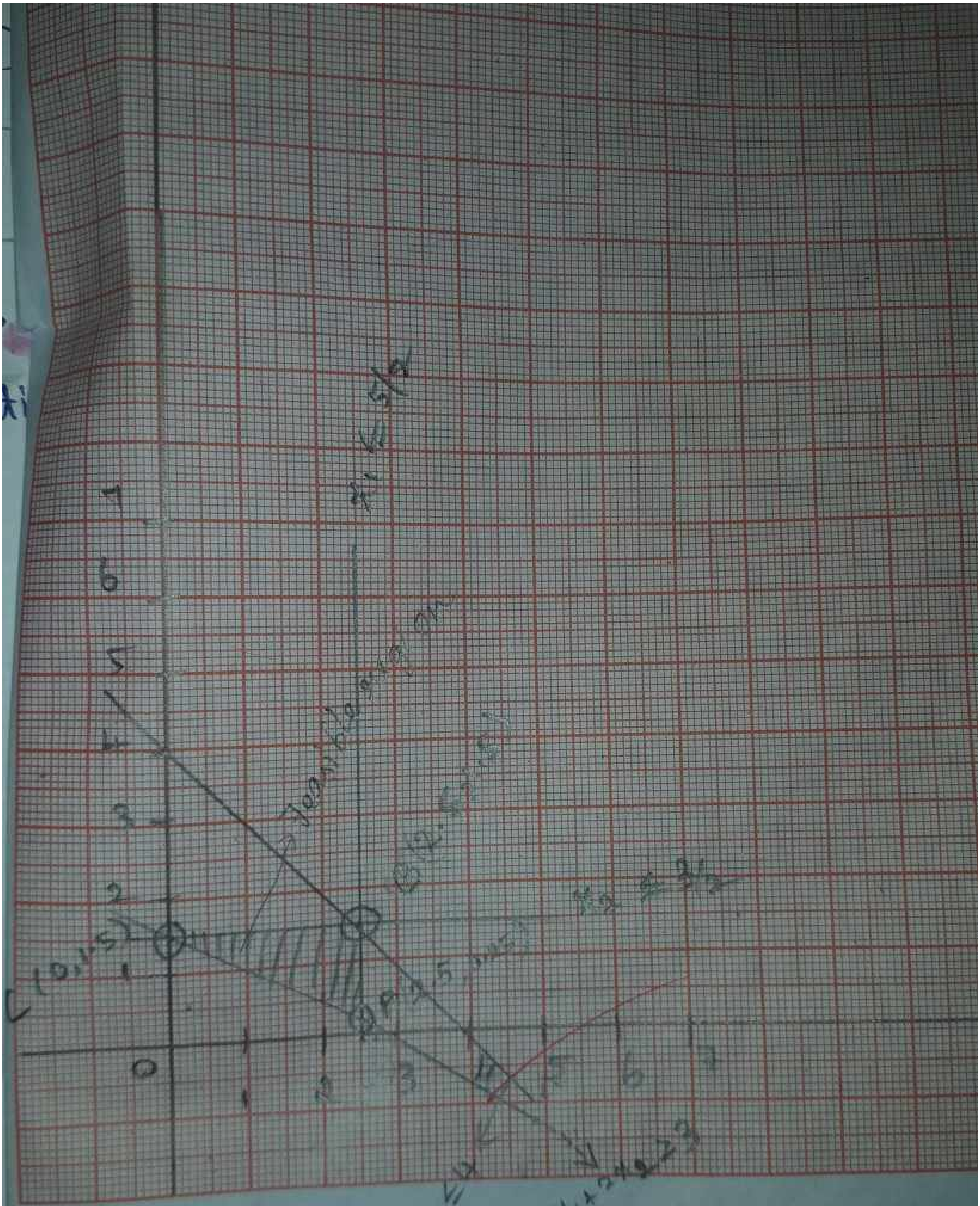
$x_1+0=4$; $x_1=4$. $[x_1=4, x_2=0]$

RESULT:-

The feasible region is ABC

Corner points	Maximum $z=7x_1+3x_2$
A(2·5, 0·25)	18·25
B(2·5, 1·5)	22·00 (maximum)
C(0, 1·5)	4·50

The maximum value of x occurs at B(2·5, 1·5). Hence the optimum solution is $x_1=2·5$, $x_2=1·5$.



$x_1 \leq 5/2$
Feasible region
$x_2 \leq 3/2$
$(0, 1.5)$
$(2.5, 1.5)$
$(2.5, 0)$
$x_1 + x_2 \leq 4$
$x_1 + 2x_2 \geq 3$

UNIT - II

1) Slack & surplus
2) optimum feasible soln
3) condition

Linear Programming Problem

A linear Programming Problem (LPP) consists of three components, namely the (i) decision variables (activities) (ii) the objective (goal), and (iii) the constraints (restrictions)

(i) The decision variables refer to the activities that are competing one another for sharing the resources available. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. All the decision variables are considered as continuous controllable and non-negative

(ii) A linear Programming Problem must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of Profit (sales) maximisations, cost (time) minimisation, and so on. The relationship among the variables representing objective must be linear

(iii) There are always certain limitations (or constrains) on the use of resources, such as labour, space, raw material, money etc that limit the degree to which an objective can be achieved.

such constraints must be expressed as linear inequalities or equalities in terms of decision variables.

Basic assumption:

The following four basic assumptions are necessary for all linear programming problems.

The standard form

The general linear programming problem in the form

Maximize or Minimize

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$$

subject to the constraints. $i = 1, 2, \dots n$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{in}x_n = b_i ,$$

$$x_1, x_2, \dots x_n \geq 0$$

is known as in a standard form.

The characteristics of this form are

i) All the constraints are expressed in the form of equations, except for the non - negative restrictions

(ii) The right hand side of each constraint equation is non - negative

The inequality constraints can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added (slack variable) if the constraint is of "$\leq$"

type and subtracted (surplus variables) if the constraint is of "$\geq$" type

In matrix notation the standard form of LPP can be expressed as

Maximize or minimize $z = cx$

subjects to the constraints

$$Ax = b$$
$$x \geq 0.$$

where $x = (x_1, x_2, \dots x_n)$ $c = (c_1, c_2, \dots c_n)$

$b^T = (b_1, b_2, \dots b_m)$ and $A = (a_{ij})$

$i = 1, 2, \dots m$; $j = 1, 2, \dots n$.

Remarks:

1. The coefficients of slack and/or surplus variables in the objective function are always assumed to the zero, so that the conversion of the constraints to a system of simultaneous linear equations does not change the objective function under consideration.

2) the linear programming form:

Maximize $z = cx$ subject to the constraints $Ax \leq b$, $x \geq 0$. is known as the canonical form of the LPP.

Theorem 2.1:

The set of feasible solutions to an L.P.P is a convex set

Proof:

Let the LPP be to determine so as to maximize the linear function $z = c^T x$ subject to the constraints.

$Ax = b$, $x \geq 0$.

Let $x^{(1)}$, $x^{(2)}$ be two feasible solutions to this problem, so that

$Ax^{(1)} = b$; $Ax^{(2)} = b$; $x^{(1)} \geq 0$ and $x^{(2)} \geq 0$

Now, consider convex combination of $x^{(1)}$ and $x^{(2)}$, namely:

$$x = \lambda x^{(1)} + (1-\lambda) x^{(2)}, \quad 0 \leq \lambda \leq 1$$

clearly,

$$Ax = A[\lambda x^{(1)} + (1-\lambda) x^{(1)}]$$

$$= \lambda A x^{(1)} + (1-\lambda) A x^{(2)}$$

$$= \lambda b + (1-\lambda) b = b$$

Again, Since $x^{(1)} \geq 0$, $x^{(2)} \geq 0$, and λ

$1 - \lambda \geq 0$, $\therefore$ $x \geq 0$.

Hence x is also feasible solution to the problem, Thus, the set

$S = \{x \mid x$ is a feasible solution to an LPP$\}$

is a convex set.

Remark

In general a convex set S is either empty or unbounded or closed.

The empty set occurs when the constraints are not satisfied simultaneously. In this case the system yields no solution. An unbounded set implies that the region of feasible solution is not constrainted in at least one direction.

A closed set implies that the region of feasible solution is a convex polyhedron, since it is defined by the intersections of finite number of linear constraints.

Definition (Basic solution)

Given a system of m simultaneous linear equations in n unknowns $(m < n)$

$$Ax = b, \quad x^0 \in R^n,$$

where A is an $m \times n$ matrix of rank m. Let B be any $m \times n$ submatrix, formed by m linearly independent columns of A. Then a solution obtained by setting $n-m$ variables not associated with the columns of B, equal to zero, and solving the resulting system, is called a basic solution to the given system of equations.

the m variables which may be all different from zero, are called basic variables. The $m \times m$ non-singular submatrix B is called a basic matrix with the columns of B as basic vectors.

<u>Remarks:</u> The ~~so~~ name basic solutions as used above merits a word of caution. If B is the basis sub-matrix chosen, then the basic solution to the system is

$$x_B = B^{-1} b.$$

But $x_B' \in R^m$ and as such cannot be called a solutions of the given system. Truly speaking, if x_B is a basic solution, then a solution to the given system is $\{x_B^T, 0\}$ where $x_B^T \in R^m$, and $0 \in R^{n-m}$ however, we shall follow the current usage and call x_B a basic solution of the given system, remembering all the time that the actual solution is $[x_B^T, 0]$.

Definition (Degenerate Solution).

A basic solution to the system is called degenerate if one or more of the basic variables vanish

Similarly, the other two solutions are:

(Basic) $x_2 = 5/3$, $x_3 = -1/3$ (non basic) $x_1 = 0$

and (Basic) $x_1 = 1$, $x_3 = 0$ (non basic) $x_2 = 0$

In each of the two basic solutions at least one of the basic variables is zero, hence two of the basic solutions are degenerate solutions.

Definition (Basic feasible solution)

A feasible solution to an LPP, which is also a basic solution to the Problem is called a basic feasible solution to the LPP

Illustrations:

1) In sample problem 401 observe that [5, 0, -1] is not a feasible solutions, only basic feasible solutions are:

(i) $x_2 = 5/3$ and $x_2 = 2/3$

(ii) $x_1 = 2$ and $x_2 = 1$

11

2) In sample problem 4.2, the non degenerate solution $[0, 5/3, -1/3]$ is not feasible, only basic feasible degenerate solutions are:

i) $x_1 = 1$ and $x_2 = 0$

(ii) $x_1 = 1$ and $x_3 = 0$

Definition (Associated cost vector)

Let x_B be a basic feasible solution to the L.pp:

Maximize $z = cx$ subject to : $Ax = b$
and $x \geq 0$

then, the vector $c_B = (c_{B1}, c_{B2}, \dots c_{Bm})$

where c_{Bi} are components of c associated with the basic variables, is called the cost vector associated with the basic feasible solutions x_B.

It is obvious that the value of the objective function for the basic feasible solutions x_B is given by $z_0 = c_B x_B$

Definition (Improved basic feasible solution)

Let x_B and $\hat{x}_B$ be two feasible solution in the standard L.p.p. Then $\hat{x}_B$ is said to be an improved basic feasible solutions as compared to x_B if

$$\hat{C}_B \hat{x}_B \geq C_B x_B$$

where $\hat{C}_B$ is constitude of cost components corresponding to $\hat{x}_B$

Definition (optimum basic feasible solution)

~~A~~ A basic feasible solutions x_B to the L.P.P:

Maximize $z = cx$ subject to $Ax = b$ and $x \geq 0$ is called an optimum basic feasible solutions if $z_0 = C_B x_B \geq z^*$ where z^* is the value of objective function for any feasible solutions.

The computational procedure:

The two fundamental conditions on which the simplex method is based are:

(i) condition of feasibility:

It assumes that if the initial (strating) solution is basic feasible, them during computation only basic feasible solutions will be obtained.

(ii) condition of optimality:

It guarantees that only better solutions will be encountered.

The computational procedure of the simplex method requires the construction of the simplex table, the initial simplex table is constructed by writing out the coefficients and the constraints of LPP in a systematic tabular form.

Following is a specimen of the simplex table which conveniently displays the basic feasible solution x_B, the associated cost vector C_B, the Basic variables, column vectors Y_i's corresponding to basic as well as non-basic variables, the corresponding cost coefficients and the net evaluations in a tabular form.

			c_1	c_2	...	c_n
C_B	Y_B	X_B	Y_1	Y_2	---	Y_n
C_{B1}	Y_{B1}	X_{B1}	Y_{11}	Y_{12}	...	Y_{1n}
C_{B2}	Y_{B2}	X_{B2}	Y_{21}	Y_{22}	..	Y_{2n}
⋮	⋮	⋮	⋮	⋮		⋮
C_{Bm}	Y_{Bm}	X_{Bm}	Y_{m1}	Y_{m2}		Y_{mn}
		Z_n	Z_1-C_1	Z_2-C_2		Z_n-C_n

The optimal solution to a general LPP (when it exist) is obtained in the following major steps

Step 1: Select an initial (starting) basic feasible solutions to initiate the algorithm. (use theorem 4-1 for the existence of a basic solutions)

Step 2: Check the objective function to see whether there is some non-basic variable that would improve the objective function if brought in the basis. If such a variable exists, go to step 3, otherwise stop (use theorem 4-5 and 4-6 for the check)

Step 3: Determine how large the variable found in step 2 can be until one of the basic variables in the current solutions becomes zero. and eliminate the latter variable the let next trial solution contain the newly found variable instead for some $j = \dots$

Step 4: check for optimality the current solution (use theorem 4.7)

Step 5: Continue the iteration until either an optimum solutions is attained or there is an indication that an unbounded solution exists

Note: To simplify the computations, we usually consider the strating basis matrix B as an identity matrix I i.e, $B = I$

Therefore $Y_j = B^{-1} a_j = I^{-1} a_j = I a_j = a_j$

$$I_{2} n_{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$$

$$= \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix} = y_1$$

$$X_B = B^{-1} b = I^{-1} b = I b = b$$

etc.

The simple Algoritam

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solutions are as follows.

Step 1: Check whether the objective function of the given L.P.P is to be maximized or minimized. If it is to be ~~max~~ minimized then we convert it into a Problem of maximization at by using the result (x) by

minimum z = - maximum (-z)

Step 2: Check whether all bi (i = 1, 2, ... m) are non-negative. If any one of bi is negative then multiply the corresponding inequation of the constraints by -1, so as to get all bi (i = 1, 2, ... m) non-negative.

Step 3: Convert all the inequations of the constraints into equations by introducing slack and /or surplus variables in the constraints. Put the costs of these variables equal to zero.

Step 4: Obtain an initial basic feasible solution to the Problem in the form $x_B = B^{-1} b$ and Put it in the first column of the simple table.

Step 5:

compute the net evaluations $z_j - c_j$ ($j = 1, 2, \ldots n$) by using the relation $z_j - c_j = c_B Y_j - c_j$, where $Y_j = B^{-1} a_j$.

Examine the sign $z_j - c_j$

(i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution x_B is an optimum basic feasible solution.

(ii) If at least one $(z_j - c_j) < 0$, proceed on to the next step

Step 6:

If there are more than one negative $z_j - c_j$ then choose the most negative of them. Let it be $z_r - c_r$ for some $j = r$

i) If all $y_{ir} \leq 0$ ($j = 1, 2, \ldots m$), then there is an unbounded solution to the given problem

(ii) If atleast one $y_{ir} > 0$ ($j = 1, 2, \ldots m$) then the corresponding vector y_r enters the basis Y_B.

Step 7

Compute the ratios $\left\{ \frac{x_{Br}}{y_{ir}} , y_{ir} > 0, j = 1, 2, \ldots m \right\}$ and choose the minimum of them. Let the minimum of these ratios be x_{BK}/y_{Kr}, then the vector Y_K will level the basis Y_B. The common element y_{Kr}, which is in the K^{th} row and the r^{th} column is known as the leading element (or pivotal element) of the table

~~UNIT - 8~~

Step 8:

Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeroes by making use of the relations:

$$\hat{y}_{ij} = y_{ij} - \frac{y_{kj}}{y_{kr}} y_{ir} \quad i = 1, 2, \ldots m+1 ; i \neq k$$

and $\hat{y}_{kj} = \frac{y_{kj}}{y_{kr}}$ $j = 0, 1, 2, \ldots n$

Step 9:

Go to step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Use of Artificial variables:

We have seen that in the computational procedure of the simplex method, it is most ~~confident~~ convenient to have the slack variables as the starting (initial) basic variables. Thus if the original constraint is an equation or is of the type (≥) we may no longer have a ready starting basic feasible solution.

In order to obtain an initial basic feasible solution, we first put the give LPP into its standard form and then a non-negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so-added variable and plays the same role as a slack variable in providing the initial basic feasible solution. However, since such artificial variables have no physical meaning from the standpoint of the original problem, the method will be valid only if we are able to force these variables to be out at zero level when the optimum solution is attained in other words, to get back to the original problem, artificial variables must be driven to zero in the final solution; otherwise the resulting solution may be infeasible

Two methods are generally employed for the solutions of linear Programming Problems having artificial variables:

1) Two - phase method ; and

2) Big - M method or method of Penalties.

Two phase Method :

In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints (known as auxilary LPP) to get a basic feasible solutions to the original L.pp second phase them optimizes the original objective function starting with the basic feasible solution obtained at the end of phase 1.

The iterative procedure of the algorithm may be summarise at below:

Step 1:

write the given L.pp into its standard form and check whether there exists a starting basic feasible solution.

a) If there is a ready starting basic feasible solution, go to phase 2

b) If there does not exist a ready starting basic feasible solutions, go on to the next step.

Phase 1.

Step 2:

Add the artificial variables to the left side of the each equation that lacks the needed starting basic variables. Construct an auxiliary objective function aimed at minimising the sum of all artificial variables.

Thus, the new objective is to

$$\text{Minimize } z = A_1 + A_2 + \cdots + A_n$$

$$\text{Maximize } z^* = -A_1 - A_2 - \cdots - A_n$$

where A_j ($j = 1, 2, \ldots m$) are the non-negative artificial variables.

Step 3: Apply simplex algorithm to the specially constructed LPP. The following three cases may arise at the last interation.

a) max $z^* < 0$ and at least one artificial variable is present in the basis with positive value. In such a case the original L.P.P does not posses any feasible solution.

b) max $z^* = 0$ and at least one artificial variables is present in the basis at zero value. In such a case, the original L.P.P posses the feasible solutions. In order to get basic feasible solution we may proceed directly to Phase 2 or else eliminate the artificial basic variable and then proceed to phase 2.

c) maximum $z^* = 0$ and no artificial variable is present in the basis. In such a case, a basic feasible solution to the original L.P.P has been found. Go to phase 2.

Phase 2

Step 4: consider the optimum basic feasible solution of phase 1 as a starting basic feasible solutions for the original L.P.P. Assign actual ~~the~~ co-efficients to the variables in the objective function and a value zero to the artificial variables that appear at zero value in the final table of phase 1.

Apply usual simplex algorithm to the modified simplex table to get the optimum solutions of the original ~~solution~~ Problems.

Note: Artificial variables that do not appear in the basic solution may be deleted from the simplex table totally

Big-M Method (Method of Penalties)

10M or 5M

The Big-M Method is an alternative method of solving a linear Programming Problem involving artificial variables. In this method we assign a very high penality (say M) to the artifical function

The iterative procedure of the algorithm is given below:

Step 1:

Write the given L.P.P into its standard form and check whether there exists a starting basic feasible solutions

a) If there is a ready starting basic feasible solution, move on to step 3.

b) If there does not exists a ready starting basic feasible solution on to step 2.

Step 2:

Add artificial variables to the left hand side of each equation that has no obvious starting basic variables. Assign a very high penalty (say M) to these variables in the objective function

Step 3:

Apply simplex method to the modified L.P.P following cases may arise at the last iteration:

a) Atleast one artificial variable is presente in the basis with a Positive value. In such a case, the given LPP does not posses an optimium basic feasible solutions. The given problem is said to have a Pseudo-optimum basic feasible solution.

PROBLEM :

Use simplex method to maximize

$z = 5x_1 + 3x_2$

subject to the constraints :

$x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12 \quad ; \quad x_1, x_2 \ge 0$

CALCULATION

By introducing slack variables $S_1, S_2, S_3 \ge 0$ respectively.

Maximize $z = 5x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3$

subject to the constraints :

$x_1 + x_2 + S_1 + 0S_2 + 0S_3 = 2$

$5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 10$

$3x_1 + 8x_2 + 0S_1 + 0S_2 + S_3 = 12$

$x_1, x_2, S_1, S_2, S_3 \ge 0$.

Matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 12 \end{bmatrix}$$

INITIAL SIMPLEX TABLE:

		C_j	5	3	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Min $= \frac{x_B}{Y_1}$
0	y_3	2	1	1	1	0	0	2
0	y_4	10	5	2	0	1	0	2
0	y_5	12	3	8	0	0	1	4
	z_j	0	0	0	0	0	0	
	$z_j - c_j$	0	-5	-3	0	0	0	

Introducing y_1 and y_3

		C_j	5	3	0	0	0
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5
5	y_1	2	1	1	1	1	0
0	y_4	0	0	-3	-5	1	0
0	y_5	6	0	5	-3	0	1
	z_j	10	5	5	5	5	0
	$z_j - c_j$	10	0	2	5	5	0

Result:

$z_j - c_j \geq 0$

Since all $z_j - c_j \geq 0$, the current basic feasible solution is optimal therefore

$Y_1 = 2$

Max $z = 10$.

PROBLEM

Use simplex method to maximize

$z = 5x_1 + 4x_2$

Subject to the constraints

$4x_1 + 5x_2 \leq 10$

$3x_1 + 2x_2 \leq 9$

$8x_1 + 3x_2 \leq 12$, $x_1, x_2 \geq 0$

CALCULATION

By introducing slack variable

$S_1, S_2, S_3 \geq 0$

Maximize $z = 5x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to the constraints

$4x_1 + 5x_2 + S_1 + 0S_2 + 0S_3 = 10$

$3x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 9$

$8x_1 + 3x_2 + 0S_1 + 0S_2 + S_3 = 12$

$x_1, x_2, S_1, S_2, S_3 \geq 0$.

INITIAL SIMPLEX TABLE

		C_j	5	4	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Min $= \frac{X_B}{Y_{ij}}$
0	Y_3	10	4	5	1	0	0	2.5
0	Y_4	9	3	2	0	1	0	3
0	Y_5	12	8	3	0	0	1	1.5
	z_j	0	0	0	0	0	0	
	$z_j - c_j$	0	-5	-4	0	0	0	

Introducing y_1, drop y_5

		c_j	5	4	0	0	0
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5
0	y_3	4	0	7/2	1	0	-1/2
0	y_4	9/2	0	7/8	0	1	-3/8
5	y_1	3/2	1	3/8	0	0	1/8
	z_j	15/2	5	15/8	0	0	5/8
	$z_j - c_j$	15/2	0	-17/8	0	0	5/8

		c_j	5	4	0	0	0
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5
4	y_2	8/7	0	1	2/7	0	-1/7
0	y_4	7/2	0	0	-1/4	1	-2/8
5	y_1	15/14	1	0	-3/28	0	5/28
	$z_j - c_j$ $z_j - c_j$	139/14	0	0	17/28	0	9/28

Result:

$$z_j - c_j \geq 0$$

Since, all $z_j - c_j \geq 0$. the current basic feasible solution is optimal therefore

$$y_2 = 8/7, \quad y_1 = 15/14$$

$$\text{Maximum } z = 139/14$$

PROBLEM

Use Penalty (or Big M) method to

Maximize $z = 3x_1 - x_2$

subject to the constraints:

$$2x_1 + x_2 \geq 2$$

$$x_1 + 3x_2 \leq 3$$

$$x_2 \leq 4, \quad x_1, x_2 \geq 0$$

CALCULATION:

By introducing surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0$ and $s_3 \geq 0$ and an artificial variable $a_1 \geq 0$;

Maximize $z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$

subject to the constraints:

$$2x_1 + x_2 - s_1 + a_1 + 0s_2 + 0s_3 = 2$$

$$x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 3$$

$$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 4$$

$$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$$

C_B	Y_B	c_j / X_B	3 / Y_1	-1 / Y_2	0 / Y_3	0 / Y_4	0 / Y_5	-M / Y_6
-M	y_6	2	2	1	-1	0	0	1
0	y_4	3	1	3	0	1	0	0
0	y_5	4	0	1	0	0	1	0
	z_j	-2M	-2M	-M	M	0	0	-M
	z_j-c_j	-2M	-2M-3	-M+1	M	0	0	0

C_B	Y_B	c_j / X_B	6 / Y_1	4 / Y_2	0 / Y_3	0 / Y_4	0 / Y_5
0	y_3	24	0	1	1	0	2
0	y_4	15	0	-1	0	1	3
6	y_1	03	1	1	0	0	-1
	z_j	18	6	6	0	0	-6
	z_j-c_j	18	0	2	0	0	-6

C_B	Y_B	c_j / X_B	6 / Y_1	4 / Y_2	0 / Y_3	0 / Y_4	0 / Y_5
0	y_3	14	0	5/3	1	-2/3	0
0	y_5	5	0	-1/3	0	1/3	1
6	y_1	8	1	2/3	0	1/3	0
	z	48	0	0	0	2	0
		z_j-c_j	0	0	0	2	0

Result:

$$z_j - c_j \geq 0$$

Since, all $z_j - c_j \geq 0$. The current basic feasible solution is optimal.

Therefore,

$y_1 = 8$ Max $z = 48$.

PROBLEM :

Use two - phase simplex method

Maximize $z = 3x_1 + 2x_2$

Subject to the constraints :

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12 \ ; \quad x_1, x_2 \geq 0 .$$

CALCULATION

Introducing slack variable $s_1 \geq 0$ and surplus variables and an artificial variable $a_1 \geq 0$, Max $z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - a_1$

subject to the constraints :

$$2x_1 + x_2 + s_1 = 2$$

$$3x_1 + 4x_2 - s_2 + a_1 = 12 ;$$

$$x_1, x_2, s_1, s_2 ; a_1 \geq 0 .$$

Initial Simplex Table

		c_j	0	0	0	0	-1
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5
0	y_2	2	2	1	1	0	0
-1	y_5	12	3	4	0	-1	1
	z_j	-12	-3	-4	0	1	-1
	$z_j - c_j$	-12	-3	-4	0	1	0

		c_j	0	0	0	0	-1
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5
0	y_2	2	2	1	1	0	0
-1	y_5	4	-5	0	-4	-1	1
	z_j	-4	5	0	4	1	-1
	$z_j - c_j$	-4	5	0	4	1	0

Result:

The given L.P.P does not ~~ex~~ posses any feasible solution

PROBLEM

Use two phase simplex method to

Minimize $z = x_1 - 2x_2 - 3x_3$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1,$$

$$x_1, x_2, x_3 \geq 0$$

CALCULATION:

Introducing artificial variables $a_1 \geq 0$ and $a_2 \geq 0$ respectively.

Minimize $z = -(\text{Maximize } z^*)$

Maximize $z^* = -x_1 + 2x_2 + 3x_3 - a_1 - a_2$

subject to the constraints

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

INITIAL ITERATION :

		C_j	0	0	0	-1	-1
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5
-1	y_4	2	-2	1	3	1	0
-1	y_5	1	2	3	4	0	1
	z_j	-3	0	-4	-7	-1	-1
	$z_j - c_j$	-3	0	-4	-7	0	0

FINAL ITERATION :

		C_j	0	0	0	-1	-1
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5
-1	y_4	5/4	-7/2	-5/4	0	1	-3/4
0	y_3	1/4	1/2	3/4	1	0	1/4
	z_j	-5/4	7/2	5/4	0	-1	3/4
	$z_j - c_j$	-5/4	7/2	5/4	0	0	~~5~~ 3/4

RESULT :

The given LPP ~~is~~ does not posses any feasible solution.