SEMESTER : **V**
CORE COURSE : **IX**

Inst Hour	**: 6**
Credit	**: 5**
Code	**: 18K5M09**

REAL ANALYSIS

UNIT 1.

Real Number system – Field axioms –Order relation in R. Absolute value of a real number & its properties –Supremum & Infimum of a set– order completeness property – countable & uncountable sets
(Chapter 1: Sections 2-7&10 of Text Book 1)

UNIT 2:

Continuous functions - Limit of a Function – Algebra of Limits – Infinite Limits– Continuity of a function – Types of discontinuities – Elementary properties of continuous functions –Uniform continuity of a function.
(Chapter 5: of Text Book 1)

UNIT 3:

Differentiability of a function– Derivability & continuity –Algebra of derivatives –Inverse Function Theorem – Daurboux's Theorem on derivatives.
(Chapter 6: Sections1-5 of Text Book 1)

UNIT 4:

Rolle's Theorem –Mean Value Theorems on derivatives– Taylor's Theorem with remainder- Power series expansion .
(Chapter 7: Sections 1-6 of Text Book 1)

UNIT 5:

Riemann integrability and integral of a bounded function over finite domain – Darboux's theorem –Another equivalent definition of Integrability and Integral – conditions for integrability – Particular classes of bounded Integrable functions - Properties of Integrable functions – Functions defined by Definite Integrals– First Mean Value Theorem of Integral Calculus – Change of variable in an Integral- Integration by parts.
(Chapter 6 of Text Book 2)

Text Book(s)

[1] M.K,Singhal & Asha Rani Singhal, A First Course in Real Analysis , R. Chand & Co June 2013

[2] Shanthi Narayan, A Course of Mathematical Analysis. 1964

Books for Reference

[1] Tom.M.Apostol, Mathematical Analysis ,II Edition.

[2] S.C.Malik , Elements of Real Analysis.

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

Verified.
1. M. Lakshmi
2. [illegible] 15/2/19

DERIVATIVES.

1.1. Derivability of an open interval:-

Let f be a real-valued function defined on an open interval $I \subset R$. If $x_0 \in I$, then we define g with domain $I \sim \{x_0\}$ by setting

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \forall \; x \in I \sim \{x_0\}.$$

If $\lim_{x \to x_0} g(x)$ exists and is finite, we denote it by $f'(x_0)$ and say that f is derivable at x_0 or that f has a derivative at x_0 or simply that $f'(x_0)$ exists. $f'(x_0)$ is called the derivative of f at x_0.

1.2. Derivability on the closed interval

Let f be defined on the closed interval $[a,b]$. If $\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a}$ exists and is finite, then we denote it by $f'(a)$ and say that f is derivable at $x = a$. If $\lim_{x \to b-0} \frac{f(x) - f(b)}{x - b}$ exists and is finite, Then we denote it by $f'(b)$ and say that f is derivable at $x - b$.

NOTE:- If f is derivable on $]a,b[$ and also at the points a and b, then we say that it is derivable on $[a,b]$.

1.3. Derivatives:

Let f be a function whose domain is an interval I. If I_1 be the set of all points x of I at which $f'(x)$ exists, and if $I_1 \neq \emptyset$, we get another function f' with domain I_1. We call f' the derivative of f. The symbol Df is also used to denote the derivative of f.

In general The n^{th} derivative $f^{n}(x_0)$ of f at a point x_0, and the function f^n may be defined similar to I_1.

1. If $f: R \to R$ is defined by $f(x) = x$ find f'.

Solution :-

Let f be the function defined on R by $f(x) = x \ \forall\, x \in R$.

If x_0 be any point of R, then $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$.

Thus we have $f'(x_0) = 1$. Since x_0 is any point of R, this means that $f'(x) = 1 \ \forall\, x \in R$. (ie.,) f' is derivable $\forall\, x \in R$ and f' is the function defined on R by $f'(x) = 1 \ \forall\, x \in R$.

2. If $f: R \to R$ is defined by $f(x) = x^n$ find f'.

Solution :-

Let n be a fixed positive integer and let f be the function defined on R by $f(x) = x^n \ \forall\, x \in R$.

If x_0 be any point of R, then $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0}$

$= \lim_{x \to x_0} (x^{n-1} + x^{n-2} x_0 + x^{n-3} x_0^2 + \dots + x_0^{n-1})$.

$= n x_0^{n-1}$. $\Rightarrow f'(x) = n x_0^{n-1}$.

2. DERIVABILITY AND CONTINUITY!

THEOREM 2-1 :-

Let f be defined on an interval I. If f be derivable at a point $x_0 \in I$, then it is continuous at x_0.

Proof:-

Since f is derivable at x_0, $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and equal $f'(x_0)$.

Now, if $x \to x_0$, then we may write

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0)$$

Taking limits as $x \to x_0$, we have

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim \left\{ \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right\}.$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} . \lim_{x \to x_0} (x - x_0),$$

$$= f'(x_0) . 0.$$

$$= 0.$$

So that $\lim_{x \to x_0} f(x) = f(x_0)$. and consequently f is continuous at x_0.

3. ALGEBRA OF DERIVATIVES:

THEOREM 3.1: If a function f is derivable at a point x_0. Then for each real number c, the function cf is also derivable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

Proof:-

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \text{ by the defintion of } f'(x_0).$$

$$\lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0} = \lim_{x \to x_0} \left\{ c . \frac{f(x) - f(x_0)}{x - x_0} \right\},$$

$$= c . \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= cf'(x_0).$$

Thus $(cf)'(x_0) = c.f'(x_0)$.

Theorem 3.2:- Let f and g be defined on an interval I. If f and g are derivable at $x_0 \in I$, then so also is $f+g$ and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

Proof:-

Since f and g are derivable at x_0,

$$\therefore \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \longrightarrow (1)$$

and

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) \longrightarrow (2)$$

we now wish to show that

$$\lim_{x \to x_0} \frac{[f(x)+g(x)]-[f(x_0)+g(x_0)]}{x-x_0} = f'(x_0)+g'(x_0) \longrightarrow (3)$$

The result follows immediately by

(i) re-writing the expression on the left of (3) in the form

$$\lim_{x \to x_0} \left\{ \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0} \right\}.$$

(ii) recalling that the limit of a sum equals the sum of the limits, and

(iii) Using (1) and (2).

Theorem 3.3:- Let f and g be defined on an interval I. If f and g are derivable at $x_0 \in I$, then so also is fg, and

$$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0).$$

Proof:

Since f and g are derivable at x_0,

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \longrightarrow (1)$$

and

$$\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = g'(x_0) \longrightarrow (2)$$

Now,

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}.$$

$$= \frac{f(x)-f(x_0)}{x-x_0} \cdot g(x) + f(x_0) \cdot \frac{g(x)-g(x_0)}{x-x_0} \longrightarrow (3)$$

By applying the theorem on limits of sums and products, and using (1) and (2) and the fact that

$\lim_{x \to x_0} g(x) = g(x_0)$ we find from (3) that

$$\lim_{x \to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$ exists and equals

$f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Hence the result.

Theorem 3-4:- Let f be derivable at x_0 and let $f(x_0) \neq 0$. Then the function $1/f$ is derivable at x_0 and $(1/f)'(x_0) = -f'(x_0)/\{f(x_0)\}^2$.

Proof:-

Since f is derivable at x_0. Therefore it is also continuous at x_0. Also, since $f(x_0) \neq 0$, therefore, $f(x_0) \neq 0$ is some neighbourhood N of f.

Now,
$$\frac{1/f(x) - 1/f(x_0)}{x - x_0} = -\frac{f(x) - f(x_0)}{x - x_0} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(x_0)}, \longrightarrow (1)$$

whenever $x \in N$, Since f is derivable at x_0, therefore,
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \longrightarrow (2)$$
Also, since f is continuous at x_0, therefore $\lim_{x \to x_0} f(x) = f(x_0) \neq 0 \rightarrow (3)$ by applying the theorem on the limit of a product to (1), and using (2) & (3), we find that
$$\lim_{x \to x_0} \frac{1/f(x) - 1/f(x_0)}{x - x_0}$$
exists and equal $-f'(x_0)/\{f(x_0)\}^2$.

Theorem: 3-5

Let f and g be defined on an interval I. If f and g be derivable at $x_0 \in I$, and if $g(x_0) \neq 0$, then the function f/g is also derivable at x_0.

Proof: Use theorem 3.3 and 3.4.

Theorem 3.6:- (CHAIN RULE) Let f and g be functions such that the range of f is contained in the domain of g. If f is derivable at x_0, and g is derivable at $f(x_0)$, then $g \circ f$ is derivable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

Proof:- Since the range of f is contained in the domain of g, therefore, $g \circ f$ has the same domain as that f.

We are required to show that,
$$\lim_{h \to 0} \frac{(g \circ f)(x_0 + h) - (g \circ f)(x_0)}{h}$$
exists and equals $g'(f(x_0)) \cdot f'(x_0)$.

For this purpose, we define a function F by setting

$$F(h) = \begin{cases} \dfrac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)}, & \text{if } f(x_0+h) - f(x_0) \neq 0, \\ g'(f(x_0)), & \text{if } f(x_0+h) - f(x_0) = 0. \end{cases}$$

In terms of the function F, we have

$$G(h) = \frac{(gof)(x_0+h) - (gof)(x_0)}{h}$$

$$F(h) \cdot \frac{f(x_0+h) - f(x_0)}{h} \longrightarrow (1)$$ whenever $h \neq 0$. It may be noted that (1) holds even when, $f(x_0+h) - f(x_0) = 0$; for then each side of (1) equals zero:

f is derivable at x_0, it follows that $\lim\limits_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ exists and equals $f'(x_0)$. From (1) we then find that in order to prove the theorem, it is enough to show that $\lim\limits_{h\to 0} F(h)$ exists and equals $g'(f(x_0))$, for if this be done, from (1) we shall find that $\lim\limits_{h\to 0} G(h)$ exists and equals $g'(f(x_0)) \cdot f'(x_0)$.

To show this, we proceed as follows:

$\lim\limits_{k\to 0} \dfrac{g(f(x_0)+k) - g(f(x_0))}{k}$ exists and equals $g'(f(x_0))$. This mean that given $\epsilon > 0$. there exists a number $\delta > 0$, such that if $0 < |k| < \delta$, then.

$$\left| \frac{g(f(x_0)+k) - g(f(x_0))}{k} - g'(f(x_0)) \right| < \epsilon \longrightarrow (2)$$

Also, since f is derivable at x_0 and consequently continuous at x_0. Therefore, we can find a number $\delta' > 0$ such that if $|h| < \delta'$, then $|f(x_0+h) - f(x_0)| < \delta \rightarrow (3)$

Let us now consider any number h such that $|h| < \delta'$; if for this h, $f(x_0+h) = f(x_0)$, then $|F(h) - g'(f(x_0))| = 0 < \epsilon \rightarrow (4)$ from the definition of F.

If on the other hand, $f(x_0+h) \neq f(x_0)$, then writing $f(x_0+h) - f(x_0) \equiv k$.

we have
$$F(h) = \frac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)},$$
$$= \frac{g(f(x_0)+k) - g(f(x_0))}{k}, \text{ so that by (2),}$$
we find that, $|F(h)| - g'(f(x_0))| < \epsilon$, provided $|k| < \delta \rightarrow (5)$

i.e., provided $|f(x_0+h) - f(x_0)| < \delta$, which is true by (3).

Thus from (4) and (5), we find that if
$$|h| < \delta'; \text{ then } |F(h) - g'f(x_0))| < \epsilon,$$
i.e., $\lim_{h \to 0} F(h)$ exists and equals $g'(f(x_0))$.

As we have already remarked above, this completes the proof.

4. INVERSE FUNCTION THEOREM FOR DERIVATIVES

Theorem 4.1:- Let f be a continuous one-to-one function defined on an interval and let f be derivable at x_0 with $f'(x_0) \neq 0$. Then the inverse of the function f is derivable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.

Proof:-

Let the domain of f be X and let its range be Y. If g be the inverse of f, then g is a function with domain Y and range X such that $f(x) = y \Rightarrow g(y) = x$.

Let now $f(x_0) = y_0$ so that $g(y_0) = x_0$, and let y_0+k be any point of Y different from y_0. Since f is one-to-one, therefore, $\exists$ a unique point, say x_0+h, different from x_0, $\ni$: $f(x_0+h) = y_0+k$. By the definition of g, we also then have $g(y_0+k) = x_0+h$.

$$f(x_0) = y_0,\ f(x_0+h) = y_0+k \longrightarrow (1)$$

$$g(y_0) = x_0,\ g(y_0+k) = x_0+h,\ \&\ k \neq 0 \Rightarrow h \neq 0 \longrightarrow (2)$$

f being derivable at x_0, it is also continuous at x_0. g is continuous at y_0 and consequently,

$$\lim_{k\to 0} [g(y_0+k) - g(y_0)] = 0, \text{ ie., } \lim_{k\to 0} [(x_0+h) - x_0] = 0,$$

ie., $\lim_{k\to 0} h = 0 \longrightarrow (3)$.

Now, let $k \neq 0$, then

$$\frac{g(y_0+k) - g(y_0)}{k} = \frac{(x_0+h) - x_0}{k},$$

$$= \frac{h}{(y_0+k) - y_0},$$

$$= \frac{h}{f(x_0+h) - f(x_0)}, \text{ by (1)},$$

$$= \frac{1}{\{f(x_0+h) - f(x_0)\}/h}.$$

This being permissible since $h \neq 0$ by (2).

Letting $k \to 0$, we find from (3) that $h \to 0$, which implies that

$$\lim_{k\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

since $f'(x_0) \neq 0$, this means that

$$\lim_{k\to 0} \frac{1}{\{f(x_0+h) - f(x_0)\}/h} = 1/f'(x_0).$$

ie., $$\lim_{k\to 0} \frac{g(y_0+k) - g(y_0)}{k} = 1/f'(x_0)$$

Thus $g'(y_0)$ exists and equals $1/f'(x_0)$.

5. DARBOUX'S THEOREM

THEOREM 5.1 :- Let f be defined and derivable on $[a, b]$. If $f'(a)f'(b) < 0$, then $\exists$ a real number c between a and $b \ni$: $f'(c) = 0$.

Proof :- case (i) :- Let $f'(b) > 0$

Step (i) :- Since $f'(a) < 0$, Therefore, $h_1 > 0 \ni$: $f(x) < f(a)\ \forall x \in]a, a+h_1[$

In fact, since f is derivable at a, $\therefore$

$$\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Taking $\epsilon = -f'(a)$ we can find $h_1 > 0 \ni$:

if $a < x < a + h_1$, then $\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$,

i.e., $f'(a) - \epsilon < \frac{f(x) - f(a)}{x - a} < f'(a) + \epsilon$.

$f'(a) + \epsilon = 0$ and $x > a$, $\therefore f(x) < f(a)$.

Step 2 :- Since $f'(b) > 0$, therefore, $\exists : h_2 > 0 \ni$: $f(x) < f(b)\ \forall x \in]b-h_2, b[$.

since f is derivable at b, $\therefore \lim_{x \to b-0} \frac{f(x) - f(b)}{x - b} = f'(b)$.

Taking $\epsilon = f'(b)$ we can find $h_2 > 0 \ni$: if $b - h_2 < x < b$, then $\left| \frac{f(x) - f(b)}{x - b} - f'(b) \right| < \epsilon$. i.e.,

$f'(b) - \epsilon < \frac{f(x) - f(b)}{x - b} < f'(b) + \epsilon$.

From the first part of the above inequality, we find that since $f'(b) - \epsilon = 0$ and $x < b$, $\therefore f(x) < f(b)$.

Step 3 :- Since f is derivable on $[a, b]$, $\therefore$ it is also continuous on $[a, b]$, and consequently, it attains its supremum as well as infimum on $[a, b]$. Now by step (i) $\inf f \neq f(a)$ and by step (2), $\inf f \neq f(b)$. This means

f does not attains its infimum at any of the end points a and b. Therefore, $\exists$ a real number c between a & b, $\ni$: inf f is attained at c.

Step 4 :- $f'(c) \not> 0$. For, if $f'(c) > 0$. then $Lf'(c) > 0$ and as in step 2, we can find $h_3 > 0$, $\ni$: $f(x) < f(c)\ \forall\ x \in]c-h_3, c[$. and this contradicts the fact that $f(c)$ is the infimum of f on $[a,b]$. Hence $f'(c) \not> 0$.

Step 5 :- $f'(c) \not< 0$. for if $f'(c) < 0$ then $Rf'(c) < 0$, and as in step 1, we can find $h_4 > 0$ such that $f(x) < f(c)\ \forall\ x \in]c, c+h_4[$. hence $f'(c) \not< 0$.

Step 6 :- By the law of trichotomy, we have from step 4 & 5 $f'(c) = 0$.

Case (ii) :- Let $f'(a) > 0$ and $f'(b) < 0$. If g be the function $-f$. then g is derivable on $[a,b]$. $g'(a) < 0$. $g'(b) > 0$. So that by case (i) $\exists$: a real number d between a & b $\ni$: $g'(d) = 0$. Now $f'(d) = -g'(d) = 0$.

1. ROLLE's THEOREM :-

Theorem : 1.1

Let f be a function defined on $[a,b]$ $\ni$:

(i) f is continuous on $[a,b]$;

(ii) f is derivable on $]a,b[$;

(iii) $f(a) = f(b)$ There $\exists$: a real number c between a & b $\ni$: $f'(c) = 0$.

Proof :- since f is continuous on $[a,b]$ and since every fn/. that is continuous on a closed interval is bounded. $\therefore$ f must be bounded on $[a,b]$. Let sup $f = M$, inf $f = m$.

Two different cases arise :-

(1) $M = m$. Then f is constant over $[a,b]$ and consequently, $f'(x) = 0 \;\forall\; x$ in $[a,b]$

(2) $M \neq m$. Since $f(a) = f(b)$. $\therefore$ at least one of the numbers M and m is different from $f(a)$ & and There fore, also from $f(b)$. For the sake of definiteness, assume that $M \neq f(a)$

Since every function that is continuous on a closed interval attains its supremum $\therefore$ $\exists$, a some real number c in $[a,b]$, $\ni$: $f(c) = M$.

since $f(a) \neq M \neq f(b)$.

$\therefore$ c is different from both a and b.

This means that c lies in the open interval $]a,b[$.

Since $f(c)$ is the supremum of f on $[a,b]$

$f(x) \leq f(c) \;\forall\; x \in [a,b]$ & $f(c-h) \leq f(c)$.

$\hookrightarrow$ (1)

for all positive real numbers $h \ni : c-h$ lies in $[a,b]$

This means that $\frac{f(c-h)-f(c)}{-h} \geq 0$.

for all positive real numbers $h \ni : c-h$ lies in $[a,b]$.

Taking limits as $h \to 0$ and observing that since $f'(x)$ exists at each point of $]a,b[$ and Therefore at $x=c$.

$Lf'(c) \geq 0 \longrightarrow$ (ii)

From (i) $f(c+h) \leq f(c)$ $\forall$ positive real numbers h $\ni : c+h$ lies in $[a,b]$. By the same argument as above we have

$Rf'(c) \leq 0 \longrightarrow$ (iii)

since $f'(x)$ exists at $x=c$,

$\therefore Lf'(c) = f'(c) = Rf'(c) \longrightarrow$ (iv)

From (ii) (iii) & (iv) we find that $f'(c)=0$.

The case $M = f(a) \neq m$ can be disposed of in the same manner as above.

2. LAGRANGE'S MEAN VALUE THEOREM :-

Theorem : 2.1 :-

Let f be a function defined on $[a,b] \ni$:

(i) f is continuous on $[a,b]$

(ii) f is derivable on $]a,b[$

Then $\exists$: a real number $c \ni : c \in]a,b[\ni$:

$$f(b) - f(a) = (b-a) f'(c).$$

Proof:-

Let F be a function defined on $[a,b]$ by setting

$F(x) = f(x) + Ax \quad \forall \; x \text{ in } [a,b] \rightarrow (1)$

where A is a constant to be suitably chosen now,

(i) Since A is continuous on $[a,b]$ and the function $x \rightarrow Ax$ is continuous on $[a,b]$ therefore F is continuous on $[a,b]$

(ii) Also, since f is derivable on $]a,b[$ and the function $x \rightarrow Ax$ is derivable on $]a,b[$ therefore F is derivable on $]a,b[$.

(iii) Let us choose A so that $F(a) = F(b)$

$$F(a) = f(a) + A.a$$
$$F(b) = f(b) = A.b$$
$$f(a) + Aa = f(b) + Ab$$
$$Aa - Ab = f(b) - f(a)$$
$$-A(b-a) = f(b) - f(a)$$
$$-A = \frac{f(b) - f(a)}{b-a} \rightarrow (ii)$$

From (1), (2) and (3) above, we find that F is satisfies all the conditions of Rolle's theorem on $[a,b]$, and consequently, $\exists$ a real number c in $]a,b[\ni$: $F'(c) = 0$ from (i) this gives $f'(c) + A = 0 \rightarrow (iii)$

From (ii) & (iii) we have $\frac{f(b) - f(a)}{b-a} = f'(c)$

i.e., $f(b) - f(a) = (b-a) f'(c)$ //.

CAUCHY'S MEAN VALUE THEOREM:-

Theorem : 3.1

Let f and g be functions defined on $[a,b]$ $\ni$:

(i) f and g are continuous on $[a,b]$

(ii) f and g are derivable on $]a,b[$ and

(iii) $g'(x)$ does not vanish at any point of $]a,b[$ Then $\exists$: a real number $c \in]a,b[$ $\ni$: $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

Proof:

Let us first observe that as a consequence of condition (iii) $g(a) \neq g(b)$. For if $g(a)$ were equal to $g(b)$,

Then the function g would satisfy all the conditions of Rolle's theorem, and consequently for some x in $]a,b[$ we would have $g'(x)=0$.

Consider the function F defined on $[a,b]$ by

$F(x) = f(x) + Ag(x)$ $\forall$ x in $[a,b]$.

where A is a constant to be suitably chosen. Now,

(1) Since f and g are continuous on $[a,b]$ therefore, F is continuous on $[a,b]$.

(2) Also, since f and g are derivable on $]a,b[$ therefore, F is derivable on $]a,b[$.

(3) Let us choose A so that $F(a)=F(b)$. This gives us

$$-A = \frac{f(b)-f(a)}{f(b)-g(a)} \longrightarrow (ii)$$

division by $g(b)-g(a)$ being permissible since we have already show that $g(b) \neq g(a)$.

From (1), (2) & (3) we find that F satisfies all the conditions of Rolle's theorem on $[a,b]$ and consequently $\exists$: a real number c in $]a,b[\ \ni$: $F'(c)=0$ From (i)

$$f'(c)+Ag'(c)=0$$

$$\text{(or)} \qquad -A=\frac{f'(c)}{g'(c)}, \longrightarrow \text{(iii)}$$

division by $g'(c)$ being permissible since $g'(x)$ is not zero for any x in $]a,b[$.

From (ii) and (iii) we have

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

TAYLOR'S THEOREM :-

Theorem : 4.1 (Taylor's theorem with Lagrange's form of remainder). Let f be a function defined on $[a,b]$ $\ni$: (i) f^{n-1} is continuous on $[a,b]$

(ii) f^{n-1} is derivable on $]a,b[$

Then $\exists$: a real number $c\in]a,b[\ \ni$:

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\cdots\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+\frac{(b-a)^n}{n!}f^n(c).$$

Proof:

Let us first observe that conditions (i) in the statement of the theorem implies that $f, f', \ldots f^{n-1}$ are all defined and continuous on $[a,b]$

Consider the function F defined on $[a,b]$ by setting

$$F(x) = f(b) - f(x) - (b-x) f'(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots$$
$$- \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) - A(b-x)^n.$$

where A is a constant to be suitably chosen.

Now

(1) Since $f, f', \ldots f^{n-1}$ are all continuous on $[a,b]$ and since $x \to (b-x)^k$ is continuous on $[a,b]$ for each positive integer k, Therefore, F is continuous on $[a,b]$

(2). Since $f, f', \ldots f^{n-1}$ are all derivable on $]a,b[$ and since $x \to (b-x)^k$ is derivable on $]a,b[$ for each positive integer k, therefore F is derivable on $]a,b[$.

(3) Let us choose A so that $F(a) = F(b)$. Since $F(b)=0$ therefore, this gives $F(a)=0$, that is

$$f(b) - f(a) - (b-a) f'(a) - \frac{(b-a)^2}{2!} f''(a) - \cdots$$
$$\text{i.e. } - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) - A(b-a)^n = 0.$$

From (1), (2) & (3) we find that F satisfies all the conditions of Rolle's theorem on $[a,b]$, and consequently $\exists$: a real number c between a and b $\ni$:

$F'(c) = 0$

Since $F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^n(x) + nA(b-x)^{n-1}$,

$\therefore$ we have

$$-\frac{(b-c)^{n-1}}{(n-1)!} f^n(c) + nA(b-c)^{n-1} = 0 \Rightarrow A = \frac{f^n(c)}{n!} \rightarrow \text{(ii)}$$

From (i) and (ii) we find that,

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{n!} f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c) \longrightarrow \text{(iii)}$$

Theorem : 4.2 (Taylor's theorem with Cauchy's form of remainder)

Let f be a function defined on $[a,b]$ such that

(i) f^{n-1} is continuous on $[a,b]$ ∍:

(ii) f^{n-1} is derivable on $]a,b[$.

Then ∃! a real number c between a & b ∍:

$$f(b) = f(a) + (b-a) f'(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-c)^{n-1}(b-a)}{(n-1)!} f^n(c).$$

Proof:

Let F and G be functions defined on $[a,b]$ by setting $F(x) = f(b) - f(x) - (b-x) f'(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x)$.

$G(x) = b - x$, $\forall\, x \in [a,b]$.

Now,

(1) The continuity of f^{n-1} on $[a,b]$ implies that F is continuous on $[a,b]$. Also, G is obviously continuous on $[a,b]$

(2) The derivability of f^{n-1} on $]a,b[$ implies that F is derivable on $]a,b[$. Also, G is obviously derivable on $]a,b[$.

(3) $G'(x)$ is not zero on $]a,b[$. (in fact $G'(x) = -1$. $\forall\, x$ in $]a,b[$.

The functions F and G satisfy all the conditions of Cauchy's mean value theorem. Therefore $\exists$ a real number c between a and b $\ni$:

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} \rightarrow (i)$$

Now $F(b) = G(b) = 0$.

$$F(b) = f(b) - f(a) - \cdots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a).$$

$$G(a) = b-a.$$

$$F'(c) = \frac{(b-c)^{n-1}}{(n-1)!} f^{n}(c).$$

$$G'(c) = -1.$$

So that from (i), $f(b)-f(a)-(b-a)f'(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$

$$= \frac{(b-c)^{n-1}(b-a)}{(n-1)!} f^{n}(c).$$

i.e. $f(b) = f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-c)^{n-1}(b-a) f^{n}(c)}{(n-1)!} \rightarrow (ii)$

Hence the proof.

POWER SERIES:

Consider Maclaurin's series expansion of the functions e^x, $\sin x$, $\cos x$, $(1+x)^m$ and $\log(1+x)$.

(a) e^x. Let $f(x) = e^x \; \forall x \in R$. Then $f^{n}(x) = e^x \; \forall x \in R$.

Proof :-

For each positive integer n, f^n is defined in the interval $[-h, h]$ whatever positive real number h may be

Also, writing Lagrange's remainder after n terms,

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x)$$

$$= \frac{x^n}{n!} e^{\theta x}.$$

We shall now show that whatever x may be $\lim_{n\to\infty} R_n(x) = 0.$

For this purpose, it is enough to show that $e^{\theta x}$ is bounded in $[-h, h]$ and $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

Since $0<\theta<1$ & $x \in [-h,h]$ $\therefore$ $|\theta x| < h.$

& $0 < e^{\theta x} < e^h$, $e^{\theta x}$ is bounded.

Let us consider

$$a_n = \frac{x^n}{n!} \text{ for all } n \in \mathbb{N}.$$

Then $\frac{a_{n+1}}{a_n} = \frac{x}{n+1}$ so that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}, = 0.$$

From above, it follows that $\lim_{n\to\infty} a_n$ exist and equals zero.

Now $\lim_{n\to\infty} R_n(x) = e^{\theta x} \left[\lim_{n\to\infty} \frac{x^n}{n!}\right] = 0.$

Thus we find that whatever h may be, the function f has a Maclaurin's series expansion for each x in $[-h, h]$. This implies that for the given function, we have,

$$f(x) = f(0) + xf'(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \cdots \rightarrow (1)$$

$\forall x \in R$

Substituting $f(x) = e^x$, $f^n(x) = e^x$ we have,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \text{ for all } x \in R.$$

UNIT – V

6.2
Definitions :-

1. The infimum of the set of the upper sums S, is called the upper integral of f over [a, b] and is denoted by
$$U = \overline{\int_a^b} f(x)\,dx .$$

2. The supremum of the set of the lower sums, s is called the lower integral of f over [a, b] and is denoted by
$$L = \underline{\int_a^b} f(x)\,dx .$$

3. A bounded function f is said to be Riemann integrable or simply integrable over [a, b], If its upper and lower integrals are equal; the common value of these integrals is called the Riemann integral or simply the integral denoted by the symbol $I = \int_a^b f(x)\,dx$.

Example : 1

If f is defined on [0, 1] by $f(x) = x \ \forall\, x \in [0,1]$, then P.T $f \in R[0,1]$ and $\int_0^1 f(x)\,dx = 1/2$.

Solution :- Let any partition of [0, 1] be $D = \{0, 0/n, 1/n, 2/n, \ldots r/n \ldots n/n\}$

Let the sub-intervals be $I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$ for $r = 1, 2, \ldots n$. If δ be the length of this interval I_r, then,
$$\delta_r = \frac{r}{n} - \frac{r-1}{n} = 1/n .$$

Also, if M_r and m_r be respectively the supremum and infimum of the function f in I_r, then $M_r = \frac{r}{n}$ and $m_r = \frac{r-1}{n}$, as $f(x) = x$.

$$\therefore S(D) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} (r/n \cdot 1/n).$$

$$= \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{1}{n^2}\left[\frac{1}{2}n(n+1)\right] = \frac{n+1}{2n}$$

$S(D) = \frac{1}{2}(1+1/n) \longrightarrow (i)$ Also

$s(D) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n}\left(\frac{r-1}{n}\right)\cdot\frac{1}{n} = \frac{1}{2}(1-1/n) \longrightarrow (ii)$

Again, $\overline{\int_0^1} f(x)dx = \inf[S(D)] = \lim_{n\to\infty}\left[\frac{1}{2}(1+1/n)\right] = 1/2 \longrightarrow (iii)$

And,

$\underline{\int_0^1} f(x)dx = \sup.[s(D)] = \lim_{n\to\infty}\frac{1}{2}(1-1/n) = 1/2 \longrightarrow (iv)$

From (iii) & (iv), we find that $\underline{\int_0^1} f dx = \overline{\int_0^1} f dx = 1/2$.

Hence $f \in R[0,1]$ and $\int_0^1 f(x)dx = 1/2$.

6.3 DARBOUX'S THEOREM:

A bounded function f is integrable over $[a,b]$ and M, m are the bounds of f, to every $\epsilon > 0$, Ǝ: there corresponds $\delta > 0$ Ǝ: (i) $S(D) < \int_a^b f(x)dx + \epsilon$. $\forall$ D with $|D| \le \delta$.

(ii) $s(D) > \int_a^b f(x)dx - \epsilon$ for every division D with $|D| \le \delta$

<u>Proof</u>:

(i) Let $|f(x)| \le k$ $\forall x \in [a,b]$,

Let δ be a positive number and D_1 a division of $[a,b]$ Ǝ: $|D_1| \le \delta$. Let D_2 be a division of $[a,b]$ consisting of all the points of D_1 and at the most some p more.

Then we shall show that, $S(D_1) - 2pk\delta \le S(D_2) \le S(D_1)$.

it follows that, $D_2 \supset D_1 \Rightarrow S(D_2) \le S(D_1)$.

Firstly suppose that $p=1$. so that only one interval, say δ_r, of D_1 is divided into two intervals, say δ_r' and δ_r''

Let M_r, M_r', M_r'' be the supremum of f in $\delta_r, \delta_r', \delta_r''$.

$\therefore S(D_1) - S(D_2) = M_r \delta_r - (M_r' \delta_r' + M_r'' \delta_r'')$

$= (M_r - M_r')\delta_r' + (M_r - M_r'')\delta_r''$ for $\delta_r = \delta_r' + \delta_r''$.

Now, $|f(x)| \leq k, \forall x \in [a,b] \Rightarrow -k \leq M_r' \leq M_r \leq k$

$\Rightarrow 0 \leq M_r - M_r' \leq 2k$. similarly we have $0 \leq M_r - M_r'' \leq 2k$

It follows that, $0 \leq S(D_1) - S(D_2) \leq 2k(\delta_r' + \delta_r'') = 2k\delta_r \leq 2k\delta$.

We now prove the main theorem,

As f is bounded, $\exists: k>0, \ni: |f(x)| \leq k \; \forall x \in [a,b]$.

since $\int_a^b f(x)dx$ is the infimum of the set of upper sums S,

$\exists$: a division $D_1 \{a = x_0, x_1, \ldots x_{p-1}, x_p = b\} \ni: S(D_1) < \int_a^b f(x)dx + \frac{\epsilon}{2}$

The points of D_1 are $(p+1)$ in number.

Let δ be the positive number such that, $2k(p-1)\delta = \frac{1}{2}\epsilon$.

Let D be any division with norm less than or equal to δ.

Let D_2 be the division consisting of the points of D_1 as well as those of D. Applying the lemma to the divisions D and D_2, we have

$$S(D) - 2(p-1)k\delta \leq S(D_2) \leq S(D).$$

Also $D_2 \supseteq D_1 \Rightarrow S(D_2) \leq S(D_1)$ Thus we obtain

$S(D) - 2(p-1)k\delta \leq S(D_1) \Rightarrow S(D) \leq 2(p-1)k\delta + S(D_1)$

$< \epsilon/2 + \int_a^b f(x)dx + \epsilon/2 = \int_a^b f(x)dx + \epsilon$.

Hence the result.

(ii) This proof is similar to that of the corresponding result on that upper integral proved above.

Conditions For integrability :-

Theorem :-

The necessary and sufficient condition for the integrability of a bounded function f is, that to every $\epsilon > 0$, there corresponds a $\delta > 0 \ni$: for every division D, whose norm is $\leq \delta$, the oscillatory sum $\omega(D)$ is $< \epsilon$.

Proof :-

The condition is necessary: The bounded function f being integrable, we have $\overline{\int_a^b} f(x)dx = \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)dx$.

Let ϵ be any positive number. By Darboux's Theorem, $\exists$: $\delta > 0 \ni$: for every division D whose norm is $\leq \delta$.

$$\begin{cases} S(D) < \overline{\int_a^b} f(x)dx + \epsilon/2 = \int_a^b f(x)dx + \epsilon/2, \\ s(D) > \underline{\int_a^b} f(x)dx - \epsilon/2 = \int_a^b f(x)dx - \epsilon/2. \end{cases}$$

$$\Rightarrow \int_a^b f(x)dx - \epsilon/2 < s(D) \leq S(D) < \int_a^b f(x)\,dx + \epsilon/2.$$

$\Rightarrow \omega(D) = S(D) - s(D) < \epsilon$ for every division D whose norm is $\leq \delta$.

The condition is sufficient :- Let ϵ be any positive number. $\exists$ a division D such that,

$$S(D) - s(D) = \left[S(D) - \overline{\int_a^b} f(x)dx\right] + \left[\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx\right] + \left[\underline{\int_a^b} f(x)dx - s(D)\right] < \epsilon.$$

Since each one of the three numbers,

$S(D) - \overline{\int_a^b} f(x)dx$, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$, $\underline{\int_a^b} f(x)dx - s(D)$,

is non-negative, $0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \epsilon$.

As ϵ is an arbitrary positive number, we see that the non-negative number, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx$, is less than every positive number, and hence

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0 \Rightarrow \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

so that f is integrable.

PROPERTIES OF INTEGRABLE FUNCTIONS :-

(i) If a bounded function f is integrable in $[a,b]$, then it is also integrable in $[a,c]$ and $[c,b]$ where c is a point of $[a,b]$. Conversely, if f is bounded and integrable in $[a,c]$, $[c,b]$ then it is also integrable in $[a,b]$. Also in either case

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a < x < b.$$

(ii) Integrability of the sums and difference :-

If f and g are two functions both bounded and integrable in $[a,b]$ then $f \pm g$ are also bounded and integrable in $[a,b]$ and $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

(iii) Integrability of product :-

If f, g are two functions, both bounded and integrable in $[a,b]$, then their product fg is also bounded and integrable in $[a,b]$.

(iv) Integrability of Quotient :-

If f, g are two functions, both bounded and integrable in $[a,b]$ and $\exists$ a number $t > 0 \ni |g(x)| \geq t \ \forall \ x \in [a,b]$ then f/g is bounded and integrable in $[a,b]$.

(v) Integrability of the Modulus of an Integrable Function:

If f is bounded and integrable in $[a,b]$, then $|f|$ is also bounded and integrable in $[a,b]$

(vi) $\int_a^b f(x)\,dx$ exists means that f is bounded and integrable in $[a,b]$.

(vii) Inequalities for an integral :-

Theorem :- If f is bounded and integrable in $[a,b]$. and M, m are the bounds of f in $[a,b]$, then,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \text{ if } b \ge a$$

$$m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a) \text{ if } b \le a.$$ & For $a = b$, the result is trivial.

Proof :-

(i) If $b > a$, then for any division D, we have,

$$m(b-a) \le \int_a^b f(x)\,dx \le S(D) \le M(b-a)$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

(ii) If $b < a \Rightarrow a > b$, then, as proved above,

$$m(a-b) \le S(D) \le \int_b^a f(x)\,dx \le M(a-b)$$

$$-m(a-b) \ge -\int_b^a f(x)\,dx \ge -M(a-b)$$

$$\Rightarrow m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a).$$

Hence the results.

Cor. 1:– If f is bounded and integrable in $[a,b]$, then $\exists$ a number, μ, lying between the bounds of f such that
$\int_a^b f(x)dx = \mu(b-a)$.

Cor 2:– If f is continuous in $[a,b]$, then $\exists$ a number, c lying between a and b $\ni$: $\int_a^b f(x)dx = (b-a)f(c)$.

Cor 3:– If f is bounded and integrable in $[a,b]$ and, k is a number such that $\forall x \in [a,b]$, $|f(x)| \le k$, then $|\int_a^b f(x)dx| \le k|b-a|$

Cor 4:– If f is bounded and integrable in $[a,b]$ and ~~$\forall x \in [a,b]$~~ $\forall x \in [a,b]$, $f(x) \ge 0$, then $\int_a^b f(x)dx \begin{cases} \ge 0, \text{ when } b \ge a, \\ \le 0, \text{ when } b \le a. \end{cases}$

Cor 5:– If $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ both exist, then,

$f \ge g \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$ when $b \ge a$.

$f \le g \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$ when $b \le a$. under the given condition $[f(x) - g(x)]$ is integrable and $\ge 0 \ \forall x \in [a,b]$.

Cor 6:– If $\int_a^b |f(x)|dx$ exists, then $|\int_a^b f(x)dx| \le |\int_a^b |f(x)|dx|$.

it has been show that $\int_a^b |f(x)|dx$ exists. we have $\forall x \in [a,b]$
$-|f(x)| \le f(x) \le |f(x)|$.

Mean Value Theorems of Integral Calculus:–

First Mean Value Theorem:– If $\int_a^b f(x)dx$ and $\int_a^b \varphi(x)dx$ both exist and $\varphi(x)$ keeps the same sign, positive or negative throughout the interval of integration, then $\exists$: a number μ lying between the bounds of f such that

$$\int_a^b f(x)dx\, \varphi(x) = \mu \int_a^b \varphi(x)dx. \longrightarrow (1)$$

First suppose that $\varphi(x)$ is positive $\forall x \in [a,b]$. If M, m be the bounds of f, we have $\forall x \in [a,b]$,

$$m \le f(x) \le M$$

$\Rightarrow m\varphi(x) \leq f(x)\varphi(x) \leq M\varphi(x)$, for $\varphi(x) \geq 0 \, \forall \, x$.

Thus $m\int_a^b \varphi(x)dx \leq \int_a^b f(x)\varphi(x)dx \leq M\int_a^b \varphi(x)dx$ if $b \geq a$.

$m\int_a^b \varphi(x)dx \geq \int_a^b f(x)\varphi(x)dx \geq M\int_a^b \varphi(x)dx$ if $b \leq a$. In either case we see that $\exists$: a number μ, lying between M and m, $\ni$: (1) is true. Hence the result.

Cor: $\int_a^b f(x)\varphi(x)dx = f(\xi)\int_a^b \varphi(x)dx$.

SECOND MEAN VALUE THEOREM:-

If $\int_a^b f(x)dx$ and $f(\xi)\int_a^b \varphi(x)dx$ both exist and φ is monotonic in $[a,b]$, then $\exists$: $\xi \in [a,b] \ni$: $\int_a^b f(x)\varphi(x)dx = \varphi(a)\int_a^{\xi} f(x)dx + \varphi(b)\int_{\xi}^b f(x)dx$

Abel's lemma:- The proof of the theorem depends upon a lemma which is due to Abel which we now state and prove.

If (i) $a_1, a_2, \dots a_n$ is a monotonically decreasing set of n +ve numbers, (ii) $v_1, v_2, \dots v_n$ is a set of any n, numbers & (iii) k, K are two numbers such that, $k < v_1+v_2+\dots+v_p \leq K$. for $1 \leq p \leq n$, then $a_1 k < \sum_{r=1}^{r=n} a_r v_r \leq a_1 K$.

$S_p = v_1 + v_2 + \dots + v_p$.

$$\sum_{r=1}^{r=n} a_r v_r = a_1 S_1 + a_2(S_2 - S_1) + \dots + a_r(S_r - S_{r-1}) + \dots + a_n(S_n - S_{n-1})$$
$$= (a_1 - a_2)S_1 + (a_2 - a_3)S_2 + \dots + (a_{n-1} - a_n)S_{n-1} + a_n S_n.$$

Now by (i), $(a_1 - a_2), (a_2 - a_3), \dots, (a_{n-1} - a_n), a_n$. are all positive, Also by (iii) $k < S_p < K \, \forall \, p \leq n$.

$$\therefore \sum_{r=1}^{r=n} a_r v_r < (a_1 - a_2)K + (a_2 - a_3)K + \dots + (a_{n-1} - a_n)K + a_n K = a_1 K.$$

$$\sum_{r=1}^{r=n} a_r v_r > (a_1 - a_2)k + (a_2 - a_3)k + \dots + (a_{n-1} - a_n)k + a_n k = a_1 k$$

Hence the lemma.

Proof of the Theorem:-

If $\int_a^b f(x)\,dx$ & $\int_a^b \psi(x)dx$ both exist. ψ is monotonically decreasing and positive in $[a,b]$ then $\exists$ a point $\xi \in [a,b] \ni$: $\int_a^b f(x)\psi(x)dx = \psi(a)\int_a^{\xi} f(x)dx$. Let $D\{a=x_0, x_1, \dots x_{r-1}, x_r \dots x_n=b\}$ be any division of $[a,b]$. Let M_r, m_r be the bounds of f in $\delta_r = [x_{r-1}, x_r]$.

Let $\xi_1 = a$ and ξ_r, when $r \neq 1$, be any point of δ_r, we have, $m_r \delta_r \leq \int_{x_{r-1}}^{x_r} f(x)dx \leq M_r \delta_r$, $m_r \delta_r \leq f(\xi_r)\delta_r \leq M_r\delta_r$.

putting $r = 1, 2, 3, \dots p$ where $p \leq n$, and adding we obtain

$$\sum_{r=1}^{n} m_r\delta_r \leq \int_a^{x_p} f(x)dx \leq \sum_{r=1}^{p} M_r\delta_r, \quad \sum_{r=1}^{p} m_r\delta_r \leq \sum_{r=1}^{p} f(\xi_r)\delta_r \leq \sum_{r=1}^{p} M_r\delta_r$$

Thus we have

$$\left|\int_a^{x_p} f(x)dx - \sum_{r=1}^{p} f(\xi_r)\delta_r\right| \leq \sum_{r=1}^{p} (M_r - m_r)\,\delta_r \leq \sum_{r=1}^{n} (M_r - m_r)\,\delta_r,$$

$$\Rightarrow \int_a^{x_p} f(x)dx - \sum_{r=1}^{n} O_r\delta_r \leq \sum_{r=1}^{p} f(\xi_r)\delta_r \leq \int_a^{x_p} f(x)dx + \sum_{r=1}^{n} O_r\delta_r.$$

where $O_r = (M_r - m_r)$ is the oscillation of f in δ_r.

Now, $\int_a^t f(x)dx$ being a continuous function with t is variable, is bounded. Let C, D be ~~its~~ the bounded.

$$\therefore C - \sum_{r=1}^{n} O_r\delta_r \leq \sum_{r=1}^{p} f(\xi_r)\delta_r \leq D + \sum_{r=1}^{n} O_r\delta_r.$$

$V_r = f(\xi_r)\delta_r$, $a_r = \psi(\xi_r)$; $k = C - \sum O_r\delta_r$, $K = D + \sum O_r\delta_r$.

& obtain $\psi(a)\left[C - \sum_{r=1}^{n} O_r\delta_r\right] \leq \sum_{r=1}^{n} f(\xi_r)\,\psi(\xi_r)\,\delta_r \leq \psi(a)\left[D + \sum_{r=1}^{n} O_r\delta_r\right]$

Let the norm of the division tend to 0. We then obtain, in the limit $C\psi(a) \leq \int_a^b f(x)\,\psi(x)dx = D\psi(a)$.

$\Rightarrow \int_a^b f(x)\psi(x)\,dx = \mu\psi(a)$. where μ is some no btn C & D.

The continuous function $\int_a^x f(x)dx$.

$\int_a^b f(x)\psi(x)dx = \psi(a)\int_a^{\xi} f(x)dx$.

We now turn to the theorem proper.

Let ϕ be monotonically decreasing so that the function ψ where $\psi(x) = \varphi(x) - \varphi(b)$, is monotonically decreasing and positive.

There exists, $\therefore$ a number, ξ, btw. a and b $\ni$:

$$\int_a^b f(x)[\varphi(x) - \varphi(b)]dx = [\varphi(a) - \varphi(b)]\int_a^{\xi} f(x)dx$$

$$\Rightarrow \int_a^b f(x)\varphi(x)dx = \varphi(a)\int_a^{\xi} f(x)dx + \varphi(b)\left\{\int_a^b f(x)dx - \int_a^{\xi} f(x)dx\right\}.$$

$$= \varphi(a)\int_a^{\xi} f(x)dx + \varphi(b)\int_{\xi}^b f(x)dx.$$

Let φ be monotonically increasing so that, $-\varphi$, is monotonically decreasing.

There exists, therefore, by the preceding, a number ξ between a & b $\ni$:

$$\int_a^b f(x)[-\varphi(x)]dx = -\varphi(a)\int_a^{\xi} f(x)dx - \varphi(b)\int_{\xi}^b f(x)dx.$$

$$\Rightarrow \int_a^b f(x)\varphi(x)dx = \varphi(a)\int_a^{\xi} f(x)dx + \varphi(b)\int_{\xi}^b f(x)dx.$$

Thus we have completely established the second mean value theorem.