UNIT - I

sets and sequences

(A part of the systerm of real numbers is called a <u>set</u> or aggregate of numbers. The numbers are known as the elaments of the set) The set said to be an infinets set if the number of elemants in it is infinied and a finied set if that numbers finied.

A squences is a particular ~~stacte~~ case of a set. If to the integers $1, 2, 3, \ldots n$. The correspond defined numbers $a_1, a_2, a_3, \ldots a_n$. The set $a_1, a_2, a_3 \ldots a_n$ is called a sequences and its denoted by sequence of $\{a_n\}$. Hence a sequence is a set of numbers which has a one to one correspontant with the set of positive integers

Example:

1) $a_n = n^2 = 1, 4, 9, \ldots$

2) $a_n = \frac{1}{n} = 1, \frac{1}{2}, \frac{1}{4}, \ldots$

let ϵ be an arbitrary positive number.

Then their exist ($\exists$) numbers N_0, N_1 depending on ϵ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq N_0$$

$$|b_n - b| < \epsilon \text{ for all } n \geq N_1$$

Let N be the grater then N_0, N_1

Then $|a_n + b_n - a - b| = |a_n - a + b_n - b|$

$$\leq |a_n - a| + |b_n - b|$$

$$\leq \epsilon + \epsilon$$

$$\leq 2\epsilon \ \forall \ n \geq N$$

since ϵ is arbitrary

$$\lim_{n \to \infty} (a_n + b_n) = a + b$$

A similar argument shows that

$$\lim_{n \to \infty} (a_n - b_n) = a - b$$

Hence $\{a_n \pm b_n\}$ convergent to $a \pm b$

Theorem : 2

If $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, then $\{a_n b_n\}$ converges to ab

proof :

write $a_n = a + \lambda_n$ and $b_n = b + \mu_n$

Since $a_n \to a$, $b_n \to b$ and λ_n μ_n both tend to zero.

$\therefore |\lambda_n| < \epsilon$ for all $n \geq N_0$

$|\mu_n| < \epsilon$ for all $n \geq N_1$

Let N be grater than N_0, N_1

$a_n b_n = (a + \lambda_n)(b + \mu_n)$

$= ab + a\mu_n + b\lambda_n + \lambda_n \mu_n$

$\therefore a_n b_n - ab = a\mu_n + b\lambda_n + \lambda_n \mu_n$

$|a_n b_n - ab| = |a\mu_n + b\lambda_n + \lambda_n \mu_n|$

$|a_n b_n - ab| \leq |a\mu_n| + |b\lambda_n| + |\lambda_n \mu_n| \to ①$

$|a\mu_n| < |a|\epsilon$; $|b\lambda_n| < |b|\epsilon$

$|\lambda_n \mu_n| < \epsilon^2$ for all $n \geq N$

From ①

$|a_n b_n - ab| < |a|\epsilon + |b|\epsilon + \epsilon^2$

$< \epsilon(|a| - |b| + \epsilon)$

$\therefore |a|, |b|, \epsilon$ are finite numbers

$|a_n b_n - ab| < A\epsilon$ where A is a +ve constant

Since ϵ is arbitrary, so also in $A\epsilon$

Hence $\lim_{n \to \infty} a_n b_n = ab$

Hence $\{a_n b_n\}$ converges to ab

30.6.15 Theorem : 3

2

If $\{a_n\}$ converges to a and $\{b_n\}$ converge to $b \neq 0$, then $\left\{\frac{a_n}{b_n}\right\}$ converges to a/b

proof:

write $a_n = a + \lambda_n$: $b_n = b + \mu_n$

Since $a_n \to a$, $b_n \to b$ and λ_n and μ_n both tent to zero

$\therefore |\lambda_n| < \epsilon$ for all $n \geq N_0$

$|\mu_n| < \epsilon$ for all $n \geq N_1$.

let N be greater than N_0, N_1

Then $\frac{a_n}{b_n} - a/b = \frac{a+\lambda_n}{b+\mu_n} - \frac{a}{b}$

$$= \frac{b(a+\lambda_n) - a(b+\mu_n)}{b(b+\mu_n)}$$

$$= \frac{ab + b\lambda_n - ab - a\mu_n}{b(b+\mu_n)}$$

$$= \frac{b\lambda_n - a\mu_n}{b(b+\mu_n)}$$

Then $|b(b+\mu_n)| = |b|\,|b+\mu_n|$

Since $|b| > 0$ and $|\mu_n| < \epsilon$, where ϵ is arbitrary ; we can assume that

$|b+\mu_n| > \frac{1}{2}|b|$

Thus $|b(b+\mu n)| > \frac{1}{2}|b|^2 = l \text{ (say)} < \epsilon l_1$

$\forall n \geq N$

then $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \leq \frac{|b\lambda n| + |a\mu n|}{|b(b+\mu n)|}$

$\leq \frac{|b|\epsilon + |a|\epsilon}{l}$

$\leq \frac{\epsilon(|b| + |a|)}{l}$

where l is a fixed positive constant

Hence $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

Hence $\left\{\frac{a_n}{b_n}\right\}$ converges to a/b

Cauchys first limit theorem

If $a_1, a_2, \ldots a_n$ denote a sequence of number such that $\lim_{n\to\infty}$ a_n exists and is equal to l

then $\lim_{n\to\infty} \frac{a_1+a_2 \ldots a_n}{n}$ also exists and is equal to l

proof:

let S_n be $a_1 + a_2 + \ldots + a_n$

since $a_n \to l$ as $n\to\infty$, we can find a number N corresponding to the arbitrary constant ϵ such that $l - \epsilon < a_n < l + \epsilon \, \forall n \geq$

Here $l-\epsilon < a_{N+1} < l+\epsilon$

$$l-\epsilon < a_{N+2} < l+\epsilon$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$l-\epsilon < a_n < l+\epsilon$$

Adding we get

$$(n-N)(l-\epsilon) < a_{N+1} + a_{N+2} + \cdots + a_n < (n-N)(l+\epsilon)$$

Adding $a_1 + a_2 + \cdots a_n$ (ie) S_N to all the terms in the equality we have

$$S_N + (n-N)(l-\epsilon) < S_n < S_N + (n-N)(l+\epsilon)$$

(ie) $\frac{S_N}{n} + \frac{(n-N)(l-\epsilon)}{n} < \frac{S_n}{n} < \frac{S_N}{n} + \frac{(n-N)(l+\epsilon)}{n}$

(ie) $\frac{S_N}{n} - \frac{N(l-\epsilon)}{n} + (l-\epsilon) < \frac{S_n}{n} < \frac{S_N}{n} - \frac{N(l+\epsilon)}{n} + (l+\epsilon)$

$\left(\frac{n}{n} - \frac{N}{n}\right)(l-\epsilon)$

(ie) $\frac{S_N}{n} - \frac{N(l-\epsilon)}{n} - \epsilon < \frac{S_n}{n} - l < \frac{S_N}{n} - \frac{N(l+\epsilon)}{n} + \epsilon$

Since N is a fixed number S_N, $N(l-\epsilon)$, $N(l+\epsilon)$ are fixed.

Hence, we can find a number. A such that S_N, $N(l-\epsilon)$ $N(l+\epsilon)$ are less then A

Hence $-\frac{2A}{n} - \epsilon < \frac{S_n}{n} - l < \frac{2A}{n} + \epsilon$

Since A is a fixed - we can find positive integer, $N_1 > N$ depending on ϵ

such that for all $n \geq N$, $\frac{2A}{n} < \epsilon$ and

$-\frac{2A}{n} > -\epsilon$

Hence the inequality becomes

$$-2\epsilon < \frac{S_n}{n} - l < 2\epsilon$$

(ie) $\left|\frac{S_n}{n} - l\right| < 2\epsilon \; \forall n \geq N$

Since ϵ is arbitrary, so also is 2ϵ

Hence $\lim\limits_{n \to \infty} \frac{S_n}{n} = l$

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = l$$

Hence the theorem.

Theorem:

Notes:

1) If $\{a_n\}$ is a convergent sequence and $\{b_n\}$ is a divergent sequence, then $\{a_n + b_n\}$ is divergent sequence

2) If $\{a_n\}$ and $\{b_n\}$ are both divergent then $\{a_n - b_n\}$ may be divergent or convergent or osscillate

3) If $\{a_n\}$ diverges to $+\infty$ and $\{b_n\}$ diverges to $-\infty$ then $\{a_n + b_n\}$ may begave in any way

ie) $|a_n - l| < \epsilon$ $\therefore a_n \to l$

This proof that condition is sufficient

13.7.15 **Cauchy's General principle of convergence**

6 A necessary and sufficient condition for the excistness of a limit to the $\{a_n\}$ is that. If any positive integer ϵ has been chosen, as small as we please there shall be a positive number m such that $|a_n - a_m| < \epsilon \; \forall \; n \geq m$

proof:-

The condition is necessary

Let the sequence converge to the limit 'l'. Having to choose ϵ, take it as $\epsilon/2$

we know that there is a positive integer m such that $|a_n - l| < \frac{1}{2}\epsilon$ for $n \geq m$ such that

But $a_n - a_m = a_n - l + l - a_m$

$$|a_n - a_m| = |a_n - l + l - a_m|$$

$$\leq |a_n - l| + |l - a_m|$$

$$< \epsilon/2 + \epsilon/2 \text{ if } n \geq m$$

$\therefore |a_n - a_m| < \epsilon$

The condition is sufficient :

For if the condition is satisfied there is a integer m such that if $n \geq m$

$|a_n - a_m| < \epsilon$

(ie) $a_m - \epsilon < a_n < a_m + \epsilon$ for $n > m$

$\therefore a_m - \epsilon$ is an inferior number and $a_m + \epsilon$ is a superior number

Hence $\mu - \lambda \leq (a_m + \epsilon) - (a_m - \epsilon)$

$\mu - \lambda \leq 2\epsilon$

(ie) $|\mu - \lambda| \leq 2\epsilon$

but $|\mu - \lambda| \geq 0$

and ϵ can be taken arbitrary small

$\therefore |\mu - \lambda| = 0$

$\therefore \overline{\text{limit}}\, a_n = \underline{\text{limit}}\, a_n$

$\therefore$ Hence the sequence $\{a_n\}$ is convergent

Monotonic sequence :

A sequence in which $a_{n+1} \geq a_n$ for all values of n is called a monotonic increasing sequence. Similarly, $a_{n+1} \leq a_n$ for all values of n, the sequence is called a monotonic decreasing sequence)

For example:-

$\{a_n\}$ define by $a_n = n$ is monotonically increasing sequence and $\{a_n\}$ define by $a_n = \frac{1}{n}$ is monotonically decreasing sequence

Also $\{a_n\}$ define by $a_n = (-1)^n$ is neither monotonically increasing (or) decreasing

Ex: 1 S.T $\left\{\frac{n}{n+1}\right\}$ is a monotonic increasing sequence.

Solution:-

Given $a_n = \frac{n}{n+1}$

$$a_{n+1} = \frac{n+1}{n+2}$$

Hence $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$

$$\Rightarrow \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{n^2+1+2n-n^2-2n}{(n+1)(n+2)}$$

$$a_{n+1} - a_n = \frac{1}{(n+1)(n+2)}$$

$a_{n+1} - a_n > 0$ for all values of n

$\therefore a_{n+1} > a_n$ for all values of n

Hence, It is a monotonic increasing sequence.

2

$n \to \infty$ $u_{n+2} = (a_1 a_2^2)^{1/3}$

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UNIT - II

Infinite series :

An expression of the form $u_1+u_2+u_3+\dots u_n \dots$ is which every term is followed another term. is called infinite series.

S_n denotes the sum of first n terms as $n \to \infty$ S_n may tends to

i) finite limit

ii) infinit infinity

iii) $-\infty$

iv) More then one limit

$\therefore x = -1$, $s_n = 1$, if n is odd

$s_n = 0$, if n is even

$\therefore x < -1$

$s_n \to \infty$, if n is even

$s_n \to -\infty$, if n is odd

$\therefore$ The series is convergent

if $|x| < 1$ divergent $x > 1$

oscillates finitely if $x = -1$

oscillates infinitely if $x < -1$

Theorem - I

If $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is convergent and has the sum (s) then $u_{m+1} + u_{m+2} + \cdots$ is convergent and has the sum $(s - (u_1 + u_2 + u_3 + \cdots + u_m))$ where m is a +ve integer

proof :-

$$\lim_{n\to\infty} (u_{m+1} + u_{m+2} + \cdots + u_{m+n})$$

$$\underset{n\to\infty}{Lt} \{(u_1 + u_2 + \cdots + u_m + u_{m+1} + \cdots + u_{m+n}) - (u_1 + u_2 + \cdots + u_m)\}$$

$$\underset{n\to\infty}{Lt} (u_1 + u_2 + \cdots + u_{m+n}) - (u_1 + u_2 + \cdots + u_m)$$

$\therefore S - (u_1 + u_2 + \cdots + u_m)$

Theorem : 2

If $u_1 + u_2 + u_3 \cdots$ is Convergent and has the sum 's' then $ku_1 + ku_2 \cdots$ is also Convergent and has the sum 'ks'

proof :-

$$\lim_{n\to\infty} (ku_1 + ku_2 + \cdots)$$

$$\lim_{n\to\infty} k(u_1 + u_2 + \cdots)$$

$$k \lim_{n\to\infty} u_1 + u_2 + \cdots$$

$\therefore ks$ is also convergent.

Theorem : 3

If $u_1 + u_2 + \cdots + u_n + \cdots$, $v_1 + v_2 + v_3 + \cdots + v_n \cdots$ are both convergent and its sum is the sum of two series

proof :-

Let the sum of two series be s and t

$$\lim_{n\to\infty} (u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n) + \cdots$$

$$= \lim_{n\to\infty} (u_1 + u_2 + \cdots + u_n + \cdots) + \lim_{n\to\infty} (v_1 + v_2 + \cdots + v_n + \cdots)$$

$$= s + t$$

$\therefore \sum (u_n + v_n)$ Convergent to $s + t$

Comparing with V_n we say ~~that~~ the series is convergent

2. Discuss the convergence of the series.

$$\sum_{n=1}^{\infty} \frac{1}{(a+n)^p (b+n)^q}, \quad a, b, p, q \text{ are positive}$$

Soln:-

$$u_n = \frac{1}{(a+n)^p (b+n)^q}$$

$$V_n = \frac{1}{n^{p+q}}$$

$$\frac{u_n}{V_n} = \frac{n^{p+q}}{(a+n)^p (b+n)^q}$$

$$= \frac{n^{p+q}}{n^{p+q}\left(\frac{a}{n}+1\right)^p \left(\frac{b}{n}+1\right)^q}$$

as $n \to \infty$ $\frac{u_n}{V_n} \to 1$

$\sum \frac{1}{n^{p+q}}$ converges if $p+q > 1$

$\sum \frac{1}{n^{p+q}}$ diverges if $p+q \leq 1$

3. Text of convergence or divergence of

$$\sum_{1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$$

Saln:-

$$\text{Let } u_n = \frac{\sqrt{n^4+1} - \sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}} \times \sqrt{n^4+1} + \sqrt{n^4-1}$$

$$= \frac{\cancel{n^4}+1-\cancel{n^4}+1}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$u_n = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$\frac{u_n}{v_n} = \frac{2n^2}{n^2\left[\left(1+1/n^4\right)^{1/2}+\left(1-1/n^4\right)^{1/2}\right]}$$

as $n\to\infty$

$$\frac{u_n}{v_n} = 2$$

Comparing with v_n, we say the series is convergent.

4) Discuss the convergence of the series

$\sqrt[3]{n^3+1}-n$

$u_n = (n^3+1)^{1/3}-n \Rightarrow (n^3)^{1/3}\left[1+1/n^3\right]^{1/3}-n$

Soln :-

let $u_n = \sqrt[3]{n^3+1}-n \Rightarrow n\left\{\{1+1/n^3\}^{1/3}-1\right\}$

$= n\left(1+\frac{1}{3}\left(\frac{1}{n^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{n^3}\right)^2\cdots\right)$

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$

$$= n\left(1+\frac{1}{3n^3}+\frac{1}{9n^6}+\cdots-1\right)$$

$n\left(1+\frac{1}{3n^3}+\frac{1}{9n^6}\cdots\right)$

$$= \frac{1}{3n^2}+\frac{1}{9n^5}\cdots$$

$$v_n = \frac{1}{n^2}$$

D'Alembert's Ratio Test:

A series is convergent, if after any particular term is ratio of each terms to proceeding is always less than some fixed quantity which itself less than unity

proof:-

case (i):

Let the ratio of each term after the r^{th} to the preceeding term to be less than where $k<1$.

Then $\frac{u_{r+1}}{u_r} < k$, $\frac{u_{r+2}}{u_{r+1}} < k$, $\frac{u_{r+3}}{u_{r+2}} < k$, ...

$u_{r+1} < k \cdot u_r$

$u_{r+2} < k \cdot u_{r+1} < k^2 u_r$

$u_{r+3} < k \cdot u_{r+2} < k^3 u_r$

adding u_r on both sides.

$u_r + u_{r+1} + u_{r+2} + \dots < u_r(1 + k + k^2 + \dots)$

$< \frac{u_r}{1-k}$, $k<1$

Hence the sum of terms begining at the r^{th} term is finite.

Since the nature of series is uneffected by omitting the finite number of terms in the begining

Σu_n is Convergent

Corallary :-

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k,$

where $k<1$, then $\sum u_n$ is Convergent

proof :-

$\because \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = k,$ we can find natural

numbers 'm' very large that

$$\left| \frac{u_{n+1}}{u_n} - k \right| < \epsilon \ \forall n \geq m$$

$$k - \epsilon < \frac{u_{n+1}}{u_n} < k + \epsilon, \ n > m$$

$$\therefore \frac{u_{n+1}}{u_n} < k + \epsilon < 1$$

$\therefore \sum u_n$ is convergent.

case (ii)

A series is convergent if after any particular term the ratio of each term to the preceeding is either equal to unity or greater than unity?

proof :-

Let all the terms after the rth term be equal to u_r.

$$u_{r+1} + u_{r+2} + \cdots + u_{n+r} = n \cdot u_r$$

$$\lim_{n\to\infty} (u_{r+1} + u_{r+2} + \cdots + u_{n+r}) = \infty$$

$\therefore \Sigma u_n$ is divergent

Then, the ratio of each term after the rth to the precceding term be greater than 1.

$u_{r+1} > u_r$; $u_{r+2} > u_{r+1} > u_r$; . . .

$u_{r+1} + u_{r+2} + \cdots + u_{n+r} > n \cdot u_r$

$$\lim_{n\to\infty} (u_{r+1} + u_{r+2} + \cdots + u_{n+r}) = \infty$$

$\therefore \Sigma u_n$ is divergent.

Corollary:

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$, where $k>1$, then Σu_n is divergent.

Proof:

$\because \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$, where $k>1$, we can find a natural numbers 'm' so large that

$$\left|\frac{u_{n+1}}{u_n} - k\right| < \epsilon, \ n>m$$

$$k-\epsilon < \frac{u_{n+1}}{u_n} < k+\epsilon, \ n>m$$

$\because k>1$, $k-\epsilon>1$

$\therefore \frac{u_{n+1}}{u_n} > k-\epsilon > 1$

$\therefore \Sigma u_n$ is divergent

If $x=1$, the series is convergent by Raabe's test

$\therefore$ The given series is convergent when $x \leq 1$ and divergent when $x > 1$

Raabe's test :-

9 If $\sum u_n$ & $\sum v_n$ are two series of +ve terms and if $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ $\forall n$ after a certain stage show that $\sum u_n$ will converge if $\sum v_n$ converges.

Since the omission of a finite number of terms from a series does not affect convergence or divergence of a series so we assume the inequality holds for all.

corollary :

· compare $\sum u_n$ with the series $\sum 1/n^p$. $\sum 1/n^p$ is convergent if $p>1$ and divergent if $p \leq 1$ if $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$

$\therefore \sum u_n$ is convergent if $\frac{u_{n+1}}{u_n} < \frac{n^p}{(n+1)^p}$

(ie) if $\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p$

(ie) if $\frac{u_n}{u_{n+1}} > (1+1/n)^p$

(ie) if $u_n/u_{n+1} > 1 + p/n + \frac{p(p-1)}{2n^2} + \dots$

(Assuming binomial theorem for rational index)

(ie) if $n\left(\frac{u_n}{u_{n+1}}-1\right) > p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \dots$

(ie) if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) \boxed{> p}$

But the auxillary series is convergent if $p>1$

$\therefore \sum u_n$ is convergent when $p>1$

$\sum u_n$ is divergent, if $\frac{u_{n+1}}{u_n} > \frac{n^p}{(n+1)^p}$

(ie) if $\frac{u_n}{u_{n+1}} < \left(\frac{(n+1)^p}{n}\right)$

(ie) if $\frac{u_n}{u_{n+1}} < (1+1/n)^p$

(ie) if $\frac{u_n}{u_{n+1}} < 1 + p/n + \frac{p(p-1)}{2} \cdot 1/n^2 + \dots$

(ie) if $\lim_{n\to\infty} \left(n \frac{u_n}{u_{n+1}} - 1\right) < p$

when $p<1$, $\sum 1/n$ is divergent

$\therefore$ In that case $\sum u_n$ is divergent.

This test can be enunciated as follows

The series whose general term is u_n is convergent or divergent, according as

$\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} > 1$ or ≤ 1

$\therefore$ This is known as Raabe's test

Similarly $\sum v_n$ is conv...

Verified M.Phil

UNIT - III

Cauchy's Condensation test :-

If $f(n)$ is positive for all positive integral values of n and continually diminishes as n increases and if a be any positive integer then the two infinite series.

$$f(1)+f(2)+f(3)+\cdots+f(n)+\cdots \text{ and}$$

$$a\,f(a)+a^2 f(a^2)+a^3 f(a^3)+\cdots+a^n f(a^n)+\cdots$$

are both convergent or both divergent.

proof :-

The terms of $\sum f(n)$ as follows,

$$\{f(1)+f(2)+\cdots+f(a)\}+\{f(a+1)+f(a+2)+\cdots+f(a^2)\}+$$

$$\{f(a^2+1)+f(a^2+2)+\cdots+f(a^3)\}+\cdots+\cdots$$

$$+\{f(a^{n-1}+1)+f(a^{n-1}+2)+\cdots+f(a^n)\}\cdots$$

If $k \le 1$

3) Discuss the convergence of the series

$$\sum \frac{1}{n(\log n)^p}$$

2

Soln :-

Here, $f(n) = \sum \frac{1}{n(\log n)^p}$

$\sum \frac{1}{n(\log n)^p}$ and $\sum \frac{a^n}{a^n(\log a^n)^p}$ converge or diverge together

$\therefore$ the second series is $\sum \frac{1}{(\log a^n)^p}$

(ie): $\frac{1}{\sum(n \log a)^p}$, (ie) $\frac{1}{(\log a)^p} \sum \frac{1}{n^p}$

This converges if $p>1$ and diverges if $p \le 1$

Cauchy's Root test:

If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms, prove that the series is convergent or divergent according as $\lim_{n \to \infty} u_n^{1/n} < 1$ or > 1

Proof :-

Case (i):

Let $\lim_{n \to \infty} u_n^{1/n}$ be l, where $l<1$.

$l<1$ and hence we can choose ϵ positive and sufficiently small so that $l+\epsilon<1$

Since $\lim_{n\to\infty} u_n^{1/n} = l$, we can find a natural number 'm' so large that u_n differs from l by less than ϵ so long as $n \geq m$

$$\therefore u_n^{1/n} < l+\epsilon$$

$$\therefore u_n < (l+\epsilon)^n$$ ϵ - is very small and sufficiently

Hence form and after the m^{th} term, the terms of the series $\sum u_n$ are less than those of the geometric series $\sum (l+\epsilon)^n$ which is convergent since $l+\epsilon<1$.

$\therefore \sum u_n$ is convergent

case (ii)

Let $\lim_{n\to\infty} u_n^{1/n}$ be l, where $l>1$

$\therefore l>1$ and hence we can choose ϵ positive and sufficiently small so that

$l-\epsilon > 1$

Now, since $\lim_{n\to\infty} u_n^{1/n} = l$, we can find a natural numbers m, so large that $u_n^{1/n}$ differs from l by less than ϵ so long as $n \geq m$.

$$\therefore u_n^{1/n} > (l-\epsilon) > 1$$

$$u_n > (l-\epsilon)^n$$

$\sum (l-\epsilon)^n$ is divergent since $l-\epsilon \geq 1$

2. Show that the series $\sum \frac{\{(n+1)r\}^n}{n^{n+1}}$ is convergent if $r<1$ & divergent if $r \geq 1$

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soln:-

Given that, $u_n = \frac{\{(n+1)r\}^n}{n^{n+1}}$

$$u_n^{1/n} = \frac{(n+1)r}{n^{(n+1)(1/n)}}$$

$$= \frac{(n+1)}{n^{1+1/n}} \cdot r$$

$$= \frac{n(1+1/n)}{n \cdot n^{1/n}} \cdot r \Rightarrow \frac{1+1/n}{n^{1/n}} \cdot r$$

$\therefore \lim_{n\to\infty} u_n^{1/n} = r$ [since $\lim_{n\to\infty} n^{1/n} = 1$]

$\therefore \sum u_n$ converges if $r<1$ and diverges if $r>1$

If $r=1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \left(\frac{n+1}{n}\right)^n \cdot 1/n$

$$= (1+1/n)^n \cdot 1/n$$

$v_n = 1/n$

Let $\sum v_n = \sum 1/n$

$$\therefore \frac{u_n}{v_n} = (1+1/n)^n$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} (1+1/n)^n = e$$

$\therefore \sum u_n$ and $\sum v_n$ behave alike but $\sum v_n$ is divergent

$\therefore$ Hence $\sum u_n$ is also divergent

3) Examine the convergence of the series

$$\sum \frac{(n+1)(n+2)\cdots\cdots(n+n)}{n^n}$$

soln:-

Let $u_n = \dfrac{(n+1)(n+2)\cdots\cdots(n+n)}{n^n}$

$$= \frac{n+1}{n} \cdot \frac{n+2}{n} \cdots\cdots \frac{n+n}{n}$$

$$u_n = (1+1/n)(1+2/n)\cdots\cdots(1+n/n)$$

$$u_n^{1/n} = \{(1+1/n)(1+2/n)\cdots\cdots(1+n/n)\}^{1/n}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \{(1+1/n)(1+2/n)\cdots\cdots(1+n/n)\}^{1/n}$$

Let this limit be l

$$\lim_{n\to\infty} \{(1+1/n)(1+2/n)\cdots\cdots(1+n/n)\}^{1/n} = l$$

$$\log l = \lim_{n\to\infty} 1/n \{\log(1+1/n) + \log(1+2/n) + \cdots\cdots + (1+n/n)\}$$

$$= \lim_{n\to\infty} 1/n \sum_{r=1}^{n} \log(1+r/n)$$

put $r/n = x$, $dx = 1/n \, dr$.

when $r=1$, $x = 1/n$, $n\to\infty$, $x=0$

where $r=n$, $x = n/n = 1 \Rightarrow x=1$

$$\log l = \int_0^1 \log(1+x)\,dx$$

Let $u = \log(1+x)$, $dv = dx$

$du = \frac{1}{1+x}\,dx$, $v = x$

$\int u\,dv = uv - \int v\,du$

$$\int_0^1 \log(1+x)\,dx = [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx.$$

$$= [1\log(1+1)] - \int_0^1 \frac{x+1-1}{1+x}\cdot dx$$

$$= \log(2) - \left\{\int_0^1 \frac{x+1}{x+1}\,dx - \int_0^1 \frac{dx}{1+x}\right\}$$

$$= \log(2) - \left\{[x]_0^1 - [\log(1+x)]_0^1\right\}$$

$$= \log(2) - \{1 - \log 2\}$$

$$= 2\log(2) - 1$$

$$= \log 2^2 - 1$$

$$\log l = \log 4 - 1$$

$$e^{\log l} = e^{\log 4 - 1}$$

$$l = 4/e$$

since e lies between 2 and 3

$\therefore \sum u_n$ is divergent. ~~condensation~~ root test.

Exercise:-

Cauchy's

Test for convergence the following

1) series $\sum \frac{x^n}{n^n} (x>0)$

soln:-

Let $u_n = \frac{x^n}{n^n} = \left(x/n\right)^n$

$u_n^{1/n} = (x/n)^{n \times 1/n}$

$\lim_{n\to\infty} u_n^{1/n} = 0 < 1$ [0 less → convergent
∞ finite → convergent]

∴ The series is convergent.

2. $\sum \frac{1}{(\log n)^n}$

soln :-

Let $u_n = \frac{1}{(\log n)^n}$

$u_n^{1/n} = \frac{1}{\log n}$

$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{\log \infty} = 1/\infty = 0$

∴ The series is convergent

3. $\sum (\sqrt{n}-1)^n$

soln :-

Let $u_n = (\sqrt{n}-1)^n$

$u_n^{1/n} = \sqrt{n} - 1$

$\lim_{n\to\infty} u_n^{1/n} = (n^{1/2}-1)$

$= \frac{1}{2n^{-1/2}}$

$= \frac{1}{2\sqrt{n}}$

∴ $\lim_{n\to\infty} u_n^{1/n} = 1/\infty = 0$

∴ The series is convergent

4) $\sum_{1}^{\infty} \frac{1}{(1+1/n)^{n^2}}$

Soln :-

Let $u_n = \frac{1}{(1+1/n)^{n^2}}$

$u_n^{1/n} = \frac{1}{(1+1/n)^{n^2 \times 1/n}} = \frac{1}{(1+1/n)^n}$

$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{(1+1/n)^n}$

$\lim_{n\to\infty} u_n^{1/n} = 1/e < 1$

$\therefore$ The series is convergent.

5) Investigate the behaviour of the series whose general terms are $\frac{n!}{n^n}$

Soln :-

Let $u_n = \frac{n!}{n^n}$

$u_n^{1/n} = \left(\frac{n!}{n^n}\right)^{1/n}$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(1/n \cdot 2/n \cdots\cdots n/n\right)^{1/n}$

Since, $\lim_{n\to\infty} u_n^{1/n} = l$

$l = \lim_{n\to\infty} \left(1/n \cdot 2/n \cdots\cdots n/n\right)^{1/n}$

Taking log on both sides

$\log l = \lim_{n\to\infty} 1/n \sum_{r=1}^{n} \log (r/n)$

put $r/n = x$, $dx = 1/n \cdot dr$

where $r = 1$, $x = 1/n$ as $n \to \infty$, $x = 0$

when $r = n$, $x = 1$

$$\log l = \int_0^1 \log x \, dx$$

$$= [x \log x]_0^1 - \int_0^1 x/x \, dx$$

$$= \{\log 1\} - [x]_0^1$$

$$= \{\log 1\} - [1 - 0]$$

$$\log l = \{0\} - 1$$

$$e^{\log l} = e^{-1}$$

$$l = 1/e$$

6. $2^n \cdot \dfrac{n!}{n^n}$.

Soln:

Given $u_n = 2^n \cdot \dfrac{n!}{n^n}$.

$$u_n^{1/n} = \left(2^n \cdot \frac{n!}{n^n}\right)^{1/n}$$

$$\lim_{n\to\infty} (u_n^{1/n}) = 2 \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n}$$

$$= 2 \lim_{n\to\infty} \left(1/n \cdot 2/n \cdots n/n\right)^{1/n}$$

Since $\lim_{n\to\infty} u_n^{1/n} = l$

$$l = 2 \lim_{n\to\infty} \left(1/n \cdot 2/n \cdots n/n\right)^{1/n}$$

6) S.T the series $\sum (-1)^n \sqrt{n^2+1} - n$ is a conditionally convergent series.

Soln:

Let $u_n = \sqrt{n^2+1} - n$

$$u_n = \sqrt{n^2+1} - n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$u_n = \frac{(n^2+1)-n^2}{\sqrt{n^2+1}+n}$$

$$u_n = \frac{1}{\sqrt{n^2+1}+n} \quad \text{and} \quad v_n = 1/n$$

$$\frac{u_n}{v_n} = \frac{n}{\sqrt{n^2+1}+n} = \frac{n}{n\left[\sqrt{1+1/n^2}+1\right]}$$

$$= \frac{1}{\sqrt{1+1/n^2}+1}$$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{\sqrt{1+1/n^2}+1} = 1$$

$n \to \infty$ tends to a finite limit

$\therefore$ The series is divergent (the series is compared to v_n) converg

$\therefore$ The series is conditionally convergent

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