UNIT - III

GENERAL PRIMAL - DUAL PAIR :

Definition 1. (Standard primal problem)

Maximize

$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ Subject to the constraints:

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i \quad : \quad i = 1.2 \dots m$$

$$x_j \geq 0 \quad : \quad j = 1.2 \dots n$$

Dual problem

minimize $z^* = b_1 W_1 + b_2 W_2 + \dots + b_m W_m$

Subject to the constrains:

$$a_{ij} W_1 + a_{2j} W_2 + \dots + a_{mj} W_m \geq c_j \ :$$

$W_i (i = 1, 2, \dots m)$ unrestricted . $j = 1.2 \dots m$

Note that x_j's are the primal variables, W_i's the dual variables and the other constants have their is meanings.

Definition . 2 (Standard primal problem)

Minimize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ Subject to the constraints

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i \quad : \quad i = 1.2 \dots m$$

$$x_j \geq 0 \quad : \quad j = 1.2 \dots n$$

Dual problem

$$\text{Maximize } z^* = b_1W_1 + b_2W_2 + \ldots + b_mW_m$$

Subject to the constraints

$$a_{j1}W_1 + a_{2j}W_m \leq c_j :$$

W_i $(i = 1, 2 \ldots m)$ unrestricted $j = 1, 2 \ldots n$

FORMULATING A DUAL PROBLEM:

Step : 1 put the given linear programming problem into its standard form. consider it as the primal problem.

Step : 2 Identify the variables to be used in the dual problem. The number of these variables equals the number of constraint equations in the primal.

Step : 3 Write down the objective function of the dual, using the right - hand side constants of the primal constraints.

If the primal problem is of maximization type. the dual Will be a minization problem and. vice - versa.

Step : 4 Making use of dual variable identified in step 2 Write the constraints for the dual problem.

a) If the primal is a maximization prob'm. the dual constraints must be all of $\geq$ type. If the primal is a minization problem, the dual constraints must be all of '$\leq$' type.

b) The column coefficients of the primal constraints become the row coefficients of the dual constraints.

c) The coefficients of the primal objective function becomes the right-hand side constans of the dual constraints.

d) The dual variables are defined to be unrestricted in sign.

Step : 5 Using steps 3 and 4 write down the dual of the given LPP.

PRIMAL - DUAL PAIR IN MATRIX FORM :

Standard primal problem

Definition : 1 (Standard primal problem). Find $x^T \in R^n$ so as to maximize $z = cx$, $c \in R^n$ subject to the constraints.

$$Ax = b \text{ and } x \geq 0, \; b^T \in R^m$$

Where A is an $m \times n$ real matrix.

Dual problem. Find $W^T \in R^m$ so as to minimize $z^* = b^T W$, $b \in R^m$ subject to the constraints:

$$A^T W \geq c^T, \; c \in R^n$$

Where A^T is the transpose of an $m \times n$ real matrix.

A and W is unrestricted in sign.

Definition 2. (Standard primal problem) Find $x^T \in R^n$

So as to minimize $z = Cx$, $C \in R^n$ subject to the constraints

$$Ax = b \text{ and } x \geq 0, \; b^T \in R^m$$

Where A is $m \times n$ real matrix.

Dual problem. Find $W^T \in R^n$, So as to maximize

$$z^* = b^T W, \; b \in R^m \text{ subject to the constraints}$$

$$A^T W \leq C^T, \; C \in R^n$$

Where A^T is the transpose of an $m \times n$ real matrix

A and W is unrestricted in sign.

DUAL SIMPLEX METHOD:

Step: 1 Write the given linear programming problem in its standard form and obtain a starting basic solution.

Step: 2 (a) If the current basic solution is feasible, use simplex method to obtain an optimum solution.

(b) If the current basic solution is infeasible, i.e, values of basic variables are ≤ 0, go to the next step.

Step : 3 check whether the solution is optimum.

(a) If the Solution is not optimum, add an artificial constraint in such a way that the condition of optimality is satisfied.

b) If the solution is optimum. go to next step.

Step : 4 Select the basic variable having the most negative value, This basic variable becomes the leaving variable and the row corresponding to it becomes the key row.

Step : 5 obtain the ratios of the net evaluations to the corresponding coefficients in the row. ignore the ratios associated with positive and zero denominators. The entering vector is the one with by Smallest absolute value of the ratios. Column corresponding do the entering vector become the key column.

Step : 6 Reduce the leading element into unity and all other entries of the key column to zero by elementary row operations.

Step : 7 Go to step 2 and repeat the procedure until an optimum basic feasible solution is attained.

UNIT → IV

THE TRANSPORTATION TABLE:

Since the T.P. is just a special case of general L.P.P, The application of Simplex method would, no doubt, give an optimum Solution to the problem. However, fortunately, Simplex, like, method for solving such problems has been devoloped, Whenever it is possible to place the given LPP. in the transportation framework. it is for Simpler to solve it by Transportation method than by the Simplex method. To facilitate presentation and Solution, the transportation problem is generally portrayed in a tabular form as shown below:

		Destination 1	2	....	n	Supply
	1	x_{11} c_{11}	x_{12} c_{12}	....	x_{1n} c_{1n}	a_1
	2	x_{21} c_{21}	x_{22} c_{22}	...	x_{2n} c_{2n}	a_2
origin	...	...	...	...	...	...
	m	x_{m1} c_{m1}	x_{m2} c_{m2}	...	x_{mn} c_{mn}	a_m
Demand		b_1	b_2	...	b_n	

The mn large squares are called the cells. The per unit cost c_{ij} of transporting from the ith origin O_i to the jth destination D_j is displayed in the lower right position of the (i, j)th cell. Any feasible

Solution to the T.P is displayed in the table by variable x_{ij} at the upper left position of the (i,j)th cell. The various origin capacities and destination requirements are listed in the right most (outer) column and the bottom (outer) row respectively. These are called rim requirements.

Solution of a Transportation problem:

The Solution to a transportation problem involves the following major steps:

Step: 1 Formulate the given L.P.P in the tabular form known as a transportation table.

Step: 2 Formulate the given problem as a linear programming problem.

Step: 3 Examine whether total supply equals total demand. If not introduce a dummy row / column having all its cost elements as zero and supply / Demand as the (+ve) differene of supply and demand.

Step: 4 Find an initial basic feasible solution that must satisfy all the supply and demand conditions.

Step: 5 Examine the Solution obtained in step 4 for optimality, i.e, examine whether an improved transportation Schedule with lower cost is possible.

Step: 6 If the Solution is not optimum, modify the shipping Schedule by including that unoccupied cell whose inclusion may result in an improved Solution.

Step: 7 Repeat step 4 until to further improvement is possible.

Finding an initial Basic feasible Solution:

There are Several methods available to obtain an initial basic feasible solution. However we shall discuss here the following Three methods:

1. North - West corner method.
2. Least - cost method, and
3. Vogel's Approximation method (or) Penalty method.

1. North - West corner method:

it is a simple and efficient method to obtain an initial basic feasiable Solution. Various steps of the methor are:

Step : 1 Select the north - west (upper left hand) corner cell of the transportation table and allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied i.e, $x_{11} = \min (a_1, b_1)$

Step : 2 If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude $x_{21} = \min (a_2, b_1 - x_{11})$ in the cell (2,1).

If $b_1 < a_1$, we move right horizonatally to the second column and make the second allocation of magnitude. $x_{12} = \min (a_1 - x_{11}, b_2)$ in the cell (1,2).

If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude.

$$x_{12} = \min (a_1 - a_1, b_1) = 0 \text{ in the cell } (1,2)$$

$$x_{21} = \min (a_2, b_1 - b_1) = 0 \text{ in the cell } (2,1)$$

Step : 3 Repeat steps 1 and 2 moving down towards the lower right corner of the transpotation table until all the rim requirments are satisfied.

2. Least - cost method or matrix minima method.

This method takes into account the minimum unit cost and can be Summarized as follows

Step: 1 Determine the Smallest cost in the cost matrix of the transpatation table. Let it be c_{ij} allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j)

Step: 2 If $x_{ij} = a_i$ cross off the ith row of the transpation table and decrease b_j by a_i Go to step 3.

If $x_{ij} = b_j$ cross off the jth column of the transpation table and decrease a_i by b_j Go to Step 3.

If $x_{ij} = a_i = b_j$ cross off either the ith row or jth column but not both.

Step: 3

Repeat steps 1 and 2 for the following resulted reduced transpatation table until all the rim requirments are satisfied whenever the minimum cost is not unique, make an arbitary choice among the minima.

3. Vogel's approximation method (VAM)

The vogel's approximation method takes into account not only the least cost C_{ij} but also the costs that just exceed C_{ij}, The steps of the method are given below.

Step: 1 For each row of the transpatation table identify the smallest and the next-to-smallest and the next. Determine the difference between them for each row. Display them alongside the transpatation table by enclosing them in parenthesis against the respective rows. Similarly, compute the differences for each column.

Step: 2

Identify the row or column with the largest difference among all the rows and columns If a tie occurs, use any arbitary tie-breaking choice. let the greatest difference correspond to ith row and let C_{ij} be the smallest cost in the ith row.

Step: 3

Recompute the column and row difference for the reduced transpatation table and to go Step. 2.

UNIT - V

MATHEMATICAL FORMULATION OF THE PROBLEM:

Consider a problem of assignment of n resoures (workers) to n activities (jobs) so as to minimize the overall cost or time in such a way that each resource can associate with one and only one job. The cost (or effectiveness) matrix (c_{ij}) is given as under:

		A_1	A_2	...	A_n	Available
Resource	R_1	c_{11}	c_{12}	...	c_{1n}	1
	R_2	c_{21}	c_{22}	...	c_{2n}	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	R_n	c_{n1}	c_{n2}	...	c_{n3}	1
	Required	1	1	...	1	

This cost matrix is same as that of a transpatation problem except that availabillity at each of the resources and the requirment at each of the destinations is unity (due to the fact that assignments are made on a one-to-one basis.

let x_{ij} denote the assignment of ith resource to jth activity, Such that

$$x_{ij} = \begin{cases} 1, & \text{if resource } i \text{ is assigned to activity } j \\ 0, & \text{otherwise} \end{cases}$$

Then, the mathematical formulation of the assignment problem is

minimize. $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} - x_{ij}$ Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{and} \quad \sum_{j=1}^{n} x_{ij} = 1 : \quad x_{ij} = 0 \text{ (or) } 1$$

for all $i = 1, 2 \ldots n$ and $j = 1, 2, \ldots n$.

Hungarian Assignment method:

Step:1) Determine the cost table from the given problem.

(i) If the number of sources is equal to the number of destinations, go to Step. 3

(ii) If the number of sources is not equal to the number of destinations, go to Step. 2

Step:2 Add a dummy source or dummy destinations, so that the cost table becomes a square matrix. The cost entries of dummy source/ destinations are always zero.

Step: 3 Locate the smallest element in each row of the given cost matrix and then subract the same from each element of that row.

Step: 4 : In the reduced matrix obtained in Step: 3 locate the smallest element of each column and then subract the same from each element of that column. each column and row now have at least one zero.

Step: 5 In the modified matrix obtained in Step: 4 Search for an optimal assignment as follows :

a) Examine the rows successively until a row with a single zero is found. Enrectage this zero (□) and cross off (x) all other zeros in its column. continue in this manner until all the rows have been taken care of.

b) Repeat the procedure for each column of the reduced matrix.

c) If a row and/or column has two or more zeros and one cannot be chosen by inspection. Then assign arbitary any one of zeros and cross successively until the chain of assigning.

d) Repeat (a) through (c) above successfily until the chain of assigning ($\square$) or cross (x) ends.

Step:6 If the number of assignment ($\square$) is equal to n (the order of the cost matrix) an optimum solution is reached.

Step:7 Draw the minimum number of horizontal and (or) vertical lines to cover all the zeros of the reduced matrix. This can be conveniently done by using a simple procedure:

(a) mark (✓) rows that do not have any assigned zero.

(b) mark (✓) columns that have zeros in in the marked rows.

(c) mark (✓) rows that have assigned zeros in the marked columns

(d) Repeat (b) and (c) above until the chain of marking is compled.

(e) Draw lines through all the unmarked rows and marked colums. This gives us the desired minimum number of lines.

Step: 8 Develop the new revised cost matrix as follows:

(a) Find the smallest element of the reduced matrix not covered by any of the lines.

(b) Subract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.

Step: 9 Go to step 6 and repeat the procedure until an optimum solution is attained.

Solve the following Assignment problem.

Tasks	Man E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

Solution:

Step 1: Subtracting the smallest element of each row from every element of the corresponding row we get the reduced matrix:

	E	F	G	H
A	7	15	6	0
B	0	15	1	13
C	23	4	3	0
D	9	16	14	0

Step 2: Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column. we get the following reduced matrix:

	E	F	G	H
A	7	11	5	0
B	0	11	0	13
C	23	0	2	0
D	9	12	13	0

Step3: Starting with row 1, we enrectangle (□) (i.e., make assignment) a single zero, if any, and Cross (x) all other zeros in the column of zero so marked. Thus, we get

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\cancel{0}$	13
23	$\boxed{0}$	2	$\cancel{0}$
9	12	13	$\cancel{0}$

In the above matrix, we arbitrarily enrectangled a zero in Column 1, because row 2 had two zeros.
It may be noted that Column 3 and row 4 do not have any assignment, So, we move on to the next step

Step4: (i) Since row 4 doesn't have any assignment, we tick this row (√)
(ii) Now there is a zero in the fourth column of the ticked row so, we tick fourth column (√)
(iii) Further there is an assignment in the first row of the ticked column. So we tick first row (√)
(iv) Draw straight lines through all unticked rows and ticked Columns. Thus, we have.

$$\begin{array}{cccc|c} 7 & 11 & 5 & \boxed{0} & \checkmark \\ \boxed{0} & 11 & \cancel{0} & 13 & \\ 23 & \boxed{0} & 2 & \cancel{0} & \\ 9 & 12 & 13 & \cancel{0} & \checkmark \\ \hline & & & \checkmark & \end{array}$$

Step5: In step4, we observe that the minimum number of lines so drawn is 3, which is less than the order of the cost matrix. Indicating that the current assignment is not optimum.
To increase the minimum number of lines, we generate new zeros in the modified matrix.
Step6: The smallest element not covered by the lines is 5. Subtracting this element from all the uncovered elements and adding the same to all the elements lying at the intersection of the lines, we obtain the following new modified cost matrix:

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Step7: Repeating step4 on the modified matrix, we get

$$\begin{bmatrix} 2 & 6 & \boxed{0} & \cancel{0} \\ \boxed{0} & 11 & \cancel{0} & 18 \\ 23 & \boxed{0} & 2 & 5 \\ 4 & 7 & 8 & \boxed{0} \end{bmatrix}$$

Now, since each row and each column has one and only one assignment, an optimal solution is reached.

The optimum assignment is : A → G, B → E, C → F, D → H.

The minimum total time for this assignment schedule is 17+13+19+10 or 59 man hours.