SEMESTER : III & IV
CORE COURSE : VI

Inst Hour	: 4 + (2)
Credit	: 5
Code	: 18K4M06

DIFFERENTIAL EQUATIONS AND TRANSFORMS

UNIT I

First order, higher degree differential equations solvable for x, solvable for y, solvablefor dy/dx, Clairaut's form – Conditions of integrability of M dx + N dy = 0 – simple problems.
(Chapter IV – Sections 1,2 & 3, Chapter II – Section 6 of Text Book 1)

UNIT II

Particular integrals of second order differential equations with constant coefficients -Linear equations with variable coefficients – Method of Variation of Parameters (Omit third & higher order equations).
(Chapter V – Sections 1,2,3,4 & 5, Chapter VIII – Section 4 of Text Book 1)

UNIT III

Formation of Partial Differential Equation – General, Particular & Complete integrals – Solution of PDE of the standard forms - Lagrange's method - Few standard forms - Solving of Charpit's method.
(Chapter XII – Sections 1 – 6 of Text Book 1)

UNIT IV

PDE of second order homogeneous equation with Constant coefficients – Particular Integrals of the forms e^{ax+by}, Sin(ax+by), Cos(ax+by), $x^r y^s$ and $e^{ax+by} f(x,y)$.
(Chapter V of Text Book 2)

UNIT V

Laplace Transforms – Standard formulae – Basic theorems & simple applications – Inverse Laplace Transforms – Use of Laplace Transforms in solving ODE with constant coefficients.
(Chapter IX – Sections 1 – 8 of Text Book 1)

Text Book(s)

[1]. T.K.Manicavachagom Pillay & S.Narayanan, Differential Equations, S.Viswanathan Publishers Pvt. Ltd., 2011.
[2]. Arumugam & Isaac, Differential Equations, New Gamma Publishing House, Palayamkottai, 2014.

Book forReference

[1]. M.D.Raisinghania , Ordinary and Partial Differential Equations, S.Chand & Co
[2]. M.K. Venkatraman, Engineering Mathematics, S.V. Publications, 1985 Revised Edition

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

9-3-18
HOD

9/3/18

UNIT - 1

1. Solve: $x^2p^2 + 3xyp + 2y^2 = 0$.

Solution:-

The given equation is a quadratic equation in p.

Solving for p, we get.

$$p = \frac{-3xy \pm \sqrt{9x^2y^2 - 8x^2y^2}}{2x^2}$$

$$= \frac{-3xy \pm \sqrt{x^2y^2}}{2x^2}$$

$$= \frac{-3xy \pm xy}{2x^2}$$

$$= \frac{-3xy + xy}{2x^2} \quad (\text{or}) \quad \frac{-3xy - xy}{2x^2}$$

i.e) $p = -y/x$ (or) $p = -2y/x$

i.e) $\frac{dy}{dx} = -y/x$ and $\frac{dy}{dx} = -2y/x$.

i.e) $\frac{dy}{y} = -\frac{dx}{x}$ and $\frac{dy}{y} = -2\frac{dx}{x}$ (separating the variables)

Integrating we get,

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \text{and} \quad \int \frac{dy}{y} = -2\int \frac{dx}{x}$$

i.e) $\log y + \log x = \log c_1$ and $\log y + 2\log x = \log c_2$

i.e) $xy = c_1$ and $x^2y = c_2$

i.e) $xy - c = 0$ and $x^2y - c = 0$

The complete solution is $(xy-c)(x^2y-c) = 0$.

2. Solve $y = (x-a)p - p^2$.

Solution :-

Given $y = xp - ap - p^2$.

This is clairaut's type. Hence the general solution is

$$y = cx - ac - c^2$$

$$y = c(x-a) - c^2 \rightarrow (1)$$

To find singular integral.

Step : 1 The general solution is $y = c(x-a) - c^2$

Step : 2 Differentiating (1) with respect to 'c',

we get,

$$0 = (x-a) - 2c$$

$$c = \frac{(x-a)}{2} \rightarrow (2)$$

Substituting (2) in (1) we get,

$$y = \left(\frac{x-a}{2}\right)(x-a) - \left(\frac{x-a}{2}\right)^2$$

$$= \frac{(x-a)^2}{4}$$

$$4y = (x-a)^2.$$

This gives the singular integral.

3. Solve $y = 2px + y^2p^3$.

solution :- The given equation is not of clairaut's type. But we can reduce the given equation to a clairaut's form as in the following way.

Multiply the given equation by y we get

$$y^2 = 2pyx + y^3p^3 \rightarrow (1)$$

put $\quad y^2 = u \rightarrow (2)$

$$\therefore 2y\frac{dy}{dx} = \frac{du}{dx}$$

ie) $\quad 2yp = \frac{du}{dx} \rightarrow (3)$

substituting (2) and (3) in (1) we get,

$$u = x\frac{du}{dx} + \left(\frac{1}{2}\frac{du}{dx}\right)^3$$

$$u = x.P_1 + \frac{1}{8}P_1^3 \rightarrow (4)$$

where $P_1 = \frac{du}{dx}$. Equation (A) is a clairaut's equation in u and x.

$\therefore$ The general solution of (A) is

$u = xc + c^3$ [By replacing P_1 by c]

Hence the general solution of the given equation is $\quad y^2 = cx + c^3 \quad [\because u = y^2]$.

To find the singular solution.

step: 1 The general solution is $y^2 = cx + c^3 \rightarrow (5)$

step: 2 Differentiating the above equation with respect to 'c', we get.

$$0 = x + 3c^2 \rightarrow (6)$$

step: 3 we have to eliminate c between equation (5) and (6)

from (6) we get, $\quad c^2 = -\frac{x}{3} \rightarrow (7)$

Squaring (5) we get,

$$y^4 = c^2x^2 + c^6 + 2c^4x \rightarrow (8)$$

substituting (7) in (8) we get,

$$y^4 = -\frac{x^3}{3} - \frac{x^3}{27} + 2.\frac{x^2}{9}.x$$

$$= -\frac{9x^3 - x^3 + 6x^3}{27}$$

$$= \frac{4x^3}{27}.$$

(i.e) $27y^4 = 4x^3$.

This gives the singular solution of the given differential equation.

Further this singular solution gives the envelope of the family of curves $y^2 = cx + c^3$.

4. solve: $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$.

solution:-

$$u = \int M\,dx + \int \left\{ N - \int \frac{\delta M}{\delta y}\,dx \right\} dy + c.$$

$$M\,dx + N\,dy = du.$$

$$M = a^2 - 2xy - y^2 \; ; \quad N = -(x+y)^2$$

$$\frac{\delta M}{\delta y} = -2x - 2y \; ; \quad \frac{\delta N}{\delta x} = -2(x+y)$$

$$\therefore \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}.$$

$$\int \frac{\delta M}{\delta y}\,dx = \int (-2x - 2y)\,dx$$

$$= -2\int (x+y)\,dx$$

$$= -2\left[\frac{x^2}{2} + xy\right]$$

$$\int \frac{\delta M}{\delta y}\, dx = -x^2 - 2xy$$

$$N - \int \frac{\delta M}{\delta y}\, dx = -(x+y)^2 - (-x^2 - 2xy)$$

$$= -(x^2+y^2+2xy) + x^2 + 2xy$$

$$= -x^2 - y^2 - 2xy + x^2 + 2xy$$

$$= -y^2.$$

$$u = \int M\, dx + \int \left\{ N - \int \frac{\delta M}{\delta y}\, dx \right\} dy + c$$

$$= \left\{ \int (a^2 - 2xy - y^2)\, dx + \int -y^2 dy + c \right\} = 0$$

$$x a^2 - \frac{2x^2}{2} y - xy^2 - \frac{y^3}{3} = c$$

$$x a^2 - x^2 y - xy^2 - y^3/3 = c.$$

5. Solve $(y^2 + 2x^2 y)\, dx + (2x^3 - xy)\, dy = 0$

Solution :-

$$(y^2 + 2x^2 y)\, dx + (2x^3 - xy)\, dy = 0 \rightarrow (1)$$

$M = y^2 + 2x^2 y$ $\qquad$ $N = 2x^3 - xy$

$My = 2y + 2x^2$ $\qquad$ $Nx = 6x^2 - y.$

$$My \neq Nx.$$

$x, x^m y^n$ on (1) we get,

$$(x^m y^{n+2} + 2x^{m+2}, y^{n+1})\, dx + (2x^{m+3} y^n - x^{m+1} y^{n+1})\, dy$$

Now, let us assume that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(ie) $M = x^m y^{n+2} + 2x^{m+2} y^{n+1}$

$N = 2x^{m+3}y^n - x^{m+1}y^{n+1}$

$N_x = 2\cdot(m+3)x^{m+2}y^n - (m+1)x^m y^{n+1}$

$\left[(n+2)y + 2(n+2)x^2 - 2(m+3)x^2 + (m+1)y\right] = 0$

$\left[(n+2+m+1)y + 2(n+1-m-3)x^2\right]x^m y^n = 0$

$\therefore x^m y^n \neq 0.$

$(n+m+3)y + 2(n-m-2)x^2 = 0.$

(ie) $(n+m+3) = 0$ & $2(n-m-2) = 0$

$\Rightarrow (a+b)$, $2n = -1$

$n = -1/2$

(ie) $2m = -5$; $m = -5/2$.

ting factories $x^{5/2}$, $y^{-1/2}$

then the given equation is

$(x^{-5/2}, y^{3/2} + 2n^{-1/2})dx + (2x^{1/2}y^{-1/2} - x^{3/2} - y^{1/2})dy$

6. Solve $(y^2e^x + 2xy)\,dx - x^2dy = 0.$

Solution:-

$(y^2e^x + 2xy)\,dx - x^2dy = 0.$

$y^2e^x dx + 2xy\,dx - x^2dy = 0.$

$\div y^2$ $\quad \dfrac{y^2e^x dx + 2xy\,dx - x^2dy}{y^2} = 0.$

$e^x dx + \dfrac{2xy\,dx - x^2dy}{y^2} = 0.$

$\left(e^x + \frac{2x}{y}\right)dx - x^2/y^2\,dy = 0.$

The equation now is exact as

$$\frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x}$$

$\therefore$ The solution is $\int \left(e^x + \frac{2x}{y} \right) dx = 0.$

(as there is no term independent of x in N and treating y as a constant).

$$\therefore e^x + \frac{x^2}{y} = c.$$

7. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

Solution :- The equation is not exact.

As the equation is homogeneous, by rule ⓒ the integrating factor is $\frac{1}{Mx + Ny}$.

$$I.F = \frac{1}{Mx + Ny}.$$

$$I.F = \frac{1}{x(x^2y - 2xy^2) + y(3x^2y - x^3)}$$

$$= \frac{1}{x^3y - 2x^2y^2 + 3x^2y^2 - x^3y}$$

$$I.F = \frac{1}{x^2y^2}.$$

Multiplying by the I.F, the equation is

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \frac{x^3 - 3x^2y}{x^2y^2}\,dy = 0$$

$$\left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2} \right) dx + \left(\frac{3x^2y}{x^2y^2} - \frac{x^3}{x^2y^2} \right) dy = 0.$$

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0.$$

As this is an exact equation, integrating we have,

$$\frac{dx}{y} - 2\frac{dx}{x} + 3\frac{dy}{y} - \frac{x}{y^2} dy = 0.$$

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

(i.e) $\log \frac{y^3}{x^2} + \frac{x}{y} = c.$

8. Solve $(y - 3x^2) dx - x(1 - xy^2) dy = 0$

Solution :- The equation can be written in the form

$$y\,dx - x\,dy - 3x^2 dx + x^2 y^2 dy = 0.$$

Dividing by x^2, we get $\frac{y\,dx - x\,dy}{x^2} - 3dx + y^2 dy = 0$

(i.e) $-d(y/x) - d(3x) + \frac{1}{3} d(y^3) = 0.$

$\therefore -y/x - 3x + \frac{1}{3} y^3 = c.$

9. Solve $\frac{dy}{dx} = \frac{2x}{x^2 + y^2 - 2y}.$

Solution :- The equation can be written in the form

$$(x^2 + y^2) dy = 2y\,dy + 2x\,dx$$

(i.e) $dy = \frac{d(y^2 + x^2)}{x^2 + y^2}$

(i.e) $dy = d \log(x^2 + y^2)$

$\therefore y = \log(x^2 + y^2) + c.$

10. Solve: $(1+xy^2)\,dx + (1+x^2y)\,dy = 0$.

Solution:-

The equation can be written in the form

$$dx + dy + xy\,(y\,dx + x\,dy) = 0.$$

$$(\text{ie})\quad d(x+y) + xy\; d(xy) = 0.$$

$$(\text{ie})\quad d(x+y) + \tfrac{1}{2}\, d\,(xy)^2 = 0.$$

$$\therefore\quad x+y+\tfrac{1}{2}(xy)^2 = c.$$

11. Solve $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$

Solution:-

This equation can be written in the form

$$x\,dx + y\,dy = a^2 \cdot \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$(\text{ie})\quad \tfrac{1}{2}\, d(x^2+y^2) = a^2\, d\left(\tan^{-1} y/x\right)$$

$$\therefore\quad \tfrac{1}{2}(x^2+y^2) = a^2 \tan^{-1}(y/x) + c.$$

12. Solve $a(x\,dy + 2y\,dx) = xy\,dy$.

Solution:-

$$a\cdot(x\,dy + 2y\,dx) = xy\,dy \rightarrow (1)$$

$$ax\,dy + 2ya\,dx - xy\,dy = 0.$$

$\div\, xy$ we get,

$$\frac{ax\,dy - xy\,dy + 2ya\,dx}{xy} = 0$$

$$(\text{ie})\quad \left(\frac{a}{y} - 1\right) dy + \frac{2a}{x}\,dx = 0 \rightarrow (2)$$

$$M = \frac{2a}{x}, \quad N = \frac{a}{y} - 1$$

$M_y = 0$, $N_x = 0$.

$\int$ ing ②, we get

$$\int\left[\left(\frac{a}{y}-1\right)dy + 2a\right]\frac{dx}{x} = c$$

$$a\log y - y + 2a\log x = c$$

$$a(\log y + \log x^2) - y = c$$

$$a(\log x^2 y) - y = c$$

$$a(\log x^2 y) = y + c.$$

UNIT - 2

Solve

1. $\frac{d^2y}{dx^2} - 4y = 6e^{5x}$.

solution :-

Given $\frac{d^2y}{dx^2} - 4y = 6e^{5x}$

$D^2 - 4 = 0$.

The auxillary equation is

$m^2 - 4 = 0$.

$m^2 = 4$

$m = \pm 2$.

$C.F = Ae^{2x} + Be^{-2x}$

$P.I = \frac{1}{D^2-4} \cdot 6e^{5x}$

$= 6 \cdot \frac{1}{D^2-4} e^{5x}$

$= 6 \cdot \frac{1}{25-4} e^{5x}$

$= \frac{6}{21} e^{5x}$

$= \frac{2}{7} e^{5x}$

$\therefore y = C.F + P.I$

$= Ae^{2x} + Be^{-2x} + \frac{2}{7} e^{5x}$.

2. solve: $(D^2 - 5D + 4)y = 0$.

solution :-

Given $(D^2 - 5D + 4)y = 0$.

The auxillary equation is

$m^2 - 5m + 4 = 0.$

$(m-1)(m-4) = 0.$

$m = 1, 4.$

$C.F = C_1 e^{x} + C_2 e^{4x}$

$\therefore y = C_1 e^{x} + C_2 e^{4x}.$

3. Solve: $(D^2+5D+6)y = 0.$

Solution:-

Given $(D^2+5D+6)y = 0.$

The auxillary equation is $m^2 + 5m + 6 = 0.$

$(m+2)(m+3) = 0.$

$m = -2, -3.$

$\therefore C.F = C_1 e^{-2x} + C_2 e^{-3x}$

The solution is $y = C_1 e^{-2x} + C_2 e^{-3x}.$

4. Solve: $(D^2+5D+6)y = e^{x}$

Solution:-

Given $(D^2+5D+6)y = 0.$

The auxillary equation is $m^2 + 5m + 6 = 6.$

$(m+2)(m+3) = 0$

$\therefore m = -2, -3.$

$C.F = C_1 e^{-2x} + C_2 e^{-3x}$

$P.I = \frac{1}{D^2+5D+6} \cdot e^{x}$

$= \frac{1}{1^2+5(1)+6} \cdot e^x$ (Replacing D by 1)

$= \frac{1}{1+5+6} e^x = \frac{1}{12} e^x$

$\therefore y = C.F + P.I$

$= c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x.$

5. Solve : $(D^2+2D+1)y = 2e^{3x}$

Solution!-

Given $(D^2+2D+1)y = 2e^{3x}$

The auxillary equation is $m^2+2m+1=0.$

$(m+1)(m+1)=0$

$\therefore m=-1,-1.$

$C.F = (c_1+c_2x)e^{-x}$

$P.I = \frac{1}{(D^2+2D+1)} \cdot 2e^{3x}$

$= \frac{1}{3^2+2(3)+1} \cdot 2e^{3x}.$

$= \frac{2e^{3x}}{9+6+1}$

$= \frac{2e^{3x}}{16}$

$= e^{3x}/8.$

$\therefore y = (c_1+c_2x)e^{-x} + \frac{e^{3x}}{8}$

Solve :-

6. $(D^2 - 6D + 13)y = 5e^{2x}$

Solution :-

Given $(D^2 - 6D + 13)y = 5e^{2x}$

The auxillary equation is $m^2 - 6m + 13 = 0$.

$$m = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-12}}{2}$$

$$= \frac{6 \pm i2\sqrt{3}}{2}$$

$$= 3 \pm i\sqrt{3}.$$

$$C.F = e^{5x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

$$P.I = \frac{1}{(D^2 - 6D + 13)} \cdot 5e^{2x}$$

$$= 5 \cdot \frac{1}{2^2 - 6(2) + 13} \cdot e^{2x}$$

$$= \frac{5}{4 - 12 + 13} e^{2x}$$

$$= \frac{5e^{2x}}{5}$$

$$= e^{2x}$$

$\therefore$ The solution is

$$y = e^{3x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + e^{2x}.$$

7. Solve: $(D^2+5D+4)y = x^2+7x+9$

Solution:-

Given $(D^2+5D+4)y = x^2+7x+9$.

The auxillary equation is $m^2+5m+4=0$.

$(m+4)(m+1)=0$.

$\therefore m=-4,-1$.

$$C.F = Ae^{-4x}+Be^{-x}$$

$$P.I = \frac{1}{(D^2+5D+4)}\cdot(x^2+7x+9)$$

$$= \frac{1}{4\left[\left(1+\left(\frac{D^2+5D}{4}\right)\right)^{-1}\right]}\cdot(x^2+7x+9)$$

$$= \frac{1}{4}\left[1+\left(\frac{D^2+5D}{4}\right)^{-1}\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2+5D}{4}\right)+\left(\frac{D^2+5D}{4}\right)^2-\cdots\right]\cdot(x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{5D}{4}-\frac{D^4}{16}+\frac{25D^2}{16}\cdots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{D^2}{4}(x^2+7x+9)+\frac{5}{4}D(x^2+7x+9)\cdots\right]$$

$$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{2}-\frac{10}{4}x-\frac{35}{4}+\frac{50}{16}\right]$$

$$= \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$$

$$P.I = \frac{1}{32}\left(8x^2+36x+23x\right)$$

$\therefore$ The solution is

$$y = Ae^{-4x}+Be^{-x}+\frac{1}{32}\left(8x^2+36x+23\right)$$

8. Solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

Solution:-

The auxillary equation is $m^2-4m+3=0$

$(m-3)(m-1)=0$

$m = 1, 3.$

$\therefore$ Roots are real and distinct.

$C.F = C_1e^{x}+C_2e^{3x}$

$P.I = \cos 2x . \sin 3x.$

$\therefore \sin 3x \cos 2x = \dfrac{\sin 5x + \sin x}{2}.$

$$P.I = \frac{\sin 5x}{2}+\frac{\sin x}{2}. \qquad \left[\because \sin A\cos B = \frac{\sin(A+B)}{2}+\frac{\sin(A-B)}{2}\right]$$

$$= \tfrac{1}{2}\left[\sin 5x\right]+\tfrac{1}{2}\left[\sin x\right]$$

$$P.I_2 = \tfrac{1}{2}\left[\frac{1}{-25-4D+3}.\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}.\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4D+22}.\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{(4D+22)(4D-22)}\right]\sin 5x$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{16D^2-484}\cdot\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{-400-484}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D-22)}{-884}\sin 5x\right]$$

$$= \frac{1}{2}\left[\frac{2(2D-11)}{884}\sin 5x\right]$$

$$= \frac{2D-11}{884}\sin 5x$$

$$= \frac{2D(\sin 5x) - 11\sin 5x}{884}$$

$$P.I_1 = \frac{10\cos 5x - 11\sin 5x}{884}$$

$$P.I_2 = \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1-4D+3}.\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4D-2}\sin x\right] = \frac{1}{2}\left[\frac{-(4D+2)}{16D^2-4}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-(4D+2)}{-16-4}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{4D+2}{20}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{2(2D+1)}{20}\sin x\right]$$

$$= \frac{(2D+1)}{20}\sin x.$$

$$P.I_2 = \frac{2D(\sin x)+\sin x}{20}.$$

$\therefore$ The solution is

$$y = Ae^{x}+Be^{3x}+\frac{10\cos 5x-11\sin 5x}{884}+\frac{\sin x+2\cos x}{20}.$$

9. Solve: $(D^3-2D+4)y = e^{x}\cos x.$

Solution:- The auxillary equation is $m^3-2m+4=0$

$$(m+2)(m^2-2m+2)=0.$$

$$m=-2. \quad m=1\pm i$$

$$\therefore C.F = C_1e^{-2x}+e^{x}+e^{x}(C_2\cos x+C_3\sin x)$$

$$P.I = \frac{1}{D^3-2D+4}e^{x}\cos x$$

$$= e^{x}.\frac{1}{(D+1)^3-2(D+1)+4}\cos x \quad \text{(by II rule)}$$

$$= e^{x}.\frac{1}{(D^3+3D^2+3D+1-2D-2+4)}.\cos x.$$

$$= e^x \cdot \frac{1}{D^3+3D^2+3D+1-2D-2+4} \cos x$$

$$= e^x \cdot \frac{1}{D^3+3D^2+D+3} \cos x$$

$$= e^x \cdot \frac{1}{D(-1)+3(-1)+D+3} \cos x \cdot (\text{Replace } D^2 \text{ by } -1)$$

$$= xe^x \cdot \frac{1}{6D-2} \cos x$$

$$= \frac{xe^x \, (6D+2)}{(6D-2)(6D+2)} \cos x$$

$$= \frac{xe^x . (6D+2)}{36D^2-4} \cos x$$

$$= \frac{xe^x . (6D+2)}{36(1)-4} \cos x$$

$$= \frac{-xe^x \, (3D(\cos x) + \cos x)}{20}$$

$$= \frac{xe^x}{20} (3\sin x - \cos x)$$

$$\therefore y = C_1 e^{-2x} + e^x (C_2 \cos x + C_2 \sin x) + \frac{xe^x}{20} (3\sin x - \cos x)$$

10. Solve :- $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

Solution :- putting $x = e^z$

$\log x = z.$

$\Rightarrow \left(D(D-1)+4D+2\right)y = e^{x}$

$(D^2-D+4D+2)y = e^{x}$

$(D^2+3D+2)y = e^{x}.$

The Auxillary equation is

$m^2+3m+2 = 0$

$(m+2)(m+1) = 0$

$m = -2, -1.$

$C.F = A_1 e^{-x} + Be^{-2x}$

To find P.I

$P.I = \dfrac{1}{(\theta+1)(\theta+2)} e^{x}$, where $D = \theta = x\dfrac{d}{dz}$.

$= \left[\dfrac{1}{(\theta+1)} - \dfrac{1}{\theta+2}\right] e^{x}$

$= x^{-1}\int e^{x} dx - x^{-2}\int x e^{x} dx$

$= x^{-1}. e^{x} - x^{-2}(xe^{x}-e^{x})$

$= x^{-2}. e^{x}$

$\therefore y = Ae^{-x} + Be^{-2x} + x^{-2}e^{x}.$

UNIT-3

1. Eliminate a and b from $z = (x+a)(y+b)$.

Solution:-

Given $z = (x+a).(y+b) \rightarrow (1)$

Differentiating with respect to x and y partially

$$\frac{\delta z}{\delta x} = 1.(y+b) = y+b \Rightarrow p = y+b$$

$$\frac{\delta z}{\delta y} = (x+a).1 = x+a \Rightarrow q = x+a.$$

Eliminating a and b, from (1)

$$z = q.p$$

$$\text{ie) } z = pq.$$

Hence the solution.

2. Eliminate the arbitrary function from $z = f(x^2+y^2)$

Solution:-

Given the function $z = f(x^2+y^2) \rightarrow (1)$

Differentiating partially with respect to x and y

$$\frac{\delta z}{\delta x} = f'(x^2+y^2)\,2x \Rightarrow p = f'(x^2+y^2).2x \rightarrow (2)$$

and $$\frac{\delta z}{\delta y} = f'(x^2+y^2)\,2y \Rightarrow q = f'(x^2+y^2).2y \rightarrow (3)$$

Eliminating $f'(x^2+y^2)$ between the latter two equations.

(ii) $\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{f'(x^2+y^2) \cdot 2x}{f'(x^2+y^2)\, 2y}$

$\Rightarrow py = qx$

Hence the problem.

3. Eliminate the arbitrary function f from $f(x^2+y^2+z^2, z^2-2xy) = 0$.
Solving $x^2+y^2+z^2 = F(z^2-2xy)$.

Solution :-

Given the function $x^2+y^2+z^2 = F(z^2-2xy)$

Differentiating partially with respect to x, and y

we get.

$$2x + 2z\frac{\delta z}{\delta x} = F'(z^2-2xy)\left(2z\frac{\delta z}{\delta x} - 2y\right)$$

$$2x + 2zp = F'(z^2-2xy)(2zp - 2y) \rightarrow (2)$$

and
$$2y + 2z\frac{\delta z}{\delta y} = F'(z^2-2xy)\cdot\left(2z\frac{\delta z}{\delta y} - 2x\right)$$

$$2y + 2zq = F'(z^2-2xy)(2zq - 2x) \rightarrow (3)$$

Dividing (2) by (3), to eliminate F'

$$\frac{2x+2zp}{2y+2zq} = \frac{F'(z^2-2xy)(2zp-2y)}{F'(z^2-2xy)\cdot(2zq-2x)}$$

$$\frac{2(x+zp)}{2(y+zq)} = \frac{2(zp-y)}{2(zq-x)}$$

$$\frac{x+zp}{y+zq} = \frac{zp-y}{zq-x} \Rightarrow (x+zp)\cdot(zq-x) = (zp-y)(y+zq) \quad (3)$$

$$\Rightarrow xzq - x^2 + z^2pq - xzp = yzp - y^2 + z^2pq - yzq$$

$$\Rightarrow y^2 - x^2 = (xz+yz)p - (xz+yz)q.$$

$$\Rightarrow (y-x)(y+x) = (x+y)zp - (x+y)zq.$$

$$\Rightarrow (y-x)(y+x) = (x+y)\boxed{zp - zq}$$

$$\Rightarrow (y-x) = z(p-q)$$

(i.e) $z(p-q) = y-x.$

Hence the problem.

4\. Eliminate f and ϕ from the relation

$$z = f(x+ay) + \phi(x-ay)$$

Solution:-

Differentiating partially with respect to x and y we get,

$$\frac{\delta z}{\delta x} = f'(x+ay) + \phi'(x-ay)$$

(i.e) $p = f'(x+ay) + \phi'(x-ay)$

and $\frac{\delta z}{\delta y} = f'(x+ay)\, a + \phi'(x-ay)(-a)$

(i.e) $q = af'(x+ay) - a\phi'(x-ay)$

Differentiating these again with respect to x and y respectively.

$$\frac{\delta p}{\delta x} = f''(x+ay) + \phi''(x-ay)$$

$$\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta^2 z}{\delta x^2} = f''(x+ay) + \phi''(x-ay)$$

and

$$\frac{\delta q}{\delta y} = a f''(x+ay) \cdot a - a\phi''(x-ay)(-a)$$

$$\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta y}\right) = a^2 f''(x+ay) + a^2 \phi''(x-ay)$$

and $\frac{\delta^2 z}{\delta y^2} = a^2\left[f''(x+ay) + \phi''(x-ay)\right]$

$$\frac{\delta^2 z}{\delta y^2} = a^2 \cdot \frac{\delta^2 z}{\delta x^2} \Rightarrow r = a^2 t \Rightarrow r - a^2 t = 0$$

where $r = \frac{\delta^2 z}{\delta y^2}$

$t = \frac{\delta^2 z}{\delta x^2}$

Hence the problem.

5. Solve $(y+z)p + (z+x)q = x+y$.

Solution:- Given $(y+z)p + (z+x)q = x+y$.

we know that the linear equation becomes

$$Pp + Qq = R.$$

Here $P = y+z$, $Q = z+x$, $R = x+y$.

The subsidary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

ie) $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx+dy+dz}{y+z+z+x+x+y}$ (5)

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{d(\Sigma x)}{2\Sigma x}.$$

They are also equivalent to

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dz-dx}{x-z} = \frac{\Sigma dx}{2\Sigma x}$$

Taking the first two ratios and integrating

$$\int \frac{dx-dy}{y-x} = \int \frac{dy-dz}{z-y}$$

$$\int \frac{dx-dy}{-(x-y)} = \int \frac{dy-dz}{y-z}$$

$$\log(x-y) = \log(y-z) + \log a$$

$$\log(x-y) = \log[(y-z).a]$$

ie) $x-y = (y-z)a$

$$\frac{x-y}{y-z} = a.$$

Taking the first and last ratios and integrating

$$\int \frac{dx-dy}{-(x-y)} = \int \frac{\Sigma dx}{2\Sigma x}$$

$$-2\int \frac{dx-dy}{(x-y)} = \int \frac{\Sigma dx}{\Sigma x}$$

$$-2\log(x-y) = \log \Sigma x + \log b$$

$$\log(x-y)^{-2} = \log(\Sigma x . b)$$

$$\frac{1}{(x-y)^2} = b\, \Sigma x \Rightarrow (x-y)^2 \Sigma x = b$$ (6)

Hence the solution required is

$$\phi\left[\frac{x-y}{y-z}, (x-y)^2 \Sigma x\right] = 0$$ where ϕ is arbitrary.

6. Solve $px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$

Find the surface that contains the straight line

$x+y=0$, $z=1$

Solution :-

Given $px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$

General form : $P = x(y^2+z)$, $Q = -y(x^2+z)$,

$R = z(x^2-y^2)$

The subsidary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy}{z(x^2-y^2)}$$

Hence taking the last two ratios and integrating

$$\int \frac{dz}{z(x^2-y^2)} = \int \frac{x\,dx + y\,dy}{z(x^2-y^2)}$$

$$\int dz = \int x\,dx + y\,dy$$

$$z + a = \frac{x^2}{2} + \frac{y^2}{2}$$

$$2z + a = x^2 + y^2 \longrightarrow (1)$$

$x^2+y^2-2z = a.$ (1)

The subsidiary equations can also be written

as $\frac{dx/x}{y^2+z} = \frac{dy/y}{-(x^2+z)} = \frac{dz}{z(x^2-y^2)}$

$$= \frac{dx/x + dy/y}{y^2+z-x^2-z}$$

$$= \frac{dx/x + dy/y}{y^2-x^2}$$

Taking the last two ratios,

$$\frac{dz/z}{x^2-y^2} = \frac{dx/x + dy/y}{y^2-x^2}$$

$$\frac{dx/x + dy/y + dz/z}{x^2-y^2+y^2-x^2} = 0$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$\int$ing on both sides, $\int \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$\log x + \log y + \log z = \log b.$

(i.e) $\log(xyz) = \log b$

$xyz = b \rightarrow$ (2).

The solution is $\phi(x^2+y^2-2z, xyz) = 0.$

$$x^2+y^2-2z = f(xyz)$$

$$(x+y)^2 - 2xy - 2z = f(xyz)$$

$x+y=0$, $z=1$ lies on this if $-2(xy+1) = f(xy)$ (8)

$\therefore f(xyz) = -2(xyz+1)$

$\therefore$ The desired integral surface is

$$x^2+y^2-2z = -2(xyz+1).$$

Hence the problem.

7. Find the integral surface of $x^2p + y^2q + z^2 = 0$ which passes through the hyperbola $xy = x+y$; $z=1$

Solution :- The subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Integrating, $\frac{1}{x} + \frac{1}{z} = a$ and $\frac{1}{y} + \frac{1}{z} = b$

$\therefore$ The solution is $\frac{1}{z} + \frac{1}{x} = f\left(\frac{1}{y} + \frac{1}{z}\right)$, where f is arbitrary,

If this surface is to pass through the hyperbola $xy = x+y$, $z=1$, we must have

$$1 + \frac{1}{x} = f\left(\frac{1}{y}+1\right)$$

From $xy = x+y$, we get $1 = \frac{1}{x} + \frac{1}{y}$

$$\therefore 1 + \frac{1}{x} = 2 - \frac{1}{y} = 3 - \left(1 + \frac{1}{y}\right)$$

$$\therefore f\left(\frac{1}{y}+1\right) = 3 - \left(1+\frac{1}{y}\right)$$

Hence the required surface is $\frac{1}{z} + \frac{1}{x} = 3 - \left(\frac{1}{y} + \frac{1}{z}\right)$

8. Solve: (i) $q = xp + p^2$.

Solution:

$$q = xp + p^2$$

Let $q = a$, Then $a = xp + p^2 \Rightarrow p^2 + xp - a = 0$.

$$\therefore p = \frac{-x \pm \sqrt{x^2+4a}}{2} = \phi(x, a).$$

Hence $dz = \frac{-x \pm \sqrt{x^2+4a}}{2} dx + a dy$.

$$\therefore z = \int \frac{-x \pm \sqrt{x^2+4a}}{2} dx + ay + b.$$

$$z = -\frac{x^2}{4} \pm \left\{ \frac{x}{4} \sqrt{4a+x^2} + a \sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) \right\} + ay + b.$$

Hence the problem.

(ii) $p = y^2 q^2$

Solution:-

Let $p = a^2$. Then $q = \pm \frac{a}{y}$

Hence $dz = a^2 dx \pm \frac{a}{y} dy$

$$\therefore z = a^2 x \pm a \log y + b$$

Hence the problem.

(iii) $p(1+q^2) = q(z-1)$

Solution:- Let $q = ap$. Then $p(1+a^2p^2) = ap(z-1)$

$\therefore 1 + a^2p^2 = a(z-1)$ (ie) $p = \pm \frac{\sqrt{az-a-1}}{a}$

Hence $dz = \pm \frac{\sqrt{az-a-1}}{a} dx \pm \frac{a\sqrt{az-a-1}}{a} dy$

(ie) $\pm \frac{a\,dz}{\sqrt{az-a-1}} = dx + a\,dy$ (ie) $\pm \int \frac{a\,dz}{\sqrt{az-a-1}} = x+ay+b$

(ie) $\pm 2\sqrt{az-a-1} = x+ay+b.$

Hence the problem,

9. Solve the equation $p+q = x+y$.

Solution!
We can write the equation in the form

$p-x = y-q.$

Let $p-x = a$, then $y-q = a$

Hence $p = x+a$, $q = y-a$.

$\therefore dz = (x+a)\,dx + (y-a)\,dy$

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b.$$

There is no singular integral and the general integral is found as usual.