Unit – I

Continuous Uniform distribution – Definition, Derivation of moments, Moment Generating function (M.G.F) Characteristic function. Exponential distribution – Mean and Variance, M.G.F. and Properties.

Unit – II

Normal Distribution – Definition, Properties,Derivation of mean and variance, moments, mode, Median, M.G.F., Characteristic function. Cumulant generating function and Additive property of Normal distribution.

Unit – III

Beta Distribution of first kind and Beta distribution of second kind – Definition, Derivation of mean, variance and Harmonic mean- properties. Gamma distribution – Definition, Moments, M.G.F. C.G.F. and Additive property.

Unit – IV

Convergence in probability – definition , Chebychev's inequality with proof, weak law of large numbers with proof. Convergence in distribution – definition , Central limit theorem (statement only). Exact sampling distribution– Chi-Square distribution – Definition, mean, variance, M.G.F., C.F., Mode and Skewness, Additive property, limiting form of Chi-Square distribution and applications.

Unit – V

Student's t distribution – definition, Derivation, Constants, Properties, limiting form of Student's t distribution. F-distribution – definition, Derivation, Constants. Relationship between t, F and Chi-Square distributions.

UNIT-I

RECTANGULAR (OR UNIFORM) DISTRIBUTION:-

Definition:-

A random variable X is said to have a continuous rectangular (uniform) distribution over an interval (a,b) i.e., $(-\infty < a < b < \infty)$, if its p.d.f is given by

$$f(x;a,b) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Remarks:-

1. a and b, $(a<b)$ are the two parameters of the distribution, the distribution is called uniform distribution on (a,b) since it assumes a constant (uniform) value for all x in (a,b)

2. The distribution is also known as rectangular distribution, since the curve $y=f(x)$ describes a rectangular over the x-axis and between the ordinates at $x=a$ and $x=b$.

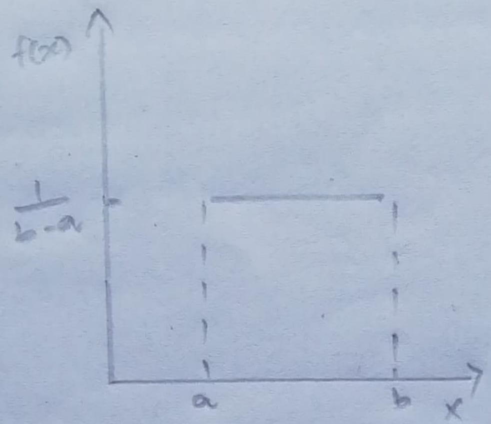
3. A uniform or rectangular variate X on the interval (a,b) is written as : $X \sim U[a,b]$ or $X - R[a,b]$

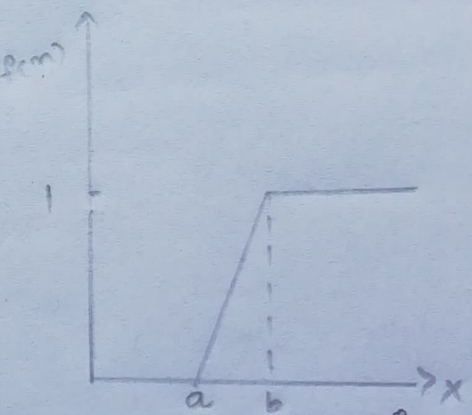
4. The cumulative distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

Since $F(x)$ is not continuous at $x=a$ and $x=b$ it is not differentiable at these points. Thus $\frac{d}{dx}F(x) = F(x) = \frac{1}{b-a} \neq 0$, exists everywhere except at the points $x=a$ and $x=b$ and consequently p.d.f $f(x)$ is given by

5. The graphs of uniform p.d.f $f(x)$ and the corresponding distribution function $F(x)$ are given below.





6. For a rectangular or uniform variate X in $(-a, a)$ the p.d.f is given by.

$$f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Moments of Rectangular Distribution on Let $X \sim U[a, b]$

$$\mu_r' = \int_a^b x^r f(x)\,dx = \frac{1}{b-a}\int_a^b x^r\,dx = \frac{1}{b-a}\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right)$$

In particular

$$\text{Mean} = \mu_1' = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{b+a}{2}$$

and

$$\mu_2' = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{3}(b^2+ab+a^2)$$

$$\therefore \text{variance} = \mu_2' - \mu_1'^2 = \frac{1}{3}(b^2+ab+a^2) - \left\{\frac{1}{2}(b+a)\right\}^2$$

$$= \frac{1}{12}(b-a)^2$$

M.G.F of Rectangular Distribution is given by

$$M_x(t) = \int_a^b e^{tx} f(x)dx = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{e^{bt} - e^{at}}{t(b-a)}, t \neq 0.$$

characteristic Function of rectangular Distribution is given by.

$$\phi_x(t) = \int_a^b e^{itx} dx = \frac{e^{ibt} - e^{iat}}{it(b-a)}, t \neq 0.$$

Mean Deviation about Mean, η of Rectangular Distribution is given by:-

$$\eta = E|X - \text{Mean}| = \int_a^b |x - \text{Mean}| f(x) dx$$

$$= \frac{1}{b-a} \int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{1}{b-a} \int_{(-b-a)}^{(b-a)/2} |t| dt, \text{ where } t = x - \frac{a+b}{2}$$

$$= \frac{1}{b-a} \cdot 2 \int_0^{(b-a)/2} t dt = \frac{b-a}{4}.$$

EXPONENTIAL DISTRIBUTION:-

Definition:-

A.r.v x is said to have an exponential Distribution with parameter $\theta > 0$, it its p.d.f is given by

$$f(x, \theta) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function $F(x)$ is given by.

$$F(x) = \int_0^x f(u)\,du = \theta \int_0^x \exp(-\theta u)\,du$$

$$F(x) = \begin{cases} 1-\exp(-\theta x), & \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Moment Generating function of Exponential Distribution:-

$$M_X(t) = E(e^{tX}) = \theta \int_0^\infty e^{tx} \cdot e^{-\theta x}\,dx = \theta \int_0^\infty \exp\{-(\theta - t)x\}\,dx$$

$$= \frac{\theta}{(\theta - t)} = \left(1 - \frac{t}{\theta}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{t}{\theta}\right)^r, \quad \theta > t$$

$\therefore \mu_r' = E(X^r) =$ coefficient of $\frac{t^r}{r!}$ in $M_X(t) = \frac{r!}{\theta^r}$; $r = 1, 2, \ldots$

$\Rightarrow$ Mean $= \mu_1' = \frac{1}{\theta}$ and variance $= \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$

Hence, if $X \sim \exp(\theta)$, then mean $= \frac{1}{\theta}$ and variance $= \frac{1}{\theta^2}$

Remark:-

$$\text{variance} = \frac{1}{\theta^2} = \frac{1}{\theta} \cdot \frac{1}{\theta} = \frac{\text{Mean}}{\theta}$$

$\therefore$ variance $>$ Mean, if $0 < \theta <$

variance $=$ Mean, if $\theta = 1$

and,

variance $<$ Mean, if $\theta > 1$

Hence for the exponential distribution, variance $>, =, <$ Mean, for different values of the parameter

Theorem :-

If $x_1, x_2, \dots, x_n$ are independent r.v's x_i having an exponential distribution with parameter θ_i ; $i = 1, 2, \dots n$; then $z = \min(x_1, x_2, \dots x_n)$ has exponential distribution with parameter $\sum_{i=1}^{n} \theta_i$

Proof:-

$$G_Z(z) = P(Z \leq z) = 1 - P(Z > z) = 1 - P[\min(x_1, x_2, \dots x_n) > z]$$

$$= 1 - P(x_i > z; i = 1, 2, \dots n) = 1 - \prod_{i=1}^{n} P(X_i > z)$$

$(\because x_1, x_2, \dots x_n$ are independent$)$

$$= 1 - \prod_{i=1}^{n} [1 - (1 - e^{-\theta_i z})] = \begin{cases} 1 - \exp\{(-\sum_{i=1}^{n} \theta_i) z\} & ; z > 0. \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore g_Z(z) = \frac{d}{dz}(G(z)) = \begin{cases} (\sum_{i=1}^{n} \theta_i) \exp\{(-\sum_{i=1}^{n} \theta_i) z\} & , z > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$\Rightarrow z = \min(x_1, x_2, \dots x_n)$ is an exponential variate with parameter $\sum_{i=1}^{n} \theta_i$.

Cor :-

If x_i; $i = 1, 2, \dots n$ are identically distributed with parameter θ, then following exponential distribution with parameter θ, then $Z = \min(x_1, x_2, \dots x_n)$ is also exponentially distributed with parameter $n\theta$

Example:-

Show that the exponential distribution lacks memory, i.e, if x has an exponential Distribution, then for every constant $a \geq 0$, one has

$$P(Y \leq x \mid X \geq a) = P(X \leq x) \text{ for all } x, \text{ where } Y = X - a$$

Solution:-

The p.d.f of the exponential distribution with parameter θ is

$$f(x) = \theta \exp(-\theta x) ; \quad \theta > 0, \quad 0 < x < \infty$$

We have

$$P(Y \le x \cap X \geq a) = P(X - a \le x \cap X \geq a)$$
$$= P(X \le a + x \cap X \geq a)$$
$$= P(a \le X \le a + x)$$
$$= \theta \int_a^{a+x} e^{-\theta x} dx = e^{-a\theta}(1 - e^{-\theta x})$$

and

$$P(X \geq a) = \theta \int_a^{\infty} e^{-\theta x} dx = e^{-a\theta}$$

$$\therefore P(Y \le x \mid X \geq a) = \frac{P(Y \le x \cap X \geq a)}{P(X \geq a)} = 1 - e^{-\theta x}$$

Also, $P(X \le x) = \theta \int_0^x e^{-\theta x} dx = 1 - e^{-\theta x}$

From (*) and (**), we get $P(Y \le x \mid X \geq a) = P(X \le x)$

i.e, exponential distribution lacks Memory.

Example:-

X and Y are independent and identically distributed their joint p.d.f is given by

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y) = \begin{cases} e^{-(x+y)} & ; x > 0, y > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Let $\begin{cases} u = x - y \\ v = y \end{cases} \Rightarrow \begin{cases} x = u + v \\ y = v \end{cases}$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Thus the joint p.d.f of u and v becomes

$$g(u, v) = e^{-(u+2v)} \; ; \; v > 0, \; -\infty < u < \infty$$

(1) $\Rightarrow u = x - v \Rightarrow v = x - u$ Thus $v > -u$, if $-\infty < u < 0$
$v > 0$, if $u > 0$

For $-\infty < u < 0$,

$$g(u) = \int_0^\infty g(u, v)\,dv = \int_{-u}^\infty e^{-(u+2v)}\,dv = e^{-u} \left| \frac{e^{-2v}}{-2} \right|_{-u}^\infty$$

$$= \frac{1}{2} e^{u}$$

and for $u > 0$,

$$g(u) = \int_0^\infty g(u, v)\,dv = e^{-u} \left| \frac{e^{-2t}}{-2} \right|_0^\infty = \frac{1}{2} e^{-u}$$

Hence the p.d.f of $u = x - y$ is given by;

$$g(u) = \begin{cases} \frac{1}{2} e^{u}, & -\infty < u < 0 \\ \frac{1}{2} e^{-u}, & u > 0 \end{cases}$$

These results can be combined to give:-

$$g(u) = \frac{1}{2} e^{-|u|}, \quad -\infty < u < \infty$$

which is the p.d.f of Standard Laplace Distribution.

Aliter:-

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{-(1-t)x}\,dx = \left| \frac{e^{-(1-t)x}}{-(1-t)} \right|_0^\infty$$

$$= \frac{1}{1-t}, \quad t < 1$$

$\therefore$ The characteristic function of x is

$\phi_x(t) = \frac{1}{1-it} = \phi_y(t)$ (since x and y are identically distributed)

Hence $\phi_{x-y}(t) = \phi_{x+(-y)}(t) = \phi_x(t) \cdot \phi_{-y}(t)$

$\therefore$ (x, y, are independent)

$= \phi_x(t) \cdot \phi_y(-t) = \frac{1}{(1-it)(1+it)} = \frac{1}{1+t^2}$

which is the characteristic function of the standard Laplace distribution.

$g(u) = \frac{1}{2} e^{-|u|}, \quad -\infty < u < \infty$

Hence by the uniqueness theorem of characteristic function $U = X - Y$ has standard Laplace distribution with the p.d.f given in (*).

UNIT - II

Normal Distribution:-

The normal was first discovered in 1733 by English mathematican De-movie who obtained this continoous distribution as a limiting case of the binomial distribution and applied it to problems assing in the game of chance. It was also known to laplace no later than 1774 but through a historial error it was credited to gauss, who first made refrence to it in the beginning of 19th century (1809) as the distribution of errors in Astronomy. Gauss used the normal curve to describe the theory of accidental errors of measurements involved bodies. Throughout the eighteenth centuries, various efforts were made to establish the normal model as the underlying law ruling all continuous random variables. Thus the name "normal" These efforts however failed because of false premises

The normal model has, nevertheless beome the most important probability model in statistical analysis.

Definition:-

A random variable X is said to have a normal distribution with parameters μ (called Mean) and σ^2 (called variance) if its p.d.f is given by the probability law;

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

$$f(x;\mu\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

Remarks:-

When a random variable is normally distributed with mean μ and Standard deviation σ, it is customary to write x is distributed as $N(\mu,\sigma^2)$ and is expressed by $X \sim N(\mu,\sigma^2)$

If $X \sim N(\mu,\sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with $E(Z)=0$ and $var(Z)=1$ and we write $Z \sim N(0,1)$

The p.d.f of standard normal variate Z is given by

$$\mu(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad - \quad -\infty < z < \infty$$

and the corresponding distribution function denoted by $\phi(z)$ is given by

$$\phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(u)du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{\mu^2}{2}} du$$

We shall prove below two important results on the distribution function $\phi(\cdot)$ of standard normal variates.

chief characteristics of the normal distribution and normal probability curve.

The normal probability curve with mean μ and standard deviation σ is given by the equation.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

and has the following properties :-

i) The curve is bell-shaped and symmetrical about the line $x=\mu$

ii) Mean, Median and mode of the distribution coincide

iii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occuring at the point $x=\mu$ and is given by :-

$$[p(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$$

iv) $\beta_1 = 0$ and $\beta_2 = 3$

v) $\mu_{2r+1} = 0$ $(r=0,1,2,\dots)$ and $\mu_{2r} = 1\cdot3\cdot5\dots(2r-1)\sigma^{2r}$

$r=(0,1,2,\dots)$

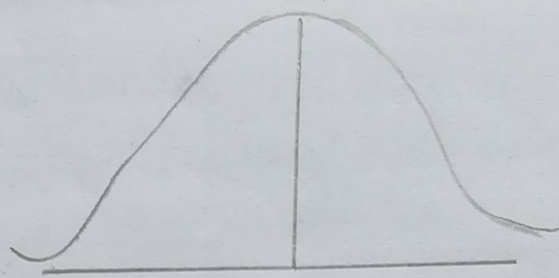
vi) Since $f(x)$ being the probability, can never be negative, no portion of the curve lies below the x-axis

vii) linear combination of independent normal variates is also an normal variate

viii) The points of inflexion of the curve are

$$x = \mu \pm \sigma, \quad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$$

Normal probability curve



$X = \mu$

x) Mean deviation about Mean :-

$$= \sqrt{\frac{2}{\pi}}\,\sigma \simeq \frac{4}{5}\,\sigma \text{ (approx)}$$

xi) Quartiles are given by

$$Q_1 = \mu - 0.6745\,\sigma$$

$$Q_3 = \mu + 0.675\,\sigma$$

xii) $Q.D = \dfrac{Q_3 - Q_1}{2} \simeq \dfrac{2}{3}\sigma$ (approximateles)

$Q:D : M:D : S:D : \frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma :: \frac{2}{3} : \frac{4}{5} : 1$

$Q.D : M.D : S:D :: 10 : 12 : 15$

xiii) Area property

$$P(\mu - \sigma < x < \mu + \sigma) = 0.6826$$

$$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9544$$

and $P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

The adjoining table gives the area under the normal probability curve for some important values of standard normal variate Z

Distances from the mean ordinates in term of $\frac{1}{\sigma}$	Area under the curve
$Z = \pm 0.745$	$50\% = 0.50$
$Z = \pm 1.00$	$68.26\% = 0.6826$
$Z = \pm 1.96$	$95\% = 0.95$
$Z = \pm 2.0$	$95.44\% = 0.9544$
$Z = \pm 2.58$	$99\% = 0.99$
$Z = \pm 3.0$	$99.73\% = 0.9973$

xiv) If x and y are idependent Standard normal variates, then it can be easily proved that if $\mu = x+y$ and $V = x-y$ are independently distributed $U \sim N(0,2)$ and $V \sim N(0,2)$

We state without proof the converse of this result which is due to D. Beenstain

Mode of normal Distribution:-

Mode is the value of x for which $f(x)$ is maximum i.e., made is the solution of

$f'(x) = 0$ and $f''(x) < 0$

$-\infty < x < \infty$

$\log(ab) = \log a + \log b$

For normal distribution with mean μ and Standard deviation σ.

$$\log f(x) = \log \frac{1}{\sigma\sqrt{2\pi}} \sigma - \frac{(x-\mu)^2}{2\sigma} \quad -\infty < \mu < \infty$$

$$= \log \frac{1}{\sigma\sqrt{2\pi}} - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\log f(x) = C - \frac{1}{2\sigma^2}(x-\mu)^2$$

where $C = \log\left(\frac{1}{\sqrt{2\pi}}\sigma\right)$ is a constant Differentiating with reference to x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\frac{1}{\sigma^2}(x-\mu) \Rightarrow f'(x) = -\frac{1}{\sigma^2}(x-\mu)f(x)$$

and

$$f''(x) = -\frac{1}{\sigma^2}\left[1 \cdot f(x) + (x-\mu)f'(x)\right]$$

$$= -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

$$\log f(x) = C - \frac{1}{2\sigma^2}(x-\mu)^2$$

$f'(x) = 0 \Rightarrow x - \mu = 0 \Rightarrow x = \mu$ at the put $x = \mu$, we have from

$$f''(x) = -\frac{1}{\sigma^2}\left[f(x)\right]_{x=\mu}$$

$$= -\frac{1}{\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} < 0$$

Hence $x = \mu$ is the mode of the normal distribution.

Median of normal Distribution:-

If M is the Median of the normal distribution

we have

$$\int_{-\infty}^{M} f(x)\,dx = 1/2$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} \exp\{-(x-\mu)^2/2\sigma^2\}\,dx = 1/2$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} \exp\{-(x-\mu)^2/2\sigma^2\}\,dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} \exp\{-(x-\mu)^2 - \mu/(2\sigma)^2\}\,dx = 1/2$$

$$z = \frac{(x-\mu)^2}{\sigma^2}$$

$$dz = x-\mu$$

$$\sigma dz = dx$$

Put $x = -\infty,\ z = -\infty$

$x = \mu,\ z = 0$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} \exp \frac{-(x-\mu)^2}{2\sigma^2}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} \exp \frac{(-z)^2}{2}\,\sigma dz$$

But

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} \exp\{-(x-\mu)^2/2\sigma^2\,dx \Rightarrow$$

$$\frac{1}{2\pi} \int_{-\infty}^{0} \exp(-z^2/2)\,dz = 1/2$$

We have,

$$\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} \exp\{-(x-\mu)^2/2\sigma^2\} dx = 1/2$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} \exp\{-(x-\mu)^2/2\sigma^2\} dx = 0 \qquad \text{i.e, } \mu = M$$

Hence, for the normal distribution Mean = Median.

MGF of normal Distribution:-

The m.g.f (about origin) is given by

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \exp\{-(x-\mu)^2/2\sigma^2\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{+(\mu+\sigma z\} \exp(-z^2/2) dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(z^2 - 2t\sigma z)\} dz$$

$$= e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\{(z-\sigma t)^2 - \sigma^2 + z^2\}\right] dz$$

$$= e^{\mu t + t^2\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-1/2(z-\sigma t)^2\} dz$$

$$= e^{\mu t + t^2\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2/2) d\mu$$

Hence $M_X(t) = e^{\mu t + t^2\sigma^2/2}$

Remark:-

MGF of standard normal variate. If $X \sim N(\mu, \sigma^2)$ then standard normal variate is given by

$$Z = (X-\mu)/\sigma$$

$$M_Z(t) = e^{-\mu t/\sigma} M_X(t/\sigma)$$

$$= \exp(-\mu t/\sigma) \cdot \exp\{(\mu t/\sigma) + (t^2/\sigma^2)(\sigma^2/2)\}$$

$$= \exp(t^2/2)$$

Aleter:-

$Z \sim N(0,1)$. Hence taking $\mu = 0$ and $\sigma^2 = 1$ in

$$M_Z(t) = \exp(t^2/2)$$

Cumulant Generating function (c.g.f) of Normal Distribution:-

The c.g.f of normal distribution is given by

$$K_X(t) = \log_e M_X(t) = \log_e(e^{\mu t + t^2\sigma^2/2}) = \mu t + \frac{t^2\sigma^2}{2}$$

Mean $= K_1 =$ coefficient of t in $K_X(t) = \mu$

Variance $= K_2 =$ coefficient of $\frac{t^2}{2!}$ in $K_X(t) = \sigma^2$

and

$K_r =$ coefficient of $\frac{t^r}{r!}$ in $K_X(t) = 0$; $r = 3, 4, \ldots$

Thus $\mu_3 = K_3 = 0$ and $\mu_4 = K_4 + 3K_2^2 = 3\sigma^4$

Hence

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3$$

Moment of Normal Distribution:-

odd order moments about mean are given by

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx$$

$$\therefore \mu_{2n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} \exp(-z^2/2)\, dz = 0$$

Since the integrand $z^{2n+1} e^{-z^2/2}$ is an odd function of z

Even order moments about mean are given by

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} z^{2n} \exp(-z^2/2)\, dz$$ (since integrand is an even function of z)

$$= \frac{2\sigma^{2n}}{2\sqrt{2\pi}} \int_{0}^{\infty} (2t)^n e^{-t} \frac{dt}{\sqrt{2t}} \quad \left(t = \frac{z^2}{2}\right)$$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty e^{-t}\, t^{(n+1/2)-1}\, dt$$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}}\, \Gamma(n+1/2)$$

changing n to $(n-1)$, we get

$$\mu_{2n-2} = \frac{2^{n-1} \cdot \sigma^{2n-2}}{\sqrt{\pi}}\, \Gamma(n-1/2)$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \frac{2\sigma^2\, \Gamma(n+1/2)}{(n-1/2)} = 2\sigma^2(n-1/2) \quad [\because \Gamma(r) = (r-1)\Gamma(r-1)]$$

$$\Rightarrow \mu_{2n} = \sigma^2(2n-1)\, \mu_{2n-2}$$

which gives the recurence relation for the moments of normal distribution we have

$$\mu_{2n} = [(2n-1)\sigma^2][(2n-3)\sigma^2]\, \mu_{2n-\cancel{4}}$$

$$= [(2n-1)\sigma^2][(2n-3)\sigma^2][(2n-5)\sigma^2]\, \mu_{2-6}$$

$$= [(2n+1)\sigma^2][(2n-3)\sigma^2][(2n-5)\sigma^2] \cdots (3\sigma^2)(1\sigma^2)\, \mu_0$$

$$= 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$$

we conclude that for the normal distribution all add moments about Mean vanish and even order moments about mean are given by

$$= 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}.$$

UNIT - III

Note :-

$$I = \int_0^1 \binom{n}{k} (n+1) x^k (1-x)^{n-k} \, dx$$

$$= \binom{n}{k} (n+1) \, [B)(k+1), (n-k+1)$$

on using beta- integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} \; ; \; m>0 ; \; n>0$$

$$I = \frac{n!\,(n+1)}{k!\,(n-k)!} \cdot \frac{k!\,(n-k)!}{(n+1)!} = 1.$$

Gamma Distribution:-

Definition:-

A random variables x is said to have a gamma distribution with parameter $\lambda > 0$ if its p.d.f is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)} & ; \; \lambda > 0, \quad 0 < x < \infty \quad \rightarrow (\lambda) \\ 0 & ; \text{ otherwise} \end{cases}$$

Remark:-

1. X is known as a Gamma variate with parameter λ and refered to as a $\gamma(\lambda)$ variate)

2. The function $f(x)$ defined above represents a probability function, since

$$\int_0^\infty f(x)dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-x} x^{\lambda-1} dx = \frac{1}{\Gamma(\lambda)} \cdot \Gamma(\lambda) = 1.$$

3. A continuous random variable X having the following p.d.f is said to have a gamma Distribution with the parameters a and λ

$$f(x) = \begin{cases} \frac{a^\lambda}{\Gamma(\lambda)} e^{-ax} x^{\lambda-1}, & a>0, \lambda>0, 0<x<\infty \\ 0, & \text{otherwise} \end{cases} \rightarrow (b)$$

Here $X \sim \gamma(a, \lambda)$. Taking $a=1$, in (a) and (b) Hence we may $X \sim \gamma(\lambda) = \gamma(1, \lambda)$

4. The cumulative distribution, called incomplete, Gamma function is defined as:-

$$F_X(x) = \begin{cases} \int_0^x f(u)du = \frac{1}{\Gamma(\lambda)} \int_0^x e^{-u} u^{\lambda-1} du, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

MGF of Gamma Distribution:-

MGF about origin is given by

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1} dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(1-t)x} x^{\lambda-1} dx$$

$$= \frac{1}{\Gamma(\lambda)} \cdot \frac{\Gamma(\lambda)}{(1-t)^\lambda}, \quad |t|<1$$

$$M_X(t) = (1-t)^{-\lambda}, \quad |t|<1.$$

Cumulant Generating Function of Gamma Distribution:-

The cumulant generating function $K_X(t)$ is given by

$$K_X(t) = \log M_X(t) = \log(1-t)^{-\lambda} = -\lambda \log(1-t); \quad |t|<1$$

$$= \lambda\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)$$

$\therefore$ Mean $= k_1 =$ co-efficient of t in $K_X(t) = \lambda$

variance $= \mu_2 = k_2 =$ co-efficient of $\frac{t^2}{2!}$ in $K_X(t) = \lambda$

Hence if $X \sim \gamma(\lambda)$, Mean $=$ variance $= \lambda$

$\mu_3 = k_3 =$ coefficient of $\frac{t^3}{3!}$ in $K_X(t) = 2\lambda$

$k_4 =$ coefficient of $\frac{t^4}{4!}$ in $K_X(t) = 6\lambda$

$$\mu_4 = k_4 + 3k_2^2 = 6\lambda + 3\lambda^2$$

Hence

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{4\lambda^2}{\lambda^3} = \frac{4}{\lambda}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{6}{\lambda}.$$

Remark:-

Like poisson distribution, the mean and variance of the Gamma distribution are also equal. However poisson distribution is discrete while Gamma distribution is continuous.

Additive Property of Gamma Distribution:-

The Sum of independent Gamma Variates is also a Gamma Variates. more precisely if $x_1, x_2, \dots x_k$ are independent Gamma variate with parameter $\lambda_1 + \lambda_2 + \dots + \lambda_k$ respectively then $x_1 + x_2 + \dots + x_k$ is also a Gamma variate with parameter $\lambda_1 + \lambda_2 + \dots + \lambda_k$

Proof:-

Since x_i is a $\gamma(\lambda_i)$ variate $M_{x_i}(t) = (1-t)^{-\lambda_i}$ The mgf of the Sum $x_1 + x_2 + \dots + x_k$ is giving

$$M_{x_1+x_2+\dots+x_k}(t) = M_{x_1}(t)\, M_{x_2}(t) \dots M_{x_k}(t)$$

$(\because x_1, x_2, \dots x_k$ are independent$)$

$$= (1-t)^{-\lambda_1} (1-t)^{-\lambda_2} \dots (1-t)^{-\lambda_k}$$

$$= (1-t)^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k)}$$

which is the m.g.f of a Gamma variate with parameter $\lambda_1 + \lambda_2 + \dots + \lambda_k$. Hence the results follows by the uniqueness theorem of m.g.f's

Remark:-

In general if $x_i \sim \gamma(a, \lambda_i)$, $i = 1, 2, \dots n$ are independent random variable then

$$\sum_{i=1}^{n} x_i \sim \gamma\left(a, \sum_{i=1}^{n} \lambda_i\right)$$

Beta distribution of first kind Definition:-

A r.v x is said to have a beta distribution of first kind with parameters μ and ν $(\mu>0, \nu>0)$ if its p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{B(\mu,\nu)} \cdot x^{\mu-1}(1-x)^{\nu-1}, & (\mu,\nu)>0,\ 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

[Where $B(\mu,\nu)$ is the Beta Function]

The random variable is known as a Beta variate of the first kind with parameters μ and ν is refered to as $\beta_1(\mu,\nu)$ variate.

Remarks:-

1. The cumulative distribution function often called the incomplete Beta function is given by

$$F(x) = \begin{cases} 0, & x<0 \\ \int_0^x \frac{1}{B(\mu,\nu)} u^{\mu-1}(1-u)^{\nu-1}\,du; & 0<x<1\ (\mu,\nu)>0 \\ 1, & x\geq 1 \end{cases}$$

2. In particular, if we take $\mu=1$ $\nu=1$ in (a) we get

$$f(x) = \frac{1}{B(1,1)} = 1, \quad 0<x<1$$

which is the p.d.f uniform distribution on $[0,1]$

3. If $X \sim B(\mu, \nu)$ then it can be easily proved that $(1-X) \sim B(\nu, \mu)$

Constants of Beta distribution of first kind:

$$\mu_r' = \int_0^1 x^r f(x)\, dx$$

$$= \frac{1}{B(\mu,\nu)} \int_0^1 x^{\mu+r-1} (1-x)^{\nu-1} dx$$

$$= \frac{1}{B(\mu,\nu)} B(\mu+r, \nu)$$

$$\mu_r' = \frac{\Gamma(\mu+r)\, \Gamma(\mu+\nu)}{\Gamma(\mu+r+\nu)\Gamma(\mu)}$$

$r = 1$

$$\mu_1' = \frac{\Gamma(\mu+1)\, \Gamma(\mu+r)}{\Gamma(\mu+1+\nu)\, \Gamma(\mu)}$$

$$= \frac{\mu+1-1\, \Gamma(\mu+1-1)\Gamma(\mu+\nu)}{(\mu+\nu+1-1)\Gamma(\mu+\nu+1-1)\Gamma(\mu)} = \frac{\mu\Gamma(\mu)\Gamma(\mu+\nu)}{(\mu+\nu)\Gamma(\mu+\nu)\Gamma(\mu)}$$

$$\mu_1' = \frac{\mu}{(\mu+\nu)}$$

$r = 2$

$$\mu_2' = \frac{\Gamma(\mu+2)\, \Gamma(\mu+\nu)}{\Gamma(\mu+2+\nu)\Gamma(\mu)}$$

$$= \frac{\mu+2-1\, \Gamma(\mu+2-1)\, \Gamma(\mu+\nu)}{\mu+\nu+2-1\, \Gamma(\mu+\nu+2-1)\, \Gamma(\mu)}$$

$$= \frac{\overline{\mu+1}\,\Gamma(\mu+1)\,\Gamma(\mu+v)}{\overline{\mu+v+1}\,\Gamma(\mu+v+1)\,\Gamma(\mu)}$$

$$= \frac{(\mu+1)(\mu+1-1)\,\Gamma(\mu+1-1)\,\Gamma(\mu+v)}{(\mu+v+1)(\mu+v+1-1)\,\Gamma(\mu+v+1-1)\,\Gamma(\mu)}$$

$$= \frac{(\mu+1)(\mu)\,\Gamma(\mu)\,\Gamma(\mu+v)}{(\mu+v+1)(\mu+v)\,\Gamma(\mu+v)\,\Gamma(\mu)}$$

$$\mu_2' = \frac{(\mu+1)(\mu)}{(\mu+v+1)(\mu+v)}$$

Variance

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{(\mu+1)(\mu)}{(\mu+v+1)(\mu+v)} - \left(\frac{\mu}{\mu+v}\right)^2 = \frac{\mu}{(\mu+v)^2(\mu+v+1)}\left[(\mu+v)(\mu+1) - \mu(\mu+v+1)\right]$$

$$= \frac{\mu}{(\mu+v)^2(\mu+v+1)} \cdot \left[\mu^2+\mu+\mu v+v-\mu^2-\mu v-\mu\right]$$

$$\mu_2 = \frac{\mu v}{(\mu+v)^2(\mu+v+1)}$$

Similarly we have

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \mu_3' - 3\left(\frac{\mu(\mu+1)}{(\mu+v)(\mu+v+1)}\right)\left(\frac{\mu}{\mu+v}\right) + 2\left(\frac{\mu}{\mu+v}\right)^3$$

$$\mu_3' = \frac{(\mu+2)(\mu+1)(\mu)}{(\mu+\nu+2)(\mu+\nu+1)(\mu+\nu)}$$

$$\mu_3 = \frac{(\mu+2)(\mu+1)(\mu)}{(\mu+\nu+2)(\mu+\nu+1)(\mu+\nu)} - 3\frac{\mu(\mu+1)}{(\mu+\nu)(\mu+\nu+1)}\left(\frac{\mu}{\mu+\nu}\right) + 2\left(\frac{\mu}{\mu+\nu}\right)^3$$

on simplification

$$\mu_3 = \frac{2\mu\nu(\nu-\mu)}{(\mu+\nu)^3(\mu+\nu+1)(\mu+\nu+2)}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_3'\mu_1'^2 - 3\mu_1'^4$$

on simplification

$$= \frac{3\mu\nu\{\mu\nu(\mu+\nu-6) + 2(\mu+\nu)^2\}}{(\mu+\nu)^4(\mu+\nu+1)(\mu+\nu+2)(\mu+\nu+3)}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$= \left(\frac{2\mu\nu(\nu-\mu)}{(\mu+\nu)^3(\mu+\nu+1)(\mu+\nu+2)}\right)^2 \times \left(\frac{(\mu\times\nu)^2(\mu+\nu+1)}{\mu\nu}\right)^3$$

$$= \frac{4\mu^2\nu^2(\nu-\mu)^2}{(\mu+\nu)^6(\mu+\nu+1)^2(\mu+\nu+2)^2} \times \frac{(\mu+\nu)^6(\mu+\nu+1)^3}{\mu^3\nu^3}$$

$$\beta_1 = \frac{4(\nu-\mu)^2(\mu+\nu+1)}{\mu\nu(\mu+\nu+2)^2}$$

$$\beta_1 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{3\mu v\{\mu v(\mu+v-6)+2(\mu+v)^2\}}{(\mu+v)^4(\mu+v-1)(\mu+v+2)(\mu+v+3)} \times \frac{(\mu+v)^4(\mu+v+1)^3}{\mu^2 v^2}$$

$$\beta_2 = \frac{3(\mu+v+1)\,\mu v(\mu+v-6)+2(\mu+v)^2}{\mu v(\mu+v+2)(\mu+v+3)}$$

The harmonic mean H is given by

$$\frac{1}{H} = \int_0^1 \frac{1}{x} f(x)\,dx$$

$$= \frac{1}{B(\mu,v)} \int_0^1 x^{\mu-2}(1-x)^{v-1}\,dx$$

$$= \frac{1}{B(\mu,v)} B(\mu-1,v)$$

$$= \frac{\Gamma(\mu-1)\Gamma(v)}{\Gamma(\mu+v-1)} \cdot \frac{\Gamma(\mu+v)}{\Gamma(\mu)\Gamma(v)}$$

$$= \frac{\Gamma(\mu-1)(\mu+v-1)\Gamma(\mu+v-1)}{\Gamma(\mu+v-1)(\mu-1)\Gamma(\mu-1)} = \frac{\mu+v-1}{\mu-1}$$

$$H = \frac{\mu-1}{\mu+v-1}$$

Beta Distribution of second kind Definition:-

A random variable x is said to have a beta distribution of the second kind with parameters μ and ν $(\mu > 0, \nu > 0)$ if its p.d.f is given by

$$f(x) = \begin{cases} \dfrac{1}{B(\mu, \nu)} \cdot \dfrac{x^{\mu-1}}{(1+x)^{\mu+\nu}} \cdot (\mu, \nu) > 0, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Remarks :-

1. The random variable x is known as a Beta variate of second kind with parameters μ and ν and is denoted as $\beta_2(\mu, \nu)$ variate.

2. Beta distribution of second distribution of first transformed to Beta distribution of first kind by the transformation

$$1+x = \frac{1}{y} \Rightarrow y = \frac{1}{1+x}$$

Thus, if $x \sim \beta_2(\mu, \nu)$ then y defined is a $\beta(\mu, \nu)$.