UNIT I

Real Numbers:

The set of rational and irrational numbers, combined together is called a set of Real numbers, denoted by R

R = {set of all rational and irrational numbers}

Natural Numbers:

The numbers (1, 2, 3, ...) used for counting are called natural numbers or positive integers. The set of natural numbers is denoted by N

N = {1, 2, 3, 4, 5, ...}

Whole Numbers:

The set of number that includes zero and all of the natural numbers. It is denoted by W

W = {0, 1, 2, 3, 4, ...}

Integers:

The numbers 0, 1, −1, 2, −2 .. are called integers of which 1, 2, 3 ..... are of positive integers and −1, −2, −3 ..... are called negative integers. The collection of all integers is denoted by the letter Z. Thus Z = {....., −3, −2, −1, 0, 1, 2, 3, ...}

Rational Numbers:

A rational Number is a number which can be expressed in the form of p/q where p and q

are integers and $q \neq 0$. It is denoted by Q
thus

$$Q = \{x \mid x = p/q \; ; p, q \in Z \text{ and } q \neq 0\}.$$

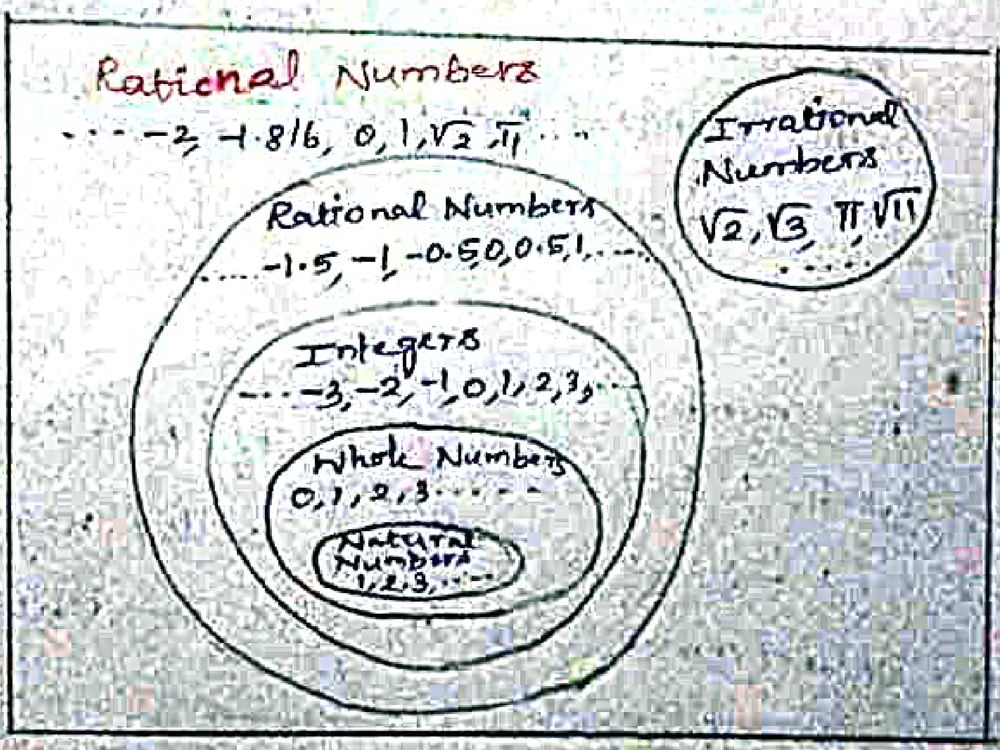
Eg: $\frac{3}{4}$, $\frac{5}{7}$

Irrational Number:

A real number which cannot be expressed as the ratio of two integers.

Eg: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc.

The Real Number system



2. THE FIELD AXIOMS:

There are two fundamental algebraic operations on the set R of real numbers - addition and multiplication. These operations obey the following laws:

A1. closure for addition:

The set R is closed with respect to addition. That is, if a and b be any two real numbers, then a+b is a unique real number

A2. Associative law of addition:

The operation of addition in R is associative. That is, for each triple of real numbers a, b, c,

$$a+(b+c) = (a+b)+c.$$

A3. Identity element for addition:

There exists a real number, namely 0 (zero), such that

$$a+0 = 0+a = a \quad \text{for all } a \in R.$$

A4. Existence of negatives:

Corresponding to each real number a, there exists a real number b (called a negative of a), such that

$$a+b = b+a = 0$$

A5. Commutative law of addition

For each pair of real numbers a and b,

$$a+b = b+a$$

M1. closure for multiplication.

The set R is closed with respect to multiplication. That is, if a and b be any two real numbers, then ab is a unique real number.

M2. Associative law of multiplication.

The operation of multiplication in R is associative. That is, for each triple of real numbers a, b, c,

$$a(bc) = (ab)c$$

M3 Identity element for multiplication.

There exists a real number, namely 1, such that

$$a \cdot 1 = 1 \cdot a = a \text{ for all } a \in R.$$

M4. Existence of inverses:

corresponding to each $a \in R$, $a \neq 0$, there exists $b \in R$ (called an inverse of a) such that

$$ab = ba = 1$$

M5. Commutative law of multiplication.

For each pair of real numbers a and b, $ab = ba$.

D Distributivity of multiplication over addition:

Multiplication distributes itself over addition. That is, for each triple of real numbers a, b, c

$$a(b+c) = ab + ac.$$

Because of the properties A1-A5, M1-M5 and D, we say that the real numbers form a field with respect to addition and multiplication.

$= (y_2 + x) + y_1$, [by associative law of addition in R]

$= 0 + y_1$ [by hypothesis]

$\boxed{y_2 = y_1}$ [by the property of 0]

Note:

The above theorem together with A4 implies that for each x in R, there exists a unique element y in R, such that $x+y=0$. We shall call this element the negative of x and denote it by $-x$. We shall frequently refer to A4 as the property of the negative.

3.3. cancellation law of addition:

Theorem 3.3:

If x, y, z be real number such that $x+z = y+z$, then $x=y$.

Proof:

Let $-z$ be the negative of z. i.e, $z+(-z)=0$.

Then $x = x+0$ [by the property of zero]

$= x+(z+(-z))$, [by the property of negative]

$= (x+z)+(-z)$, [by A.L.A in R]

$= (y+z)+(-z)$, [by hypothesis]

$= y+(z+(-z))$, [by A.L.A in R]

$= y+0$, [by the property of negative]

$\boxed{x=y}$ [by the property of zero]

Theorem 3.4:

For each real number x,

i) $-(-x) = x$

ii) $-(x+y) = (-x)+(-y)$

proof:

$-(-x)$

(i) If the negative of $-x$ be denoted by z,

then $-x + z = 0$

Now $x = x+0$, [by the property of zero]

$= x + (-x+z)$, [Proved]

$= (x+(-x))+z$, [by A.L.A. in R]

$= 0+z$, [by the property of negative]

$= z$, [by the property of 0]

$\boxed{x = -(-x)}$

ii) $-(x+y) = 0+0+[-(x+y)]$ [by the property of 0]

$= [x+(-x)]+[y+(-y)]+[-(x+y)]$
[by the property of negative]

$= [x+y]+[(-x)+(-y)]+[-(x+y)]$
[by A.L.A in R],

$= [x+y]+[-(x+y)]+[(-x)+(-y)]$
[by C.L.A in R],

$= 0+[(-x)+(-y)]$
[by the property of 0]

$= (-x)+(-y)$

$\boxed{-(x+y) = (-x)+(-y)}$

Note

The above theorem implies that for each $x \in R$, $x \neq 0$, there exists a unique element y in R such that $xy = 1$. We shall call this element the inverse of x or reciprocal of x and denote it by x^{-1}. We shall frequently refer to M4 as the property of the inverse.

3.6 Cancellation law for multiplication

Theorem 3.7:

$x \cdot 0 = 0$ for all x in R.

Proof:

$x \cdot 0 = x \cdot (0+0)$, [by the property of zero]

$x \cdot 0 = x \cdot 0 + x \cdot 0$ [by D.M.A in R]

Thus $x \cdot 0 + x \cdot 0 = x \cdot 0$

$x \cdot 0 + x \cdot 0 = x \cdot 0 + 0$ [by the property of zero]

Therefore,

$-(x \cdot 0) + (x \cdot 0 + x \cdot 0) = -(x \cdot 0) + (x \cdot 0 + 0)$

[by adding $-(x \cdot 0)$ to both sides]

so that $(-(x \cdot 0) + x \cdot 0) + x \cdot 0 = (-(x \cdot 0) + x \cdot 0) + 0$

[by A.L.A in R]

or $0 + x \cdot 0 = 0 + 0$ [by the property of negative]

or $x \cdot 0 = 0$ [by the property of zero]

$$\boxed{x \cdot 0 = 0}$$

Hence proved.

Theorem 3.8:

If x, y be real numbers such that $xy=0$, then either $x=0$ or $y=0$

Proof.

If $x=0$, we have finished.

If $x \neq 0$, then by Existence of inverses, x^{-1} exists

Now $xy=0 \Rightarrow x^{-1}(xy) = x^{-1}.0 = 0$ [by $x.0=0$ for all x in R]

$\Rightarrow (x^{-1}x)y = 0$ [by A.L.M in R]

$\Rightarrow 1y = 0$ [by the property on inverse]

$\Rightarrow y = 0$ [by the property of 1]

Theorem 3.9:

(Cancellation Law of Multiplication):

If x, y, z be real numbers such that $xz = yz$ and $z \neq 0$, then $x=y$.

Proof

$[x+(-y)]z = xz + (-y)z$, [by D.M.A in R]

$= yz + (-y)z$, [$\because xz = yz$]

$= [y+(-y)]z$, [by D.M.A. in R]

$= 0 \cdot z$, [by the property of negative]

$[x+(-y)]z = 0$ [by $x.0=0 \; \forall \; x$ in R]

But $z \neq 0$. We know that, "If x, y be real numbers st $xy=0$, then either $x=0$ or $y=0$". $\therefore x+(-y)=0$

$x+(-y) = y+(-y)$ [by the property of negative]

Hence $x = y$ [by cancellation L.A in R]

Theorem 3.10

For all x, y in R,

(i) $x(-y) = -(xy)$;

(ii) $(-x)y = -(xy)$;

(iii) $(-x)(-y) = xy$

Proof

(i) The result is equivalent to saying that $x(-y)$ is the negative of xy. We must therefore show that the sum of xy and $x(-y)$ is zero.

Now $x(-y) + xy = x[(-y) + y]$ [by D.M.A. in R]

$= x \cdot 0$, [by the property of negative]

$x(-y) + xy = 0$ [by $x \cdot 0 = 0 \; \forall \; x$ in R]

Therefore $x(-y) = -(xy)$

(ii) $(-x)y = y(-x)$, [by C.L.M in R]

$= -(yx)$ [by (i)]

$(-x)y = -(xy)$ [by C.L.M in R]

(iii) $(-x)(-y) = -[(-x)y]$ [by (i)]

$= -[-(xy)]$ [by (ii)]

We know that, "For each real number x, $-(-x) = x$"

$\therefore (-x)(-y) = xy$.

A ORDER IN R

O1 Law of Trichotomy.

Given any two real numbers a, b, one and only one of the following holds:

$a > b,\ a = b,\ b > a.$

O2 Law of Transitivity.

For each triple of real numbers a, b and c, if $a > b$, $b > c$ then $a > c$.

O3 Monotone property for addition:

For all real numbers a, b and c,

$a > b$ and $c > 0 \Rightarrow a + c > b + c$

O4 Monotone property for multiplication:

For all real numbers a, b and c, $a > b$ and $c > 0 \Rightarrow ac > bc$.

Because of the properties O1 - O4, we say that the field of real numbers is ordered. In fact, any field satisfying the properties O1 - O4 is called an ordered field.

Example:

(i) The set Q of all rational numbers is an ordered field.

(ii) The set C of all complex numbers is not an ordered field.

A.1 Positive numbers

Definition A.1:

A real number a is said to be positive if $a > 0$

The following theorem gives another version of the law of trichotomy.

Theorem A.1:

For each real number a one and only one of the following holds:

$$a > 0, \quad a = 0, \quad -a > 0$$

Proof

By the law of Trichotomy, it suffices to prove that $0 > a \Leftrightarrow -a > 0$

Now $0 > a \Rightarrow 0 + (-a) > a + (-a) \Rightarrow -a > 0$

$$0 > a \Rightarrow -a > 0$$

Also, $-a > 0$

$$\Rightarrow (-a) + a > 0 + a$$

$$\Rightarrow 0 > a$$

Theorem A.2:

If a, b be positive real numbers, then $a+b$ is a positive real number.

Proof

$a > 0 \Rightarrow a + b > 0 + b$, [by Monotone property for addition]

$\Rightarrow a + b > b > 0$ [since $b > 0$]

$\Rightarrow a + b > 0$

THEOREM A.5

For all real numbers a, b and c,

$a < b \Leftrightarrow a+c < b+c.$

$a < b$ and $c < 0 \Leftrightarrow ac > bc.$

THEOREM A.6

For each real number a, one and only one of the following holds:

$a < 0, \ a = 0, \ -a < 0.$

PROOF:

By O1, it suffices to prove that $0 < a \Leftrightarrow -a < 0$

Now $0 < a \Rightarrow 0 + (-a) < a + (-a) \Rightarrow (-a) < 0.$

Also, $-a < 0 \Rightarrow -a + a < 0 + a$

$0 < a$

THEOREM A.7

$a < 0, b < 0 \Leftrightarrow a+b < 0$ and $a.b > 0.$

PROOF:

$a < 0 \Rightarrow a+b < 0+b$, [by O3]

$\Rightarrow a+b < b < 0$ [since $b < 0$]

$\Rightarrow a+b < 0.$

$a < 0$ and $b < 0 \Rightarrow ab < 0 \cdot b$ [by O1]

But $0 \cdot b = 0.$

Therefore, we have $ab < 0.$

Definition 4.3

A real number a is said to be greater than or equal to b (written $a \geq b$) if either $a > b$ or $a = b$.

A real number a is said to be less than or equal to b (written $a \leq b$) if either $a < b$ or $a = b$.

The relations '$\geq$' and '$\leq$' are called weak inequalities

5 Absolute value

Definition 5.1:

If x be a real number, then its absolute value, denoted by $|x|$, is defined by the rule

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We may observe that $|x|$ is defined for every $x \in R$

Also $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

Theorem 5.1:

For every $x \in R$, $|x| = \max\{-x, x\}$.

Proof

By the law of trichotomy, one and exactly one of the following is true:

(i) $x > 0$ (ii) $x = 0$ (iii) $x < 0$

If $x \geq 0$, then

$$|x| = x, \text{ and } x \geq -x$$

If $x < 0$, then

$$|x| = -x, \text{ and } -x > x$$

Thus in either case, $|x|$ is the greater of the two numbers x and $-x$, that is,

$$|x| = \max\{x, -x\}$$

Corollary

For every $x \in R$, $x \leq |x|$

Proof

$$|x| = \max\{x, -x\} \geq x$$

$$\therefore\ x \leq |x|$$

Theorem 1.2:

For every $x \in R$, $|x|^2 = x^2 = |-x|^2$

Proof

By definition,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In either case,

$$|x|^2 = x^2 \longrightarrow (i)$$

Also, similarly

$$|-x|^2 = (-x)^2 = x^2 \longrightarrow (ii)$$

From (i) & (ii), $|x|^2 = x^2 = |-x|^2$.

Theorem 5.3

For every $x \in R$, $|x| = |-x|$

Proof

$|-x| = \max\{-x, -(-x)\}$ $\quad \max\{x, -x\}$

$= \max\{-x, x\}$

$= |x|$

Theorem 5.4 :

For all $x, y \in R$, $|x \cdot y| = |x| \cdot |y|$

Proof

$|x \cdot y|^2 = (xy)^2$

$= x^2 y^2$

$= |x|^2 \cdot |y|^2$ $[\because |x|^2 = x^2, |y|^2 = y^2]$

$= (|x| \cdot |y|)^2$

Since $|x \cdot y| = |x| \cdot |y|$ are both non-negative, therefore, taking the positive square roots of both sides, we have

$|x \cdot y| = |x| \cdot |y|$

Theorem 5.5 (The triangle inequality).

For all real numbers x and y, $|x+y| \leq |x| + |y|$

proof.

case 1 :

$x+y \geq 0$. In this case

$|x+y| = x+y \longrightarrow ①$

Since $x \leq |x|$ and $\longrightarrow ②$

$y \leq |y| \longrightarrow ③$

Adding ② & ③, we get $x+y \leq |x| + |y| \longrightarrow ④$

From ① & ④, $|x+y| \leq |x| + |y|$

Case 2:

$x+y<0$. In this case,

$-(x+y)>0$, that is, $(-x)+(-y)>0$

Now $|x+y| = |-(x+y)|$ [$\because |x| = |-x|$, for every $x \in \mathbb{R}$]

$= |(-x)+(-y)|$

$\leq |-x| + |-y|$

Since $|-x| = |x|$, $|-y| = |y|$

$\therefore$ It follows that

$|x+y| \leq |x| + |y|$.

Theorem 5.6:

For all real numbers x and y,

$$|x-y| \geq ||x|-|y||$$

Proof

By the triangle inequality, we have

$$|x| = |(x-y)+y| \leq |x-y| + |y|,$$

so that $|x|-|y| \leq |x-y| \longrightarrow (1)$

Again $|y| = |(y-x)+x| \leq |y-x| + |x|$,

so that $|y|-|x| \leq |y-x|$

i.e., $-(|x|-|y|) \leq |x-y|$, since $|y-x| = |x-y|$ $\longrightarrow (2)$

Now, $||x|-|y|| = \max\{|x|-|y|, -(|x|-|y|)\} \leq |x-y|$ by (1) & (2)

$\therefore |x-y| \geq ||x|-|y||$

Example 1:

If x, y be any real numbers, show that

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$$

Solution:

$$|x+y|^2 + |x-y|^2 = (x+y)^2 + (x-y)^2$$
$$= x^2 + y^2 + 2xy + x^2 + y^2 - 2xy$$
$$= 2x^2 + 2y^2$$
$$= 2(x^2 + y^2)$$
$$= 2(|x|^2 + |y|^2)$$

Example 2:

If x, l, ε be real numbers, and $\varepsilon > 0$, show that $|x-l| < \varepsilon \iff l-\varepsilon < x < l+\varepsilon$

Solution:

$$|x-l| < \varepsilon \iff \max\{(x-l), -(x-l)\} < \varepsilon$$
$$\iff x-l < \varepsilon \text{ and } l-x < \varepsilon$$
$$\iff x < l+\varepsilon \text{ and } l-\varepsilon < x$$
$$\iff l-\varepsilon < x < l+\varepsilon.$$

6. COMPLETENESS

Definition 6·1.

If for a set S of real numbers, there exists a real number u, such that:

$$x \in S \Rightarrow x \le u,$$

then u is called an upper bound of S. If there exists an upper bound for a set S, then S is said to be bounded above.

Illustrations:

1 The set of negative real numbers is bounded above, 0 being an upper bound.

2 The set of positive real numbers is not bounded above. For, if we assume that the set R^+ of positive real numbers is bounded above, and that u is an upper bound, then we are immediatly led to a contradiction by observing that

(i) since $1 \in R^+$, therefore, $1 \le u$, which means $u > 0$;

(ii) $u+1 > 0$ and consequently $u+1 \in R^+$;

(iii) $u < u+1$, so that there is an x in R^+, namely $u+1$, which does not satisfy $x \le u$;

(iv) since u is assumed to be an upper bound of R^+, therefore, we should have $x \le u$ for all x in R^+.

From illustration 2, above, we find, that it is not necessary that a set should be bounded above. However, if a set has one upper bound, then it has many upper bounds. For, if u be an upper bound of a set S, then every real number u' greater than u is also an upper bound of S.

Definition 1.2

If the set of all upper bounds of a set S of real numbers has a smallest member, say w, then w is said to be a least upper bound or a supremum of S (written sup S).

Result:

A set cannot have more than one supremum.

proof:

If w, w' be two suprema of a set S, then

(i) w and w' are both upper bound of S;

(ii) Since w is a supremum of S and w' is an upper bound of S, therefore, $w \le w'$, that is, $w \not> w'$;

(iii) Since w' is a supremum of S and w is an upper bound of S, therefore, $w' \le w$, that is, $w' \not> w$;

(iv) by the law of trichotomy, $w \not> w'$ and $w' \not> w$ together imply $w = w'$.

$\Rightarrow$ Supremum of a set if it exists is unique

Order completeness property:

Every non-empty set of real numbers which is bounded above has a supremum.

Definition:

A field which is also complete is called a complete ordered field.

Theorem 6.1:

(Archimedean property of Real numbers)

If x and y be any positive real numbers, the there exists a positive integer n such that $ny > x$.

Proof:

Suppose the statement of the theorem is false. Then, for each positive integer n, we must have $ny \leq x$. This means that x is an upper bound of the set

$$S = \{y, 2y, 3y, \cdots\}$$

By the completeness property of R, S must have a supremum, say s. Then, $ny \leq s$ for all positive integers n, and consequently $(n+1)y \leq s$ for all positive integers n. This implies that $ny \leq s-y$ for all positive integers n, so that $s-y$ is an upper bound of S. Thus, we have an upper bound of S, namely $s-y$, which is less than the supremum of S. since this contradicts the definition of s, therefore, the statement of the theorem must be true.

Corrollaries 1:

If x be any real number, then there exists a positive integer n such that $n > x$.

Proof:

Take $y = 1$ in the proof of the theorem

2. If x be any real number, and y be any positive real number, then there exists a positive integer n such that $ny > x$.

Proof:

If $x > 0$, the corollary is a re-statement of the theorem 6.1. If $x \leq 0$, then $n=1$ suffices.

For, $1 \cdot y = y > 0 \geq x$

3. If x be any real number, then there exists a positive integer n such that $n > x$

proof.

Take $y = 1$ in corollary 2.

Theorem 6.2

Let S be a non-empty set of real numbers bounded above. Then, a real number s is the supremum of S iff the following two conditions hold:

(i) $x \leq s$ for all $x \in S$

(ii) For each positive real number ε, there is a real number $x \in S$ such that $x > s - \varepsilon$.

Proof:

The conditions are necessary. In fact, since s is the supremum of S, therefore, for all $x \in S$, we must have $x \leq s$. Also, if ε be any positive real number whatever, then $s - \varepsilon$ cannot be an upper bound of S (for it is less than the supremum) and therefore, for some $x \in S$, we must have $x > s - \varepsilon$.

The conditions are sufficient as well. Suppose there exists a real number s' satisfying the conditions (i) and (ii). By (i) it follows that s is an upper bound of S. Also if s' is any real number less than s, then $s - s' > 0$. Letting $\varepsilon = s - s'$, we find by (ii) that there exist an $x \in S$ such that $x > s - \varepsilon$, i.e., $x > s'$, showing that s' is not an upper bound of S. Thus we find that s is an upper bound of S and any number less than s is not an upper bound of S. Hence s is the supremum of S.

6.1 Lower bounds

Definition 6.3

If, for a set S of real numbers, there exists a real number v such that $x \in S \Rightarrow x \geq v$, then v is called a lower bound of S. If there exists a lower bound for the set S, then S is said to be bounded below.

Illustrations

1. The set of positive real numbers is bounded below, 0 being a lower bound.

2. The set of negative real numbers is not bounded below.

From illustration 2 above, we find that it is not necessary that a set should be bounded below. However, if a set has one lower bound, then it has many lower bounds. For, if v be a lower bound of a set S, then every real number v' less than v is also a lower bound of S.

Definition 6.4

If the set of all lower bounds of a set S of real numbers has a greatest member, say t, then t is said to be a greatest lower bound or an infimum of S (written $\inf S$).

Theorem 6.3

Any non-empty set of real numbers which is bounded below has an infimum.

Proof

Let S be any non-empty set of real numbers and let v be a lower bound of S. Let us denote by T the set of negatives of members of S. That is,

$$T = \{-x : x \in S\}$$

We shall show that T is bounded above. In fact, if y be an arbitrary member of T, then $y = -x$ for some $x \in S$. Since v is a lower bound of S, therefore, it follows that $x \geq v$, and consequently $y \leq -v$.

Since $y \leq -v$ for all $y \in T$, therefore T is bounded above, $-v$ being an upper bound. By completeness property, T has supremum, say t. It can be shown that $-t$ is the infimum of S. This is equivalent to showing that if w be any lower bound of S, then $-t \geq w$. Now, w is a lower bound of $S \Rightarrow -w$ is an upper bound of $T \Rightarrow t \leq -w \Rightarrow -t \geq w$. Hence the theorem.

THEOREM 6.4

Let S be a non-empty set of real numbers bounded below. A real number t is the infimum of S iff the following conditions hold:

(i) $x \geq t$ for all $x \in S$;

(ii) for each positive real number ε, there is a real number $x \in S$ such that $x < t+\varepsilon$

Proof

The conditions are necessary. In fact, since t is the infimum of S, therefore for all $x \in S$, we must have $x \geq t$. Also if $\varepsilon > 0$ be given, then $t+\varepsilon$ is greater than the infimum of S and cannot therefore, be a lower bound of S. This implies that for some $x \in S$, we must have $x < t+\varepsilon$.

The conditions are sufficient as well. Suppose there exists a real number t satisfying the conditions (i) and (ii). By (i) it follows that t is a lower bound of S. Also, if t' be any real number greater than t, then $t'-t > 0$. Setting $\varepsilon = t'-t$, we find by (ii) that there exists an $x \in S$ such that $x < t+\varepsilon$, i.e., $x < t'$, showing that t' is not a lower bound of S.

Thus we find that t is a lower bound of S and no number greater than t is a lower bound of S. Hence t is the infimum of S.

Theorem 7.1

There is no rational number whose square is 2.

Proof

If possible, let p/q, where p and q are integers and $q \neq 0$, be a rational number whose square is 2.

Let g be the G.C.D of p and q, and

$\frac{p}{g} = m$, $p = gm$, $\frac{q}{g} = n$, $q = gn$

let $p = gm$, $q = gn$

Then m and n are co-prime, and $(m/n)^2 = 2$. Since $(m/n)^2 = 2$, therefore $m^2 = 2n^2$, so that m^2 is even and consequently m is even. Let $m = 2u$. Then $m^2 = 2n^2$ gives $n^2 = 2u^2$, whence n must be even, say $n = 2v$.

Now $m = 2u$, $n = 2v$, where u and v are integers, so that 2 is a common divisor of m and n. Since this contradicts our choice of m and n, therefore, 2 cannot be the square of a rational number.

THEOREM 7.2

The set of rational numbers is not order-complete.

Proof

It is enough to show that there exists a non-empty subset of Q which is bounded above but which does not have a supremum.

Let us consider the set of all those positive rational numbers whose squares are less than 2. That is, let

$$S = \{x : x \in Q^+, 0 < x^2 < 2\}.$$

The set S is clearly non-empty ($1 \in S$), and is bounded above (2 being an upper bound). We shall

Show that S does not have any rational number as its least upper bound.

Consider any rational number x. The following cases arise:

(i) $x \leq 0$. Since every member of S is positive, therefore, x cannot be an upper bound of S, and consequently it cannot be the least upper bound of S.

(ii) $x > 0$ and $0 < x^2 < 2$

Let $$y = \frac{4+3x}{3+2x} \longrightarrow (1)$$

Then $$y^2 - 2 = \frac{(4+3x)^2}{(3+2x)^2} - 2$$

$$= \frac{16+9x^2+24x}{9+4x^2+12x} - 2$$

$$= \frac{16+9x^2+\cancel{24x}-18-8x^2-\cancel{24x}}{(3+2x)^2}$$

$$= \frac{x^2-2}{(3+2x)^2} \longrightarrow (2)$$

and $$y - x = \frac{4+3x}{3+2x} - x$$

$$= \frac{4+\cancel{3x}-\cancel{3x}-2x^2}{3+2x}$$

$$y - x = \frac{2(2-x^2)}{3+2x} \longrightarrow (3)$$

Since x is a positive rational number, therefore from (1) we find that y is also a positive rational number. Again since $x^2 < 2$, therefore from (2) we find that $y^2 < 2$, and from (3) we find that $y > x$. Since y is a positive rational number such that $0 < y^2 < 2$, therefore it follows that $y \in S$. Also, since $x < y$, therefore, it follows that x cannot be an upper bound of S.

(iii) $x > 0$ and $x^2 = 2$. This is not possible since there is no rational number whose square is 2.

(iv) $x > 0$ and $x^2 > 2$ Let y be defined as in case (ii)
From ① we find as before, that y is a positive rational number
From ② we find that $y^2 > 2$ and from ③ we find that $y < x$.

If p be any member of S, then we have $0 < p^2 < 2 < y^2$ and $y < x$ i.e., $0 < p < y < x$. This shows that both x and y are upper bounds of S and that x is not the least upper bound (because y is an upper bound of S that is less than x)

Since, by the law of trichotomy, the above possibilities are mutually exclusive and exhaustive, therefore, it follows that x cannot be the least upper bound of S since x is any rational number, therefore, we conclude that no rational number can be the least upper bound of S.

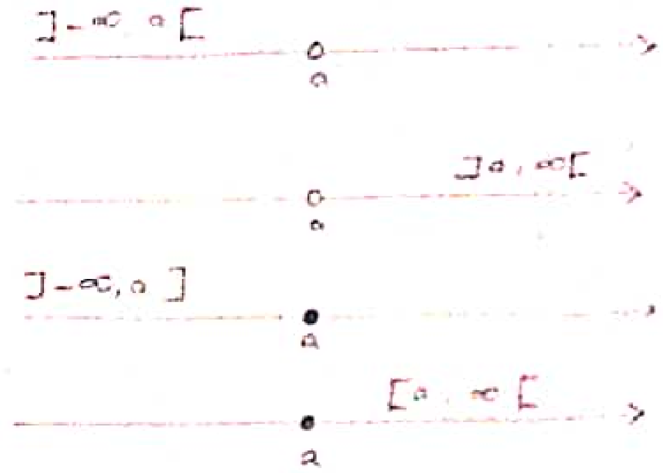
The sets

$\{x : x < a\}$,

$\{x : x > a\}$,

$\{x : x \leq a\}$

and $\{x : x \geq a\}$



are called rays. They are denoted by $]-\infty, a[$, $]a, \infty[$, $]-\infty, a]$ and $[a, \infty[$ respectively. In each of the four cases, the point 'a' is called an end-point of the ray. The first two rays are said to be open rays. The last two are said to be closed rays.

Since $R = \{x : x \leq 0\} \cup \{x : x \geq 0\}$

$$=]-\infty, 0] \cup [0, \infty[,$$

therefore, it is customary to denote the set of all real numbers $]-\infty, \infty[$. In symbols, we write

$$\{x : x \in R\} =]-\infty, \infty[.$$

10 COUNTABLE AND UNCOUNTABLE SETS

DEFINITION 10·1

A set S is said to be finite if either it is empty, or for some natural number n, there exists a one-to-one mapping from the set $\{1, 2, \ldots, n\}$ onto the set S.

If a set is not finite, then it is said to be Infinite.

Illustrations

1. The set ϕ is a finite set.
2. The set of all natural numbers is an infinite set

3. The set of all rational numbers is an infinite set.

4. The set $\{x : x \in R \text{ and } 0 \le x \le 1\}$ is an infinite set.

THEOREM 10.1 :

a, Every subset of a finite set is a finite set.

b, Every superset of an infinite set is an infinite set.

c, The intersection of every non-empty family of finite sets is a finite set.

d, The union of every non-empty family of infinite sets is an infinite set.

DEFINITIONS 10.2:

A set S is said to be enumerable, if there exists a one-to-one mapping from the set N of all natural numbers onto the set S.

A set S is said to be countable if it is either finite or enumerable. If a set is not countable then it is said to be uncountable.

Illustrations

1. The set N of all natural numbers is enumerable the identity mapping being a desired one-to-one mapping.

2. The empty set is countable.

3. The set $\{4, -7, e, \pi\sqrt{5}\}$ is a countable set.

THEOREM 10.2

Every subset of a countable set is countable.

Proof

Let A be a countable set and let B be a subset of A. If B is finite, we have nothing to prove. We may, therefore, assume without loss of generality that A is an infinite countable set and that B is an infinite subset of A.

Let $A = \{a_1, a_2, a_3, \dots\}$. Each element of B is an a_i for some index i. Let n_1 be the smallest index for which $a_{n_1} \in B$. Consider now the set $A \sim \{a_{n_1}\}$. Let n_2 be the smallest index for which a_{n_2} belongs to B as well as to $A \sim \{a_{n_1}\}$. Consider now the set $A \sim \{a_{n_1}, a_{n_2}\}$. Let n_3 be the smallest index for which a_{n_3} belongs to B as well as to $A \sim \{a_{n_1}, a_{n_2}\}$. Proceeding in this manner, we find that $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$. Then $k \to a_{n_k}$ is a one-to-one function from N onto B, and consequently B is countable.

THEOREM 10.3

Every superset of an uncountable set is uncountable.

Proof:

Let A be an uncountable set and let $B \supset A$. If B is countable, then the set A must also be countable (For, it is a subset of a countable set B). Since A is given to be uncountable, it follows B must also be uncountable.

THEOREM 10.4 :

If $A_1, A_2, \ldots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

PROOF :

Let us write

$$A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, \ldots \}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, \ldots \}$$
$$\vdots$$
$$A_n = \{a_{n1}, a_{n2}, a_{n3}, a_{n4}, \ldots \}$$
$$\vdots$$

Here a_{ij} stands for the j^{th} element of the i^{th} set as listed above. Let us define the height of the element a_{ij} to be $i+j$. With this definition the height of a_{11} is 2, and this is the only element of height 2 similarly, the height of each of the elements a_{12} and a_{21} is 3 and these are the only elements of height 3. Similar observations can be made about other elements since each element will have a unique height (the element a_{ij} has height $i+j$, and there are exactly $m-1$ elements of height m, these being $a_{1,m-1}$; $a_{2,m-2}$; $\ldots$ $a_{m-1,1}$). therefore, we can arrange the elements according to their height as $a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, \ldots$

leaving out any element has already occurred. Thus all elements will be counted out and consequently $\bigcup_{n=1}^{\infty} A_n$ is countable.

The following diagram gives a visual picture of the above counting process.

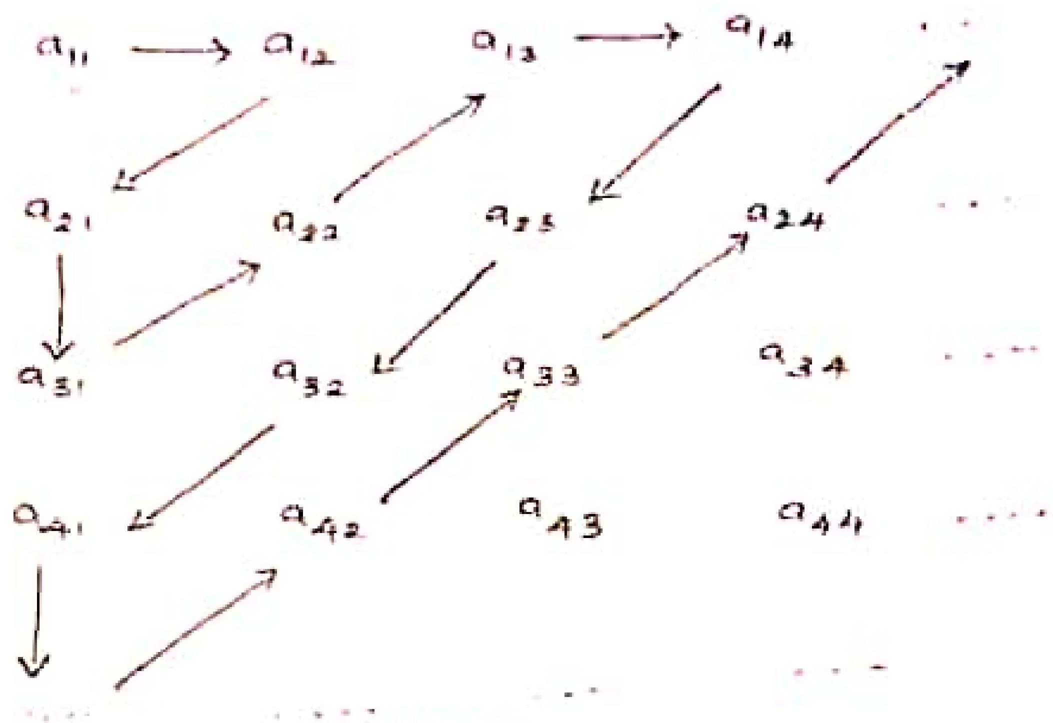


Fig 1.2

According to this scheme, the elements are to be counted as $a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, \ldots$

THEOREM 10.5:

The set $N \times N$ is countable.

PROOF

We may arrange the set $N \times N$ as shown in Fig 1.3

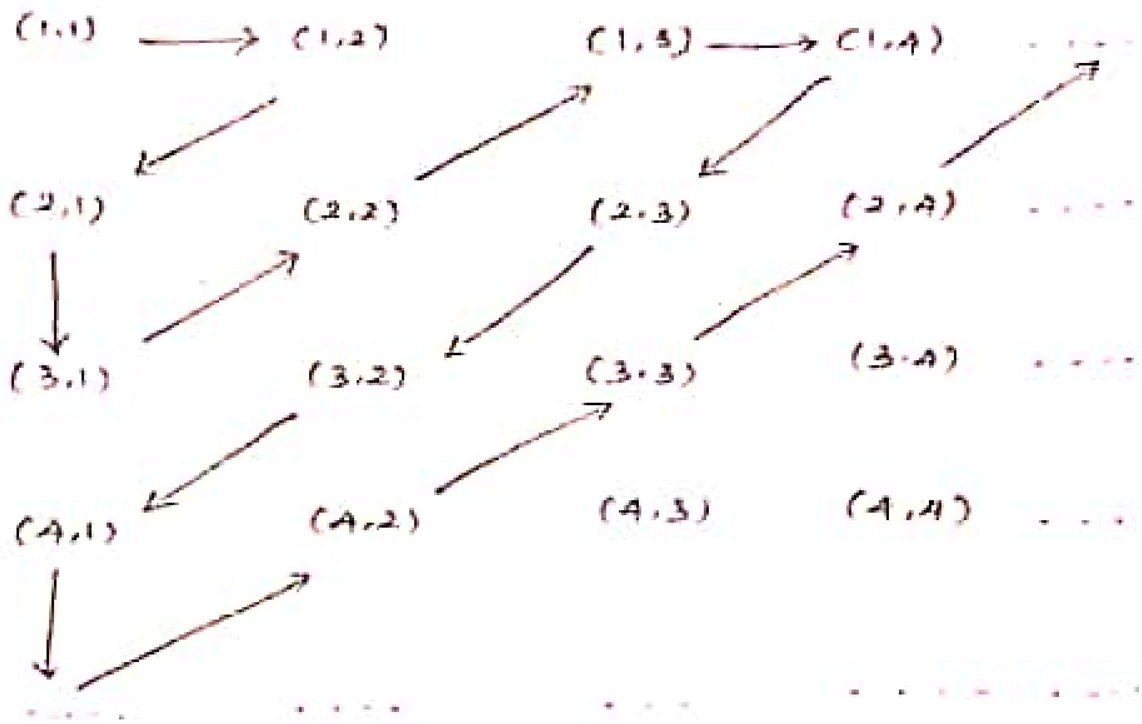


Fig 1.3

This scheme arranges all the elements of $N \times N$ into a sequence and consequently shows that $N \times N$ is countable.

COROLLARIES

1. The set of all positive rational numbers is countable.

PROOF

Every positive rational number is expressible as p/q, where p and q are positive integers prime to each other. Let us denote the set of all positive rational numbers by A and let B be the set defined as $B = \{(p,q) : (p,q) \in N \times N,$ p and q are prime to each other$\}$

It is obvious that the elements of A and B are in one-to-one correspondence, and therefore, A is countable if and only if B is countable. Since the set B is a subset of the countable set $N \times N$, therefore, it is countable. Hence A is countable.

(2). The set of all negative rational numbers is countable.

Proof

The set C of all negative rational numbers can be put in one-to-one correspondence with the set A of all positive rational numbers. Since A is countable, therefore, C must be countable.

(3) The set Q of all rational numbers is countable.

Proof:

THEOREM 10.6

The set $[0,1]$ is uncountable.

PROOF:

We are now ready to show that the set $[0,1]$ is uncountable. Suppose that $[0,1]$ is countable. Then there exists a one-to-one mapping from N onto $[0,1]$. This means that if this mapping be f, then the set $[0,1]$ can be written as

$$\{f(1), f(2), \ldots\ldots, f(n), \ldots\}$$

Expressing each $f(n)$ as a decimal, we have

$$f(1) = 0.a_{11}a_{21}a_{31}\ldots\ldots,$$
$$f(2) = 0.a_{12}a_{22}a_{32}\ldots\ldots,$$
$$f(3) = 0.a_{13}a_{23}a_{33}\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f(n) = 0.a_{1n}a_{2n}a_{3n}\ldots\ldots,$$

all the a_{ij}'s being integers belonging to the set $\{0, 1, \ldots, 9\}$. Let us choose for each $n \in N$, a positive integer b_n as follows:

$$b_n = 1 \text{ if } a_{nn} \neq 1,$$
$$b_n = 2 \text{ if } a_{nn} = 1.$$

That is, if $a_{11} = 1$, we choose $b_1 = 2$ and if $a_{11} \neq 1$ we choose $b_1 = 1$, and likewise for $b_2, b_3, \ldots$. This choice means that for each n, $b_n \neq a_{nn}$.

Let now $y = .b_1b_2b_3\ldots\ldots$

Now y is a real number in $[0,1]$. Also it is not in the set $\{f(1), f(2), \dots\}$. In fact, it differs from $f(1)$ in the first decimal place because $b_1 \neq a_{11}$, it differs from $f(2)$ in the second decimal place, ... it differs from $f(n)$ in the n^{th} decimal place, ... Also, the decimal expansion of y is unique, since no b_n is equal to 0 or 9. This means that $y \neq f(n)$ for any n. We have thus found a real number y which is in $[0,1]$, but which is not in $\{f(1), f(2), \dots f(n), \dots\}$. This contradicts the assumption that the set $[0,1]$ is countable.

Hence the set $[0,1]$ is uncountable.

COROLLARY:

1 The set of irrational numbers is uncountable

PROOF

Let S be the set of irrational numbers. If S be countable, then the set $S \cup Q$ (where Q is the set of rational numbers) will be countable. Since $S \cup Q = R$ and since the set R is uncountable, therefore, we have a contradiction. Hence the set S is uncountable.

THEOREM 10.7:

Let P_n be the set of polynomial functions f of degree n defined by relations of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots\cdots + a_n$$

where n is a fixed non-negative integer, the coefficients $a_0, a_1, a_2, \ldots, a_n$ are all integers and $a_0 \neq 0$. The set P_n is countable.

PROOF:

We shall prove the result by induction on n, the degree of f. The result is true for n = 0. For the set of all polynomials of degree zero is in one-to-one correspondence with the set $Z \sim \{0\}$ of a non-zero integers and is, therefore, countable. Let us now assume that the set P_k is countable for some fixed positive integer k.

For each positive integer m, let

$$S_m = \{f : f = mx^{k+1} + g,\ g \in P_k\}$$

$$S_{-m} = \{f : f = -mx^{k+1} + g,\ g \in P_k\}$$

The sets S_m and S_{-m} are both countable, these being in one-to-one correspondence with the countable set P_k.

Since the union of two countable sets is countable, therefore, the set $T_m = S_m \cup S_{-m}$ is countable.

Again, since the union of a countable family of countable sets is countable, therefore, $\bigcup_{m=1}^{\infty} T_m$ is countable. Since $P_{k+1} = \bigcup_{m=1}^{\infty} T_m$, therefore, P_{k+1} is countable.

The proof is now complete by induction.

COROLLARY:

The set P of polynomial functions with integer coefficients is countable.

PROOF

If P_n be the set of polynomial functions of a degree n with integral co-efficients, then P_n is countable.

Since $P = \bigcup_{n=0}^{\infty} P_n$,

and since the union of countably many countable sets is countable; therefore, it follows that P is countable.

REMARK:

i) For each fixed non-negative integer n, the set Q_n of polynomial functions of the form

$a_0 x^n + a_1 x^{n-1} + \cdots + a_n$

where $a_0, a_1, a_2, \ldots, a_n$ are rational numbers and $a_0 \neq 0$, is countable.

ii) If Q_n be as in (i) above, then

$$\bigcup_{n=0}^{\infty} Q_n \text{ is countable.}$$

DEFINITION 10.3

A real number is said to be algebraic if it is the root of some polynomial equation with rational coefficients.

THEOREM 10.8

The set of algebraic numbers is countable.

PROOF:

Let n be an arbitrary but fixed positive integer. The set Q_n is countable. We may, therefore, write it as

$$\{f_{n_1}, f_{n_2}, f_{n_3}, \ldots\},$$

where each f_{nk} is a polynomial of degree n with rational coefficients.

If A_{nk} denotes the set of real roots of the equation $f_{nk} = 0$, then A_{nk} is a countable set (in fact, it is a set consisting of almost n elem

$$\text{Let } \bigcup_{k=1}^{\infty} A_{nk} = A_n$$

The set A_n is clearly the set of all those algebraic numbers which are the roots of polynomial equations of degree n with rationa coefficients. Since the union of a countable family of countable sets is countable, therefore, A_n is countable.

Let us now write $\bigcup_{n=1}^{\infty} A_n = A$

A is clearly the set of algebraic numbers. Since A is the union of a countable family of countable sets, and since the union of every countable family of countable sets is countable, therefore A is countable.

Hence, the set of algebraic numbers is countable.

DEFINITION 10.4

A real number is said to be transcendental if it is not algebraic.

THEOREM 10.9:

The set of transcendental numbers is uncountable.

PROOF:

Let T be the set of transcendental numbers and let A be the set of algebraic numbers. If T be countable, then the set $T \cup A$ will be countable. Since by definition, $T \cup A = R$, and R is known to be uncountable, therefore, we have a contradiction. Hence the set T must be uncountable.

UNIT - II

LIMITS AND CONTINUITY

1. LIMITS

1.1. One-sided Limits

DEFINITION 1·1 (Right Limit)

A function f defined on a set S containing $]c, d[$ is said to tend to (or approach) a number l as x tends to (or approaches) c from the right (or from above), if given $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$c < x < c + \delta \Rightarrow |f(x) - l| < \varepsilon$$

In symbols, we then write

$$\lim_{x \to c+0} f(x) = l, \text{ or } f(c+0) = l,$$

Or $$\lim_{\substack{x \to c \\ x > c}} f(x) = l.$$

The negation of the above definition can be easily seen to be the following:

If there is some $\varepsilon > 0$, such that for every $\delta > 0$ there is an x for which $c < x < c + \delta$ and $|f(x) - l| \geq \varepsilon$, then f does not approach l as x approaches c from the right.

DEFINITION 1.2 (Left Limit)

A function defined on a set S containing $]b, c[$ is said to tend to (or approach) a number l as x tends to (or approaches) c from the left (or from below) if given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$c - \delta < x < c \Rightarrow |f(x) - l| < \varepsilon$$

In symbols, we then write

$$\lim_{x \to c-0} f(x) = l, \text{ or } f(c-0) = l \text{ or } \lim_{\substack{x \to c \\ x < c}} f(x) = l.$$

The negation of the above definition can be easily seen to be the following:

If there exists an $\varepsilon > 0$ such that for each $\delta > 0$, there is some x for which $c - \delta < x < c$ and $|f(x) - l| \geq \varepsilon$, then $f(x)$ does not tend to l as x tends to c from the left.

Illustrations.

1\. Let f be the function defined on $R - \{0\}$ by setting

$$f(x) = \frac{|x|}{x}, \text{ whenever } x \neq 0.$$

Then prove that $f(0+0) = 1$, $f(0-0) = -1$

Solution:

(i) For when $x > 0$

$$f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$$

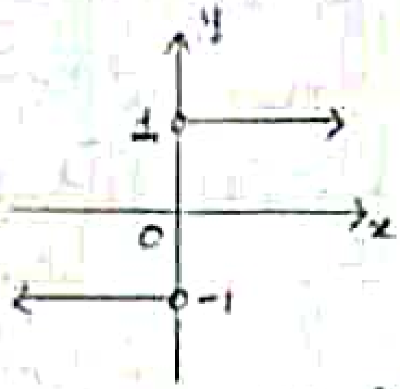


Fig. 5.1 Graph of $f(x) = \frac{|x|}{x}, x \neq 0$

(ii) when $x < 0$

$$f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$$

So that $f(x) = \begin{cases} 1, & \text{when } x > 0 \\ -1, & \text{when } x < 0 \end{cases}$

Given any $\varepsilon > 0$, taking $\delta = \varepsilon$, we have

$$0 < x < \delta \Rightarrow |f(x) - 1| = |1 - 1| = 0 < \varepsilon$$

So that $\lim\limits_{x \to 0+0} f(x) = 1$

That is, $f(0+0) = 1$

and $0 - \delta < x < 0 \Rightarrow |f(x) - (-1)| = |-1+1| = 0 < \varepsilon$

$-\delta < x < 0 \Rightarrow |f(x) - (-1)| < \varepsilon$

So that $\lim\limits_{x \to 0-0} f(x) = -1$

That is, $f(0-0)$

(5). Let f be the function defined on R as follows:

$$f(x) = \begin{cases} 1-2x, & \text{when } x<0 \\ 0, & \text{when } x=0 \\ 1+3x, & \text{when } x>0 \end{cases}$$

Then prove that $f(0+0) = f(0-0) = 1$.

Solution:

Case (i)

Let $x<0$.

Given $\varepsilon>0$, $|f(x)-1| = |1-2x-1| = |-2x| = -2x < \varepsilon$ whenever $x > -\frac{\varepsilon}{2}$

Taking $\delta = \frac{1}{2}\varepsilon$, we find that

$$-\delta < x < 0 \Rightarrow |f(x)-1| < \varepsilon$$

Hence $\lim\limits_{x\to 0-0} f(x) = 1$

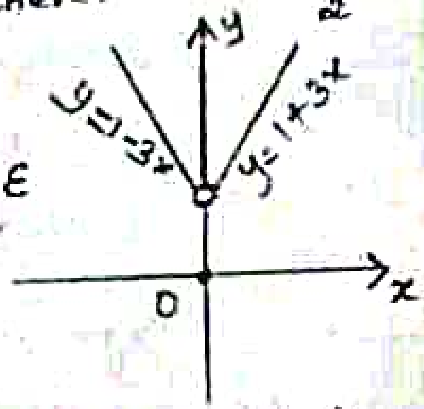


Fig 5.3 Graph of f.

Case (ii)

Let $x>0$.

$|f(x)-1| = |1+3x-1| = |3x| = 3x < \varepsilon$ whenever $x<\frac{\varepsilon}{3}$

Taking $\delta = \frac{1}{3}\varepsilon$, we find that

$$0<x<\delta \Rightarrow |f(x)-1| < \varepsilon$$

Hence $\lim\limits_{x\to 0+0} f(x) = 1$

THEOREM 1.1:

Let f be defined on $]c,d[$. Then $\lim_{x\to c+0} f(x) = l$ iff for every sequence $\langle x_n \rangle$, where $x_n > c$ for all $n \in N$, converging to c, the sequence $\langle f(x_n) \rangle$ converges to l

PROOF:

Assume that $\lim_{x\to c+0} f(x) = l$,

and $\langle x_n \rangle$ is a sequence with $x_n > c$ for $n \in N$ ... such that $x_n \to c$.

Since $\lim_{x\to c+0} f(x) = l$, therefore given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$c < x < c+\delta \Rightarrow |f(x) - l| < \epsilon \longrightarrow (1)$$

Again, since $x_n \to c$, therefore corresponding to the above δ we can find a positive integer m such that

$$|x_n - c| < \delta, \text{ whenever } n \geq m,$$

i.e., $c - \delta < x_n < c + \delta$, whenever $n \geq m \longrightarrow (2)$

From (1) and (2), it follows that

for all $n \geq m$, $|f(x_n) - l| < \epsilon$ whence $f(x_n) \to l$.

Conversely, assume that $f(x)$ does not tend to l as x tends to c from the right. We shall then show that there exists a sequence $\langle x_n \rangle$ of real numbers greater than c which converges to c, and is such that $\langle f(x_n) \rangle$ does not converges to c.

Since $f(x)$ does not tend to l as x tends to c from the right, therefore there exists an $\varepsilon > 0$ for which no δ works in the sense that whatever $\delta > 0$ we may choose, there is at least one x in $]c, c+\delta[$ such that $|f(x) - l| \geq \varepsilon$.

For each $n \in N$, choose an x_n in $]c, c+\frac{1}{n}[$ such that $|f(x_n) - l| \geq \varepsilon$. Clearly x_n tends to c, but $f(x_n)$ does not tend to l.

THEOREM 1.2

Let f be defined on $]b, c[$. Then $\lim_{x \to c-0} f(x) = l$ if for every sequence $\langle x_n \rangle$, where $x_n < c$ for all $n \in N$, converging to c, the sequence $\langle f(x_n) \rangle$ converges to l.

PROOF:

Imitate the proof of theorem 1.1 above.

1.2 LIMIT AS x APPROACHES c.

DEFINITION 1.3:

Let f be a function defined on some neighbourhood N of c, except possible at $x = c$. f is said to approach a limit l as x approaches c if for every $\varepsilon > 0$, there is some $\delta > 0$

such that

$$0 < |x-c| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

In symbols, we then write

$$\lim_{x \to c} f(x) = l.$$

If there is some $\varepsilon > 0$, such that for every $\delta > 0$ there is an x for which $0 < |x-c| < \delta$ and $|f(x) - l| \geq \varepsilon$, then f does not approach l as x approaches c.

THEOREM 1.3

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$, then $l = m$.

PROOF:

Let us first translate our hypothesis into ε's and δ's.

Let ε be any positive number.

Since $\lim_{x \to a} f(x) = l$, therefore we can find a $\delta_1 > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x-a| < \delta_1$ — (1)

Again, since $\lim_{x \to a} f(x) = m$, therefore, we can find a $\delta_2 > 0$ such that $|f(x) - m| < \varepsilon$ whenever $0 < |x-a| < \delta_2$ — (2)

If we choose $\delta_0 =$ [illegible], then $0 < |x-a| < \delta_0$

implies that $0<|x-a|<\delta_1$, and $0<|x-a|<\delta_2$ both hold, and therefore, we have

$|f(x)-l|<\varepsilon$ and $|f(x)-m|<\varepsilon$, whenever $0<|x-a|<\delta_0$.

The crux of the proof lies in showing that it is possible to choose an $\varepsilon>0$ such that

$|f(x)-l|<\varepsilon$ and $|f(x)-m|<\varepsilon$

cannot both hold if $l\neq m$. This can be done as follows:

Let $l\neq m$, so that $|l-m|>0$. Choose

$\varepsilon=\frac{1}{2}|l-m|$.

There exists a positive number δ_0 such that

$|f(x)-l|<\frac{1}{2}|l-m|$ and $|f(x)-m|<\frac{1}{2}|l-m|$

whenever $0<|x-a|<\delta_0$

This implies that if $0<|x-a|<\delta_0$, then

$$|l-m|=|f(x)-m-(f(x)-l)|$$
$$\leq|f(x)-m|+|f(x)-l|$$
$$<\frac{1}{2}|l-m|+\frac{1}{2}|l-m|=|l-m|$$
$$<|l-m|$$

and this is a contradiction.

Hence $l=m$.

Note:

The above theorem says that if $f(x)$ tends to a limit as x ends to a then this limit must be unique.

The following theorem shows the relationship between the limit of a function f as x tends to c with the one-sided limits as x tends to c either side.

THEOREM 1.4

Let f be defined on a deleted neighbourhood N^* of c. $\lim_{x \to c} f(x)$ exists and equals l iff $f(c+0)$, $f(c-0)$ both exist and are equal to l.

PROOF:

It is trivial to show that if $\lim f(x)$ exists, then $\lim_{x \to c+0} f(x)$ and $\lim_{x \to c-0}$ both exist and are equal. In fact, let us assume that $\lim_{x \to c} f(x)$ exists and equals l, and let $\epsilon > 0$ be given.

Y. Manishabhavani
III BSc mathematics. (1)

Then there exists a $\delta > 0$ such that

$$0 < |x-c| < \delta \Rightarrow |f(x)-l| < \varepsilon \longrightarrow (1)$$

Since each of the relations $c-\delta < x < c$ and $c < x < c+\delta$ implies $0 < |x-c| < \delta$, therefore (1) implies that

$$|f(x)-l| < \varepsilon \text{ whenever } c-\delta < x < c \longrightarrow (2)$$

and $$|f(x)-l| < \varepsilon \text{ whenever } c < x < c+\delta \longrightarrow (3)$$

Since (2) $\Rightarrow \lim\limits_{x \to c-0} f(x)$ exists and equal l,

and (3) $\Rightarrow \lim\limits_{x \to c+0} f(x)$ exists and equals l,

Therefore we find that $\lim\limits_{x \to c-0} f(x)$ and $\lim\limits_{x \to c+0} f(x)$ both exist and are equal to l.

Conversely, let us assume that

$$\lim_{x \to c+0} f(x) = l = \lim_{x \to c-0} f(x)$$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{x \to c+0} f(x) = l$, there exists $\delta_1 > 0$ such that $c < x < c+\delta_1 \Rightarrow |f(x)-l| < \varepsilon \longrightarrow (4)$

Again, since $\lim\limits_{x \to c-0} f(x) = l$, there exists $\delta_2 > 0$ such that

$$c-\delta_2 < x < c \Rightarrow |f(x)-l| < \varepsilon \longrightarrow (5)$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$

Then $c-\delta_0 < x < c \Rightarrow c-\delta_2 < x < c$,

$$\Rightarrow |f(x)-l| < \varepsilon \text{ by } (5) \longrightarrow (6)$$

Also $c < x < c+\delta_0 \Rightarrow c < x < c+\delta_1$

$$\Rightarrow |f(x)-l| < \varepsilon, \text{ by } (4) \quad (7)$$

From ⑥ and ⑦, we have

$$0 < |x-c| < \delta_0 \Rightarrow |f(x)-l| < \varepsilon,$$

and consequently,

$$\lim_{x \to c} f(x) = l.$$

THEOREM 1.5:

Let f be defined in some interval $]c-\delta, c+\delta[$ except possibly at $x=c$. Then $\lim_{x \to c} f(x)$ exists and equals l iff for every sequence $\langle x_n \rangle$, where $0 < |x_n - c| < \delta$ for $n \in N$, converging to c, the sequence $\langle f(x_n) \rangle$ converges to l.

EXAMPLE 1:

Let f be a function defined on $R \sim \{0\}$ by setting $f(x) = \sin \frac{1}{x}$, whenever $x \neq 0$. Show that f does not tend to any limit as x tends to zero.

SOLUTION:

For each positive integer n, let

$$x_n = \frac{1}{2n\pi}$$

$$y_n = \frac{1}{2n\pi + \frac{1}{2}\pi}$$

Clearly the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ both tend to zero as $n \to \infty$. Consider now the sequences $\langle f(x_n) \rangle$ and $\langle f(y_n) \rangle$

Further, if $g(x) \neq 0$ wherever $x \in D_1 \subset D$, then the reciprocal of g is the function $1/g$ defined on D_1 by setting

$$(1/g)(x) = 1/g(x), \text{ for all } x \in D_1.$$

Finally, if $g(x) \neq 0$ wherever $x \in D_1 \subset D$, then the quotient f/g is the function defined on D_1 by setting

$$(f/g)(x) = f(x)/g(x), \text{ for all } x \in D_1.$$

THEOREM 1.6:

Let f and g be defined on some neighborhood of c. If $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$, then $\lim_{x \to c} (f+g)(x) = l+m$.

PROOF:

The hypothesis means that given any $\varepsilon > 0$, there exist δ_1 and $\delta_2 > 0$ such that,

$$0 < |x-c| < \delta_1 \Rightarrow |f(x)-l| < \varepsilon/2$$

and $0 < |x-c| < \delta_2 \Rightarrow |g(x)-m| < \varepsilon/2$

Now let $\delta = \min\{\delta_1, \delta_2\}$

If $0 < |x-c| < \delta$, then $0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2$ both hold, and consequently,

$$|f(x) - l| < \varepsilon/2$$

and $|g(x) - m| < \varepsilon/2$,

both are true. Hence if $0 < |x-c| < \delta$, then

$$|(f+g)(x) - (l+m)| = |f(x) - l + g(x) - m|$$
$$\leq |f(x) - l| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$< \varepsilon,$$

so that $0 < |x-c| < \delta \Rightarrow |(f+g)(x) - (l+m)| < \varepsilon$, and this implies that

$$\lim_{x \to c} (f+g)(x) = l+m.$$

THEOREM 1.7:

Let f and g be defined on some neighbourhood of c. If $\lim_{x\to c} f(x) = l$, $\lim_{x \to c} g(x) = m$, then $\lim_{x \to c} (fg)(x) = lm$.

PROOF:

We wish to show that $|f(x)\, g(x) - lm|$ can be made arbitrarily small by taking x sufficiently near c: and this has to be done by using the facts that $|f(x) - l|$ can be made arbitrarily small by taking x sufficiently near c, and $|g(x) - m|$ can be made arbitrarily small by taking x sufficiently near c.

Let $\varepsilon > 0$ be given.

$$|(fg)(x) - lm| = |f(x) g(x) - lm|$$
$$= |f(x) g(x) - m f(x) + m f(x) - lm|$$

$$= |f(x)(g(x)-m) + m(f(x)-l)|$$

$$\leq |f(x)(g(x)-m)| + |m(f(x)-l)|$$

$$\leq |f(x)|\cdot|g(x)-m| + |m|\,|f(x)-l| \longrightarrow (i)$$

Since $\lim_{x\to c} f(x) = l$, therefore, we can so choose $\delta_1 > 0$ such that

$$|f(x)-l| < 1 \text{ whenever } 0 < |x-c| < \delta_1,$$

which implies that

$$|f(x)| = |f(x)-l+l| \leq |f(x)-l| + |l| < 1+|l|$$

whenever $0 < |x-c| < \delta_1$. $\longrightarrow$ (ii)

Since $\lim_{x\to c} g(x) = m$, therefore, we can choose $\delta_2 > 0$ such that

$$|g(x)-m| < \frac{\varepsilon}{2(1+|l|)}, \text{ whenever } 0<|x-c|<\delta_2 \rightarrow (iii)$$

Also, since $\lim_{x\to c} f(x) = l$, therefore, we can choose $\delta_3 > 0$ such that

$$|f(x)-l| < \frac{\varepsilon}{2(1+|m|)}, \text{ whenever } 0<|x-c|<\delta_3 \rightarrow (iv)$$

If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then from (i), (ii), (iii) and (iv), we have

$$|f(x)g(x) - lm| < \frac{(1+|l|)\varepsilon}{2(1+|l|)} + |m|\frac{\varepsilon}{2(1+|m|)} < \varepsilon,$$

whenever $0 < |x-c| < \delta$.

Hence $\lim_{x\to c} (fg)(x) = lm$.

THEOREM 4.8:

If $\lim_{x \to c} g(x) = m$, and $m \neq 0$, then

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}.$$

PROOF:

We shall first show that since $\lim_{x \to c} g(x) \neq 0$, therefore we can find a deleted neighbourhood of c in which $g(x)$ does not vanish. (If N be a neighbourhood of c, then $N \sim \{c\}$ is called a deleted neighbourhood of c).

Since $\lim_{x \to c} g(x) = m$, therefore, given $\varepsilon > 0$, we can choose a $\delta > 0$, such that

$$|g(x) - m| < \varepsilon, \text{ whenever } 0 < |x - c| < \delta$$

Let us take $\varepsilon = |m|/2$. (This is possible since $m \neq 0$ and consequently $|m| > 0$). We can then find a $\delta_0 > 0$ such that

$$|g(x) - m| < |m|/2, \text{ whenever } 0 < |x - c| < \delta_0.$$

Now $\quad |m| = |m - g(x) + g(x)|$

$$|m| \leq |m - g(x)| + |g(x)| < |m|/2 + |g(x)|,$$

whenever $0 < |x - c| < \delta_0$, and this implies that

$$|m| - \frac{|m|}{2} < |g(x)|$$

$$\frac{2|m| - |m|}{2} < |g(x)|$$

$|m|/2 < |g(x)|$

i.e., $|g(x)| > |m|/2 > 0$, whenever $0 < |x-c| < \delta_0$ ... (i)

From (i) we find that there is a deleted neighbourhood of c in which g(x) does not vanish. This means that $\frac{1}{g(x)}$ is defined in a deleted neighbourhood of c.

Let ε_0 be any positive number. We have to find a $\delta > 0$ such that

$$\left|\frac{1}{g(x)} - \frac{1}{m}\right| < \varepsilon_0 \text{ whenever } 0 < |x-c| < \delta.$$

$$\text{Now } \left|\frac{1}{g(x)} - \frac{1}{m}\right| = \frac{|m-g(x)|}{|m|\,|g(x)|} = \frac{|m-g(x)|}{|m|}\cdot\frac{1}{|g(x)|}$$

$$= \frac{|m-g(x)|}{|m|}\cdot\frac{1}{|g(x)|} < \frac{|m-g(x)|}{|m|}\cdot\frac{2}{|m|}$$

$$< \frac{2|m-g(x)|}{|m|^2} \longrightarrow \text{(ii)}$$

whenever $0 < |x-c| < \delta_0$, by (i)

Take $\varepsilon = \varepsilon_0 |m|^2/2$

We can then find a positive number δ_1 such that

$|g(x) - m| < \varepsilon_0 |m|^2/2$, whenever $0 < |x-c| < \delta_1$ $\longrightarrow$ (iii)

Taking $\delta_2 = \min(\delta_0, \delta_1)$, we find that

$0 < |x-c| < \delta_2 \Rightarrow 0 < |x-c| < \delta_0$ and $0 < |x-c| < \delta_1$

$$\Rightarrow \left|\frac{1}{g(x)} - \frac{1}{m}\right| < \frac{2|m-g(x)|}{|m|^2} \quad \text{, by (ii)}$$

$$< \frac{\cancel{2}\, \varepsilon_0 \frac{\cancel{|m|^2}}{\cancel{2}}}{\cancel{|m|^2}}$$

$$< \varepsilon_0 \quad \text{by (iii)}$$

Hence $\lim\limits_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

THEOREM 1.9:

If $\lim\limits_{x\to c} f(x) = l$, $\lim\limits_{x \to c} g(x) = m$, then

$\lim\limits_{x\to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$, provided $m \neq 0$.

PROOF:

$\lim\limits_{x\to c} \left(\frac{f}{g}\right)(x) = \lim\limits_{x\to c} \left\{f(x) \cdot \frac{1}{g(x)}\right\}$

Since $m \neq 0$, we know that, "If $\lim\limits_{x\to c} g(x) = m$, and $m \neq 0$, then $\lim\limits_{x\to c} \frac{1}{g(x)} = \frac{1}{m}$"

Again, since $\lim\limits_{x\to c} f(x) = l$, $\lim\limits_{x\to c} \frac{1}{g(x)} = \frac{1}{m}$,

We know that,

" Let f and g be defined on some neighbourhood of c. If $\lim\limits_{x\to c} f(x) = l$, $\lim\limits_{x\to c} g(x) = m$; then $\lim\limits_{x\to c} (fg)(x) = lm$."

$$\Rightarrow \lim_{x \to c} \left\{ f(x) \cdot \frac{1}{g(x)} \right\} = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)}$$

$$= l \cdot \frac{1}{m}$$

$$= \frac{l}{m}.$$

That is $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$.

5. U.Q

THEOREM 1.10:

Let f be defined on D and let $f(x) \geq 0$ for all $x \in D$. If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c} f(x) \geq 0$.

PROOF:

Let $\lim_{x \to c} f(x) = l$.

Suppose that l is negative.

Choosing $\varepsilon = -\frac{1}{2} l$, we can find a $\delta > 0$ such that

$|f(x) - l| < -\frac{1}{2} l$, whenever $0 < |x - c| < \delta$.

This implies that

$$l - \left(-\tfrac{1}{2}\right) l < f(x) < l - \tfrac{1}{2} l$$

$$l + \tfrac{1}{2} l < f(x) < l - \tfrac{1}{2} l$$

$$\frac{2l + l}{2} < f(x) < \frac{2l - l}{2}$$

$$\frac{3l}{2} < f(x) < \frac{l}{2} < 0, \text{ whenever } 0 < |x - c| < \delta.$$

This is not possible since we are given that $f(x) \geq 0$ for all $x \in D$. Consequently, l cannot be negative.

Hence $\lim_{x \to c} f(x) \geq 0$.

CORRLLARY:

Let f be defined on D and let $f(x) > 0$ for all $x \in D$. If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c} f(x) \geq 0$

PROOF:

$f(x) > 0 \Rightarrow f(x) \geq 0$. Now apply theorem 1.10.

U.Q

THEOREM 1.11:

Let f and g be defined on D and let $f(x) \geq g(x)$ for all $x \in D$. Then $\lim_{x \to c} f(x) \geq \lim_{x \to c} g(x)$, provided these limits exist.

PROOF:

Let $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$

If h be the function defined by setting $h(x) = f(x) - g(x)$, for all $x \in D$, then

(1) $h(x) \geq 0$ for all $x \in D$

(2) $\lim_{x \to c} h(x)$ exist and equals $l - m$.

(3) applying theorem 1.10 to the function

$\lim_{x \to c} h(x) \geq 0$.

[illegible] on D and let $f(x) \geq 0$ to all $x \in D$

$\lim$ $f(x) \geq 0$

and these imply that

$$l-\varepsilon < h(x) \le g(x) \le f(x) < l+\varepsilon.$$

Then $0<|x-c|<\delta \Rightarrow l-\varepsilon < g(x) < l+\varepsilon$.

This means that $\lim\limits_{x\to c} g(x)$ exists and equals l.

$$\therefore \lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$$

THEOREM 1·13

If $\lim\limits_{x\to c} f(x) = l$, then $\lim\limits_{x\to c} |f(x)| = |l|$.

PROOF:

If p and q be any real numbers, then

$$|p-q| \ge \big||p|-|q|\big| \longrightarrow (i)$$

Setting $p=f(x)$ and $q=l$ in (i), we have

$$|f(x)-l| \ge \big||f(x)|-|l|\big| \text{, for all } x \longrightarrow (ii)$$

Since $\lim\limits_{x\to c} f(x) = l$, therefore, given $\varepsilon>0$, we can choose a $\delta>0$ such that

$$|f(x)-l| < \varepsilon, \text{ whenever } 0<|x-c|<\delta \longrightarrow (iii)$$

From (ii) and (iii), we find that

$$\big||f(x)|-|l|\big| < \varepsilon, \text{ whenever } 0<|x-c|<\delta.$$

This implies that $\lim\limits_{x\to c} |f(x)|$ exists and equals $|l|$

$$\text{i.e. } \lim_{x\to c} |f(x)| = |l|$$

2. CONTINUOUS FUNCTIONS

DEFINITION 2.1:

Let f be a function whose domain I is an open interval and whose range is contained in R, and let $x_0 \in I$. f is said to be continuous at x_0, if given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If f is continuous for each $x_0 \in I$, then we say that f is continuous on I.

It can be easily seen that the above definition of continuity is equivalent to the following:

A function defined on an open interval I is said to be continuous at $x_0 \in I$ if $\lim\limits_{x \to x_0+0} f(x)$ exists and equals $f(x_0)$.

DEFINITION 2.2:

A function f defined on an open interval I is said to be continuous from the left at $x_0 \in I$ if $\lim\limits_{x \to x_0-0} f(x)$ exists and equals $f(x_0)$. Also, f is said to be continuous from the right at x_0 if $\lim\limits_{x \to x_0+0} f(x)$ exists and equals $f(x_0)$

DEFINITION 2.3.

A function f defined on the closed interval $[a,b]$ is said to be continuous at a if it is continuous from the right at a. Also, f is said to be continuous at b if it is continuous from the left at b. Further, f is said to be continuous on $[a,b]$ if (i) it is continuous on $]a,b[$, (ii) continuous from the right at a (iii) continuous from the left at b.

Note:

If a function is not continuous at a point, then it is said to be discontinuous at that point.

THEOREM 2.1:

*: Imp U.Q A function f defined on $I \subset \mathbb{R}$ is continuous at $p \in I$ iff for every sequence $\langle p_n \rangle$ in I which converges to p, we have $\lim_{n\to\infty} f(p_n) = f(p)$

PROOF:

Let us first suppose that f is continuous at p and that $\langle p_n \rangle$ is a sequence in I such that $\lim_{n\to\infty} p_n = p$.

Let ε be a positive number. Since f is continuous at p, therefore, we can find $\delta > 0$ such that

$$|x-p| < \delta \Rightarrow |f(x)-f(p)| < \varepsilon \longrightarrow (1)$$

Also, since $\lim_{n\to\infty} p_n = p$ therefore, we can find a positive integer m such that

$$n > m \Rightarrow |p_n - p| < \delta \longrightarrow (2)$$

Putting $x = p_n$ in (1), we have

$$|p_n - p| < \delta \Rightarrow |f(p_n) - f(p)| < \varepsilon \longrightarrow (3)$$

From (2) and (3), we have

$$n > m \Rightarrow |f(p_n) - f(p)| < \varepsilon,$$

i.e., $\lim_{n\to\infty} f(p_n) = f(p)$.

Let us now suppose that f is not continuous at p. We shall then show that there is a sequence $\langle p_n \rangle$ such that $\lim_{n\to\infty} p_n = p$, but $\lim_{n\to\infty} f(p_n) \neq f(p)$.

Since f is not continuous at p, therefore, there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is an x such that $|x-p| < \delta$ and $|f(x) - f(p)| \geq \varepsilon$. By taking $\delta = \frac{1}{n}$, we find that for each positive integer n, there is a p_n such that

$$|p_n - p| < \frac{1}{n} \text{ and } |f(p_n) - f(p)| \geq \varepsilon$$

Then $\lim p_n = p$, but $f(p_n)$ does not converge to $f(p)$.

f has a discontinuity of the second kind at p if neither of $\lim_{x\to p-0} f(x)$ and $\lim_{x\to p+0} f(x)$ exists

Illustrations

Let f be the function defined on R by setting

$$f(x) = \frac{\sin x}{x}, \text{ if } x \neq 0$$

$$f(0) = 0$$

Here $\lim_{x\to 0} f(x) = 1$

Since $\lim_{x\to 0} f(x)$ exists but is not equal to $f(0)$, therefore, f is not continuous at $x=0$.

In fact, f has a removable discontinuity at $x=0$.

2. Let f be the function defined on R by setting

$$f(x) = e^{\frac{1}{x}} - e^{-\frac{1}{x}}, \text{ if } x \neq 0$$

$$f(0) = 1$$

Here $f(0+0) = \lim_{\substack{h\to 0 \\ h>0}} \dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$

$$= \lim_{\substack{h\to 0 \\ h>0}} \frac{e^{1/h}\left[1 - \frac{e^{-1/h}}{e^{1/h}}\right]}{e^{1/h}\left[1 + e^{-1/h}/e^{1/h}\right]}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{1 - e^{-\frac{1}{h}} \cdot e^{-\frac{1}{h}}}{1 + e^{-\frac{1}{h}} \cdot e^{-1/h}}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{1 - e^{-2/h}}{1 + e^{-2/h}}$$

$$= \frac{1 - e^{-2/0}}{1 + e^{-2/0}} = \frac{1 - e^{-\infty}}{1 + e^{-\infty}} = 1$$

$$[\because e^{-\infty} = 0]$$

$f(0+0) = 1$

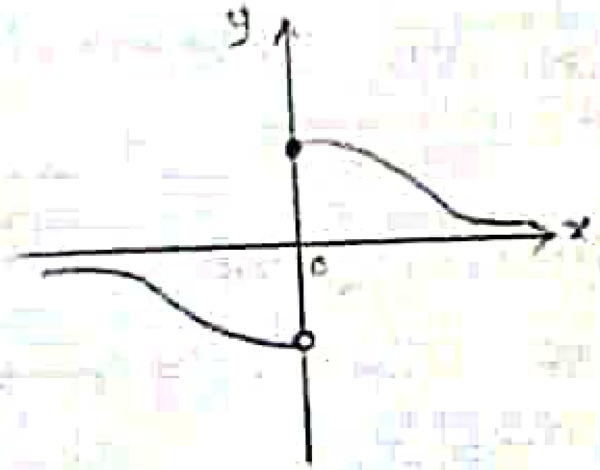


Fig 5.15

$$\text{Also } f(0-0) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{1/h}\left[\frac{e^{-1/h}}{e^{1/h}} - 1\right]}{e^{1/h}\left[\frac{e^{-1/h}}{e^{1/h}} + 1\right]}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{-1/h} \cdot e^{-1/h} - 1}{e^{-1/h} \cdot e^{-1/h} + 1}$$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{-2/h} - 1}{e^{-2/h} + 1}$$

$$= -1$$

Since $f(0+0) = f(0) \neq f(0-0)$, therefore, f is continuous from the right at $x=0$ and has a discontinuity of the first kind from the left at $x=0$.

3. Let f be the function defined on R by setting

$$f(x) = \frac{e^{1/x}}{1+e^{1/x}} \quad , \text{if } x \neq 0$$

$$= 0 \quad , \text{if } x = 0$$

Here, $f(0+0) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{e^{1/h}}{1+e^{1/h}}$

$$= \lim_{\substack{h \to 0 \\ h > 0}} \frac{\cancel{e^{1/h}}}{\cancel{e^{1/h}}\left[\frac{1}{e^{1/h}} + 1\right]}$$

4. ALGEBRA OF CONTINUOUS FUNCTIONS

THEOREM 4·1:

Let f and g be defined on an interval I. If f and g are continuous at $p \in I$, then $f+g$ is continuous at p

PROOF:

Let us assume that f and g are both continuous at $p \in I$. If $\langle p_n \rangle$ be any sequence converging to p, then

$$\lim_{n \to \infty} f(p_n) = f(p),$$

and $$\lim_{n \to \infty} g(p_n) = g(p),$$

Since f and g are both continuous at p.

Now, $$\lim_{n \to \infty} (f+g)(p_n) = \lim_{n \to \infty} \{f(p_n) + g(p_n)\},$$

$$= \lim_{n \to \infty} f(p_n) + \lim_{n \to \infty} g(p_n),$$

$$= f(p) + g(p),$$

$$= (f+g)(p),$$

Showing that $f+g$ is continuous at p.

THEOREM 4·2:

Let f and g be defined on an interval I. If f and g are continuous at $p \in I$, then fg is continuous at p

PROOF:

Let us assume that f and g are both continuous at $p \in I$. If $\langle p_n \rangle$ be any sequence converging to p, then

$$\lim_{n\to\infty} f(p_n) = f(p),$$

and $\lim_{n\to\infty} g(p_n) = g(p)$,

since f and g are both continuous at p

$$\text{Now } \lim_{n\to\infty} (fg)(p_n) = \lim_{n\to\infty} \{f(p_n) g(p_n)\}$$

$$= \lim_{n\to\infty} f(p_n) \lim_{n\to\infty} g(p_n)$$

$$= f(p)\, g(p)$$

$$= (fg)(p)$$

showing that fg is continuous at p

THEOREM 4.3:

If f is continuous at a point p and $c \in R$, then cf is continuous at p

PROOF:

Let us assume that f is continuous at p

If $\langle p_n \rangle$ be any sequence converging to p, then

$$\lim_{n\to\infty} f(p_n) = f(p),$$

since f is continuous at p

$$\text{Now, } \lim_{n\to\infty} (cf)(p_n) = \lim_{n\to\infty} cf(p_n),$$

$$= c \lim_{n\to\infty} f(p_n)$$

$= c f(p)$.

$= (cf)(p)$.

Showing that cf is continuous at p

THEOREM: 4.4.

Let f and g be defined on an interval I and let $g(p) \neq 0$. If f and g are continuous at $p \in I$, then f/g is continuous at p.

PROOF:

Let $\langle p_n \rangle$ be any sequence converging to p.

Since g is continuous at p, therefore,

$$\lim_{n \to \infty} g(p_n) = g(p)$$

Again, since $g(p) \neq 0$, therefore, there is a positive integer m such that $g(p_n) \neq 0$, whenever $n > m$

Also, since f is continuous at p, therefore,

$$\lim_{n \to \infty} f(p_n) = f(p)$$

Now, $\lim_{n \to \infty} (f/g)(p_n) = \lim_{n \to \infty} \{f(p_n) / g(p_n)\}$,

$$= \lim_{n \to \infty} f(p_n) \Big/ \lim_{n \to \infty} g(p_n)$$

$$= f(p) / g(p)$$

$$= (f/g)(p)$$

Showing that f/g is continuous at

THEOREM 4.5:

If f is continuous, then $|f|$ is continuous.

PROOF:

Let p be any point belonging to the domain of f and let $\langle p_n \rangle$ be a sequence converging to p.

Then $\lim_{n\to\infty} |f|(p_n) = \lim_{n\to\infty} |f(p_n)| = \left|\lim_{n\to\infty} f(p_n)\right|$

$= |f(p)| = |f|(p)$.

So that $|f|$ is continuous at p.

THEOREM: 4.6

Let f and g be defined on an interval I. If they are both continuous at $p \in I$, then the functions $\max.\{f, g\}$ and $\min.\{f, g\}$ are both continuous at p.

PROOF:

$$\max.\{f.g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

$$\min.\{f.g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

$$\max(f.g)(x) = \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) < g(x) \end{cases}$$

and $\min(f.g)(x) = \begin{cases} f(x) & \text{if } f(x) \leq g(x) \\ g(x) & \text{if } f(x) > g(x) \end{cases}$

We know that if f and g are continuous at p.

$f+g$, $f-g$ and $|f-g|$ are all continuous at p

$\therefore$ $\max\{f,g\}$ and $\min\{f,g\}$ are also continuous at p.

THEOREM 4.7:

Let f and g be defined on intervals I and J respectively, and let $f(I) \subset J$. If f is continuous at $p \in I$ and g is continuous at $f(p)$, then $g \circ f$ is continuous at p.

PROOF:

Let $\langle p_n \rangle$ be any sequence in I converging to p. Since f is continuous at p. therefore, $\langle f(p_n) \rangle$ Converges to $f(p)$. Also, since $f(I) \subset J$, therefore $\langle f(p_n) \rangle$ is a sequence in J.

Now g is continuous at $f(p)$ and $\langle f(p_n) \rangle$ is a sequence in J converging to $f(p)$. Therefore, $\langle g(f(p_n)) \rangle$ converges to $g(f(p))$, i.e., $\langle (g \circ f)(p_n) \rangle$ converges to $(g \circ f)(p)$

Since $\langle p_n \rangle$ converges to p implies that $\langle (g \circ f)(p_n) \rangle$ converges to $(g \circ f)(p)$, therefore $g \circ f$ is continuous at p

THEOREM 6.6

(Intermediate value Theorem).

If f be continuous on $[a,b]$ and c be any real number x in $[a,b]$ such that $f(x)=0$.

PROOF:

Let g be a function defined on $[a,b]$ by setting

$$g(x) = f(x) - c, \text{ for all } x \text{ in } [a,b]$$

clearly, g is continuous on $[a,b]$. Also $g(a) = f(a) - c$, $g(b) = f(b) - c$. Since c lies between $f(a)$ and $f(b)$, therefore $g(a)$ and $g(b)$ have opposite signs. We know that,

" If f be continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists a point x in $]a,b[$ such that $f(x) = 0$". It follows that there exists x_0 in $]a,b[$ such that $g(x_0) = 0$, that is $f(x_0) = c$.

COROLLARY:

If f be continuous on $[a,b]$ and c be any real number between $\sup f$ and $\inf f$, then there exists a real number x in $]a,b[$ such that $f(x) = c$

PROOF:

If $\sup f = \inf f$, then f is constant on

$[a,b]$, so that $\sup f = \inf f = c$, and the result is obvious. Let us, therefore, consider the case when $\sup f \neq \inf f$.

Since f is continuous on $[a,b]$ and since every function defined and continuous on a closed interval attains its supremum and infimum, therefore, there exist real numbers p and q in $[a,b]$ such that $f(p) = \sup f$, $f(q) = \inf f$.

Since $\sup f \neq \inf f$, therefore, $p \neq q$.

If $p < q$, then $a \leq p < q \leq b$, so that f is continuous on $[p,q]$. Since c lies between $f(p)$ and $f(q)$, therefore, by the intermediate value theorem, there exists a real number x_0 in $]p,q[$ and therefore, in $]a,b[$ such that $f(x_0) = c$.

If $q < p$, then $a \leq q < p \leq b$, so that f is continuous on $[q,p]$. Since c lies between $f(q)$ and $f(p)$, therefore, by the intermediate value theorem there exists a real number x_1 in $]q,p[$ and hence in $]a,b[$ such that $f(x_1) = c$.

Thus in each case we have $f(x) = c$ for some x in $]a,b[$.

8. UNIFORM CONTINUITY.

DEFINITION 8.1

A function f defined on an interval I is said to be **uniformly continuous** on I, if given $\epsilon > 0$ there exists a $\delta > 0$ such that if x, y are in I and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

THEOREM 8.1

If f be uniformly continuous on an Interval I, then it is continuous on I.

PROOF:

Let f be uniformly continuous on I, let x_0 be any point of I, and let $\epsilon > 0$ be given.

Since f is uniformly continuous on I, therefore, there exists a $\delta > 0$ such that if x and y are in I and $|x-y| < \delta$, then

$$|f(x) - f(y)| < \epsilon$$

Setting $y = x_0$, we have in particular, if $x \in I$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

This means that f is continuous at x_0.

THEOREM 8.2

If f is continuous on the closed and bounded interval I $(=[a,b])$, then f is uniformly continuous on I.

PROOF:

Let f be not uniformly continuous on I. Then there is a positive number ε such that whatever $\delta > 0$ we take, we can find real number x and y in I, such that

$$|x-y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon.$$

In particular, for each positive integer n, we can find real numbers x_n and y_n in I such that

$$|x_n - y_n| < 1/n \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon \rightarrow ①$$

Since $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences in I, and since every sequence in a closed interval has a convergent subsequence, therefore, there exist subsequences $\langle x_{n_k} \rangle$ and $\langle y_{n_k} \rangle$ of $\langle x_n \rangle$ and $\langle y_n \rangle$ respectively, and points x_0 and y_0 in I, such that

$$x_{n_k} \rightarrow x_0 \text{ and } y_{n_k} \rightarrow y_0 \longrightarrow ②$$

From ①, we find that

$$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \text{ and } |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon \rightarrow ③$$

for all k

From the first of the inequalities (3) we find that

$$\lim_{n\to\infty} x_{n_k} = \lim_{n\to\infty} y_{n_k} \text{, that is } x_0 = y_0$$

From the second of the inequalities (3), we find that in case $\langle f(x_{n_k})\rangle$ and $\langle f(y_{n_k})\rangle$ converge converge, the limits to which they converge are different!

We thus have two sequences $\langle x_{n_k}\rangle$ and $\langle y_{n_k}\rangle$ both of which converge to x_0, but $\langle f(x_{n_k})\rangle$ and $\langle f(y_{n_k})\rangle$ do not converge to the same limit. This means that f is not continuous at x_0 (for, if f were continuous at x_0, then $\langle f(x_{n_k})\rangle$ and $\langle f(y_{n_k})\rangle$ would have converged to the same limit, namely $f(x_0)$).