Semester : V STATICS Code : 18K5M10

Unit I : Forces acting at a Point – Parallel forces (Chapter 2 & 3 : Section 1 to 6)

Unit II : Moment of a Force about a Point on a Line – Theorem on Moments & Couples (Chapter 3 : Section 7 to 14 & Chapter 14)

Unit III : Equilibrium of three forces acting on a Rigid body – Coplanar forces (Simple Problems only. (Chapter 5 : Section 1 to 7, Chapter 6 : Sec 1 to 13

Unit IV : Equilibrium of strings under gravity – Common Catenary – Parabolic Catenary – Suspension Bridge (Chapter 11)

Unit V : Friction – Laws of Friction – Coefficient of Friction, Angle and Cone of Friction – Equilibrium of a Particle on a rough inclined Plane under a force Parallel to the Plane and under any Force – Problems on Friction (Simple Problems only) (Chapter 7 : Sections 1 to 13)

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Text Book :

M.K Venkataraman, Statics, Agasthiar Publication, 18th Edition 2016.

Reference Book :

(1) S. Narayanan Statics

(2) A.V. Dharmapadham, Statics .

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Introduction :

Mechanics : Mechanics is the Science that deals with the action of forces on bodies. Under the influence of forces, bodies may be either in Motion or at rest.

Statics : That branch of mechanics which is concerned with the Conditions under which bodies remain at rest when acted on by forces is Called statics.

Force : Force as any Cause which Produces or tends to Produce a Change in the existing state of rest of a body or of its uniform motion in a straight line.

A force will be Completely known when we know

(i) its magnitude

(ii) its direction

(iii) its Point of application.

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(iv) the Point of the body at which the force acts.

Since the st. line has both magnitude and direction, a force can be conveniently represented by a st. line through the point of application. Such a st. line representing a force is called Vector. The direction of the force is indicated by $\overline{AB}$ represents a forces acting from A to B.

<u>Types of Forces</u>: The forces on a body with which we concerned in Statics can be classified as follows.

(i) an attraction

(ii) a tension

(iii) a reaction.

<u>Attraction and Repulsion</u>: These are forces acting between two bodies which are not necessarily connected. When the bodies tend to approach each other, the force is called <u>attraction</u> and when they tend to separate out the force is called repulsion.

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Tension or Thrust : When we Push or Pull a body by means of a string or rod, we exert some force on the body through the string or the rod. Such a force is called tension.

Reaction : Forces Produced by direct Contact are called action and reaction. When two bodies are in actual Contact with each other each will exert a force on the other. The force exerted by one of the bodies upon the other may be called action and the force exerted by the Second body on the first is called reaction.

Equilibrium : When a number of forces act on a body and keep it at rest the forces are called equilibrium.

Equilibrium of two forces : If two forces be equal and opposite (ie) if two forces acting on a body ∋ (i) equal magnitude (ii) Same line of action and (iii) opposite direction.

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Then these two forces are in equilibrium.

<u>Principle of the transmissibility of a force</u>

If a force acts at any Point of a rigid body it may be Considered to act at any other Point in its line of action Provided this later Point is rigidly Connected with the body.

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Unit I : Forces acting at a Point

Resultant : If two or more forces F_1, F_2 --- act on a rigid body and if a single force R can be found whose effect on the body is the same as that of all the forces F_1, F_2 --- put together, then the single force R is called **resultant** of the forces F_1, F_2 --- etc.

Note: The forces F_1, F_2 --- are called **Components**

Simple Case of finding the resultant :

(a) **Case (i)** : If two forces P and Q act in the same direction simultaneously on a particle, the resultant is $P+Q$ (ie) acting in the same direction.

Case (ii) : If two forces P and Q act in the opposite direction the resultant is equal to $P-Q$ (ie) acts in the direction of greater force.

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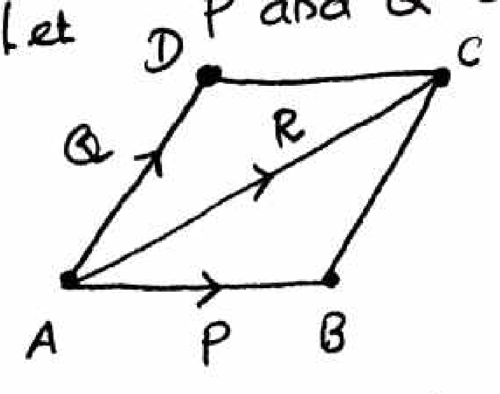
Note: When two forces acting at a Point are in different directions their resultant can be found with the help of fundamental theorem of statics. It is known as Law of the Parallelogram of Forces.

Theorem: Parallelogram of Forces

Statement: If two forces acting at a Point be represented in magnitude and direction by the sides of a Parallelogram drawn from the Point their resultant is represented both in magnitude and direction by the diagonal of the Parallelogram drawn through that Point.

Proof: Let P and Q be the forces.



If the forces P and Q acting at A are represented in magnitude and direction by

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the Straight lines AB and AD and if the Parallelogram BAD is Completed, then the diagonal AC will represent in magnitude and direction the resultant of P and Q.

(ie) $\overline{AB} + \overline{AD} = \overline{AC}$

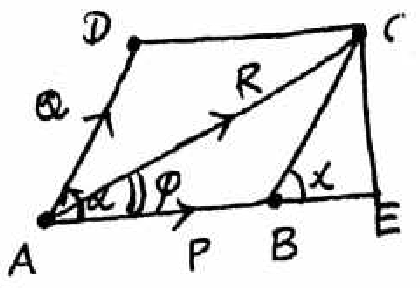
Analytical expression for the resultant of two forces acting at a Point.

Let the two forces P and Q acting at e A be represented by AB and AD.

Let the angle between AB and AD is α.

(ie) $\angle BAD = \alpha$.

Complete the Parallelogram BAD



Then, the diagonal AC is resultant. Say R. be the magnitude of the resultant and make it an angle φ with The force P.

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(i.e) $\angle CAB = \phi$

Draw $CE \perp$ to AB. $BC = AD = Q$

From the right angled triangle BCE

$$\sin\alpha = \frac{CE}{BC} \Rightarrow \sin\alpha = \frac{CE}{Q}$$

$$\therefore CE = Q\sin\alpha \longrightarrow (1)$$

$$\cos\alpha = \frac{BE}{BC} = \frac{BE}{Q}$$

$$\therefore BE = Q\cos\alpha \longrightarrow (2)$$

In Δ ACE

$$AC^2 = AE^2 + CE^2$$

(i.e) $AC^2 = (AB + BE)^2 + CE^2$

(i.e) $R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$ [$\because$ Using eqn. (1) & (2)]

$$= P^2 + 2PQ\cos\alpha + Q^2\cos^2\alpha + Q^2\sin^2\alpha$$

$$= P^2 + Q^2 + 2PQ\cos\alpha$$

$$\therefore R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \rightarrow (3)$$

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Now $\tan\phi = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha} \rightarrow (4)$

Equation (3) gives the magnitude and

Equation (4) gives the direction of resultant.

<u>Result 1:</u> If the forces P and Q are at <u>right angles</u> to each other.

(ie) If $\alpha = 90°$ then $\sin\alpha = \sin 90° = 1$ and

$\cos\alpha = \cos 90° = 0$.

Put $\cos\alpha = 0$ in eqn. (3) and (4) we get

$$R = \sqrt{P^2 + Q^2}$$

$$\tan\phi = Q/P$$

<u>Result 2:</u> If the forces are <u>equal</u>

(ie) If $P = Q$ then the eqn. (3) & (4) is

$$R = \sqrt{P^2 + 2P^2\cos\alpha + P^2} = \sqrt{2P^2(1+\cos\alpha)}$$

$$= \sqrt{2P^2 \cdot 2\cos^2 \alpha/2} = 2P\cos\alpha/2$$

$$R = 2P\cos\alpha/2$$

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and $\tan\phi = \dfrac{P\sin\alpha}{P+P\cos\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha} = \dfrac{2\sin\alpha/2\,\cos\alpha/2}{2\cos^2\alpha/2}$

(ie) $\tan\phi = \tan\alpha/2 \Rightarrow \phi = \alpha/2$

<u>Result 3:</u> Let the magnitudes P and Q of two forces acting at an angle α be given. Then their resultant R is greatest when $\cos\alpha$ is greatest.

(ie) When $\cos\alpha = 1$ then the eqn. (3) is

$R^2 = \sqrt{P^2+2PQ+Q^2} = (P+Q)^2$

(ie) $R = P+Q$ [The forces act along the same line and same direction].

When $\cos\alpha = -1$ then $R^2 = P^2 - 2PQ + Q^2$

$R^2 = (P-Q)^2 \Rightarrow R = P-Q$ [the forces act along the same line but is opposite direction]

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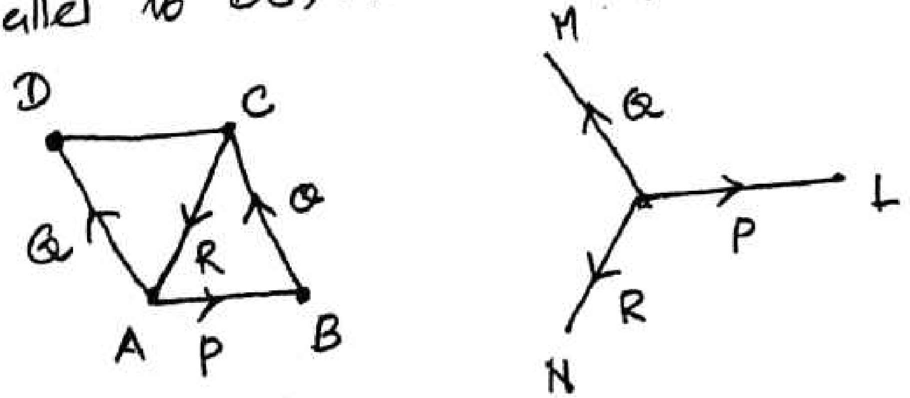
Triangle of Forces:

Theorem: If three forces acting at a Point Can be represented in magnitude and direction by the Sides of a triangle taken in order, They will be in equilibrium.

Proof: Let the forces P, Q, R act at a Point O and be represented in magnitude and direction by the Sides AB, BC and CA of the triangle ABC.
Now we have to Prove that the forces are in equilibrium.

Complete the Parallelogram BADC. As AD is equal and Parallel to BC, AD also represents Q in



magnitude and direction.

$$P+Q = \overline{AB} + \overline{AD} = \overline{AC} \text{ (using Parallelogram Law)}$$

(ie) $P+Q = \overline{AC}$ shows that the resultant of the forces P and Q at O is represented

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in magnitude and direction by AC.

The third force R acts at O and it is represented in magnitude and direction by CA.

Hence $P + Q + R = \overline{AC}$ at $O + \overline{CA}$ at O

(ie) $P + Q + R = 0$ (as the two vectors at O are equal and opposite)

Hence the forces are in equillibrium.

Note: If two forces acting at a Point are Such that represented in magnitude and direction by two sides of a triangle taken in the same order. The resultant will be represented in magnitude and direction by the third side taken in the reverse order.

(ie) $\overline{AB} + \overline{BC} = \overline{AC}$

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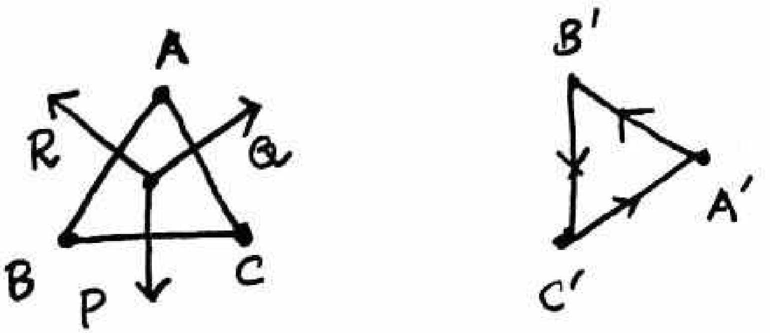
Perpendicular Triangle of Forces

Statement : If three forces acting at a point are such that their magnitudes are proportional to the sides of a triangle and the directions are perpendicular to the corresponding sides, all inwards or all outwards then also the forces will be in equilibrium.

Proof : Let the forces P, Q, R meet at O ABC is a triangle such that magnitudes of P, Q, R are proportional to the sides BC, CA and AB respectively of triangle ABC and their directions are perpendicular to the corresponding sides all outwards.



To prove that the forces P, Q and R in equilibrium.

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If we rotate the ΔABC through 90° in its own Plane We get a new triangle $A'B'C'$ whose sides are Parallel to the given forces and represent the forces both in magnitude and direction. Hence the triangle of forces P, Q and R is equilibrium.

Result: The above result will be true, if the direction of the forces instead of being Perpendicular to the Corresponding sides make equal angles in the Same Sense with them.

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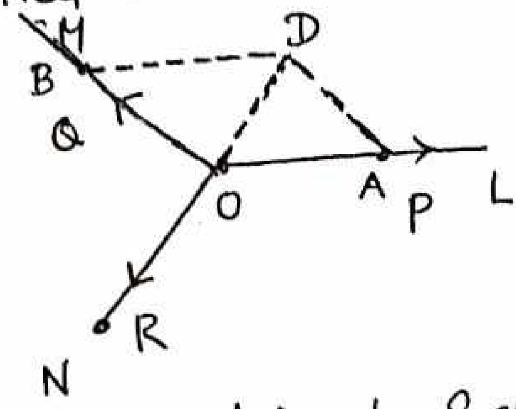
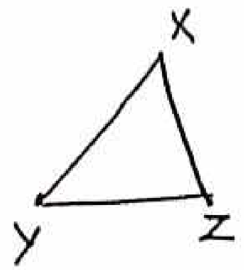
Converse of the Triangle of Forces:

Statement: If three forces acting at a Point are in equilibrium then any triangle drawn so as to have its sides Parallel to the directions of the forces shall represent them in magnitude also.

Proof: Let three forces P, Q and R acting at O along the directions OL, OM and ON respectively and keep it in equilibrium.





XYZ is a triangle such that the sides YZ, ZX and XY are Parallel to the directions of P, Q and R respectively.

T.P.T the sides of a triangle XYZ are Proportional to the magnitudes of P, Q and R.

(ie) $\frac{YZ}{P} = \frac{ZX}{Q} = \frac{XY}{R}$

Given: $P + Q + R = 0$

Along OL cut off OA to represent the magnitude of P on some scale.

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(ie) $\overline{OA} = P$

On the same scale let $\overline{OB} = Q$

To get the resultant of P and Q.

Complete the $\parallel^{gm}$ AOB

$P + Q = \overline{OA} + \overline{OB} = \overline{OD}$

Since $P + Q + R = 0$

(ie) $\overline{OD} + R = 0 \Rightarrow R = \overline{DO}$ shows that the third force R is represented in magnitude on the same scale by DO and DON is a straight line.

Hence the three forces P, Q and R are parallel and proportional to the sides of a Δ OAD.

Now take a Δxyz whose sides are parallel to the directions of P, Q and R respectively it is similar to the ΔOAD.

(ie) $\frac{yz}{OA} = \frac{zx}{AD} = \frac{xy}{DO}$

But $\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{DO}$

$$\therefore \boxed{\frac{yz}{P} = \frac{zx}{Q} = \frac{xy}{R}}$$

Hence the sides of Δxyz will be proportional to P, Q and R.

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Polygon of Forces :

Statement : If any number of forces acting at a Point can be represented in magnitude and direction by the sides of a Polygon taken in order, the forces will be in equilibrium.

Proof : Let the forces $P_1, P_2 -- P_n$ acting at O represented in magnitude and direction by the sides $B_1B_2, B_2B_3 ---$ B_nB_1 of the Polygon $B_1, B_2 -- B_n$.

T.P.T the forces will be in equilibrium.

(ie) $P_1 + P_2 -- + P_n = 0$.

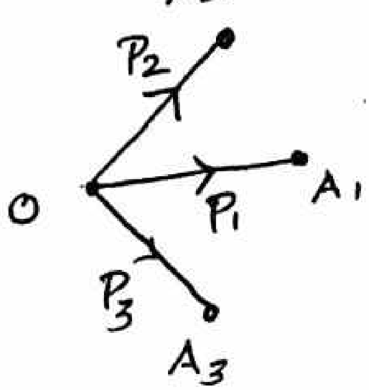
Compounding the forces by the Vector law, We get

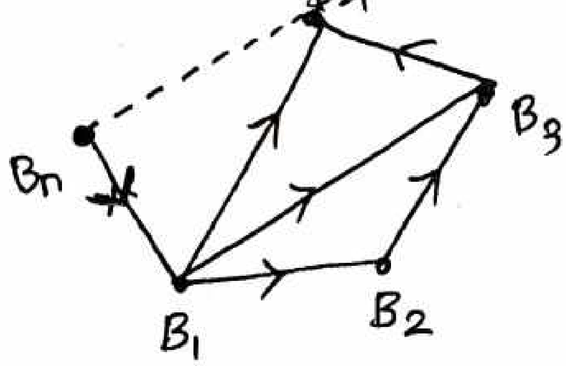
$$P_1 + P_2 = \overline{B_1B_2} + \overline{B_2B_3} = \overline{B_1B_3}$$

$$P_1 + P_2 + P_3 = \overline{B_1B_3} + \overline{B_3B_4} = \overline{B_1B_4}$$

and

$$P_1 + P_2 + -- + P_{n-1} = \overline{B_1B_{n-1}} + \overline{B_{n-1}B_n} = \overline{B_1B_n}$$





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In each of the equations above,

The resultant of the right side, of the forces named on the left side act at the Point O.

The last force is represented by $\overline{B_nB_1}$

$\therefore P_1 + P_2 + \dots + P_{n-1} + P_n = \overline{B_1B_n}$ at O + $\overline{B_nB_1}$ at O

(ie) $\boxed{P_1 + P_2 + \dots + P_n = 0}$

Hence the forces will be in equilibrium.

Note:

① The Polygon of the forces is true even when the forces acting at O are not in the same Plane.

② The Converse of the Polygon of Forces is not true.

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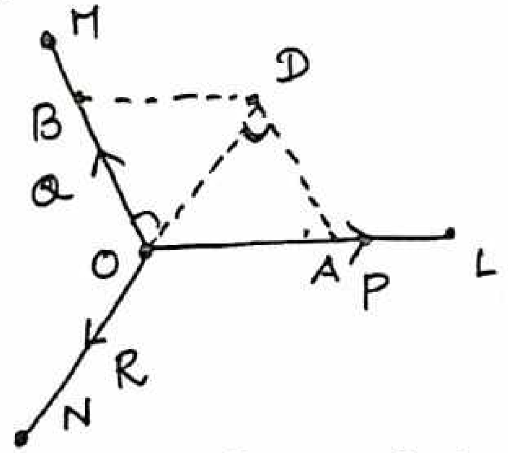
Lami's Theorem :

Statement : If three forces acting at a Point are in equilibrium, each force is Proportional to the sine of the angle between the other two.

Proof :



Let the three forces P, Q, R acting at O along the directions OL, OM and ON respectively to keep it in equilibrium.

T.P.T

$$\frac{P}{\sin(Q,R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$$

Using the sine rule to ΔOAD, we get

$$\frac{OA}{\sin\angle ODA} = \frac{AD}{\sin\angle DOA} = \frac{DO}{\sin\angle OAD} \rightarrow (1)$$

But $\angle ODA$ = alternative angle of $\angle BOD$

$= 180^\circ - \angle MON$

Taking sine on both side, we get.

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$\sin \angle ODA = \sin(180^\circ - \angle MON) = \sin \angle MON \rightarrow (2)$

|||ly $\angle DOA = 180^\circ - \angle LON$

(ie) $\sin \angle DOA = \sin(180^\circ - \angle LON) = \sin \angle LON \rightarrow (3)$

$\angle OAD = 180^\circ - \angle MOL$

(ie) $\sin \angle OAD = \sin(180^\circ - \angle MOL) = \sin \angle MOL \rightarrow (4)$

Substituting (2), (3) and (4) in (1), We get

$$\frac{OA}{\sin \angle MON} = \frac{AD}{\sin \angle LON} = \frac{DO}{\sin \angle MOL}$$

(ie) $$\frac{P}{\sin \angle MON} = \frac{Q}{\sin \angle LON} = \frac{R}{\sin \angle MOL}$$

[∵ OA, AD and DO acting of the forces P, Q and R at O].

$$\therefore \boxed{\frac{P}{\sin(Q,R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}}$$

Hence the theorem is Proved.

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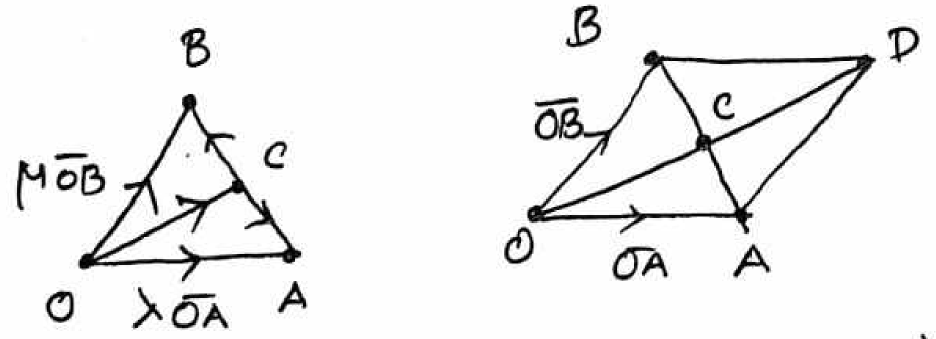
Extended Form of The Parallelogram Law of forces

Statement : If the forces $\lambda\overline{OA}$ and $\mu\overline{OB}$ act at a Point O along the lines OA and OB, their resultant will be the force $(\lambda+\mu)\overline{OC}$ Where C is the Point on AB Such that $\lambda \cdot AC = \mu \cdot CB$.

Proof :

Let the forces represend by λOA and μOB act along the lines OA and OB.

Take the Point C on AB $\ni \lambda \cdot AC = \mu CB$



In ΔO . $\overline{OA} = \overline{OC} + \overline{CA}$; Multiplying λ, We get

$$\therefore \lambda\overline{OA} = \lambda\overline{OC} + \lambda\overline{CA} \longrightarrow (1)$$

In the ΔOCB $\overline{OB} = \overline{OC} + \overline{CB}$

Multiplying μ, We get

$$\mu\overline{OB} = \mu\overline{OC} + \mu\overline{CB} \longrightarrow (2)$$

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$(1) + (2) \Rightarrow \lambda \overline{OA} + \mu \overline{OB} = (\lambda+\mu)\overline{OC} + \lambda \overline{CA} + \mu \overline{CB}$ $\rightarrow (3)$

Since $\lambda AC = \mu CB$

$\therefore$ The forces $\lambda \overline{CA}$ and $\mu \overline{CB}$ are equal and opposite forces acting at c.

(ie) $\lambda \overline{CA} + \mu \overline{CB} = 0 \rightarrow (4)$

Substituting (4) in (3) we get.

$$\boxed{\lambda . \overline{OA} + \mu \overline{OB} = (\lambda+\mu) . \overline{OC}}$$

Hence the theorem is Proved.

Note:

① Take $\lambda = \mu = 1$ and c is the mid Point of AB then the above eqn. gives

$$\boxed{\overline{OA} + \overline{OB} = 2\overline{OC}}$$

(ie) The resultant of two forces completely represented by $\overline{OA}$ and $\overline{OB}$ is $2\overline{OC}$ where c is the midPoint of AB.

② $\overline{OA} + \overline{OB} = \overline{OD}$ where c is the mid Point of the diagonal OD $[\because \overline{OD} = 2\overline{OC}]$

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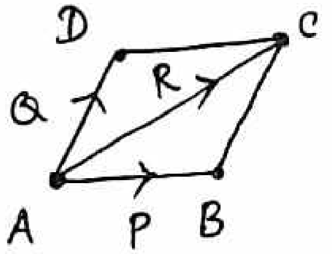
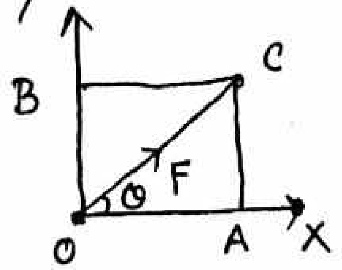
Resolution of a Force: Two forces given in magnitude and direction have only one resultant, for we can construct only one parallelogram when two adjacent sides are given. Conversely, a single force can be resolved into two components in an infinite number of ways since any number of parallelograms can be constructed on a given line AC as diagonal.



If BADC is any one of these, the force AC is equivalent to the two component of the forces AB and AD.

The resolution of a force occurs.
When a given force is to be resolved in two directions at right angles and one of these directions being given.



Let OC be the given force it is represented by F.

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OX be a line inclined at an angle θ to OC.

Let OY is Perpendicular to OX.

Draw $CA \perp OX$ and Complete the Parallelogram OABC

$\therefore$ The force OC is equivalent to two Component forces OA and OB.

$OA = OC \cos\theta \Rightarrow OA = F \cos\theta$

$OB = AC \Rightarrow OB = OC \sin\theta$ [$\because$ In ΔOAC

$\cos\theta = \frac{OA}{OC}$

$\sin\theta = \frac{AC}{OC}$]

When a given force is resolved into two Components in two mutually Perpendicular directions, the Components are referred to as the resolved Parts in the Corresponding directions.

OA is resolved Part of F along OX and

OB is resolved Part of F along OY [$\because$ as shown in fig]

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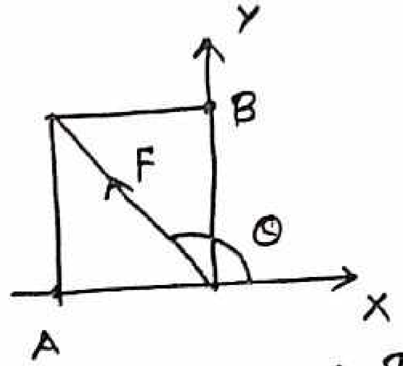
When θ is obtuse and OA is in a direction opposite to OX.



In this case the resolved part of F along OX is negative. (ie) OA = $F\cos\theta$ is negative, as θ is obtuse.

Result : ① A Force F is equivalent to a force $F\cos\theta$ along a line making an angle θ with its own direction together with a force $F\sin\theta$ perpendicular to the direction of the first component

② When $\theta = 0$, $\cos\theta = 1$ ∴ Resolved Part = F

(ie) the resolved part of a force in its own direction is the force itself.

③ When $\theta = 90°$, $\cos 90° = 0$ ∴ Resolved Part = 0

ie) A force has no resolved part in a direction perpendicular to itself.

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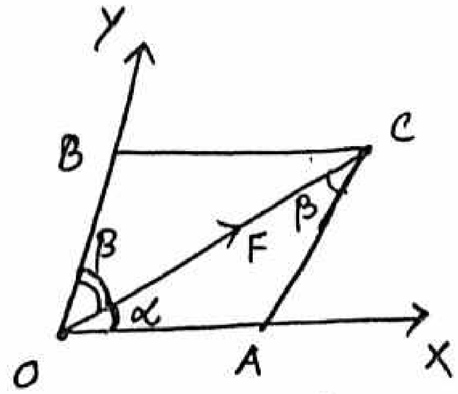
Components of a force along two given directions

Let OC be a given force F and OX, OY be the two lines make an angle α and β with OC.



Draw CA Parallel to OY and CB Parallel to OX we get a Parallelogram OACB.

$\therefore$ OA and OB are Components of the force OC along OX and OY.

From the triangle OAC

Using Lami's Theorem, we get

$$\frac{OA}{\sin\angle OCA} = \frac{AC}{\sin\angle AOC} = \frac{OC}{\sin\angle OAC}$$

$$\text{(i.e.)}\quad \frac{OA}{\sin\beta} = \frac{AC}{\sin\alpha} = \frac{OC}{\sin[180^\circ - (\alpha+\beta)]}$$

$$\text{(i.e.)}\quad \frac{OA}{\sin\beta} = \frac{AC}{\sin\alpha} = \frac{F}{\sin(\alpha+\beta)} \Rightarrow \boxed{OA = \frac{F\sin\beta}{\sin(\alpha+\beta)}\ ;\ OB = AC = \frac{F\sin\alpha}{\sin(\alpha+\beta)}}$$

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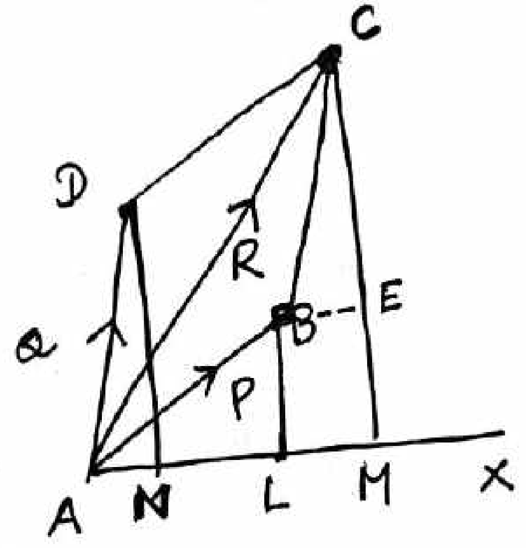
Theorem on Resolved Parts:

Statement: The algebraic sum of the resolved Parts of two forces in any direction is equal to the resolved Part of the resultant in the Same direction.

Proof: Let AB and AD represent the forces P and Q and Ax be the direction in which the forces are to be resolved. Complete the Parallelogram ABCD, the resultant R is represented by AC.



Draw BL, DN and CM Perpendicular to OX. and BE $\perp^r$ to CM.

$\therefore$ AL, AN. and AM are resolved Parts of the forces P, Q and R along AX.

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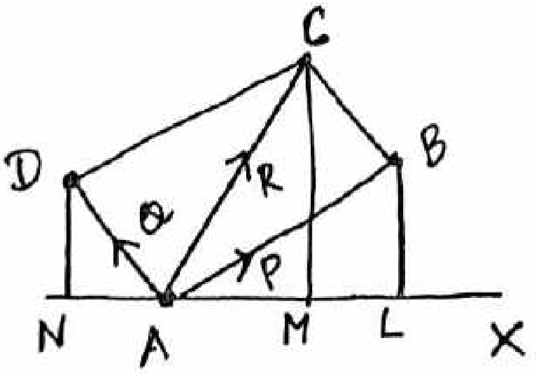

In this fig. AD makes an obtuse angle along AX then the resolved Part of Q is $-AN$

To Prove that $AL \pm AN = AM$

The triangles DAN and CBE are Congruent

$\therefore AN = BE$

$\therefore AL \pm AN = AL \pm BE = AL \pm LM = AM$

(ie) $\boxed{AL \pm AN = AM}$

Obviously.

The above theorem can be extended to the resultant of any number of forces acting at a Point.

Let P_1, P_2 and P_3 are three forces acting at O.

Let R_1 be the resultant of P_1 and P_2

and R_2 be the resultant of R_1 and P_3

Applying the theorem to the two sets of forces

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Resultant of any number of forces acting at a Point in Graphical Method.

Let P, Q, R and S be the forces acting at O.

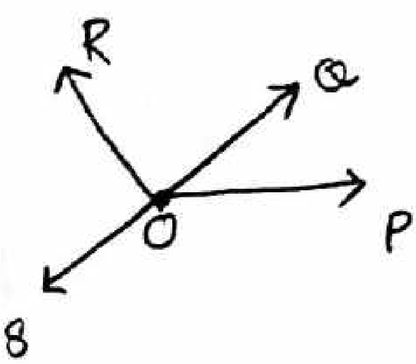
Take a Point A and draw lines AB, BC, CD and DE to represent the forces P, Q, R and S in magnitude and direction.

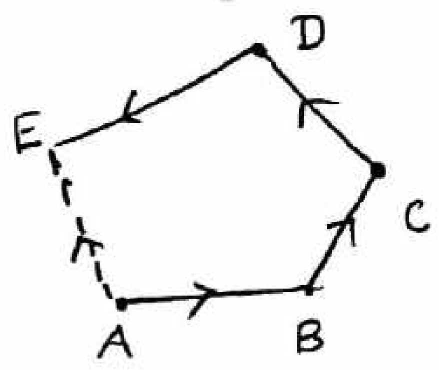
The forces by the vector law we get

$$P+Q = \overline{AB} + \overline{BC} = \overline{AC}$$

$$P+Q+R = \overline{AC} + \overline{CD} = \overline{AD}$$

$$P+Q+R+S = \overline{AD} + \overline{DE} = \overline{AE}$$





Hence the required resultant is represented in magnitude and direction by the line AE.

The same construction can apply for any number of forces.

The figure ABCDE is called Force - Polygon.

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P_1, P_2, R_1 and R_1, P_3, R_2 We get

resolved Part of R_1 along OX = resolved Part of P_1 + resolved Part of P_2 → (1)

and resolved Part of R_2 along OX = resolved Part of R_1 + resolved Part of P_3 → (2)

From the equations (1) & (2), We get

resolved Part of R_2 = resolved Part of P_1 + resolved Part of P_2 + resolved Part of P_3

and so on.

Hence the algebraic sum of the resolved Parts of number of forces in any direction is equal to the resolved Part of the resultant in the same direction.

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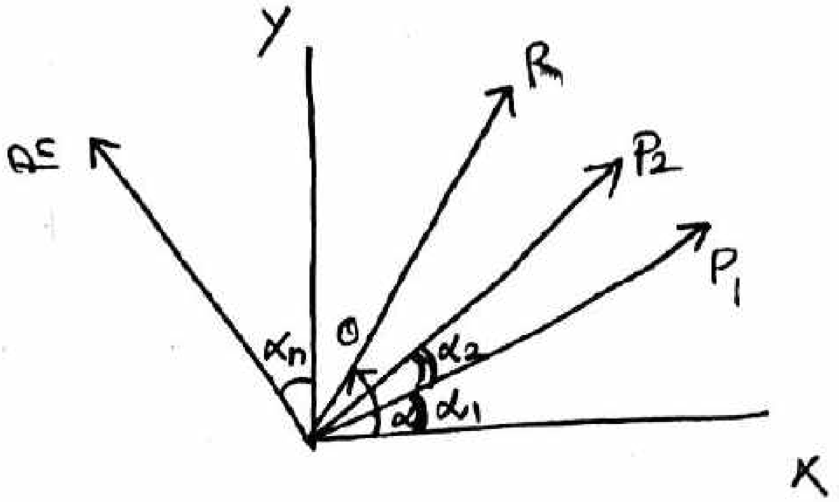
Resultant of any number of Coplanar forces acting at a Point in analytical Method

Let the forces $P_1, P_2 -- P_n$ act at o.

Through o, draw two lines ox and oy are at right angles to each other.

Let the lines of action of forces $P_1, P_2 -- P_n$ make an angles $\alpha_1, \alpha_2 -- \alpha_n$ with ox.

Let R be the resultant inclined at an angle θ to ox.



$\therefore R \cos\theta$ = resolved Part of the resultant along ox

= algebraic Sum of the resolved Parts of $P_1, P_2 -- P_n$ along ox.

$$R \cos\theta = P_1 \cos\theta_1 + -- + P_n \cos\theta_n = X \text{ (Say)} \rightarrow (1)$$

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$R\sin\theta$ = resolved Part of the resultant along oy

= algebraic sum of the resolved Parts of

$P_1, P_2 \ldots P_n$ along oy.

$$R\sin\theta = P_1\sin\theta_1 + P_2\sin\theta_2 + \ldots + P_n\sin\theta_n = y \text{ (say)} \rightarrow (2)$$

Squaring and adding the eqns. (1) & (2), We get

$$R^2(\cos^2\theta + \sin^2\theta) = P_1^2(\cos^2\theta_1 + \sin^2\theta_1) + \ldots + P_n^2(\cos^2\theta_n + \sin^2\theta_n)$$

$$R^2 = x^2 + y^2$$

$$\therefore R = \sqrt{x^2+y^2} \rightarrow (3)$$

$$\frac{(2)}{(1)} \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}(y/x) \rightarrow (4)$$

∴ The analytical expression for the resultant of any number of Coplanar acting at a Point is

$$\boxed{R = \sqrt{x^2+y^2}\ ; \tan\theta = \frac{y}{x}}$$

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Conditions of equilibrium of any number of forces acting upon a Particle

Forces acting at a Point are in equilibrium when their resultant is zero.

(ie) A number of forces acting at a Point of a rigid body or a Particle in order that the body or the Particle may be at rest.

Geometrical or graphical Conditions:

If forces acting at a Point are represented in magnitude and direction by the lines joining the Successive Sides of a Polygon, then for equilibrium, The Polygon must be closed.

Analytical Conditions:

W.K.T $R^2 = x^2 + y^2$

Since the forces are in equilibrium

(ie) $R = 0 \Rightarrow \boxed{x^2 + y^2 = 0}$

The sum of the squares of two real

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Quantity Cannot be Zero unless each Quantity is Separately Zero.

$\therefore$ $\boxed{X=0 \text{ and } Y=0}$

Hence if any number of forces acting at a Point are in Equilibrium, the algebraic Sums of the resolved Parts of the forces in any two Perpendicular direction must be Zero Separately.

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Problems :

(1) The resultant of two forces P, Q acting at a Certain angle is X and that of P, R acting at the Same angle is also X. The resultant of Q, R again acting at the Same angle is Y. Prove that

$$P = (X^2 + QR)^{1/2} = \frac{QR(Q+R)}{Q^2+R^2-Y^2}$$

Prove also that, if $P+Q+R=0$, $Y=X$.

Soln: Let P and Q act at an angle α

Given : The resultant of two forces P, Q acting at a Certain angle is X and that P, R acting at the Same angle is X, Q, R acting at the Same angle is Y

(ie) $X^2 = P^2 + Q^2 + 2PQ\cos\alpha \rightarrow (1)$

$X^2 = P^2 + R^2 + 2PR\cos\alpha \rightarrow (2)$

$Y^2 = Q^2 + R^2 + 2QR\cos\alpha \rightarrow (3)$

(1) − (2) $\Rightarrow$ $0 = Q^2 - R^2 + 2P(Q-R)\cos\alpha$

$0 = (Q-R)(Q+R) + 2P(Q-R)\cos\alpha$

$0 = (Q-R)\left[Q+R+2P\cos\alpha\right]$

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But $Q \neq R$ So $Q - R \neq 0$

$\therefore Q + R + 2P\cos\alpha = 0$

(ie) $\cos\alpha = -\dfrac{Q+R}{2P} \rightarrow (4)$

Now Substituting (4) in (1) we get

$$x^2 = P^2 + Q^2 + 2PQ\left[-\frac{(Q+R)}{2P}\right]$$

$$= P^2 + Q^2 - Q^2 - QR = P^2 - QR$$

(ie) $x^2 + QR = P^2 \Rightarrow \boxed{P = (x^2 + QR)^{1/2}}$

Subs. (4) in (3) we get

$$y^2 = Q^2 + R^2 + 2QR\left[-\frac{(Q+R)}{2P}\right]$$

$$= Q^2 + R^2 - \frac{QR(Q+R)}{P}$$

$\therefore \dfrac{QR(Q+R)}{P} = Q^2 + R^2 - y^2 \Rightarrow \boxed{P = \dfrac{QR(Q+R)}{Q^2 + R^2 - y^2}}$

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If $P+Q+R=0$ then $Q+R=-P$

Put $Q+R=-P$ in eqn. (4) we get

$$\cos\alpha = \frac{P}{2P} = \frac{1}{2}$$

Now Substitute $\cos\alpha = \frac{1}{2}$ in eqn. (2) & (3), we get

$$x^2 = P^2 + R^2 + PR \rightarrow (5)$$

$$y^2 = Q^2 + R^2 + QR \rightarrow (6)$$

$$(5)-(6) \Rightarrow x^2 - y^2 = P^2 - Q^2 + PR - QR$$

$$= (P-Q)(P+Q) + R(P-Q)$$

$$= (P-Q)[P+Q+R]$$

$$= (P-Q)\cdot 0 \quad [\because P+Q+R=0]$$

$$\therefore \quad x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow \boxed{x=y}$$

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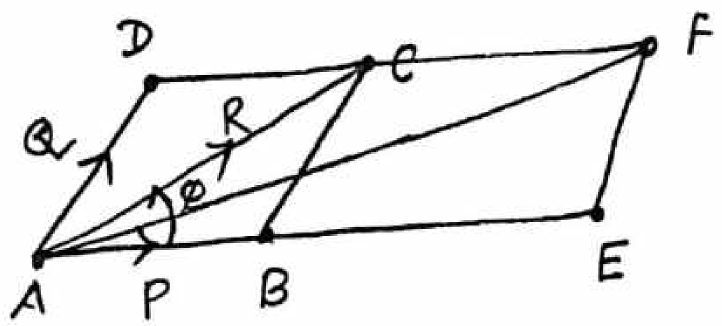
(2) If the resultant R of two forces P and Q inclined to one another at any given angle makes an angle φ with the direction of P. Show that the resultant of forces (P+R) and Q acting at the same angle will make an angle $\varphi/2$ with the direction of P+R.

Soln. Let $\overline{AB} = P$ and $\overline{AD} = Q$



From a Parallelogram ABCD

$$\overline{AB} + \overline{AD} = \overline{AC} = R$$

To mark the force P+R Produce AB to E

So that BE = AC

In the ||gm DAEF

AF gives the new resultant

In Δ CAF

CA = CF (each representing R in magnitude)

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$\therefore \angle CAF = \angle CFA$

$= \angle FAE$ [$\because$ alternate angles are equal]

(ii) AF bisects $\angle CAB$.

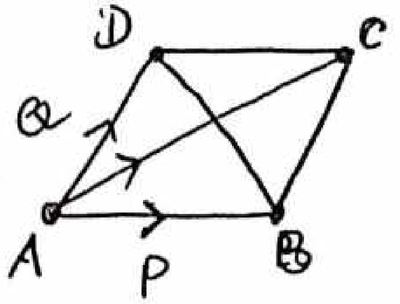
Hence the resultant of forces (P+R) and Q acting at the same angle will make an angle $\theta/2$ with the direction of P+R

③ Two forces act on a Particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.

Soln: Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and AD. Complete the Parallelogram BAD.

Then $P + Q = \overline{AB} + \overline{AD} = \overline{AC}$ [Parallelogram Law]

$\therefore \overline{AC}$ is the sum of the two forces.



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$P - Q = \overline{AB} - \overline{AD}$

$= \overline{AB} + \overline{DA} = \overline{DA} + \overline{AB} = \overline{DB}$

$P - Q = \overline{DB}$ (by triangle Law)

$\therefore \overline{DB}$ is the difference of two forces.

<u>Given</u> : $\overline{AC}$ and $\overline{DB}$ are at right angles.

(ie) In Parallelogram ~~know~~ ABCD, the diagonals AC and BD cut at right angles.

$\therefore$ ABCD is a rhombus.

(ie) $AB = AD \Rightarrow \boxed{P = Q}$

Hence the sum and difference of the forces are at right angles to each other then the forces are of equal magnitude.

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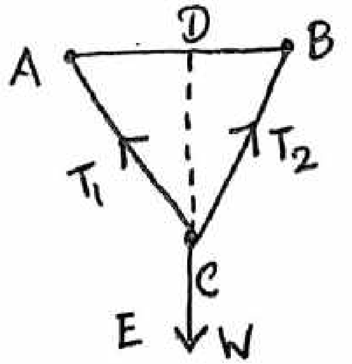
(4) A and B are two fixed Points on a horizontal line at a distance c apart. Two fine light Strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the Strings are in the ratio

$b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$

Soln. Let T_1 and T_2 be the tensions along the Strings CA and CB and W be the Weight of the mass at C acting Vertically downwards along CE.



Produce EC to meet AB at D.

Since C is at rest under the action of the three forces

By Lami's Theorem

$$\frac{T_1}{\sin\angle ECB} = \frac{T_2}{\angle ECA} \rightarrow (1)$$

[∵ If the three forces acting at a Point are in Equilibrium, each forces is Proportional to the Sine of the angle between the other two]

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Now.

$$\sin\angle ECB = \sin(180^\circ - \angle DCB) = \sin\angle DCB$$

$$= \sin(90^\circ - \angle ABC) = \cos\angle ABC$$

III'ly $\sin\angle ECA = \sin(180^\circ - \angle ACD)$

$$= \sin\angle ACD$$

$$= \sin(90^\circ - \angle BAC) = \cos\angle BAC.$$

$\therefore$ Eqn. (1) gives

$$\frac{T_1}{\cos\angle ABC} = \frac{T_2}{\cos\angle BAC} \Rightarrow \frac{T_1}{T_2} = \frac{\cos\angle ABC}{\cos\angle BAC} = \frac{\cos B}{\cos A} \rightarrow (2)$$

In ΔABC W.K.T

$$\cos B = \frac{c^2+a^2-b^2}{2ca} \text{ and } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

Subs/. $\cos A$ and $\cos B$ is eqn. (2) We get

$$\frac{T_1}{T_2} = \frac{c^2+a^2-b^2}{2ca} \cdot \frac{2bc}{b^2+c^2-a^2} = \frac{b(c^2+a^2-b^2)}{a(b^2+c^2-a^2)}$$

(ie) $\boxed{b(a^2+c^2-b^2) = a(b^2+c^2-a^2)}$

Hence it is Proved.

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(5) ABC is a given triangle. Forces P, Q, R acting along the lines OA, OB, OC are in Equilibrium. Prove that

(i) $P : Q : R = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$ if O is the Circumcentre of the triangle.

(ii) $P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$ if O is the incentre of the triangle.

(iii) $P : Q : R = a : b : c$ if O is the ortho Centre of the triangle

(iv) $P : Q : R = OA : OB : OC$ if O is the Centroid of the triangle.

<u>Soln.</u>

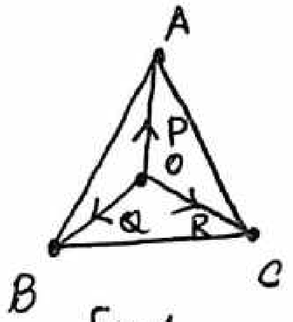

fig 1

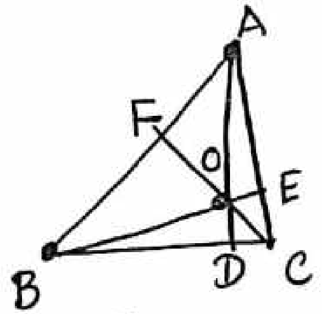

fig 2

By Lami's Theorem, W.K.T

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB} \rightarrow (1)$$

(i) <u>Given</u>: O is the CircumCentre of ΔABC

$\angle BOC = 2\angle BAC$; $\angle COA = 2\angle ABC$; $\angle AOB = 2\angle BCA$

(ie) $\angle BOA = 2A$; $\angle COA = 2B$; $\angle AOB = 2C$

$$(1) \Rightarrow \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

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(ie) $\dfrac{P}{2\sin A\cos A} = \dfrac{Q}{2\sin B\cos B} = \dfrac{R}{2\sin C\cos C} \rightarrow (2)$

But $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $\sin A = \dfrac{2\Delta}{bc}$ Where Δ is the area.

$$\therefore 2\sin A\cos A = 2\cdot\frac{2\Delta}{bc}\cdot\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2\Delta\,(b^2+c^2-a^2)}{b^2c^2}$$

lll'ly $2\sin B\cos B = \dfrac{2\Delta\,(c^2+a^2-b^2)}{c^2a^2}$ and

$$2\sin C\cos C = \frac{2\Delta\,(a^2+b^2-c^2)}{a^2b^2}$$

$$(2) \Rightarrow \frac{Pb^2c^2}{2\Delta(b^2+c^2-a^2)} = \frac{Qc^2a^2}{2\Delta(c^2+a^2-b^2)} = \frac{Ra^2b^2}{2\Delta(a^2+b^2-c^2)}$$

Multiplying by $\dfrac{2\Delta}{a^2b^2c^2}$, We get

$$\boxed{\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}}$$

Hence Condition (1) is Proved.

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(ii) <u>Given</u>: O is the in-Centre of the triangle.

OB and OC are the bisectors of ∠B and ∠C.

$$\therefore \angle BOC = 180^\circ - B/2 - C/2 = 180^\circ - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180^\circ - \left(90^\circ - \frac{A}{2}\right) = 90^\circ + \frac{A}{2}$$

III'ly $\angle COA = 90^\circ + B/2$. and $\angle AOB = 90^\circ + C/2$

eqn (1) ⇒ $$\frac{P}{\sin\left(90^\circ + \frac{A}{2}\right)} = \frac{Q}{\sin(90^\circ + B/2)} = \frac{R}{\sin(90^\circ + C/2)}$$

(ie) $$\boxed{\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}}$$

Hence Condition (ii) is Proved.

(iii) <u>Given</u>: O is the Ortho Centre of the triangle.

In fig 2 AD, BE, CF are the altitudes.

Quadrilateral AFOE is Cyclic.

Since $\angle AFO + \angle AEO = 90^\circ + 90^\circ = 180^\circ$

$\therefore \angle FOE + A = 180^\circ \Rightarrow \angle FOE = 180^\circ - A$

$\angle BOC$ = Vertically opposite $\angle FOE$

$\angle BOC = 180^\circ - A$

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III'ly $\angle COA = 180^\circ - B$ and $\angle AOB = 180^\circ - C$

Hence eqn. (i) gives.

$$\frac{P}{\sin(180^\circ - A)} = \frac{Q}{\sin(180^\circ - B)} = \frac{R}{\sin(180^\circ - C)}$$

(ie) $\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$

(ie) $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$ $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right]$

Hence $\boxed{P : Q : R = a : b : c}$

$\therefore$ The Condition (iii) is Proved.

(iv) Given: O is the Centroid of the triangle.

W.K.T $\Delta BOC = \Delta COA = \Delta AOB$ and each is equal to $\frac{1}{3}\Delta ABC$

$$\Delta BOC = \frac{1}{2} \cdot OB \cdot OC \cdot \sin\angle BOC = \frac{1}{3}\Delta ABC$$

$$\therefore \sin\angle BOC = \frac{2\Delta ABC}{3 \cdot OB \cdot OC}$$

III'ly $\sin\angle COA = \frac{2\Delta ABC}{3 \cdot OC \cdot OA}$ and $\sin\angle AOB = \frac{2\Delta ABC}{3\, OA\, OB}$

$\therefore$ Eqn. (i) gives

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$$\frac{P \cdot 3\, OB\, OC}{2\, \Delta ABC} = \frac{Q \cdot 3\, OC \cdot OA}{2\, \Delta ABC} = \frac{R \cdot 3\, OA\, OB}{2\, \Delta ABC}$$

(i.e) $P\, OB\, OC = Q\, OC\, OA = R\, OA\, OB$

÷ by $OA \cdot OB \cdot OC$ we get

$$\boxed{\frac{P}{OA} = \frac{Q}{OB} = \frac{R}{OC}}$$

Hence Condition (iv) is Proved.

⑥ Weights W, w, W are attached to Points B, C, D respectively of a light String AE, Where B, C, D divide the String into 4 equal lengths. If the String hangs in the form of 4 Consecutive Sides of a regular Octagon with the ends A and E attached to Points on the Same level Prove that $W = (\sqrt{2}+1)w$

Soln:

Let ABCDE is a Part of a regular Octagon.

W.K.T each angle of a regular Polygon of n Sides

$$= \left(\frac{2n-4}{n}\right) \times 90^\circ$$

$$= \left(\frac{2 \cdot 8 - 4}{8}\right) \times 90^\circ \quad [\because n = 8]$$

$$= \frac{12}{8} \times 90^\circ = 135^\circ$$

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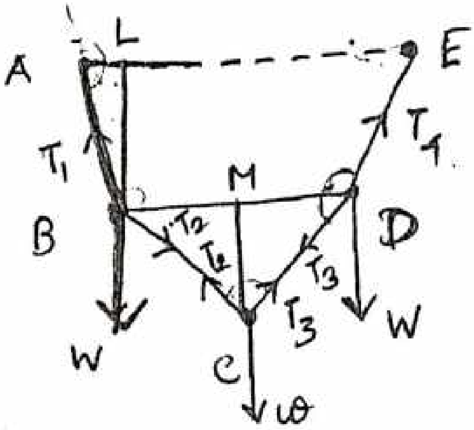

Let the tensions in the Portions AB, BC, CD, DE be T_1, T_2, T_3 and T_4 respectively.

The String BC Pulls B towards C and Pulls C towards B.

the tension being the Same throughout its length.

In ΔBCD $\angle BCD = 135^\circ$

$\therefore \angle CBD = \angle CDB = 45^\circ = 22\frac{1}{2}^\circ$

$\therefore \angle ABD = \angle ABC - \angle CBD = 135^\circ - 22\frac{1}{2}^\circ = 112\frac{1}{2}^\circ$

W.K.T every regular Polygon is Cyclic.

$\therefore$ A, B, C, D and E lie on the same Circle.

$\angle EAB = 180^\circ - \angle BDE = 180^\circ - \{\angle CDE - \angle BDC\}$

$= 180^\circ - \{135^\circ - 22\frac{1}{2}^\circ\} = 180^\circ - 135 + 22\frac{1}{2}^\circ = 45^\circ + 22\frac{1}{2}^\circ$

$\angle EAB = 67\frac{1}{2}^\circ$

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$\therefore \angle EAB + \angle ABD = 67\frac{1}{2}^{\circ} + 112\frac{1}{2}^{\circ} = 180^{\circ}$

$\therefore AE \parallel BD$ and BD is horizontal.

Let the Vertical line through B meet AE at L

and the Vertical line through C meet BD at M.

Using Lami's theorem for the three forces at B, we get

$$\frac{W}{\sin\angle ABC} = \frac{T_2}{\sin(180^{\circ} - \angle ABL)}$$

$$\frac{W}{\sin\angle ABC} = \frac{T_2}{\sin(180^{\circ} - 22\frac{1}{2}^{\circ})}$$

[$\because \Delta ABL$
$\angle A = 67\frac{1}{2}^{\circ}$; $\angle L = 90^{\circ}$
$\angle B = 90^{\circ} - 67\frac{1}{2}^{\circ} = 22\frac{1}{2}^{\circ}$]

$$\frac{W}{\sin 135^{\circ}} = \frac{T_2}{\sin 22\frac{1}{2}^{\circ}}$$

$$\boxed{T_2 = \frac{W\sin 22\frac{1}{2}^{\circ}}{\sin 35^{\circ}}} \rightarrow (1)$$

Using Lami's theorem for three forces at C, we get

$$\frac{W}{\sin\angle BCD} = \frac{T_2}{\sin(180^{\circ} - \angle MCD)}$$

[$\because$ In ΔMCD
$\angle M = 90^{\circ}$; $\angle D = 22\frac{1}{2}^{\circ}$
$\angle C = 90^{\circ} - 22\frac{1}{2}^{\circ}$]

$$\frac{W}{\sin\angle BCD} = \frac{T_2}{\sin(90^{\circ} - 22\frac{1}{2}^{\circ})}$$

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$$\frac{w}{\sin 135^\circ} = \frac{T_2}{\cos 22\tfrac{1}{2}^\circ}$$

(ie) $$\boxed{T_2 = \frac{w \cos 22\tfrac{1}{2}^\circ}{\sin 35}} \rightarrow (2)$$

From (1) & (2), We get

$$\frac{W}{\sin 135^\circ} \sin 22\tfrac{1}{2}^\circ = \frac{w}{\sin 135^\circ} \cos 22\tfrac{1}{2}^\circ$$

$$\frac{w}{W} = \tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1$$

$$\therefore W = \frac{w}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{w(\sqrt{2}+1)}{(\sqrt{2})^2 - 1}$$

Hence $$\boxed{W = w(\sqrt{2}+1)}$$

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⑦ A Weight is Supported on a Smooth Plane of inclination α by a String inclined to the horizon at an angle θ. If the Slope of the Plane be increased to β and the Slope of the String unaltered, the tension of the string is doubled. Prove that $\cot\alpha - 2\cot\beta = \tan\theta$

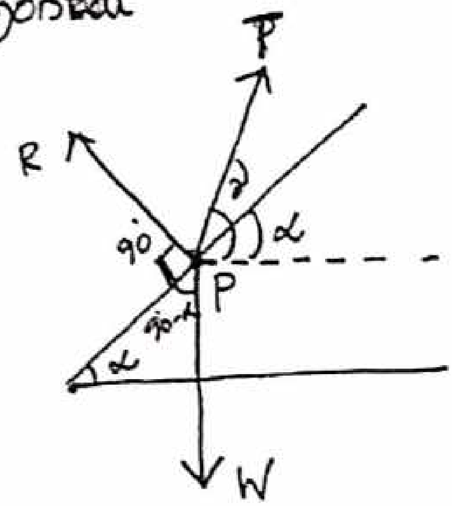
Solution: Let P is the Position of the Weight.

The forces acting at P are

(i) its Weight W downwards

(ii) the normal reaction R Perpendicular to inclined Plane

(iii) The tension T along the string at an angle θ to the horizontal



Using Lami's theorem. for the three forces acting at P.

$$\frac{T}{\sin(180^\circ-\alpha)} = \frac{W}{\sin\left[90^\circ-(\theta-\alpha)\right]}$$

$$\frac{T}{\sin\alpha} = \frac{W}{\cos(\theta-\alpha)} \Rightarrow T = \frac{W\sin\alpha}{\cos(\theta-\alpha)} \longrightarrow (1)$$

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In the Second Case the inclination of the Plane is β

There is no change in θ and T_1 is the string, we get

$$T_1 = \frac{W \sin \beta}{\cos(\theta - \beta)} \longrightarrow (2)$$

Since $T_1 = 2T$

$$(1) \quad \frac{W \sin \beta}{\cos(\theta - \beta)} = 2 \cdot \frac{W \sin \alpha}{\cos(\theta - \alpha)}$$

$$\Rightarrow \sin\beta \cdot \cos(\theta - \alpha) = 2 \sin\alpha \cos(\theta - \beta)$$

$$\Rightarrow \sin\beta \left[\cos\theta \cos\alpha + \sin\theta \sin\alpha\right] = 2 \sin\alpha \left[\cos\theta \cos\beta + \sin\theta \sin\beta\right]$$

$$\Rightarrow \sin\beta \cos\theta \cos\alpha + \sin\alpha \cdot \sin\beta \sin\theta = 2 \sin\alpha \cdot \cos\theta \cos\beta + 2 \sin\alpha \sin\beta \sin\theta$$

$$\Rightarrow \sin\beta \cdot \cos\alpha \cdot \cos\theta = 2 \sin\alpha \cdot \cos\theta \cdot \cos\beta + \sin\alpha \cdot \sin\beta \sin\theta$$

$$= \sin\alpha \left[2 \cos\theta \cos\beta + \sin\beta \sin\theta\right]$$

$$\Rightarrow \frac{\cos\alpha}{\sin\alpha} = \frac{2 \cos\theta \cos\beta + \sin\beta \sin\theta}{\sin\beta \cdot \cos\theta}$$

$$\Rightarrow \cot\alpha = 2 \cot\beta + \tan\theta$$

$$\Rightarrow \boxed{\tan\theta = \cot\alpha - 2\cot\beta}$$

Hence the result is Proved.

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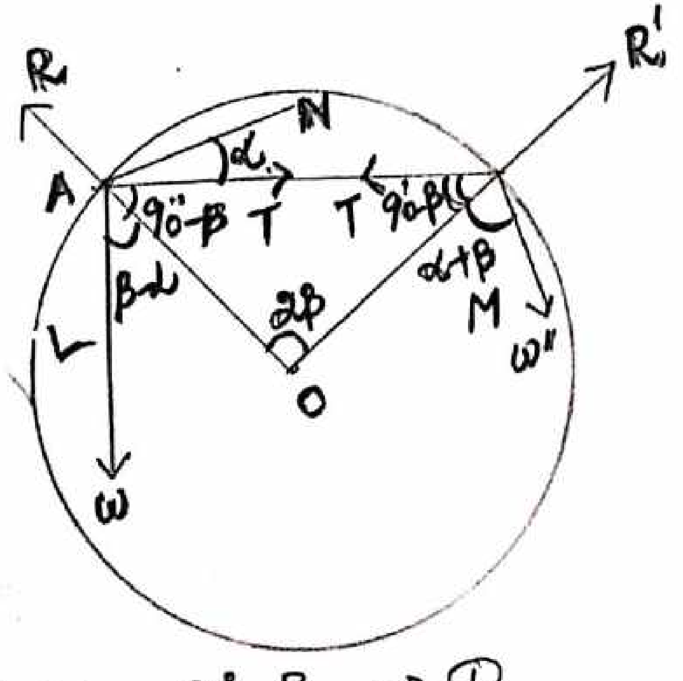
(8) Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of the circle when the beads are in equilibrium on the upper half of the wire. Prove that the inclination of the string to the horizontal is $\tan\alpha = \frac{w-w'}{w+w'}\tan\beta$ (10)

<u>Soln</u>:- Let A and B be the beads of weights w and w' connected by a light string and sliding on a circular wire.

In equilibrium position, $\angle AOB = 2\beta$, O being the centre of the circle.



$\therefore \angle OAB = \angle OBA = 90° - \beta \rightarrow (1)$

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Let AB make an angle α with the horizontal AN.
AL and BM are the vertical lines through A and B

$$\angle OAL = 90^\circ - \angle OAN$$
$$= 90^\circ - [\angle OAB + \angle NAB]$$
$$= 90^\circ - [90^\circ - \beta + \alpha] = \beta - \alpha$$

$$\therefore \angle OAL = \beta - \alpha . \rightarrow (2)$$

Since AL ∥ BM, $\angle ABM + \angle BAL = 180^\circ$

$$\therefore \angle ABM = 180^\circ - \angle BAL = 180^\circ - (90^\circ - \beta + \beta - \alpha)$$
$$= 180^\circ - (90^\circ - \alpha)$$
$$\angle ABM = 90^\circ + \alpha \rightarrow (3)$$

$$\therefore \angle OBM = \angle ABM - \angle ABO$$
$$= 90^\circ + \alpha - (90^\circ - \beta) \quad [\because \text{Using } (1) \,\&\, (3)]$$

$$\angle OBM = \alpha + \beta \rightarrow (4)$$

The forces acting on the bead W at A are

(i) Weight W acting vertically downward along AL.
(ii) Normal reaction R due to contact with the wire along the radius OA outwards.
(iii) tension T in the string along AB.

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III'y The forces acting on the bead ω' at B are

(i) Weight ω' acting Vertically downward along BM.

(ii) Normal reaction R' along the radius OB outwards.

(iii) tension T in the string along BA.

Using Lami's theorem for the three forces acting at A.

$$\frac{\omega}{\sin[180^\circ-(90-\beta)]} = \frac{T}{\sin[180^\circ-(\beta-\alpha)]}$$

(ie) $\dfrac{\omega}{\sin(90-\beta)} = \dfrac{T}{\sin(\beta-\alpha)} \Rightarrow \dfrac{\omega}{\cos\beta} = \dfrac{T}{\sin(\beta-\alpha)} \longrightarrow (1)$

III'y For the three forces at B

$$\frac{\omega'}{\sin[180^\circ-(90-\beta)]} = \frac{T}{\sin[180^\circ-(\beta+\alpha)]}$$

(ie) $\dfrac{\omega'}{\cos\beta} = \dfrac{T}{\sin(\beta+\alpha)} \rightarrow (2)$

From (1) & (2) we get

$$\frac{\omega'}{\omega} = \frac{\sin(\beta-\alpha)}{\sin(\beta+\alpha)}$$

$$\frac{\omega'+\omega}{\omega-\omega'} = \frac{\sin(\beta-\alpha)+\sin(\beta+\alpha)}{\sin(\beta+\alpha)-\sin(\beta-\alpha)} = \frac{2\cos\alpha\sin\beta}{2\sin\alpha\cos\beta}$$

$\left[\because \begin{array}{l}\sin(A+B)-\sin(A-B)=2\cos A\sin B \\ \sin(A+B)+\sin(A-B)=2\sin A\cos B\end{array}\right]$

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$$\frac{\omega'+\omega}{\omega-\omega'} = \frac{\cos\alpha \sin\beta}{\sin\alpha \cos\beta} = \frac{\tan\beta}{\tan\alpha}$$

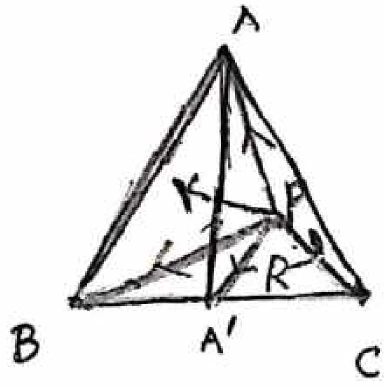
$$\therefore \boxed{\tan\alpha = \left[\frac{\omega-\omega'}{\omega+\omega'}\right] . \tan\beta}$$

Hence the result is Proved.

(9) ABc is a triangle. G is its Centroid and P is any Point in the Plane of the triangle. Show that the resultant of forces represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ is $3\overline{PG}$ and find the Position of P, if the three forces are in equilibrium.

Soln: let A′ be the mid Point of C.

Then $\overline{PB} + \overline{PC} = 2PA'$ [∵ In Δ OAB → $\overline{OA} + \overline{OB} = 2\overline{OC}$ Where C is the mid Pt. of AB]



$$\therefore \overline{PA} + \overline{PB} + \overline{PC} = \overline{PA} + 2\overline{PA'}$$

$$= 1 . \overline{PA} + 2 . \overline{PA'}$$

$$= (1+2)\, \overline{PK} \longrightarrow (1)$$

Where K is the Point on AA′ ∋ 1 · AK = 2 KA′ [∵ using extended form of II^{nd} Law of forces]

(ie) $\dfrac{AK}{KA'} = \dfrac{2}{1}$

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(ie) K divides the median AA' in the ratio 2 : 1

∴ K is same as the centroid of the triangle.

$$(1) \Rightarrow \boxed{\overline{PA} + \overline{PB} + \overline{PC} = 3\,\overline{PG}} \rightarrow (2)$$

Hence the condition is proved.

Since the three forces are in equilibrium.

(ie) $\overline{PA} + \overline{PB} + \overline{PC} = 0$ in (2), we get

$$\overline{PG} = \overline{0} \Rightarrow \boxed{PG = 0}$$

(ie) P must be taken at the centroid G of the triangle.

Note: If G is a centroid of a ΔABC forces are GA, GB, GC respectively it is an equilibrium then $\overline{GA} + \overline{GB} + \overline{GC} = \overline{0}$

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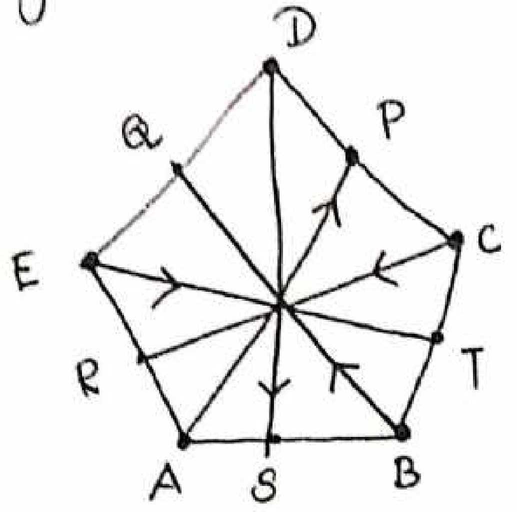
(10) Five forces acting at a Point are represented in magnitude and direction by the lines joining the Vertices of any Pentagon to the midpoints of their opposite sides. Show that they are in equilibrium.

Sol: Let ABCDE is a Pentagon and P, Q, R, S and T are mid Points of the sides CD, DE, EA, AB and BC respectively.



T.P.T the five forces will be in equilibrium.

(i.e) $\overline{AP} + \overline{BQ} + \overline{CR} + \overline{DS} + \overline{ET} = \overline{0}$

W.K.T In $\Delta OAB \Rightarrow \overline{OA} + \overline{OB} = 2\overline{OC}$ Where C is the midPoint of AB.

(ie) $\overline{AD} + \overline{AC} = 2\overline{AP} \Rightarrow \overline{AP} = \frac{1}{2}\left[\overline{AD} + \overline{AC}\right] \rightarrow (1)$

III'y $\overline{BQ} = \frac{1}{2}\left[\overline{BD} + \overline{BE}\right] \rightarrow (2)$

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$$\overline{CR} = \frac{1}{2}\left[\overline{CE} + \overline{CA}\right] \rightarrow (3)$$

$$\overline{DS} = \frac{1}{2}\left[\overline{DA} + \overline{DB}\right] \rightarrow (4)$$

and $$\overline{ET} = \frac{1}{2}\left[\overline{EB} + \overline{EC}\right] \rightarrow (5)$$

Adding these eqns/. from (1) to (5) we get

$$\overline{AP} + \overline{BQ} + \overline{CR} + \overline{DS} + \overline{ET} = \frac{1}{2}\left[\overline{AD} + \overline{AC} + \overline{BD} + \overline{BE} + \overline{CE} + \overline{CA} + \overline{DA} + \overline{DB} + \overline{EB} + \overline{EC}\right]$$

$$= \frac{1}{2} \times \overline{0} \quad [\because \text{the Vectors are equal and opposite}]$$

$$= \overline{0}$$

Hence $$\boxed{\overline{AP} + \overline{BQ} + \overline{CR} + \overline{DS} + \overline{ET} = \overline{0}}$$

(11) OA, OB, OC are the lines of action of two forces P and Q and their resultant R respectively. Any transversal meets the lines in L, M and N respectively

Prove that $\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$

Soln/ Let $\overline{OA} = P$ and $\overline{OB} = Q$

Complete the Parallelogram AOB. $\overline{OC} = R$

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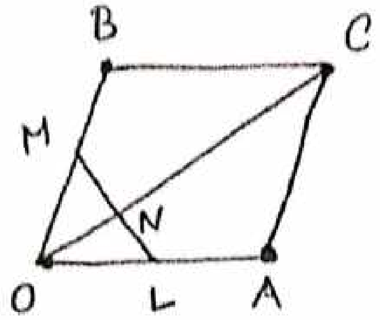

Let $\frac{OA}{OL} = \lambda$ and $\frac{OB}{OM} = \mu$

$\therefore$ $OA = \lambda OL$ and $OB = \mu OM$

(ie) $\overline{OA} + \overline{OB} = \lambda \overline{OL} + \mu \overline{OM}$

$\overline{OA} + \overline{OB} = (\lambda + \mu)\overline{OK} \rightarrow (1)$ where K is the mid Point of LM.

W.K.T $\overline{OA} + \overline{OB} = \overline{OC} \rightarrow (2)$

From (1) & (2)

$(\lambda+\mu)\overline{OK}$ and $\overline{OC}$ must be the same.

(ie) K is a Point on OC.

$\therefore$ K is a intersection of OC and LM it is same as N.

$(\lambda+\mu)\overline{OK} = \overline{OC}$

(ie) $(\lambda+\mu) ON = OC \Rightarrow \lambda + \mu = \frac{OC}{ON}$

$\therefore \frac{OA}{OL} + \frac{OB}{OM} = \frac{OC}{ON} \Rightarrow \boxed{\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}}$

Hence the Condition is Proved.

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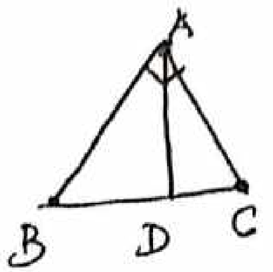
(12) ABC is a triangle, With a right angle at A: AD is the Perpendicular on BC. Prove that the resultant of the forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC is $\frac{1}{AD}$ acting along AD.

Soln.

W.K.T

$$\left.\begin{aligned} AB^2 &= BC \cdot BD \\ AC^2 &= BC \cdot CD \\ AD^2 &= BD \cdot DC \end{aligned}\right\} \rightarrow (1)$$



The forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC.

It is Considered as the forces $\frac{1}{AB^2} \cdot AB$ acting along AB and $\frac{1}{AC^2} \cdot AC$ acting along AC.

Let $\lambda = \frac{1}{AB^2}$ and $\mu = \frac{1}{AC^2}$

(ie) $\lambda AB^2 = \mu AC^2$

$\lambda [BC \cdot BD] = \mu [BC \cdot CD]$ [$\because$ Using eqn. (1)]

$$\boxed{\lambda BD = \mu CD}$$

Hence the resultant of forces $\lambda \overline{AB}$ and $\mu \cdot \overline{AC}$ is $(\lambda + \mu) \cdot \overline{AD}$

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(ii) The resultant of forces $\frac{1}{AB}$ along AB and $\frac{1}{AC}$ along AC is the force $(\lambda+\mu)\,AD$ acting along AD.

Magnitude of the resultant

$$= (\lambda+\mu)\,AD$$

$$= \left(\frac{1}{AB^2}+\frac{1}{AC^2}\right)AD$$

$$= \left[\frac{AC^2+AB^2}{AB^2 \cdot AC^2}\right] \cdot AD$$

$$= \frac{BC^2}{AB^2 \cdot AC^2} \cdot AD = \frac{BC^2}{BC \cdot BD \cdot BC \cdot CD} \cdot AD \quad [\text{Using eqn. (1)}]$$

$$= \frac{AD}{BD \cdot CD} = \frac{1}{AD^2} \cdot AD = \frac{1}{AD}$$

Hence the forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC is $\frac{1}{AD}$ acting along AD.

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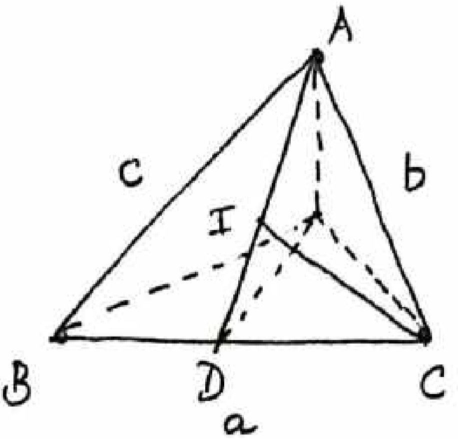
(13) P is a Point in the Plane of the triangle ABC and I is the incentre. Show that the resultant of forces represented by PA Sin A, PB Sin B and PC Sin C is $4PI \cdot \cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}$

Soln.



Let AD be the internal bisector of ∠A and I is the incentre.

W.K.T $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} = \frac{\sin C}{\sin B}$

$\therefore BD \sin B = DC \sin C \rightarrow (1)$ $\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right]$

$\Rightarrow \frac{\sin C}{\sin B} = \frac{c}{b}$

Also,

BI bisects ∠B

(ie) $\frac{AI}{ID} = \frac{AB}{BD}$

CI bisects ∠C

(ie) $\frac{AI}{ID} = \frac{AC}{CD}$

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$$\therefore \frac{AI}{ID} = \frac{AB}{BD} = \frac{AC}{CD}$$

$$\therefore \frac{AI}{ID} = \frac{AB+AC}{BD+CD} = \frac{AB+AC}{BC} = \frac{c+b}{a} = \frac{\sin B + \sin C}{\sin A}$$

$$AI \sin A = (\sin B + \sin C) \cdot ID \qquad \left[\frac{c}{a} = \frac{\sin C}{\sin A} \; ; \; \frac{b}{a} = \frac{\sin B}{\sin A} \right]$$

Now

$$\overline{PB} \sin B + \overline{PC} \sin C = (\sin B + \sin C) \cdot \overline{PD} \rightarrow (3)$$

$$\therefore \overline{PA} \sin A + \overline{PB} \sin B + \overline{PC} \sin C = \overline{PA} \sin A + (\sin B + \sin C) \cdot \overline{PD}$$

(Using eqn. (3))

$$= [\sin A + \sin B + \sin C] \, \overline{PI} \longrightarrow (4)$$

W.K.T

$$\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2} \rightarrow (5)$$

Substituting the eqn. (5) in (4) we get

The resultant is

$$\boxed{4 \cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2} \cdot \overline{PI}}$$

Hence the Condition is Proved.

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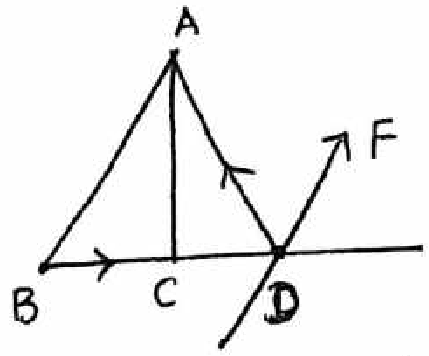
(14) Show that a given force may be resolved into three Components acting in three given lines which are not all Parallel or all Concurrent.

<u>Soln.</u> Let the three lines form a ΔABC and let the given force F meet the side BC in D.

$\therefore$ F Can be resolved into two Components acting along BC and DA respectively.

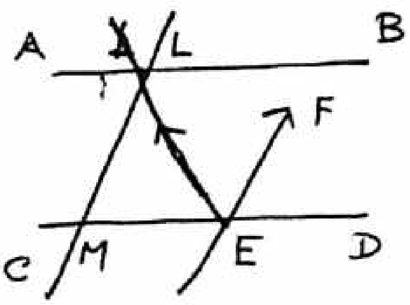


The Component along DA Can be resolved into two Components along AB and AC respectively.

Suppose two lines AB and CD are Parallel and LM is the third line.

Let the given force F meet CD at E.

F Can be resolved into two Components along CD and EL



The Component along EL Can be resolved into two Components along BA and ML respectively

Hence it is Proved.

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(15) ABCD is a quadrilateral and forces acting at a point are represented in direction and magnitude by BA, BC, CD and DA. Find their resultant.

Soln.

W.K.T $\overline{BC} + \overline{CD} + \overline{DA} = \overline{BA}$

$\therefore \overline{BA} + (\overline{BC} + \overline{CD} + \overline{DA}) = \overline{BA} + \overline{BA} = 2\overline{BA}$

Hence the resultant is 2BA, both in magnitude & direction.

(16) ABCDEF is a regular hexagon and at A, act forces represented by $\overline{AB}$, $2\overline{AC}$, $3\overline{AD}$, $4\overline{AE}$ and $5\overline{AF}$. Show that the magnitude of the resultant is $AB\sqrt{351}$ and it makes an angle $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ with AB.

Soln. Let 'a' be the side of the hexagon.

Each interior angle of a regular hexagon is

$= \frac{2n-4}{n} \times 90°$ where $n = 6$

$= \frac{12-4}{6} \times 90° = \frac{8}{6_3} \times 90° = 120°$

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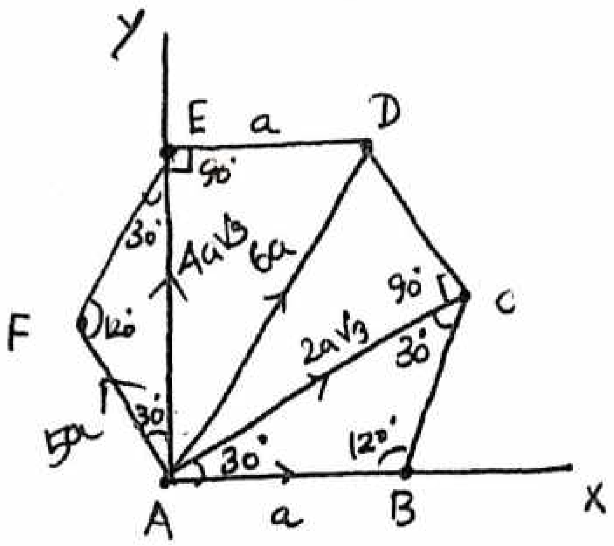

$\therefore \angle CAB = \angle ACB = 30^\circ$ From the isoceles ΔABC.

$\angle FAE = \angle FEA = 30^\circ$ $\quad AC = 2AB\cos 30^\circ = 2a\frac{\sqrt{3}}{2}$

$\boxed{AC = a\sqrt{3} = AE} \rightarrow ①$

$\angle AED = 90^\circ$ $\quad \therefore AD^2 = AE^2 + ED^2 = (a\sqrt{3})^2 + a^2 = 3a^2 + a^2 = 4a^2$

[$\because$ using eqn.(1)]

$\therefore \boxed{AD = 2a} \rightarrow ②$

Since the Vertices of a regular hexagon lie on a Circle.

$\angle DAB + \angle DCB = 180^\circ \Rightarrow \angle DAB = 180^\circ - \angle DCB$

$\angle DAB = 180^\circ - 120^\circ = 60^\circ$ [$\because \angle DCB = 120^\circ$]

$\therefore \angle DAC = 30^\circ$ and $\angle EAD = 30^\circ$

Since the magnitude of the forces acting at A are

$a,\ 2a\sqrt{3},\ 3(2a),\ 4(a\sqrt{3}),\ 5(a)$ ($\because$ using (1) & (2))

(ie) $a, 2a\sqrt{3}, 6a, 4a\sqrt{3}$ and $5a$ (as shown in the fig)

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Take AB and AE as axes of x and y and R be the resultant inclined at an angle θ to AB.

Resolving the forces along AB and AE, we get

$$R\cos\theta = a + 2a\sqrt{3}\cos 30^\circ + 6a\cos 60^\circ + 4a\sqrt{3}\cos 90^\circ + 5a\cos 120^\circ$$

$$= a + 2a\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) + 6a\cdot\left(\frac{1}{2}\right) + 4a\sqrt{3}(0) + 5a\cos(90^\circ+30^\circ)$$

$$= a + 3a + 3a + 0 - 5a\sin 30^\circ \quad [\cos(90^\circ+\theta) = -\sin\theta]$$

$$= a + 3a + 3a - 5a\left(\frac{1}{2}\right) = 7a - \frac{5a}{2} = \frac{14a-5a}{2} = \frac{9a}{2}$$

$$\boxed{R\cos\theta = \frac{9a}{2}} \rightarrow (3)$$

and $R\sin\theta = 2a\sqrt{3}\cdot\sin(30^\circ) + 6a\sin 60^\circ + 4a\sqrt{3}\sin 90^\circ + 5a\sin 120^\circ$

$$= 2a\sqrt{3}\left(\frac{1}{2}\right) + 6a\left(\frac{\sqrt{3}}{2}\right) + 4a\sqrt{3}(1) + 5a\sin(90^\circ+30^\circ)$$

$$= a\sqrt{3} + 3a\sqrt{3} + 4a\sqrt{3} + 5a\cos 30^\circ$$

$$= 8a\sqrt{3} + 5a\left(\frac{\sqrt{3}}{2}\right) = \frac{16a\sqrt{3}+5a\sqrt{3}}{2}$$

$$\boxed{R\sin\theta = \frac{21a\sqrt{3}}{2}} \rightarrow (4)$$

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Squaring and adding the eqn. (3) & (4)

$$R^2 = \frac{81a^2}{4} + \frac{441 a^2 \cdot 3}{4} = \frac{81a^2 + 1323a^2}{4} = \frac{1404 a^2}{4}$$

$$R^2 = 351a^2 \Rightarrow R = a\sqrt{351} \Rightarrow \boxed{R = AB\sqrt{351}}$$

$$\frac{(4)}{(3)} \Rightarrow \frac{21a\sqrt{3}}{2} \cdot \frac{2}{9a} = \frac{21\sqrt{3}}{9} = \frac{21 \cdot \sqrt{3}}{3 \cdot 3} = \frac{7}{\sqrt{3}}.$$

$$\boxed{\tan\theta = \frac{7}{\sqrt{3}}}$$

Hence the resultant is a force of magnitude $AB\sqrt{351}$ and a direction makes an angle is $\tan^{-1}\left(\frac{7}{\sqrt{3}}\right)$ with AB.

(17) Forces acting at a Point are represented in magnitude and direction by $\overline{AB}$, $2\overline{BC}$, $2\overline{CD}$, $\overline{DA}$ and $\overline{DB}$ where ABCD is a Square. Prove that forces are in equilibrium.

Soln: $\overline{AB} + 2\overline{BC} + 2\overline{CD} + \overline{DA} + \overline{DB}$

$$= (\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}) + (\overline{BC} + \overline{CD} + \overline{DB}) = \overline{0}$$

[∵ The first set of forces are in equilibrium by the Square ABCD and the Second set of forces are in equilibrium by the triangle BCD]. Hence the forces are in equilibrium.

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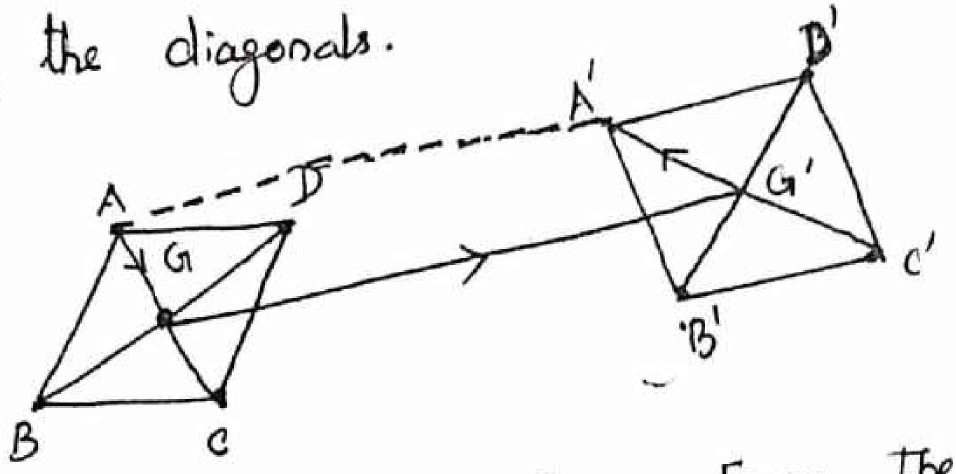
(18) ABCD and A'B'C'D' are Parallelograms. Prove that forces $\overline{AA'}$, $\overline{B'B}$, $\overline{CC'}$ and $\overline{D'D}$ acting at a Point will keep it at rest.

<u>Sol:</u> Let G and G' be the Points of intersection of the diagonals.



By the Polygon of forces, from the Quadrilateral AGG'A'

$$AA' = \overline{AG} + \overline{GG'} + \overline{G'A'} \rightarrow (1)$$

|||ly $$\overline{B'B} = \overline{B'G'} + \overline{G'G} + \overline{GB} \rightarrow (2)$$

$$\overline{CC'} = \overline{CG} + \overline{GG'} + \overline{G'C'} \rightarrow (3)$$

$$\overline{D'D} = \overline{D'G'} + \overline{G'G} + \overline{GD} \rightarrow (4)$$

from the equations (1) to (4), we get.

$$\overline{AA'} + \overline{B'B} + \overline{CC'} + \overline{D'D} = \overline{O}$$ [∵ the forces are equal and opposite]

Hence the forces are in equilibrium.

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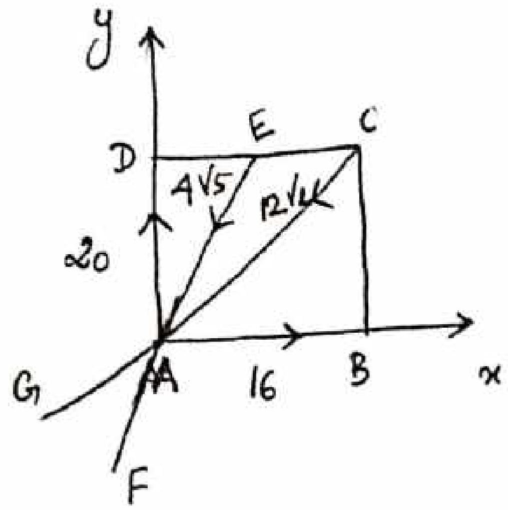
(19) E is the middle Point of the Side CD of a Square ABCD. Forces 16, 20, $4\sqrt{5}$, $12\sqrt{2}$ kg wt. act along AB, AD, EA, CA in the directions indicated by the order of the letters. Show that they are in Equilibrium.

Soln.



Take AB and AD be the axis of x and y.

Produce EA to F. Take $\angle BAF = \theta$

Produce CA to G.

(ie) $\angle BAG = \angle BAC + \angle CAG = 45^\circ + 180^\circ = 225^\circ$

Let R be the resultant of the forces inclined at an angle θ to AB.

Resolving the forces AB and AD We get

$R \cos\theta = 16 + 12\sqrt{2} \cos 225^\circ + 4\sqrt{5} \cos\theta$

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$= 16 + 12\sqrt{2}\cos(180^\circ + 45^\circ) + 4\sqrt{5}\cos\theta$

$= 16 + 12\sqrt{2}\left[-\cos 45^\circ\right] + 4\sqrt{5}\cos\theta \quad \left[\because \cos(180^\circ + \theta) = -\cos\theta\right]$

$= 16 + 12\sqrt{2} \cdot \left(\frac{-1}{\sqrt{2}}\right) + 4\sqrt{5}\cos\theta$

$= 16 - 12 + 4\sqrt{5}\cos\theta$

$$\boxed{R\cos\theta = 4 + 4\sqrt{5}\cos\theta} \longrightarrow (1).$$

$\angle BAE = \angle BAF - \angle EAF = \theta - 180^\circ$

$\angle DEA =$ alternate angle of $\angle BAE$

$\therefore \boxed{\angle DEA = \theta - 180^\circ}$

In a right angle $\triangle ADE$

$AE^2 = AD^2 + DE^2$

$= a^2 + \left(\frac{a}{2}\right)^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$

$AE^2 = \frac{5a^2}{4} \Rightarrow \boxed{AE = \frac{\sqrt{5}}{2}a}$

$\cos(\theta - 180^\circ) = \frac{DE}{AE} = \frac{\left(\frac{a}{2}\right)}{\frac{\sqrt{5}}{2} \cdot a} = \frac{1}{\sqrt{5}}$

(ie) $\cos(180^\circ - \theta) = \frac{1}{\sqrt{5}} = -\cos\theta \Rightarrow \boxed{\cos\theta = \frac{-1}{\sqrt{5}}} \rightarrow (2)$

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Subs/. (2) in (1) We get

$$R\cos\theta = 4 + 4\sqrt{5}\left(\frac{-1}{\sqrt{5}}\right) = 0$$

$$\boxed{R\cos\theta = 0} \rightarrow (3)$$

$$R\sin\theta = 12\sqrt{2}\sin 225^\circ + 4\sqrt{5}\sin\theta + 20$$

$$= 12\sqrt{2}\sin(180^\circ + 45^\circ) + 4\sqrt{5}\sin\theta + 20$$

$$= -12\sqrt{2}\sin 45^\circ + 4\sqrt{5}\sin\theta + 20$$

$$= -12\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + 4\sqrt{5}\sin\theta + 20$$

$$= 8 + 4\sqrt{5}\sin\theta$$

$$\therefore \boxed{R\sin\theta = 8 + 4\sqrt{5}\sin\theta} \rightarrow (4)$$

From a right angle triangle $\triangle AED$

$$\sin(\theta - 180^\circ) = \frac{AD}{AE} = \frac{a}{\left(\frac{a}{2}\right)\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$-\sin(180^\circ - \theta) = \frac{2}{\sqrt{5}}$$

$$\boxed{\sin\theta = \frac{-2}{\sqrt{5}}} \rightarrow (5)$$

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Subst. (5) in (4), We get

$$R\sin\theta = 8 + 4\sqrt{5}\left(\frac{-2}{\sqrt{5}}\right) = 8 - 8 = 0$$

$$\boxed{R\sin\theta = 0} \rightarrow (6)$$

Squaring and adding the eqns. (3) & (6), We get

$$R^2(\cos^2\theta + \sin^2\theta) = 0 \Rightarrow R^2 = 0 \Rightarrow \boxed{R = 0}.$$

Hence the forces are in Equilibrium.

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Parallel forces and Moments:

Two Parallel forces are Said to be like When they act in the Same direction, they are Said to be unlike When they act in opposite Parallel directions.

To find the resultant of two like Parallel forces acting on a rigid body.

Let like Parallel forces P and Q act at the Points A and B of the rigid body respectively and let them represented by the lines AD and BL.

At A and B, we have introduce two equal and opposite forces F of arbitrary magnitude along AB and let them represented by AG and BN.

These two forces will balance each other and will not affect the resultant of the system.

The two forces F and P acting at A can be compounded into a Single force R_1 represented by the diagonal AE of the Parallelogram ADEG.

Similarly two forces F and Q acting at B will have a resultant R_2 represented by the diagonal BM of the Parallelogram BLMN.

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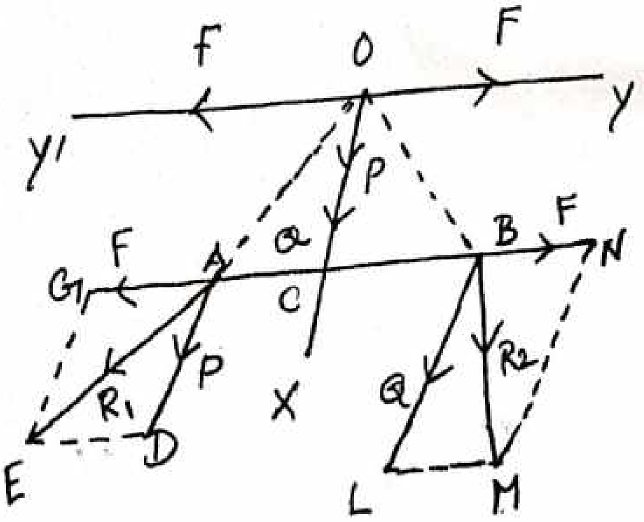

Produce EA and MB and meet at O.

The resultant R_1 and R_2 can be considered to act at O.

At O draw $Y'OY \parallel AB$ and OX parallel to the direction of P and Q.

Resolve R_1 and R_2 at O into their original components.

R_1 at O is equal to a force F along OY' and a force P along OX.

R_2 at O is equal to a force F along OY and a force Q along OX.

Two F's at O cancel each other being equal and opposite.

Now we are left with the two forces P and Q along OX.

Hence the Resultant is a force (P+Q) acting along OX.

(i.e) acting in a direction parallel to original directions of P and Q.

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(ii) The magnitude of the resultant of two like Parallel forces is their Sum. The direction of the resultant is Parallel to the Components and in the Same Sense.

To find the Position of the resultant

Let OX to meet AB at C.

Triangles OAC and AED are Similar.

$\therefore \frac{OC}{AD} = \frac{AC}{ED} \Rightarrow \frac{OC}{P} = \frac{AC}{F} \Rightarrow F.OC = P.AC \rightarrow (1)$

Triangles OCB and BLM are Similar.

$\therefore \frac{OC}{BL} = \frac{CB}{LM} \Rightarrow \frac{OC}{Q} = \frac{CB}{F} \Rightarrow F.OC = Q.CB \rightarrow (2)$

From (1) & (2) We get

$$P.AC = Q.CB \Rightarrow \boxed{\frac{AC}{CB} = \frac{Q}{P}}$$

Hence the Point C divides AB internally in the inverse ratio of the forces.

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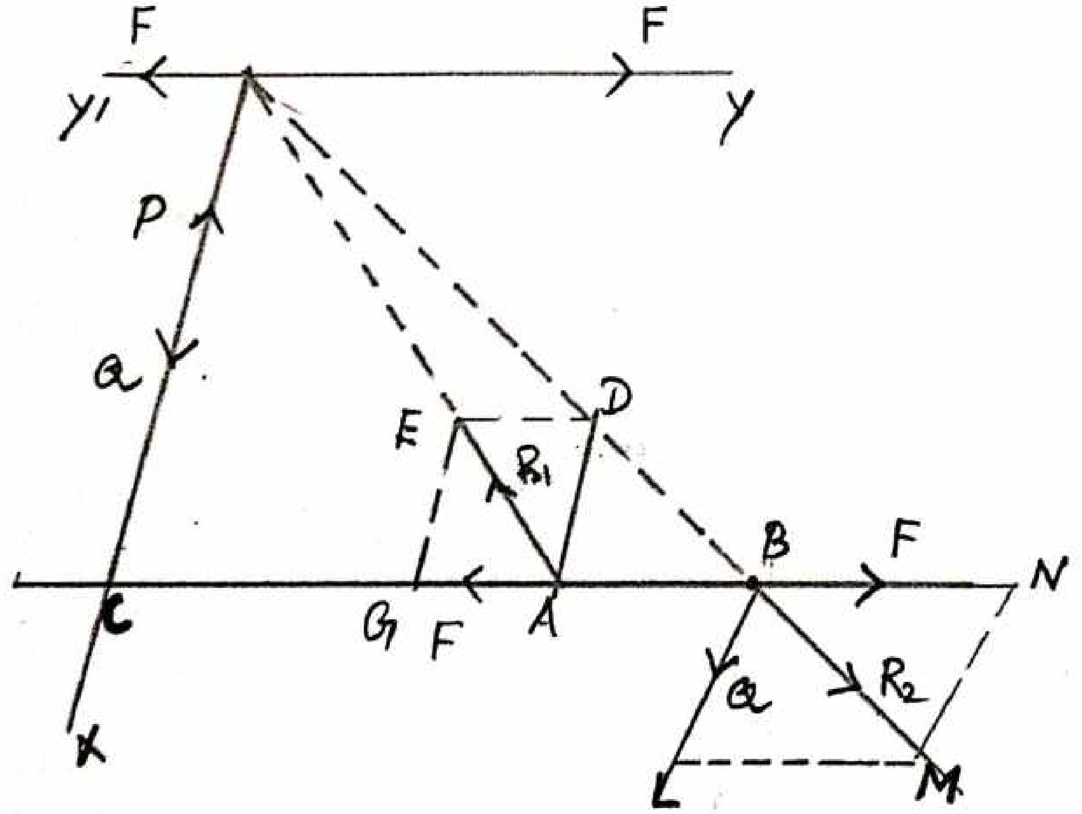
To find the resultant of two unlike and unequal Parallel forces acting on a rigid body.

Let P and Q be unequal and unlike Parallel forces acting at the Points A and B of the rigid body. Let P > Q and let them represented by AD and BE.



At A and B we introduce two equal and opposite forces F of arbitrary magnitude along the line AB. and let them represented by AG and BN.

These two forces will balance each other and will not affect the resultant of the System.

The two forces F and P acting at A can be Compounded into a single force R_1 represented by the diagonal AE in the Parallelogram AGED.

Similarly the two forces F and Q acting at B a resultant R_2 can be represented by the diagonal BM of the Parallelogram BLMN.

Produce AE and MB and let them meet at O.

The resultant R_1 and R_2 to act at O.

At O, draw $y'oy$ ∥ to AB and

ox Parallel to the directions of P and Q.

Resolve R_1 and R_2 at O into their original Components.

R_1 at O is equal to a force F along oy' and a force P along xo.

R_2 at O is equal to a force F along oy and a force Q along ox.

The two forces at O Cancel each other being equal and opposite.

Now we have left with a force P along xo and Q along ox.

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∴ The resultant is a force P−Q acting along XO.
(ie) acting in a direction Parallel to that of P.
∴ The magnitude of the resultant of two unlike Parallel forces is their difference.
The direction of the resultant is Parallel to the Sense of the greater Component.

<u>To find the Position of the resultant</u>:

Let OX meet AB at C

Triangles OCA and EGA are Similar.

$$\therefore \frac{OC}{EG} = \frac{CA}{GA} \Rightarrow \frac{OC}{P} = \frac{CA}{F} \Rightarrow F \cdot OC = P \cdot CA \rightarrow (1)$$

Triangles OCB and BLM are Similar.

$$\therefore \frac{OC}{BL} = \frac{CB}{LM} \Rightarrow \frac{OC}{Q} = \frac{CB}{F} \Rightarrow F \cdot OC = Q \cdot CB \rightarrow (2)$$

From (1) & (2) We get

$$P \cdot CA = Q \cdot CB \Rightarrow \boxed{\frac{CA}{CB} = \frac{Q}{P}}$$

(ie) The Point C divides AB <u>externally</u> in the inverse ratio of the forces.
When $P > Q$, $CB > CA$.
Hence the resultant Passes nearer the greater force.

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Note: The resultant of two unlike Parallel force P and Q will fail if $P = Q$

(ie) if the forces are in equal magnitude.

In that case Δ^{les} AGE and BNM are Congruent.

(ie) $\angle GAE = \angle NBM$ and the lines AE and MB are $\parallel^{el}$.

There will be no Point as O.

Hence the effect two equal and unlike Parallel forces cannot be replaced as a Single force.

Resultant of number of Parallel forces acting on a rigid body

Let P, Q, R --- be a numbers of Parallel forces act on a rigid body.

First we find the resultant R_1 of P and Q ~~and~~

Second we find the resultant R_2 of R_1 and R

This Process is Continued until the final resultant is obtained.

Case (i) : If the Parallel forces are all like, the magnitude of the final resultant is Sum of the forces.

Case (ii) If the $\parallel^{el}$ forces are not all like the magnitude of the resultant is Sum of the forces each taken with its proper Sign.

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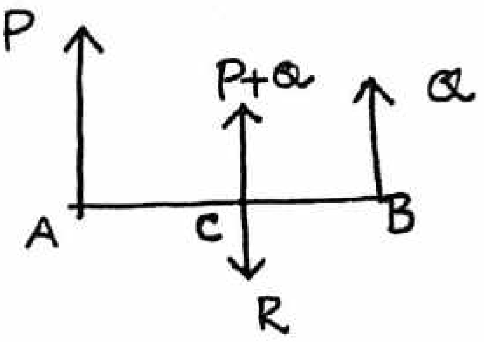
Conditions of Equilibrium of Three Coplanar Parallel forces

Let P, Q and R be the three forces Parallel in one Plane and be in equilibrium.

Draw a line to meet the lines of action of these forces at A, B and C respectively.



If all the forces are in the Same Sense and be in equilibrium.

∴ Two forces P and Q must be like and the third force R unlike.

The resultant of P and Q is P+Q Parallel to P or Q and for equilibrium R must be equal and opposite.

∴ $R = P+Q$ the line of action of P+Q must Passes Through C.

$$\therefore P \cdot AC = Q \cdot CB \Rightarrow \frac{P}{CB} = \frac{Q}{AC}$$

and each is equal to $\frac{P+Q}{CB+AC} = \frac{P+Q}{AB} = \frac{R}{AB}$

(i.e) $\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$.

Hence if three Parallel forces are in Equilibrium each is Proportional to distance between the other two.

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Centre of two Parallel Forces

Let P and Q be two Parallel forces acting at two Points A and B then their resultant R Passes Through a Point C which divides AB internally or externally in the ratio Q : P

(ie) $\frac{AC}{CB} = \frac{Q}{P} \rightarrow (1)$

The Position of C is given by eqn. (1). depends only upon the Positions of A and B and then magnitude of the forces P and Q. It does not depend on the actual direction.

In otherwords, the Common direction of Parallel of the forces P and Q their resultant will always Passes through a Certain fixed Point. This Point is Called the Centre of two Parallel forces.

∴ The Centre of two Parallel forces is a fixed Point through which their resultant always Passes Whatever be the direction of Parallelism.

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In generally

The resultant of a system of Parallel forces of given magnitude acting at given Points of a body. will always Pass through a fixed Point, for all directions of Parallelism. This Point is called Centre of Parallel forces.

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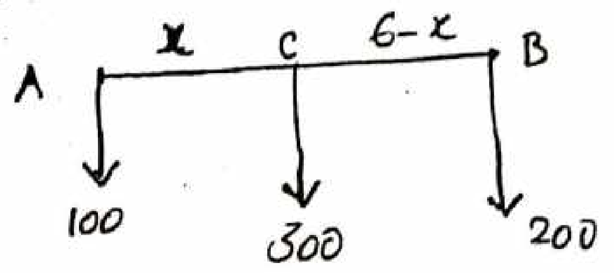
Problems :

(1) Two men, One Stronger than the other have to remove a block of Stone Weighing 300 kgs with a light Pole whose length is 6 meter. The weaker man Cannot Carry more than 100 kgs. Where must the stone be fastened to the Pole So as just to allow him his full Share of weight?

Soln. Let A be the weaker man bearing 100 kgs, his full share of the weight of the Stone and B be the stronger man bearing 200 kgs. Let C be the Point on AB where the stone is fastened to the Pole ∋ AC = x.

Then the weight of the stone acting at C is resultant of the Parallel forces 100 and 200 at A and B respectively.



$\therefore \; 100 \cdot AC = 200 \cdot BC$

$\Rightarrow \; 100x = 200(6-x)$

$\Rightarrow \; 1200 - 200x = 100x$

$\Rightarrow \; 300x = 1200$

$\Rightarrow \; \boxed{x = 4}$

Hence the Stone must be fastened to the Pole at the Point distance 4 metres from the weaker man.

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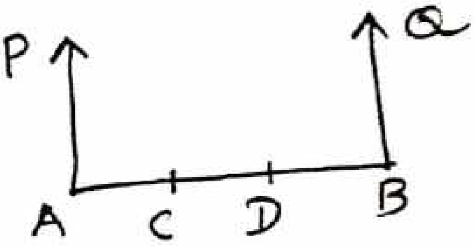
(2) Two like Parallel forces P and Q act on a rigid body at A and B respectively.

(a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

(b) If P and Q be interchanged in Position, show that the point of application of the resultant will be displaced along AB through a distance d where $d = \frac{P-Q}{P+Q}.AB$.

Soln.

(a) Let C be the Centre of two Parallel forces with P at A and Q at B.



Then $P.AC = Q.CB \rightarrow (1)$

If Q is changed to $\frac{P^2}{Q}$

Let D be the new Centre of Parallel forces.

$\therefore P.AD = \frac{P^2}{Q} DB \rightarrow (2)$

(ie) $PQ.AD = P^2.DB$

$Q.AD = P.DB \rightarrow (3)$

The eqn. (3) shows that D is the Centre of two like Parallel forces with Q at A and P at B.

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(b) When the forces P and Q are interchanged in Position D is the new Centre of Parallel forces.

$CD = d$

From the equation (3)

$Q.(AC+CD) = P.(CB-CD)$

(ie) $Q.AC + Q.d = P.CB - P.d$

$(Q+P).d = P.CB - Q.AC$

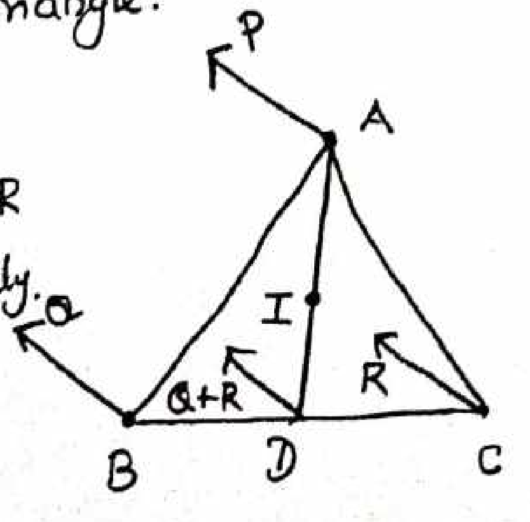
$= P(AB-AC) - Q(AB-CB)$

$= PAB - PAC - QAB + QCB$

$= (P-Q).AB$ [∵ Using eqn. (1)]

$\therefore \boxed{d = \frac{P-Q}{P+Q}.AB}$

(3) Three like Parallel forces, acting at the Vertices of a triangle have magnitude Proportional to the opposite Sides. Show that their resultant Passes through the incentre of the triangle.



Soln

Let like Parallel forces P, Q, R act at A, B and C respectively.

(ie) $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \longrightarrow (1)$

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Let the resultant of Q and R meet BC at D

W.K.T the magnitude of the resultant is Q + R.

Also $\frac{BD}{DC} = \frac{\text{force at } C}{\text{force at } B} = \frac{R}{Q} = \frac{c}{b}$ [from the eqn. (1)]

$$\frac{BD}{DC} = \frac{AB}{AC}$$

∴ AD is the internal bisector of ∠A.

Now to find the resultant of the two like Parallel forces Q+R at D and P at A.

Let this resultant meet AD at I

$$\therefore \frac{AI}{ID} = \frac{\text{force at } D}{\text{force at } A} = \frac{Q+R}{P} = \frac{b+c}{a} \quad \text{[from the eqn. (1)]}$$

$$\frac{AI}{ID} = \frac{b+c}{a} \rightarrow (2)$$

From the eqn. (2) shows that I is the incentre of Δ.

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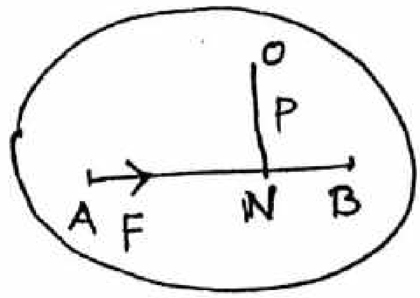
Unit - 2

Moment of a Force :

When forces act on a Particle, the only motion that can occur is a motion of translation. But a force acting on a rigid body may produce either a motion of translation or rotation or of translation and rotation combined. When there is a motion of translation alone, the force is measured by the product of the mass of the Particle and the acceleration produced on it by the force. In case of rotation the idea of the turning effect or moment of a force is introduced.

Consider a sheet of cardboard Pivoted freely at a fixed Point O. If a force F acts along a St. line AB, it is clear that there will be no rotation if AB Passes through O.
If AB does not Pass through O the force will tend to rotate the sheet about O.



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This tendency to rotate the body will increase as the magnitude of the force increases and also as the Perpendicular distance from O on the line of action of the force increases.

Let ON be the length of the $\perp^r$ from O on the line of action of F. This tendency to rotate varies as F when ON is constant. It also varies as ON when F is constant. Hence it varies as F × ON. That is the Product of F and ON, when both these quantities vary. This Product is called <u>moment of a force F about O.</u>

Thus the <u>moment of a force about a Point</u> is defined to be the Product of the force and the $\perp^r$ distance of the Point from the line of action of the force.

<u>Note:</u> If the moment of a force about a Point is zero either

(i) the force itself is zero (ie) F = 0 or

(ii) the line of action of the force Passes through the Point.

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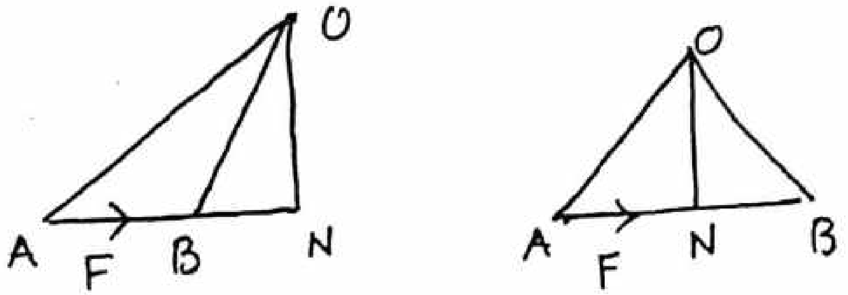
Physical Significance of the moment of a Force

The Physical meaning for the moment of a force about a Point is that it measures the tendency to rotate the body about that Point.

Geometrical Representation of a Moment

Let a force F acting on a body be represented in magnitude, direction and line of action by the line AB.

Let O be any given Point and ON the Perpendicular from O on AB or AB Produced.



The moment of a force about O is

$$F \times ON = AB \times ON = 2\Delta AOB$$

Hence if a force is represented completely by a St. line its moment about any Point is given by twice the area of the Δ^{le} which the St. line subtends at that Point.

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Sign of the moment :

When the force F acts along AB, it will tend to rotate the lamina in the anticlockwise direction.

(ie) In a direction opposite to that in which the hands of clock move. In such cases the moment is said to be Positive.

If the force tends to turn the body in a Clockwise direction its moment is said to be negative

Unit of moment :

The moment of a unit force about a Point at a unit Perpendicular distance from the line of action of the force is defined as the unit for the measurement of moments. If the unit of force be Poundal and unit of distance be one foot the unit of moment is a Poundal foot.

If the unit of force be dyne and unit of distance be one Centimetre the unit of moment is a dyne-cm.

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Varigon's Theorem of Moments

Statement: The algebraic Sum of the moments of two forces about any Point in their Plane is equal to the moment of their resultant about that Point.

Proof:

Case (i) Let the forces be Parallel.

Let P and Q be two Parallel forces and O any Point in their Plane.

Draw AOB ⊥r to the forces to meet their line of action in A and B.

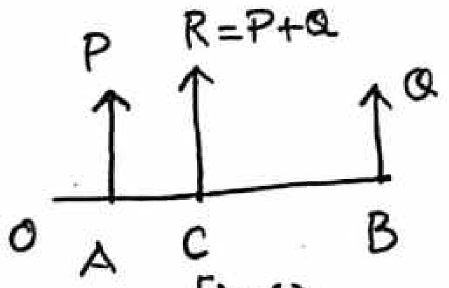

fig (1)

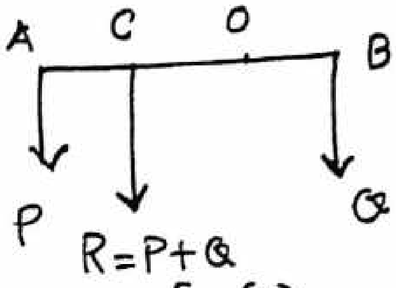

fig (2)

The resultant of P and Q is a force R = P+Q acting at C ∋ P. AC = Q . CB → (1)

In fig (1)

The algebraic Sum of the moments of P and Q about O

$= P.OA + Q.OB$

$= P(OC - AC) + Q(OC + CB)$

$= (P+Q)OC - P.AC + Q.CB$

$= (P+Q)OC - Q.CB + Q.CB$ [∵ Using eqn. (1)]

$= (P+Q).OC$

$= R.OC$

= Moment of R about O.

In fig (2) O is within AB and the algebraic Sum of the moments of P and Q about O

$= P.OA - Q.OB$

$= P(OC + CA) - Q(CB - CO)$

$= (P+Q)OC + P.CA - Q.CB$

$= (P+Q)OC + Q.CB - Q.CB$ (∵ Using eqn. (1))

$= (P+Q).OC$

$= R.OC$

= moment of R about O.

<u>Case (ii)</u> Let the forces are unlike meet at a Point.

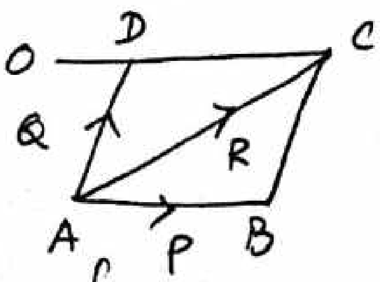

fig (1)

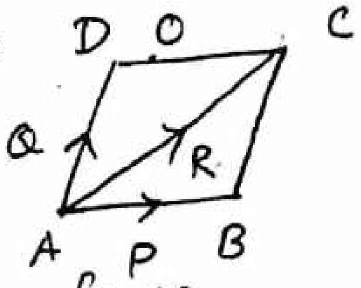

fig (2)

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Let the forces P and Q act at A as shown in the fig.

Let O be any Point in their Plane.

Through O, draw a line $\parallel^{el}$ to P meeting the the line of action of Q at D.

Choose the scale of representation ∋ length AD represents Q in magnitude.

On the same scale let length AB represent P.

Complete the Parallelogram BAD so that the diagonal AC represents the resultant R of P and Q.

The moments of P, Q, R about O are represented by $2\Delta AOB$, $2\Delta AOD$, $2\Delta AOC$ resp.

In fig (1)

O lies outside the ∠BAD and the moments of P and Q are Positive.

The algebraic sum of the moments of P and Q

$= 2\Delta AOB + 2\Delta AOD$

$= 2\Delta ACB + 2\Delta AOD$ $[\because \Delta AOB = \Delta ACB]$

$= 2\Delta ADC + 2\Delta AOD$ [∵ diagonal AC bisects the $\parallel^{gm}$]

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$= 2(\Delta ADC + \Delta AOD)$

$= 2\Delta AOC$

= moment of R about O.

In fig (2) O lies inside the ∠BAD.

The moment of P about O is Positive while that of Q is negative.

The algebraic Sum of the moments of P and Q

$= 2\Delta AOB - 2\Delta AOD$

$= 2\Delta ACB - 2\Delta AOD$

$= 2\Delta ADC - 2\Delta AOD$

$= 2[\Delta ADC - \Delta AOD]$

$= 2\Delta AOC$

= moment of R about O.

Hence the algebraic Sum of the moments of two forces about any Point in their Plane is equal to the moment of their resultant about That Point.

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Generalised Theorem of moments

Statement: If any number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.

Proof: Let P_1, P_2 .. be any no. of coplanar forces and O be any point in their plane.

Let R_1 be the resultant of P_1 and P_2

R_2 be the resultant of R_1 and P_3

R_3 be the resultant of R_2 and P_4

and so on until the final resultant is obtained.

Applying Varigon's thm. to the forces P_1, P_2 and R_1

We have

Moment of P_1 about O + Moment of P_2 about O

= Moment of R_1 about O ⟶ (1)

III'ly applying the thm. to the forces R_1, P_3 and R_2

We have

Moment of R_1 about O + moment of P_3 about O

= Moment of R_2 about O ⟶ (2)

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Combining the eqns. (1) & (2) we get

Moment of P_1 about O + Moment of P_2 about O + Moment of P_3 about O = Moment of R_2 about O

Proceeding thus till all the forces are exhausted.

Let p_1, p_2, p_3 -- be the $\perp^r$ distances of O from the lines of action of the forces P_1, P_2 -- respectively and p be the $\perp^r$ distance of O from the line of action of the resultant R.

$\therefore$ The above theorem can be written as

$$P_1 p_1 + P_2 p_2 + \cdots = pR$$

(ie) $$\boxed{\Sigma P_1 p_1 = pR}$$

<u>Result 1</u> : If the basic point O about which moment is taken, happens to lie on the line of action of the resultant R, then p = 0

$$\therefore \Sigma P_1 p_1 = 0.$$

Hence the algebraic sum of the moment of any number of coplanar forces about any point on the line of action of their resultant is zero.

Result 2: Suppose $\Sigma P_1 p_1 = 0$

(ie) $PR = 0$

$\therefore$ Either $P = 0$ or $R = 0$

If $P = 0$ it means that the basic Point about O which moment is taken lie on the line of action of the resultant.

If $R = 0$ it means that there is no resultant for the system.

(ie) the forces are in equilibrium.

Hence the algebraic Sum of the moments of any number of forces about any Point in their Plane is Zero, then either their resultant Passes through the Point about which moments are taken or the resultant is zero.

Result 3: Suppose $R = 0$ (ie) The forces are in equilibrium

(ie) $\Sigma P_1 p_1 = 0$

Hence if a system of Coplanar forces in equilibrium the algebraic Sum of their moments about any Point in their Plane is Zero.

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Moment of a force about an axis:

We have considered only Coplanar forces and their moments about a Point in their plane.

Let us consider a rigid body which is capable of turning about some axis fixed in the body. For instance a door capable of turning about the line of hinges.

Now any force whose line of action is not Parallel or does not Pass through this axis, will tend to turn the body about it.

To measure the tendency of rotation in such cases we introduce the idea of the moment of a force about an axis.

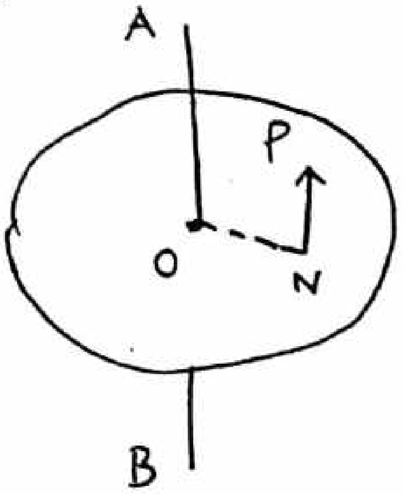

Fig 1

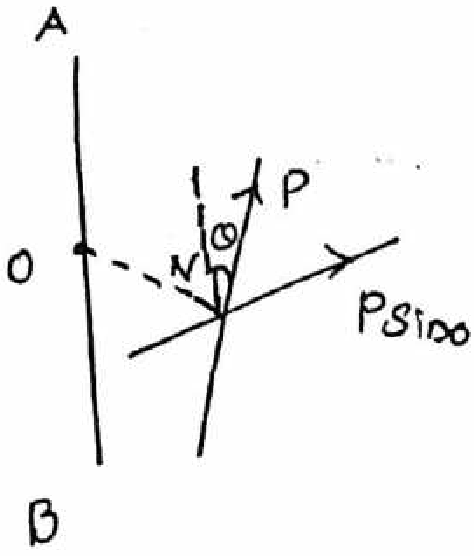

Fig 2

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In fig (1)

a force P acts on a body in a direction Perpendicular to a line AB in the body, but not intersecting it.

(ie) P acts in a Plane $\perp^r$ to AB.

The moment of the force P about the line AB is defined to be P.ON where ON is the $\perp^r$ distance between the line of action of P and the line AB.

In fig (2)

a force P acts in any direction (not necessary $\perp^r$ to AB)

Let ON be the Shortest distance between AB and the line of action of P.

The force P Can be Considered to act at N along its line of action.

It Can be resolved into two Components

(i) $P\cos\theta$ $\parallel^{el}$ to AB

(ii) $P\sin\theta$ $\perp^r$ to AB.

The Component $P\cos\theta$ being $\parallel^{el}$ to AB has zero moment about AB.

The Component $P\sin\theta$ $\perp^r$ to AB is $P\sin\theta \cdot ON$

This is moment of P about AB.

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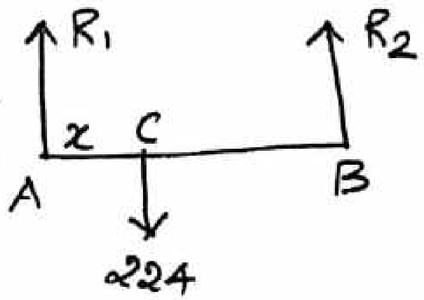
Problems :

① Two men Carry a load of 224 Kg wt Which hangs from a light Pole of length 8m each end of which rests on a shoulder of one of the men. The Point from which the load is hung is 2m. nearer to one man than the other. What is the Pressure on each Shoulder ?



Soln

Let AB is the light Pole of length 8m. and C is the Pt/. from which the load of 224 kgs is hung.

Let $AC = x$ the $BC = 8 - x$.

It is given That $(8-x) - x = 2 \Rightarrow 2x = 6 \Rightarrow x = 3$

$\therefore AC = 3$ and $BC = 5$

Let the Pressures at A and B be R_1 and R_2 kg.

Since the Pole is an equilibrium

the algebraic Sum of the moments of the three forces R_1, R_2 and 224 kg about any Pt/. must be equal to Zero.

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Taking moment about B

$224\,CB - R_1\,AB = 0$ (as the moment of R_2 about B is zero)

$224(5) - R_1(8) = 0$

$$R_1 = \frac{224 \times 5}{8} = 140 \Rightarrow \boxed{R_1 = 140}$$

Taking moments about A

$R_2\,AB - 224\,AC = 0$

$8R_2 - 224(3) = 0 \Rightarrow \boxed{R_2 = 84}$

(2) A uniform plank of length 2a and weight W is supported horizontally on two vertical props at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_2 respectively.

Show that

$$\frac{W_1}{W+W_1} + \frac{W_2}{W+W_2} = \frac{b}{a}.$$

Soln

Let AB be the plank placed upon two vertical props at C and D. CD = b. The weight W of the plank acts at G the mid pts/. of AB.

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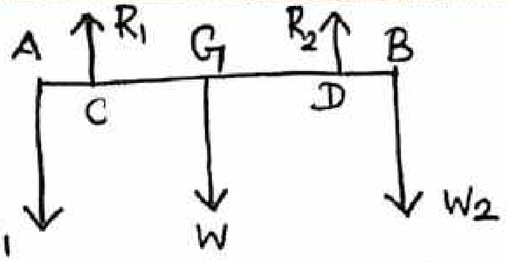
$AG = GB = a$



When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D then is zero.

There is upward reaction R_1 at C

Taking moments about C, we get

$$W_1 \cdot AC = W \cdot CG$$

(ie) $W_1 (AG - CG) = W \cdot CG$

Or $W_1 \cdot AG = (W + W_1) \cdot CG$

(ie) $W_1 a = (W + W_1) CG$

$$CG = \frac{W_1 a}{W + W_1} \rightarrow (1)$$

When the weight W_2 is attached at B, there is loose contact at C. The reaction at C is zero.

There is upward reaction R_2 about D.

Taking moments about D, we get

$$W \cdot GD = W_2 \cdot BD$$

(ie) $W \cdot GD = W_2 (GB - GD)$

(ie) $GD (W + W_2) = W_2 GB = W_2 a$

(ie) $GD = \frac{W_2 a}{W + W_2} \rightarrow (2)$

But $CG + GD = CD = b$ $\therefore$ $\boxed{\frac{W_1}{W + W_1} + \frac{W_2}{W + W_2} = \frac{b}{a}}$ [Using eqn (1) & (2)]

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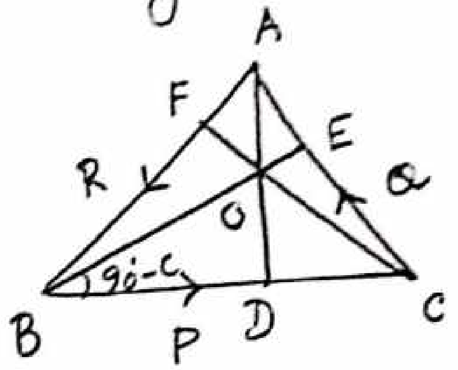
③ The resultant of three forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled. If $\angle A = 120^\circ$ and B = C show that

$Q + R = P\sqrt{3}$

Soln: Let AD, BE and CF be the altitudes of the triangle intersecting at O be the orthocentre.



As the resultant passes through O, moment of the resultant about O = 0

resultant about O is zero.

∴ Sum of the moments about P, Q, R about O is zero.

∴ Taking moments about O, we get

$$P.OD + Q.OE + R.OF = 0 \rightarrow (1)$$

In right angled triangle BOD

$\angle OBD = \angle EBC = 90^\circ - C$

$$\therefore \tan(90^\circ - C) = \frac{OD}{BD}$$

(ie) $$\cot C = \frac{OD}{BD}$$

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$OD = BD \cot C \rightarrow (2)$

From right angled triangle ABD

$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \cos B \Rightarrow BD = c \cos B \rightarrow (3)$

Subst. (3) in (2), we get

$OD = c \cos B \cot C = c . \cos B . \frac{\cos C}{\sin C}$

$= \frac{c}{\sin C} \cos B \cos C = 2R' \cos B \cos C$ $\left[\because \frac{c}{\sin C} = 2R' \right]$

R' is circumradius of the Δ^{le}

$|||^{ly}$ $OE = 2R' \cos C \cos A$

and $OF = 2R' \cos A \cos B$

$\therefore$ (1) implies

$P . 2R' \cos B \cos C + Q\, 2R' \cos C \cos A + R\, 2R' \cos A \cos B = 0$

$\div$ by $2R' \cos A \cos B \cos C$, we get

$\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0 \rightarrow (4)$

Now P, Q, R being magnitudes of the forces all are Positive.

Hence in order that the eqn. (3) may hold good, at least one of the terms must be negative.

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∴ One of the Cosines must be negative.

(ie) The triangle must be Obtuse angled.

If $A = 120^\circ$ and the other angles equal then $\angle B = \angle C = 30^\circ$

(A) $\Rightarrow \dfrac{P}{\cos 120^\circ} + \dfrac{Q}{\cos 30^\circ} + \dfrac{R}{\cos 30^\circ} = 0$

(ie) $\dfrac{P}{\cos(90^\circ + 30^\circ)} + \dfrac{Q}{\cos 30^\circ} + \dfrac{R}{\cos 30^\circ} = 0$

$$\frac{P}{-\sin 30^\circ} + \frac{Q+R}{\cos 30^\circ} = 0$$

$$\frac{P}{-1/2} + \frac{Q+R}{\sqrt{3}/2} = 0$$

(ie) $\boxed{P\sqrt{3} = Q + R}$

④ Forces P, Q, R act along the Sides BC, AC, BA respectively of an equilateral triangle. If their resultant is a force Parallel to BC through the Centroid of the triangle Prove that $Q = R = \frac{1}{2}P$

Soln Let ABC be equilateral triangle the medians AA′, BB′ and CC′ are the altitudes meeting at G it is the Centroid.

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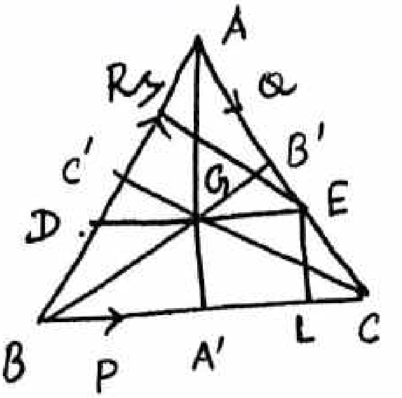

Let DE ∥ BC through G

It is given that DGE is the line of action of the resultant.

As the resultant passes through G its moment about G is zero.

∴ Sum of the moments of P, Q, R about G is zero.

(i.e) $P\,GA' - Q\,GB' - R\,GC' = 0$

(i.e) $P - Q - R = 0 \rightarrow (1)$ $(\because GA' = GB' = GC')$

Since the resultant passes through E also, sum of the moments of P, Q, R about E is zero.

Draw $EL \perp BC$ and $EM \perp AB$

$\therefore P.EL - R.EM = 0 \rightarrow (2)$

From the similar Δ^{les} ELC and AA'C

$\frac{EL}{AA'} = \frac{EC}{AC} = \frac{1}{3}$ $\left[\because DGE \parallel BC \text{ and } \frac{AD}{DB} = \frac{AE}{EC} = \frac{AG}{GA'} = \frac{2}{1}\right]$

$\therefore EL = \frac{1}{3}AA' \rightarrow (3)$

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From the Similar triangles AME and AC'C

$$\frac{EM}{CC'} = \frac{AE}{AC} = \frac{2}{3}$$

$$\therefore EM = \frac{2}{3} CC' \rightarrow (4)$$

Subst. (3) & (4) in eqn. (2) We get

$$P \cdot \frac{1}{3} AA' - R \cdot \frac{2}{3} CC' = 0$$

(ie) $\frac{P}{3} - \frac{2R}{3} = 0 \quad [\because AA' = CC']$

$$P - 2R = 0 \Rightarrow P = 2R \Rightarrow \boxed{R = \frac{P}{2}} \rightarrow (5)$$

Put $R = \frac{P}{2}$ in eqn. (1) We get

$$P - Q - \frac{P}{2} = 0$$

(ie) $\frac{P}{2} - Q = 0 \Rightarrow \boxed{\frac{P}{2} = Q} \rightarrow (6)$

From the eqns. (5) & (6) We get

$$\boxed{\frac{P}{2} = Q = R}$$

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(12)

⑤ A uniform Circular Plate is Supported horizontally at three Points A, B, C of its Circumference. Show that the Pressures on the Supports are in the ratio $\sin 2A : \sin 2B : \sin 2C$.

<u>Soln:</u> Let $BC = a$; $CA = b$ and $AB = c$

W be the Weight of the Plate acts at O, the Centre of the Circle and which is also the Circumcentre of the triangle.

Let OD be $\perp^r$ to BC.

W.K.T $\angle BOD = A$

From the right angled Δ^{le}

$OD = OB \cdot \cos \angle BOD = R \cdot \cos A \rightarrow (1)$

R being Circum radius of the Δ^{le}.

Let AE $\perp^r$ to BC

$AE = AC \sin \angle ACE = b \sin C \rightarrow (2)$

Let R_1 be the reaction at A

Taking moments about BC (to avoid the reactions at B and C)

$R_1 \cdot AE = W \cdot OD$

$\therefore R_1 = \dfrac{W R \cos A}{b \sin C}$ [$\because$ Using eqns. (1) & (2)]

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$$= \frac{WR\cos A}{2R\sin B\sin C} \quad (\because b = 2R\sin B)$$

$$= \frac{W\cos A}{2\sin B\sin C} = \frac{2W\sin A\cos A}{4\sin A\sin B\sin C}$$

$$= \frac{W\sin 2A}{4\sin A\sin B\sin C}$$

Illy the reactions R_2 and R_3 are

$$R_2 = \frac{W\sin 2B}{4\sin A\sin B\sin C} \quad ; \quad R_3 = \frac{W\sin 2C}{4\sin A\sin B\sin C}$$

Hence

$$\boxed{R_1 : R_2 : R_3 = \sin 2A : \sin 2B : \sin 2C}$$

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Couples

Couple : The effect of two equal and unlike Parallel forces Cannot be replaced by a Single force, A Pair of Such forces is Called a Couple.

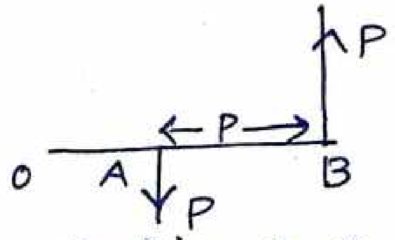
Two equal and unlike Parallel forces not acting at the Same Point are Said to be Constitute a Couple.

Example : A Couple are the forces used in winding a Clock or turning a tap. Such forces acting upon a rigid body Can have only a rotary effect on the body and they Cannot Produce a motion of translation.

Let P be the magnitude of the forces forming a Couple and O any Point in their Plane.

Draw OAB Perpendicular to the forces to meet their lines of action in A and B.

The algebraic Sum of the moments of the forces about O is P.OB − P.OA.



$$= P(OB - OA) = P.AB$$

and this Value is independent of the Position of O.

∴ The algebraic Sum of the moments of the two forces forming a Couple about any Point in their Plane is Constant and is equal to the Product

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of either of the forces and the Perpendicular distance between them.

This algebraic sum measures the total turning effect of the forces of the Couple upon the body and it is called the <u>moment of the Couple</u>.

∴ The moment of a couple is the product of either of the two forces of the Couple and the Perpendicular distance between them.

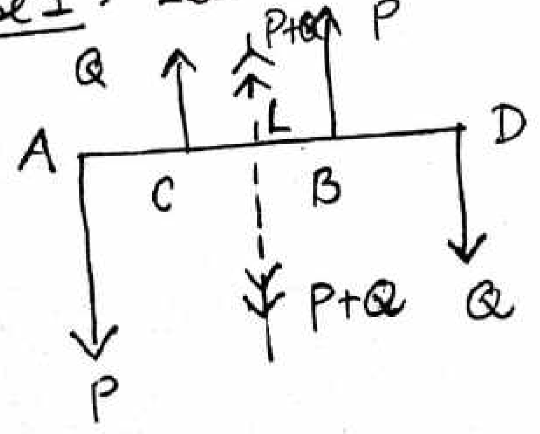
<u>Equilibrium of two Couples: Theorem 1</u>

<u>Statement</u>: If two Couples whose moments are equal and opposite, act in the same plane upon a rigid body they balance one another.

<u>Proof</u>:

Let (P,p) and (Q,q) be two given Couples such that $Pp = Qq$ in magnitude but opposite in sign.

<u>Case 1</u>: Let the forces P and Q be Parallel.



Draw a St. line $\perp^r$ to the lines of action of the forces, meeting them at A, B, C, D as show in this fig.

Since the ~~mo~~moments of the Couples are equal

$$P.AB = Q.CD \rightarrow (1)$$

The downward like $\parallel^{el}$ forces P at A and Q at D can be compound into a single force P+Q acting at L ∋

$$P.AL = Q.DL \rightarrow (2)$$

(1) − (2) ⇒

$$P(AB-AL) = Q(CD-DL)$$

$$P.BL = Q.CL \rightarrow (3)$$

The eqn. (3) shows that the resultant of upward like parallel forces P at B and Q at C will also passes through L.

The magnitude of this resultant is also (P+Q) but it is opposite in direction to the previous resultant

(ie) The resultant balance each other.

Hence the four forces forming the couples are in equilibrium

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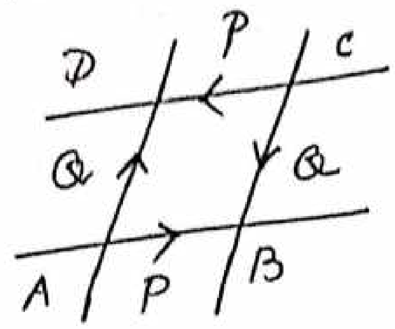
Case 2 : Let the forces P and Q intersect.



Let the two forces p of the Couple (P, P) meet the two forces Q of the Couple (Q, q) at the Points A, B, C, D. (i.e) ABCD is a Parallelogram.

Let AB represent P on some scale.

The moments of the two Couples are equal.

We get $P.p = Q.q \rightarrow (1)$

Also,

$AB \cdot P = AD \cdot q$ (each being equal to the area of the Parallelogram ABCD) $\rightarrow (2)$

$(1) \div (2) \Rightarrow \dfrac{P}{AB} = \dfrac{Q}{AD} \rightarrow (3)$

Eqn. (3) Shows that the Side AD will represent Q on the Same Scale in which the Side AB represent P

The two forces P and Q meeting at A Can be Compounded by llgm law

$(P+Q)$ at $A = \overline{AB} + \overline{AD} = \overline{AC}$

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Similarly

$(P+Q)$ at $C = \overline{CD} + \overline{CB} = \overline{CA}$

The two resultants $\overline{AC}$ and $\overline{CA}$ being equal and opposite cancel each other.

Hence the four forces forming the couples are in equilibrium.

<u>Result:</u>

(1) The perpendicular distance AB between the two equal forces P of a couple is called the <u>arm of the couple.</u>

(2) A couple each of whose forces is P and whose arm is p then the couple is denoted by (P, p)

(3) A couple is positive when the moment is positive

(i) If the forces of the couple tend to produce rotation is anticlockwise direction.

(4) A couple is negative when the forces tend to produce rotation in clockwise direction.

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Equivalence of two Couples:

Theorem: Two Couples in the Same Plane whose moments are equal and of the Same Sign are equivalent to One another.

Pf: Let (P,P) and (Q,Q) be two Couples in one Plane having the Same equal moments in magnitude and direction. Let (R, r) be a third Couple in the Same Plane, whose moment is equal to the moment of either (P,P) or (Q,Q) only in magnitude but opposite in direction.

By the thm. of equilibrium of two Couples the Couple (R, r) will balance the Couple (P,P).

It will also balance the Couple (Q,Q)

Hence the effects of the Couples (P,P) and (Q,Q) must be the same otherwise they are equivalent.

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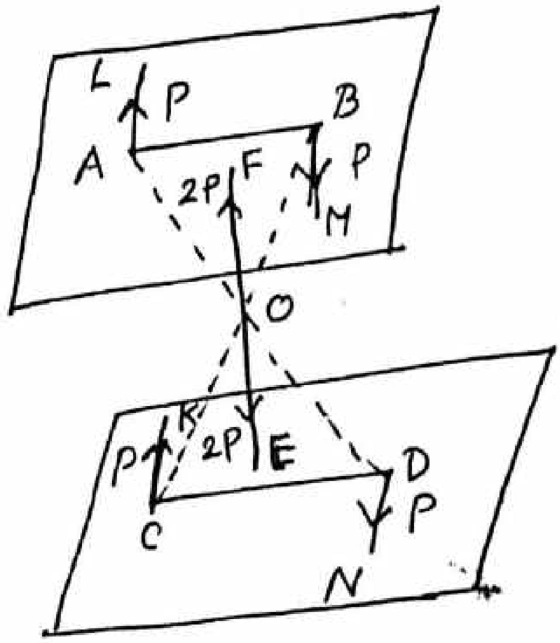
Couples in Parallel Planes : Thm 3 :

Statement : The effect of a Couple upon a rigid body is not altered if it is transferred to a Parallel Plane Provided its moment remains unchanged in magnitude and direction.

Proof: Consider a Couple of forces P at the ends of arm AB in a given Plane.

Let AL and BM be the lines of action of the forces.

In any Parallel Plane take a st. line CD equal and Parallel to AB.



Then ABCD will be Parallelogram. The diagonals AD and BC will bisect each other at O.

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At O introduce two equal and opposite forces of magnitude 2P along EF, Parallel to the forces P at A and B. By this the effect of the given Couple is not altered.

Now the unlike Parallel forces P along AL and 2P along OE can be Compounded into a Single force P acting at D. Since $\frac{AD}{OD} = \frac{2}{1} = \frac{2P}{P}$

This resultant force P acts along DN in the Second Plane.

III'y the unlike Parallel forces P along BM and 2P along OF can be Compounded into a Single force P acting at C along CK.

Therefore left with a Couple of forces P at the end of the arm CD in a Plane Parallel to that of Original Couple.

(ie) The given Couple with the arm AB is equivalent to another Couple of the Same moment in a Parallel Plane, having its arm CD equal and $\parallel^{el}$ to AB.

Now this Couple with arm CD can be replaced in its own Plane by another Couple, Provided the moment is unchanged in magnitude and direction as in equivalence of two Couples.

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Hence a Couple in any Plane Can be replaced by another Couple acting in a Parallel Plane, Provided that moments of the two Couples are the Same in magnitude and Sign.

Representation of a Couple by a Vector

A Couple is specified if we know

(i) the direction of the Set of Parallel Planes.

(ii) the magnitude of its moment

(iii) the Sense in which it acts.

These three aspects of a Couple can be conveniently represented by a St. line drawn

(i) Perpendicular to the set of Parallel Planes to indicate the direction.

(ii) of a measured length to indicate the moment of the Couple.

(iii) in a definite direction to indicate the Sense of the moment.

Such a Vector which is represent a Couple is Called the Axis of the Couple.

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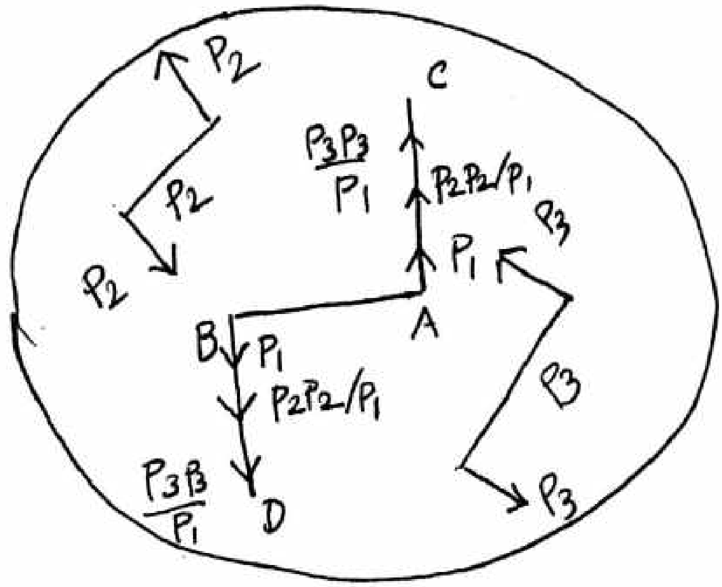
Resultant of Coplanar Couples:

Thm 4: The resultant of any number of couples in the same plane on a rigid body is a single couple whose moment is equal to the algebraic sum of the moments of the several couples.

Proof:



Let (P_1, P_1), (P_2, P_2), (P_3, P_3) etc. be a number of couples acting in the same plane upon a body.

Let AB be the arm p_1 of the first couple (P_1, P_1) whose component forces P_1 act along AC and BD.

The moment of the second couple $(P_2, P_2) = P_2 \cdot p_2$

This couple can be replaced by an equivalent couple having its arm along AB and having its forces along AC and BD.

If F is the force of such a replacing couple

We have $F \cdot P_1 = P_2 p_2 \Rightarrow F = \frac{P_2 p_2}{p_1}$

Thus the couple (P_2, P_2) is replaced by another couple whose arm coincides with AB and whose component force along AC and BD are magnitude $\frac{P_2 p_2}{p_1}$

III'ly the couple (P_3, P_3) is replaced by a couple $\left(\frac{P_3 p_3}{p_1}, p_1\right)$ with the forces $\frac{P_3 p_3}{p_1}$ along AC and BD.

This process is repeated for the other couples.

At last we get a single couple with the arm AB each of whose component forces is

$$P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \cdots$$

The moment of this resultant couple

$$= \left[P_1 + \frac{P_2 p_2}{p_1} + \frac{P_3 p_3}{p_1} + \cdots\right] \cdot p_1$$

$$= P_1 p_1 + P_2 p_2 + \cdots$$

= algebraic sum of the moments of the several couples.

Hence the thm. is proved.

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Results:

① If all the Component Couples have not the Same Sign we have merely to give each its Proper Sign and the Same Proof will apply.

② If all the Couples do not lie in the Same Plane but in different ||el Planes, they Can all be transferred into equivalent Couples in One Plane ||el to the given Planes and then their resultant Can be found.

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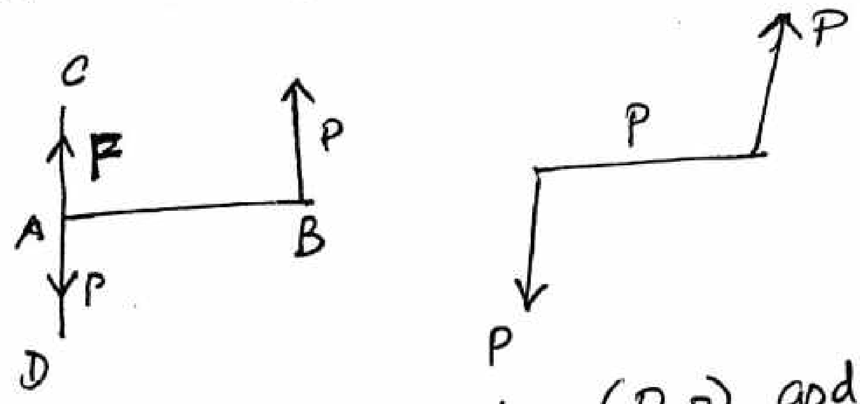
Resultant of a Couple and a Force

Thm 5: A Couple and a Single force acting on a body Cannot be in equilibrium but they are equivalent to the Single force acting at Some other Point Parallel to its original direction.

Pf:



Let the given Couple be (P, P) and the given force be F lying in the Same Plane.

Let F act along AC

Replace the Couple (P, P) by another Couple whose each force is equal to F.

If x be the length of the arm of this new Couple its moment is $F \cdot x = Pp$

$$\therefore x = \frac{Pp}{F}$$

Place this Couple such that one of its Component forces F acts at A along the line of action of the given force F but in the opposite direction.

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(ie) it acts along AD

The original force F along AC and the force F along AD balance. We are left with a Force F acting at B parallel to AC, as the statical equivalent of the system.

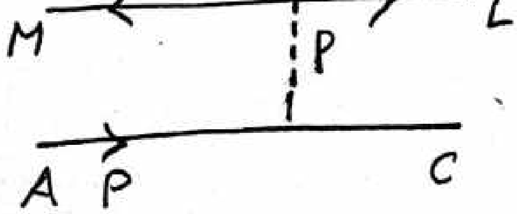
$$AB = x = \frac{Pp}{F}$$

Hence the Couple (P, P) and the force F are equivalent force F, Parallel to its original direction at a distance $\frac{Pp}{F}$ from its original line of action.

Thm 6: A force acting at any Point A of a body is equivalent to an equal and Parallel force acting at any other arbitrary Point B of the body together with a Couple.

Pf/ Let **P** be a force acting at A along AC and B any arbitrary Point.

Let p be the distance of B from AC



At B apply two equal and opposite forces each equal and parallel to P along BL and BM These two new forces being equal and opposite will have no effect on the body. Now the three forces acting on the body, the forces P along BM and P along AC form a Couple and the remaining is the force P acting at B $\parallel^{el}$ to the original force. Thus the ~~[illegible]~~ statical equivalent of the original force P at A is an equal and parallel force P at B together with a couple whose moment is Pp where p is the $\perp^r$ distance of B from AC.

Thm 7: If three forces acting on a rigid body be represented in magnitude and direction and line of action by the sides of a triangle taken in order, they are equivalent to a couple whose moment is twice the area of the triangle.

Proof: Let P, Q, R be three forces acting on a rigid body and represented in magnitude, direction and line of action by the sides

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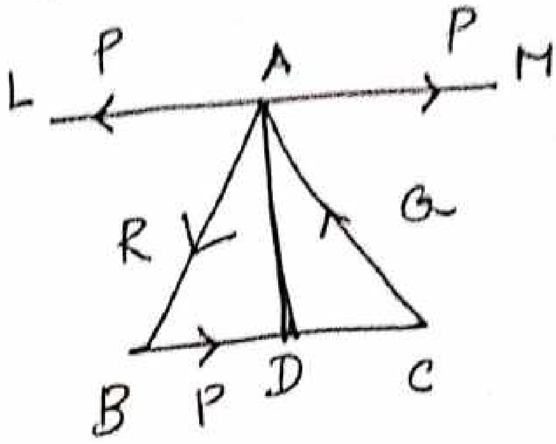
BC, CA, AB of the Δ^{le} ABC. Through A draw LAM $\parallel^{el}$ to BC and AD $\perp^{r}$ BC



At A, along AL and AM introduce two equal and opposite forces each equal to P. These two new forces being equal and opposite have no effect on the body.

Now the three forces P along AM, Q along CA, and R along AB act at the Point A and they are completely represented by the sides of the ΔABC taken in order. Hence the triangle of the forces are in equilibrium.

We are left with a force P along AL and a force P along BC.

These being two equal and opposite forces form a couple whose moment is

$$P \cdot AD = BC \cdot AD = 2\Delta ABC.$$

Hence the Thm/. is Proved.

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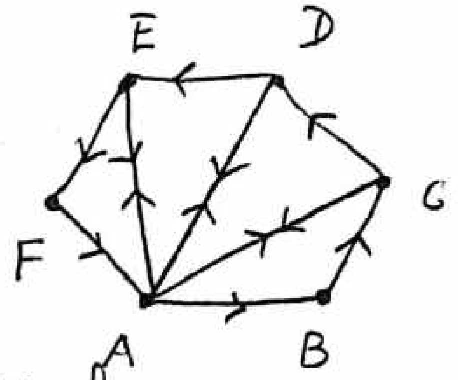
Thm 8: If any number of forces acting on a rigid body be represented in magnitude, direction and line of action by the sides of a Polygon taken in order they are equivalent to a couple whose moment is twice the area of the Polygon.

Proof: Let the forces be represented by the sides of the closed Polygon AB, BC, CD, DE, EF and FA of the closed Polygon ABCDEF. Join AC, AD and AE.



Introduce along AC, AD and AE Pairs of equal and opposite forces represented completely by these lines.

These new forces do not affect the resultant of the system.

Applying the 7 we get

$\overline{AB} + \overline{BC} + \overline{CA}$ = a Couple whose moment is equal to $2\Delta ABC$

$\overline{AC} + \overline{CD} + \overline{DA}$ = a Couple whose moment is equal to $2\Delta ACD$

$\overline{AD} + \overline{DE} + \overline{EA}$ = a Couple whose moment is equal to $2\Delta ADE$

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$\overline{AE} + \overline{EF} + \overline{FA}$ = a Couple whose moment is equal to $2\Delta AEF$

Now $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} + \overline{FA}$ = resultant of four Couples

= a single Couple whose moment is equal to

$2(\Delta ABC + \Delta ACD + \Delta ADE + \Delta AEF)$

Hence the resultant is a Couple whose moment is equal to twice the area of the Polygon ABCDEF.

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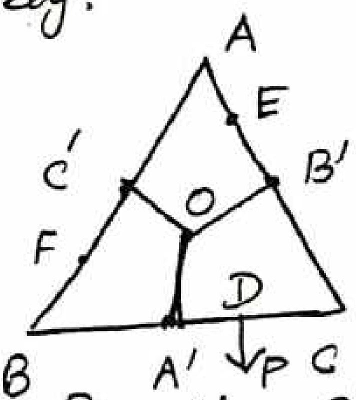
Problems:

① ABC is an equilateral triangle of side a. D E F divide the sides BC, CA, AB respectively in the ratio 2:1. Three forces each equal to P act at D, E, F perpendicularly to the sides and outward from the triangle. Prove that they are equivalent to a couple of moment $\frac{Pa}{2}$.

Soln: Let O be the circumcentre of the equilateral triangle and A′, B′, C′ be the mid points of the sides BC, CA and AB respectively.



Now $OA' \perp^r$ to BC

Applying Thm. 6

The force P acting at D $\perp^r$ to BC is equivalent to a parallel force P acting at O along OA′ together with a couple whose moment

$$= P.A'D = P.[A'C - DC]$$

$$= P.\left[\frac{a}{2} - \frac{a}{3}\right] = \frac{Pa}{6}$$

lll'y the force P acting at E $\perp$ to CA is replaced by a parallel force P acting at O along OB′ together with a couple whose moment is $\frac{Pa}{6}$

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The force P acting at F $\perp^r$ to AB is replaced by a Parallel force P acting at O along OC' together with a Couple whose moment is $\frac{Pa}{6}$

The three equal forces P acting O $\perp^r$ to the sides of the Δ^{le} are in equilibrium by the $\perp^r$ triangle of forces.

$\therefore$ The three Couples having the Same moment $\frac{Pa}{6}$ each in the Same direction are equivalent to a Single Couple whose moment is

$$3 \cdot \frac{Pa}{6} = \frac{Pa}{2}.$$

Hence the Condition is Proved.

(2) Five equal forces act along the Sides AB, BC, CD, DE, EF of a regular hexagon. Find the sum of the moments of these forces about a Point Q on AF at a distance x from A. Interpret the result and explain why it is so.

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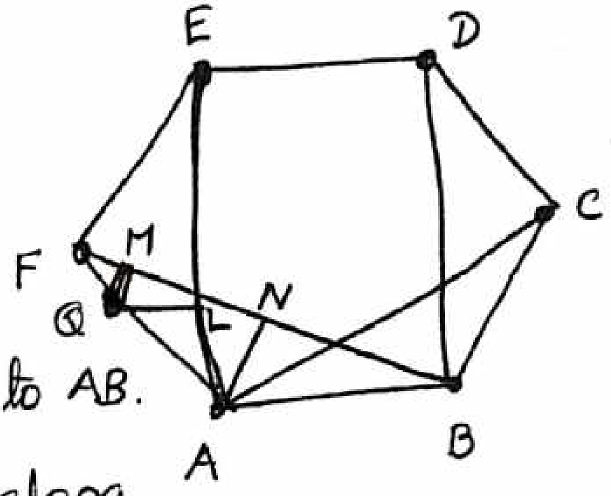
Soln Let a be the length of each side of the regular (ii)
hexagon. Each interior angle of the regular
hexagon is 120°



W.K.T

AB ∥ DE ; BC ∥ EF ;

DC ∥ EF

FB ⊥ BC, AE and DB ⊥r to AB.

Let equal force P act along
the sides AB, BC, CD, DF and EF.

Q is a point on AF ∋ AQ = x.

From Q draw QL ⊥r to EA and QM ⊥r to BF.

Let AN ⊥r to BF

$FB = FN + NB = a\cos 30° + a\cos 30°$ $\left[\because \Delta AFN \quad \cos 30° = \frac{FN}{AF} = \frac{FN}{a}\right.$

$= 2a\cos 30° = 2a \cdot \sqrt{3}/2$ $\left.\Delta ABN \cdot \quad \cos 30° = \frac{BN}{AB} = \frac{BN}{a}\right]$

Moment of P along AB about Q

$= P \cdot AL = P \cdot x \cos 30°$ (from the right angle ΔAQL)

$= Px\frac{\sqrt{3}}{2} \rightarrow (1)$

Moment of P along BC about Q

$= P \cdot MB = P[FB - FM]$

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$= P\left[2a\cos 30^\circ - (a-x)\cos 30^\circ\right]$ $\left[\because AF = a \quad AQ = x \quad QF = AF - AQ = a - x\right.$

In a right angle Δ^{le} FQM

$\cos 30^\circ = \frac{FM}{QF} = \frac{FM}{a-x}$

(ie) $\left.FM = (a-x)\cos 30^\circ\right]$

$= P(2a - a + x)\cos 30^\circ$

$= P(a+x)\cdot\frac{\sqrt{3}}{2} \longrightarrow (2)$

Moment of P along CD about Q

$= P.\ AC$ $[\because AF \parallel CD$ and $AC \perp CD]$

$= P.\ 2a\cos 30^\circ = P.\ 2a\cdot\frac{\sqrt{3}}{2} = Pa\frac{\sqrt{3}}{2} \rightarrow (3)$

Moment of P along DE about Q

$= P.\ EL = P[AE - AL]$

$= P\left[2a\cos 30^\circ - x\cos 30^\circ\right]$

$= P(2a - x)\cdot\frac{\sqrt{3}}{2} \rightarrow (4)$

Moment of P along EF about Q

$= P.\ MF = P(a-x)\cos 30^\circ = P(a-x)\cdot\frac{\sqrt{3}}{2} \rightarrow (5)$

Add the eqns. from (1) to (5) we get

$= Px\frac{\sqrt{3}}{2} + P(a+x)\frac{\sqrt{3}}{2} + P2a\frac{\sqrt{3}}{2} + P(2a-x)\frac{\sqrt{3}}{2}$
$+ P(a-x)\frac{\sqrt{3}}{2}$

$= P\frac{\sqrt{3}}{2}\left[x + a + x + 2a + 2a - x + a - x\right] = P.\frac{\sqrt{3}}{2}.6a$ (3)

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$= 3\sqrt{3}\,Pa$ = a Constant.

∴ The sum of the moments of the five forces about any point on the sixth side AF is Constant.

Introduce two equal and opposite forces each equal to P along the sixth side.

These new forces do not affect the resultant of the system.

We have seven forces. The moment of the new force P introduce along AF about Q is zero.

The other six forces act along the sides of a hexagon and are represented in magnitude, direction and line of action by the sides of a hexagon.

Hence by Thm. 8. They are equivalent to a couple whose moment is 2 × area of the hexagon

$$= 2 \times 6 \times a^2 \frac{\sqrt{3}}{4}$$

$$= 3a^2\sqrt{3} = 3\sqrt{3}\,aP$$

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