ALLIED COURSE - VI - OPERATIONS RESEARCH - III

SYLLABUS

UNIT - I

Introduction - definition - payoff - types of games - the maximin minimax principles saddle point - Game with saddle point - without saddle point - mixed strategies - 2x2 games - graphical method for 2xn or mx2 games - dominance property - Resolving games by L.P.P - simple problems.

UNIT - II

Sequencing - Basic terms - processing n jobs through two machines, processing n jobs through k machines, processing 2 jobs through k machines.

UNIT - III

Queuing system - elements of queuing system - operating characteristics of a queuing systems - deterministic queuing system - probability distribution in queuing system.

UNIT - IV

classification of queuing models - definition of transient and steady states - poisson queuing system - Model I :{(M/M/1) : (/FIFO)} and Model II :{(M/M/1) : (/SIRO)} - simple problems.

UNIT - V

Network analysis - Basic concepts - constraints in Network - construction of network - critical path method (CPM) - program evaluation Review techniques (PERT) - simple Problems.

UNIT - I

GAMES AND STRATEGIES

INTRODUCTION :-

If it has the following properties :-

(i) There are a finite number of competitors (participants) called players

(ii) Each player has a finite number of strategies (alternatives) available to him.

(iii) A play of the game take place when each player employs his strategy.

(iv) Every game results in an outcome. e.g loss or gain or a draw, usually called payoff to some player.

TWO - PERSON ZERO - SUM GAMES

When there are two competititors playing a game. It is called "a two person game". In case the number of competititors exceeds two, say n. then the game is terred as a "n-person game".

Games having the "zero-sum" character that the algebraic sum of gains and lossess of all the players is zero are called "zero - sum games". The play does not add a single paisa to the total wealth of all the players It merely results in a new distribution. of initial money among them. zero sum games with two players called two - person zero - sum games. In this cases · the loss (gain) of one player is exactly equal to the gain (loss) of the other.

THE MAXIMIN - MINIMAX PRINCIPLE :

We shall now explain the so called maximin-minimax Principle for the selection to the optimal strategies by the two - players

For player A, Minimum value in each now represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by now minutes. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the maximin Principle, and the corresponding gain is called the maximin value of the game.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him If he chooses his particular energy these are written in the represents by column maxima.

If the maximin value equals to minimax value, then the game is said to have a saddle point and the corresponding strategis are called optimum strategies. The amount of payoff at an equilibrium point is known as the value of the game.

RULE FOR DETERMINING A SADDLE POINT

Step 1 : select the minimum element of each row of the payoff matrix and mark them [*].

Step 2 :- select the greatest element of each column of the payoff matrix and mark them [+].

Step 3 :- If there appears an elements in the payoff matrix marked [*] and [+] both, the poistion of that elements is a saddle point of the payoff matrix.

GAMES WITHOUT SADDLE POINTS – MIXED STRATEGIES

AS determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position - saddle point

In all such cases to solve games. both the players must determine an optimal mixture of strategies to find a saddle point. the optimal energy mixture for each the optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilities combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents the least player A expect to win and the least which player B can lose. the expected payoff to a player in a game arbitrary payoff matrix (a_{ij}) of order $m \times n$ is defined as :-

$$E(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q$$

where p and q denote the mixed strategies for players A and B respectively.

MAXIMIN - MINIMAX CRITERION :-

consider an $m \times n$ game (a_{ij}) without any saddle point strategies are mixed let $p_1, p_2 \cdots p_n$ be the probabilities with which player A will play his move $A_1, A_2, \ldots A_m$ respectively ; and let $q_1, q_2 \cdots q_n$ be the probabilities with which player B will put his move $B_1, B_2 \cdots B_n$ respectively, obviously $p \geq 0$ $(i = 1, 2, \ldots m)$ $q_j \geq 0$ $(j = 1, 2, \ldots n)$

and $p_1 + p_2 + \dots + p_m = 1 : q_1 + q_2 + \dots + q_n = 1$

The expected payoff function for player A. therefore will be given by

$$E(p,q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i, a_{ij}, q_{ij}$$

Making use of maximin - minimax criterion, we have

for player A

$$\underline{v} = \max_{p} . \min_{q} E(p,q) = \max_{p} \left[\min_{j} \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_{p} . \left[\min_{j} \left\{ \sum_{i=1}^{m} p_i a_{i1}, \sum_{i=1}^{m} p_i a_{i2}, \dots \sum_{i=1}^{m} p_i a_{in} \right\} \right]$$

Here $\min \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the expected gain to player A,

for player A.

$$\bar{v} = \min_{q} \left[\max_{i} . \left\{ \sum_{j=1}^{n} q_j a_{1j} \sum_{j=1}^{n} q_j a_{2j} \dots \sum_{j=1}^{n} q_j a_{mj} \right\} \right]$$

DEFINITION :

A pair of strategies (p, q) for which $\underline{v} = \bar{v} = v$ is called a saddle point of $E(p, q)$

Theorem 17.2

For any 2x2 two person zero sum game without any saddle point having payoff matrix for

Player A

$$\begin{array}{c c} & \begin{array}{cc} B_1, & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$ the value v of the sum to A is given by

$$v = \frac{a_{11} a_{22} - a_{21} a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

DOMINANCE PROPERTY

Sometimes it is observed that one of the pure strategies of either player is always inferior to at least one of the remaning ones. The superior strategies are said to dominate the inferior ones. clearly a player would have no incentive to use inferior strategies which are dominated by the superior ones. In such cases of dominance we can reduce the size of the payoff matrix by deleting those strategies which are dominated by others thus if each element one row say kth of the payoff matrix (a_{ij}) is less than or equal to the corresponding elements in some other row, say rth then player A, will never choose the kth strategy. In other words, probability $P_k = P$ (choosing the kth strategy) $a_{kj} \leq a_{rj}$ for all $j = 1, \ldots n$.

The value of the game and the non-zero choice of probabilities remain unchanged even after the deletion of kth row from the payoff matrix. In such a case the kth strategy is said to be dominated by the rth one.

GENERAL RULES FOR DOMINANCE ARE :-

(a) If all the elements of a row, say kth, are less than or equal to the corresponding elements of the any other row, say rth, then kth row is dominated by the rth row. say rth then kth row is dominated by rth row.

(b) If all the elements of a column may be deleted say kth are greater than or equal to the corresponding elements of any other columns say rth, then kth column is dominated by the r^{th} column.

(c) Dominated rows or columns may be deleted to reduce the size of payoff matrix, as the optimal strategies will remain unaffected.

MODIFIED DOMINANCE PROPERTY:-

The dominance property is not always based on the superiority of pure strategies only. A given strategy can also be said to be dominated if it is inferior to an average of two or more other pure strategies. More generally, if some convex linear combination of some rows dominates the ith row, then ith row will be deleted. Similar arguments follow for column.

UNIT - II

SEQUENCING PROBLEM

PROBLEM OF SEQUENCING

The general sequencing problem may be defined as: let there be n jobs to be performed one at time on each of m machines. The sequence (order) of the machines in which each job should be performed is given. The actual or expected time required by the jobs on each of the machines is also given. The general sequencing problem therefore is to find the sequence out of $(n!)^m$ possible sequence which minimize that total elapsed time between the start of the job in the first machine and the completion of the last job on the last machine.

Given below are the assumption underlying a sequencing problem :-

1. Each job, once started on a machine is to be performed upto completion on that machine.

2. The processing time on each machine is known. such a time is independent of the order of the jobs in which they are to be proceed

3. The time taken by each job in changing over from one machine to another is negligible.

4. A job starts on the machine as soon as the job and the machine both are idle to job is next to the machine is also next to the job.

5. No machine may process more than the one job simultaneously.

BASIC TERMS USED IN SEQUENCING :-

1. Number of machines :- It refers to the number of service facilities through which a job must pass before it is assumed to be completed

2. Processing order :- It refers to the order in which given machines are required for completing the job.

3. processing time :- It indicates the time required by a job on each machine.

4. Total elapsed time :- It is time interval between starting the first job and completeing the last job including the idle time (if any) in a particular order by the given set of machines

5. Idle time on a machine :- It is the time for which a machine does not have a job to process idle time from the end of job (i-1) to the start of job i.

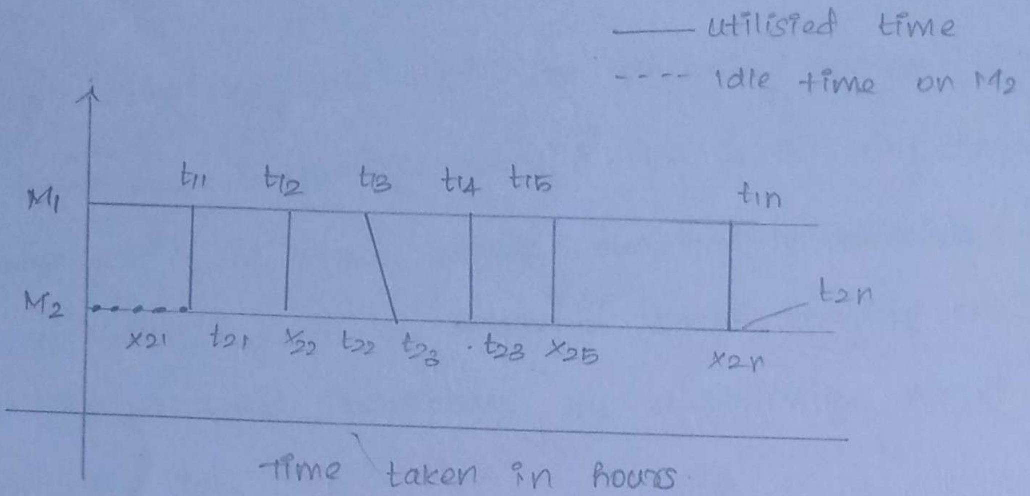
PROCESSING n JOBS THROUGH TWO MACHINES

Let there be n, jobs, each of which is to be proceed the two machines, say M_1 and M_2 in the order $M_1 M_2$. That is each jobs has to pass through the same sequence of operations In other words a job is assigned on machine M_1, First and after it has been completely processed on machine M_1. It is assigned to machine M_2. If the machine M_2 is not free at the moment for processing the same job, assigumed to machine M_2. If the machine M_2, is not free at the moment for processing the same job. passing is not allowed Let t_{ij} $(i=1,2; j=1,2,\dots n)$ be the time required for processing ith job on the jth machine.

Since passing is not allowed, therefore machine M_1 will remain busy in processing all the n jobs one-by-one while machines M_2, may remain idle after completion of one job and before starting of another jobs. thus the objective is to minimize the idle time of the second machine. Let x_{2j} be the time for which machine M_2, remains idle after finishing $(j-1)$th job and before starting processing the jth job $(j=1,2,\dots n)$ clearly the total explained time T is given by

$$T = \sum_{j=1}^{n} t_{2j} + \sum_{j=1}^{n} x_{2j}$$

where some of the x_{2j}'s may be zero's.



From the chart, it is apparent that

$$x_{21} = t_{11}$$

$$x_{22} = \begin{cases} t_{11} + t_{12} - x_{21} - t_{21} & \text{if } t_{11} + t_{12} > x_{21} + t_{21} \\ 0 & \text{otherwise} \end{cases}$$

The expression for x_{22} may be rewritten as

$$x_{22} = \text{Max}\{t_{11} + t_{12} - x_{21} - t_{21}, 0\}$$

$$x_{21} + x_{22} = \max\{t_{11} + t_{12} - t_{21}, t_{11}\} \quad \text{since } x_{21} = t_{11}$$

$$x_{23} = \text{Max}\{t_{11} + t_{12} + t_{13} - t_{21} - t_{22} - x_{21} - x_{22}, 0\}$$

$$x_{21} + x_{22} + x_{23} = \max\left\{\left(\sum_{j=1}^{3} t_{1j} - \sum_{j=1}^{2} t_{2j}\right), \sum_{j=1}^{2} x_{2j}\right\}$$

$$= \text{Max}\left\{\left(\sum_{j=1}^{3} t_{1j} - \sum_{j=1}^{2} t_{2j}\right), \left(\sum_{j=1}^{2} t_{1j} - t_{2j}\right), t_{11}\right\}$$

$$\sum_{j=1}^{n} x_{2j} = \text{Max}\left\{\left(\sum_{j=1}^{n-1} t_{1j} - \sum_{j=1}^{n-1} t_{2j}\right), \left(\sum_{j=1}^{n-1} t_{1j} - \sum_{j=1}^{n-2} t_{2j}\right) \ldots t_{11}\right\}$$

$$= \underset{1 \le n \le n}{\text{Max}} \left\{\sum_{j=1}^{n} t_{1j} - \sum_{j=1}^{n-1} t_{2j}\right\}.$$

Now if we denote $\sum_{j=1}^{n} x_{2j}$ by $D_n(S)$ then the problem becomes that of finding the sequence $\langle S^* \rangle$ for processing the jobs 1, 2, ... n so as to have the inequality $D_n(S^*) \leq D_n(S_0)$ for any sequence $\langle S_0 \rangle$ other than $\langle S^* \rangle$ in other words, one has to determine an optimal sequences so as to minimize $D_n(S)$ applying the above rule. this can be achieved iteratively by successively interchanging the consecutive jobs.

PROCESSING n JOBS THROUGH K MACHINES

there is no general method available by which we can obtain optimal sequences in problems involving processing of n jobs on K machines. they can be handled only by enumerations, which is it very lengthy and time consuming. exercise because a total of $(n!)^k$ different sequences would require consideration in such a case. However we do have a method applicable under the condition that no passing of of jobs is permissible and if either or both of the conditions stipulated below is /are satisfied

Let there be n jobs each of which is to be processed through K machines say $M_1, M_2 \ldots M_K$ in the order $M_1, M_2, \ldots M_K$. the list of jobs with their processing times is :

	Job Number	:	1	2	3	....	n
Processing time on machine	M_1	:	t_{11}	t_{12}	t_{13}	....	t_{1n}
	M_2	:	t_{21}	t_{22}	t_{23}	....	t_{2n}
	M_3	:	t_{31}	t_{32}	t_{33}	....	t_{3n}
	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	M_k	:	t_{k1}	t_{k2}	t_{k3}	....	t_{kn}

An optimum solution to this problem can be obtained. If either or both of the following conditions hold:

(a) $\min t_{ij} \geq \max. t_{ij}$, for $i = 2, 3 \cdots k-1$,

or

(b) $\min t_{kj} \geq \max. t_{ij}$, for $i = 2, 3, \cdots k-1$

OPTIMAL SEQUENCE ALGORITHM

The iterative procedure for determining the optimal sequence for 'n' jobs on 'k' machines can be summarized as follow

Step 1: Find $\min t_{ij}$, $\min t_{kj}$ and maximum each of

$t_{2j}, t_{3j} \ldots t_{k-1,j}$ for all $j = 1, 2, \ldots n$.

Step 2: Check the following:

(a) $\min t_{ij} \geq \max, t_{ij}$, for $i = 2, 3, \cdots k-1$,

(or)

(b) $\min, t_{kj} \geq \max. t_{ij}$, for $i = 2, 3, \cdots k-1$.

Step 3: If the inequalities of Step 2 are not satisfied method fails. Otherwise go to next step.

Step 4: convert the k machine problem into a two-machine problem into a two-machine problem by Introducing two fictious machines G and H, such that

$$t_{Gj} = t_{1j} + t_{2j} + \dots + t_{k-1\,j}$$

$$t_{Hj} = t_{2j} + t_{3j} + \dots + t_{kj}$$

PROCESSING 2 JOBS THROUGH k MACHINES

Let there be two jobs 1 and 2 each of which is to processed on k machines say $M_1, M_2, \dots M_k$ in two different orders. The technological ordering of each of the two jobs through k machines is known in advance. Such ordering may not be same for both the jobs. The exact or expected processing times on all the given machines are known. Each machine can perform only one job at time. The objective is to determine an optimal sequence of processing the jobs so as to minimize total elapsed time.

The optimal sequence in this case can be obtained by making use of graph. The solution procedure can be summarised in the following steps.

Step 1: Draw two perpendicular lines, horizontal one representing the processing time for job 1, while job 2 remains idle, and the vertical one representing the processing time for job 2 while job 1 remains idle.

Step 2:- Mark the processing time for jobs 1 and 2 on the horizontal and vertical lines respectively according to the given order of machines

Step 3: construct various blocks starting from the origin by pairing the same machines untill the end point.

Step 4: Draw the line starting from the origin to end point by moving horizontally, vertically and diagonally along a line which makes an angle of 45° with the horizontal line (base). the diagonal segment of the line shows that the both the jobs are under process simultaneously

Step 5:- An optimum path is one that minimizes the idle time for both the jobs. Thus the must choose the path on which diagonal movement is maximum.

Step 6:- The total elapsed time is obtained by adding the idle time for either job to the processing time for that job.

UNIT - III

QUEUEING - THEORY

QUEUEING SYSTEM

The mechanism of a queueing process is very simple customers arrive at a service counter and are attended to by one or more of the servers. As soon as a customer is served. It departs from the system. Thus a queueing system can be described as consisting of customers arriving for service waiting for service if it not immediate. and leaving the system after being served.

ELEMENTS OF A QUEUEING SYSTEM

The basic elements of a queueing system are follows :-

* INPUT (OR ARRIVAL PROCESS) :-

This element of queueing system is concerned with the pattern in which the customers arrive for service. Input source can be described by following factors.

a) Size of the queue : If the total number of potential customer requiring service are only few then size of the input source is said to be finite. on the other hand, if potential customers requires service are sufficiently large in number, then the input source is considered to be infinite

Also the customers may arrive at the service facility in batches of fixed size or of variable or one by one. bulk arri

b) Pattern of arrivals :- customers may arrive in the system at known (regular or otherwise) times or they may arrive in a random way. In case the arrival times are known with certainty the queueing problems are categorized as determinstic model. on the other hand - If the time between successive arrivals (inter-arrival times) is uncertain, the arrival pattern is measured by either arrival rate or inter-arrival time. these are characterised by the probability distribution associated this random process. the most common stochastic queueing models assume that arrival rate follow poisson distribution and/or the inter-arrival times follow an exponential distribution.

* QUEUE DISCIPLINE :-

It is rule according to which customers are selected for service when queue has been formed. The most common queue discipline is the first come first served (FCFS) or the first in first out (FIFO) rule under which the customers are serviced in the strict order of their arrivals. other queue discipline include. "last in first out (LIFO) rule according to which the last arrival in the system is serviced first

This discipline is practised in most cargo handing situations where the last item located is removed first. Another example may be from the production process, where items arrive at workplace and are stacked one on top of the other Item on the top of the stack is taken first for processing which is the last one to have arrived service. and a variety of Priority schemes- according to which a customers service other customer

* SERVICE MECHANISM

The service mechanism is concerned with service time and service facilities. service time and service facilities of the completion of service. If there are infinite number of servers then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite, then the customers are served according to a specific order, further, the customers, may be served batches of fixed size or of variable size rather than individually by the same server, such as computer with parallel processing or people boarding a bus. the service system in this case is termed as bulk service system.

* CAPACITY OF THE SYSTEM :-

The source from which customers are generated may be finite or infinite. A finite source limits the customers arriving for service. there is finite limit to the maximum queue size. the queue can also be viewed as one with forced balking. where a customer is forced to balk if he arrives at a time. an infinite source is forever "abundant" as in the case of telephone calls arriving at a telephone exchange.

OPERATING CHARACTERISTICS OF A QUEUEING SYSTEM

Some of the operational characteristics of a queuing system. the are of general interset for the evaluation of the performance of an existing queueing system and to design a new system are as follows:-

1. Expected Number of customers in the system denoted by $E(n)$ or L is the average number of customers in the system, both waiting and his service. Here n stands for the number of customers in the queueing system.

2. Expected number of customers in the queue denoted by $E(m)$ or L_q is the average number of customers waiting in the queue. Here $m = n - 1$ excluding the customer being served.

3. Expected waiting time in the system denoted by $E(v)$ or W is the average total time spent by a customer in the system. It is generally taken to be the waiting time plus servicing time.

4. Expected waiting time in queue denoted by $E(w)$ or W_q is the average time spent by a customer in the queue before the commencement of his service

The server utilization factor is also known as traffic intensity or the clearing ratio.

DETERMINISTIC QUEUEING SYSTEM

A queueing system wherein the customers arrive at regular intervals and the service time for each customer is known and constant. is known as deterministic queueing system.

Let the customers come at the teller counter of a bank for withdrawl every 3 minutes. thus the interval between

Futher, suppose that the incharge of that particular teller takes exactly 3 minute to server a customer. this implies that the arrival and service rates are both equal to 20 customers per hour. In this suitation there shall never be a queue and the incharge of the teller shall always be busy with serving work.

Now suppose instead, that the incharge of the teller can serve 30 customers per hour, i.e. he takes 2 minutes to serve a customer and then has to wait for one minute for the next customers to come for service. Here also there would be no queue. but the teller is not always busy

Further, suppose that the incharge of the teller can serve 30 customers per hour, i.e, he takes 4 minutes to serve a customer clearly in his suitation he would be always busy and the queue length will increase commonly without limit the passage of time. this implies that when the service rate is less than the arrival time the service facility cannot cope with all the arrivals and eventually the system leads to an explosive situation. In such situations the problem can be responsibility providing additional service facilities, like opening parallel. counters, we can summarize the above as follows

Let the arrive rate be λ customers per unit time and the service rate be μ customers per unit time. then

(i) If $\lambda > \mu$, the waiting line (queue) shall be formed and will increase indefinitely; the service facility would always be busy and the service system will eventually fail.

(ii) If $\lambda \leq \mu$, there shall be no queue and hence no waiting time; the proportion of time the service facility would be idle is $1 - \lambda/\mu$.

However it is easy to visualize that the condition of uniform arrival and uniform service rates has a very limited practicability. Generally the arrival, and servicing time are both variable and uncertain thus, variable aates and servicing times are the more realistic assumptions. the probabilistic queueing models are based on these assumptions.

PROBABILITY DISTRIBUTIONS IN QUEUEING SYSTEMS

It is assumed that customers joining the queueing system arrive in a random manner and follow a poisson distribution or equivalently the inter-arrival times obey exponential distributi

In most of the cases, service times are also assumed to be exponentially distributed. It implies that the probability of service completion in any short time period is constant and independent of the length of time that the service has been in progress.

In this section, the arrival and service distributions for poisson queues are derived. the basic assumptions (axioms) governing this type of queues are stated below :-

Axiom 1: The number of arrivals in non-overlapping intervals are statistically independent that is, the process has independent increments

Axiom 2 :- The probability of more than one arrival between time t and time $t+\Delta t$ is $o(\Delta t)$ that is the probability of two or more arrivals during the small time interval Δt is negligible thus

$$P_0(\Delta t) + P_1(\Delta t) + o(\Delta t) = 1$$

Axiom 3 :- The probability that an arrival occurs between time t and time $t + \Delta t$ is equal to $\lambda \Delta t + o(\Delta t)$ thus

$$P_1(\Delta t) = \lambda \Delta t + o(\Delta t)$$

where λ is a constant and is independent of the total number arrivals upto time t. Δt is an incremental element, and $o(\Delta t)$ represents the terms such that $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.