SEMESTER : V
CORE COURSE : X

Inst Hour	: 7
Credit	: 6
Code	: 18K5M10

STATICS

UNIT1:
Forces Acting at a Point – Parallel forces.
(Chapter2: & Chapter 3: Sections 1 to 6)

UNIT 2:
Moment of a Force about a point on a line – Theorem on Moments & Couples
(Chapter 3: sections 7 to 14 & Chapter 4)

UNIT 3:
Equilibrium of three forces acting on a Rigid body – Coplanar forces (Simple Problems only).
(Chapter 5: section 1to7, Chapter 6: Section 1to 13)

UNIT 4:
Equilibrium of Strings under gravity – Common Catenary –Parabolic Catenary –Suspension Bridge.
(Chapter 11)

UNIT 5:
Friction – Laws of Friction –Coefficient of Friction, Angle & Cone of Friction – Equilibrium of a particle on a rough inclined plane under a force parallel to the plane and under any force – Problems on Friction. (Simple Problems only)
(Chapter 7: Sections 1to13)

Text Book

[1] M.K. Venkataraman, Statics, Agasthiar Publication,18th Edition, 2016

Books for Reference

[1] S.Narayanan., Statics.

[2] A.V.Dharmapadham, Statics.

Question Pattern (Both in English & Tamil Version)

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

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UNIT - 3

Equilibrium of three forces Acting on a Rigid body. (1)

1. Two trigonometrical theorems!

If D is any point on the base BC of triangle ABC such that $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$.

then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta \rightarrow (1)$

and $(m+n)\cot\theta = n\cot B - m\cot C \rightarrow (2)$.

Proof :-

(1) $\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{DA} \cdot \frac{DA}{DC}$

$= \frac{\sin\angle BAD}{\sin\angle ABD} \cdot \frac{\sin\angle ACD}{\sin\angle DAC}$

$= \frac{\sin\alpha}{\sin(\theta-\alpha)} \times \frac{\sin(\theta+\beta)}{\sin\beta}$.

$[\because \angle ACD = 180° - (\theta+\beta)]$

$= \frac{\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)}{\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)}$

$= \frac{\cot\beta + \cot\theta}{\cot\alpha - \cot\theta}$ (dividing the numerator and denominator by $\sin\alpha\sin\beta\sin\theta$).

$\therefore n(\cot\beta + \cot\theta) = m(\cot\alpha - \cot\theta)$

(or) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(2) Again,

$$\frac{m}{n} = \frac{\sin \angle BAD}{\sin \angle ABD} \cdot \frac{\sin \angle ACD}{\sin \angle DAC}$$

$$= \frac{\sin(\theta - B) \cdot \sin C}{\sin B \cdot \sin(C+\theta)} \quad (\because \angle DAC = 180^\circ - \overline{\theta + C})$$

$$= \frac{\sin C (\sin\theta \cos B - \cos\theta \sin B)}{\sin B \cdot (\sin C \cos\theta + \cos C \sin\theta)}$$

$$= \frac{\cot B - \cot\theta}{\cot\theta + \cot C}$$

(dividing the numerator and denominator by $\sin B \sin C \sin\theta$).

(i.e) $m(\cot\theta + \cot C) = n(\cot B - \cot\theta)$

(or) $(m+n)\cot\theta = n\cot B - m\cot C$.

2. A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length l, tied to one end of the rod, the other end of the string being attached to a point in the wall. Show that the rod can rest inclined to the wall at an angle θ given by

$$\cos^2\theta = \frac{l^2 - a^2}{3a^2}$$

What are the limits of the ratio of $a : l$ in order that equilibrium may be possible ?

AB is the rod of length a, with G its centre of gravity and BC is the string of length l.

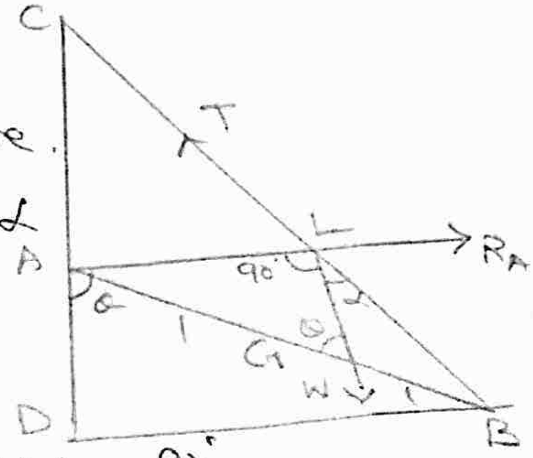
The forces acting on the rod are.

(i) its weight w acting vertically downwards through G.

(ii) its reaction R_A at A which is normal to the wall and therefore horizontal.

(iii) the tension T of the string along BC.

These three forces in equilibrium not being all parallel, must meet in a point L, as shown in the figure.



Let the string make an angle α with the vertical.

$\therefore \angle ACB = \alpha = \angle GLB$

Also $\angle LGB = 180° - \theta$ and $\angle ALG = 90°$

Using the first trigonometrical theorem to $\triangle ALB$ and noting that $AG : GB = 1:1$, we have

$(1+1) \cot(180° - \theta) = 1 . \cot 90° - 1 . \cot \alpha$

(i) $\quad -2 \cot\theta = -\cot\alpha$

(or) $2 \cot\theta = \cot\alpha \rightarrow (1)$

Draw BD ⊥ to CA.

From rt ∠d △ CDB, $BD = BC.\sin\alpha$
$= l.\sin\alpha$ and

from rt ∠d △ ABD $BD = AB\sin\theta . a\sin\theta$.

$\therefore l\sin\alpha = a\sin\theta \rightarrow (2)$

Eliminate α between (1) and (2)

We know that $\operatorname{cosec}^2\alpha = 1+\cot^2\alpha \rightarrow (3)$

From (2), $\sin\alpha = \dfrac{a\sin\theta}{l}$

$\therefore \operatorname{Cosec}\alpha = \dfrac{l}{a\sin\theta} \rightarrow (4)$.

Substituting (4) and (1) in (3), we have

$$\frac{l^2}{a^2\sin^2\theta} = 1+4\cot^2\theta.$$

(i) $\dfrac{l^2}{a^2} = \sin^2\theta + 4\cos^2\theta = 1+3\cos^2\theta$.

$\therefore 3\cos^2\theta = \dfrac{l^2}{a^2} - 1 = \dfrac{l^2-a^2}{a^2}$

$\therefore \cos^2\theta = \dfrac{l^2-a^2}{3a^2} \rightarrow (5)$

For the above equilibrium position to be possible, $\cos^2\theta$ must be positive and less than 1.

$\therefore l^2-a^2 > 0$

(i) $l^2 > a^2$ (or) $a^2 < l^2$

Also $\dfrac{l^2-a^2}{3a^2} < 1$

(i) $l^2 - a^2 < 3a^2$ (or) $l^2 < 4a^2$

(ii) $a^2 > \frac{l^2}{4}$

$\therefore a^2$ lies between $\frac{l^2}{4}$ and l^2

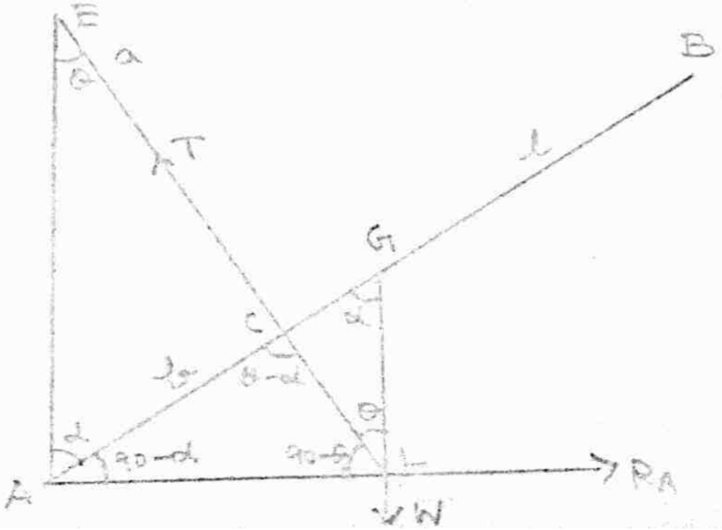
(i) $\frac{a^2}{l^2}$ lies between $\frac{1}{4}$ and 1

(or) $\frac{a}{l}$ must lie between $1/2$ and 1.

8. A uniform rod of length $2l$ rests with its lower end in contact with a smooth vertical wall. It is supported by a string of length a, one end of which is fastened to a point in the wall and the other end to a point in the rod at a distance b from its lower end. If the inclination of string to the vertical be θ, show that $\cos^2\theta = \frac{b^2(a^2-b^2)}{a^2 l(2b-l)}$

The forces acting on the rod AB are

(i) reaction R_A at A perpendicular to the wall and hence horizontal.



(ii) its weight W acting vertically downwards through G, the mid point of AB and. (6)

(iii) the tension T of the string along CE.

Let the forces (i) and (ii) meet at L. For equilibrium, T also must pass through L as shown in the figure.

$AC = b$, $CE = a$ and $AG = l = GB$.

Let $\angle EAB = \alpha$, Then $\angle CGL = \alpha$. and $\angle CAL = 90° - \alpha$

$\angle CLA = 90° - \theta$ and $\angle CLG = \theta$

From ΔACE, we have.

$$\frac{AC}{\sin\theta} = \frac{EC}{\sin\alpha} \quad (\text{ie}) \quad \frac{b}{\sin\theta} = \frac{a}{\sin\alpha}$$

(or) $b\sin\alpha = a\sin\theta \rightarrow$ (1).

From ΔACL,

$$\frac{CL}{\sin(90° - \alpha)} = \frac{AC}{\sin(90° - \theta)}$$

ie) $$\frac{CL}{\cos\alpha} = \frac{b}{\cos\theta} \quad (\text{or}) \quad CL = \frac{b\cos\alpha}{\cos\theta} \rightarrow (2)$$

From ΔCGL,

$$\frac{CL}{\sin\alpha} = \frac{CG}{\sin\theta}$$

$$\therefore CL = CG \cdot \frac{\sin\alpha}{\sin\theta} = (AG - AC)\frac{\sin\alpha}{\sin\theta}$$

$$= (l-b)\frac{\sin\alpha}{\sin\theta} \rightarrow (3)$$

Equating (2) and (3), we get.

$$(l-b)\frac{\sin\alpha}{\sin\theta} = b\cdot\frac{\cos\alpha}{\cos\theta}.$$

(i.e) $$\frac{\cos\alpha}{\sin\alpha} = \left(\frac{l-b}{b}\right)\frac{\cos\theta}{\sin\theta}$$

(i.e) $$\cot\alpha = \frac{l-b}{b}\cot\theta \rightarrow (4)$$

we have now to eliminate α between (1) and (4).

$$\operatorname{cosec}^2\alpha = 1+\cot^2\alpha.$$

(i.e) $$\frac{1}{\sin^2\alpha} = 1+\cot^2\alpha,$$

(i.e) $$\frac{1}{\left(\frac{a\sin\theta}{b}\right)^2} = 1+\left(\frac{l-b}{b}\cot\theta\right)^2$$

[using (1) and (4)]

(i.e) $$\frac{b^2}{a^2\sin^2\theta} = 1+\frac{(l-b)^2}{b^2}\cdot\frac{\cos^2\theta}{\sin^2\theta}.$$

$$\frac{b^2}{a^2} = \sin^2\theta + \frac{(l-b)^2}{b^2}\cos^2\theta.$$

$$= 1-\cos^2\theta + \frac{(l-b)^2}{b^2}\cos^2\theta.$$

$$\therefore \cos^2\theta\left[1-\frac{(l-b)^2}{b^2}\right] = 1-\frac{b^2}{a^2}$$

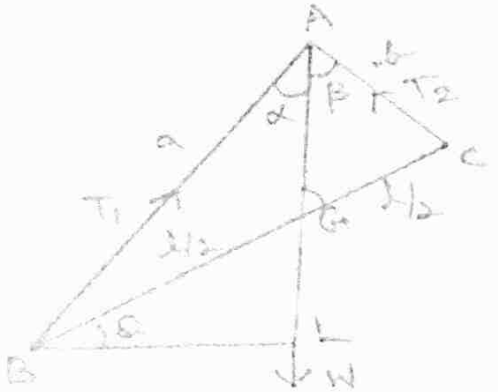
$$\frac{\cos^2\theta \cdot (b^2 - (l-b)^2)}{b^2} = \left(\frac{a^2-b^2}{a^2}\right)$$

$$\cos^2\theta = \frac{b^2 \cdot (a^2-b^2)}{a^2\left[b^2-(l-b)^2\right]}$$

$$= \frac{b^2 (a^2-b^2)}{a^2 (2bl - l^2)}.$$

4. A uniform beam of length l and weight W hangs from a fixed point by two strings of lengths a and b. Prove that the inclination of the rod to the horizon is $\sin^{-1} \frac{a^2-b^2}{l\sqrt{2(a^2+b^2)-l^2}}$

Find also that tensions of the strings.



G is the mid. Pt. of the beam BC. BA = a and AC = b are the strings. The forces acting on the beam are.

(i) Tension T_1 along BA

(ii) Tension T_2 along CA and

(iii) weight W acting vertically downwards through G. The forces (i) and (ii) meet at A.

$\therefore$ For equilibrium, AG is vertical.

Let BC make an angle θ with the horizontal

Draw $BL \perp$ to AG.

We shall calculate BL by two methods.

From rt $\angle$d Δ BGL, $BL = BG.\cos\theta = \frac{l}{2}\cos\theta \rightarrow$ ①

From rt $\angle$d Δ ABL, $BL = AB.\cos\angle ABL$

$$= a\cos(B+\theta)$$

$$= a(\cos B\cos\theta - \sin B\sin\theta) \rightarrow ②$$

Equating ① and ②

$$\frac{l}{2}\cos\theta = a\cos B\cos\theta - a\sin B\sin\theta.$$

$$\therefore a\sin B\sin\theta = \cos\theta\left(a\cos B - l/2\right) \rightarrow ③$$

$$\frac{\sin\theta}{\cos\theta} = \frac{a\cos B - l/2}{a\sin B} = \frac{2a\cos B - l}{2a\sin B}.$$

$$\text{(ie) } \tan\theta = \frac{2a\cos B - l}{2a\sin B} \rightarrow ④.$$

We have to calculate $\sin\theta$ from ④.

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}}$$

using ④

$$= \frac{1}{\sqrt{1+\frac{4a^2\sin^2 B}{(2a\cos B - l)^2}}}$$

$$= \frac{2a\cos B - l}{\sqrt{(2a\cos B - l)^2 + 4a^2\sin^2 B}}$$

$$= \frac{2a\cos B - l}{\sqrt{4a^2(\cos^2 B + \sin^2 B) - 4al\cos B + l^2}}$$

$$= \frac{2a\cos B - l}{\sqrt{4a^2 - 4al\cos B + l^2}} \rightarrow (5)$$

From Δ ABC, we have $b^2 = a^2 + l^2 - 2al\cdot\cos B$.

$$\therefore \cos B = \frac{a^2 + l^2 - b^2}{2al} \rightarrow (6)$$

putting (6) in (5), we have

$$\sin\theta = \frac{\frac{a^2 + l^2 - b^2}{l} - l}{\sqrt{4a^2 - 2(a^2 + l^2 - b^2) + l^2}}$$

$$= \frac{a^2 + l^2 - b^2 - l^2}{l\sqrt{2a^2 + 2b^2 - l^2}}$$

$$= \frac{a^2 - b^2}{l\sqrt{2(a^2 + b^2) - l^2}} \rightarrow (7).$$

$$\therefore \theta = \sin^{-1}\frac{a^2 - b^2}{\sqrt{2(a^2 + b^2) - l^2}}$$

Let $\angle BAG = \alpha$ and $\angle CAG = \beta$.

Applying Lami's theorem,

$$\frac{T_1}{\sin\beta} = \frac{T_2}{\sin\alpha} = \frac{W}{\sin A} \rightarrow (8)$$

$$\therefore T_1 = \frac{W\sin\beta}{\sin A} \rightarrow (9)$$

From ΔAGC, $\frac{GC}{\sin\beta} = \frac{AC}{\sin(90^\circ-\theta)}$

i.e.) $\frac{l}{2\sin\beta} = \frac{b}{\cos\theta}$

$$\therefore \sin\beta = \frac{l\cos\theta}{2b}.$$

$$\therefore (9) \text{ gives } T_1 = \frac{Wl\cos\theta}{2b\sin A} \rightarrow (10)$$

But $\frac{l}{\sin A} = \frac{b}{\sin B}$

$$\therefore b\sin A = l\sin B.$$

$$\therefore (10) \text{ becomes } T_1 = \frac{Wl\cos\theta}{2l\sin B}.$$

$$= \frac{W}{2}\frac{\cos\theta}{\sin B}$$

$$= \frac{W}{2}\cdot\frac{a\sin\theta}{(a\cos B - l/2)} \quad \text{using (3)}$$

$$= \frac{W}{2}\cdot\frac{a(a^2-b^2)}{l\cdot\sqrt{2(a^2+b^2)-l^2}} \times \frac{1}{\left[a\left(\frac{a^2+l^2-b^2}{2al}\right)-\frac{l}{2}\right]}$$

using (7) and (6)

$$= \frac{Wa(a^2-b^2)}{2l\sqrt{2(a^2+b^2)-l^2}} \cdot \frac{2l}{(a^2+l^2-b^2-l^2)}$$

$$= \frac{Wa}{\sqrt{2(a^2+b^2)-l^2}}$$

Similarly we can show that

$$T_2 = \frac{Wb}{\sqrt{2(a^2+b^2)-l^2}}.$$

UNIT - 4

Equilibrium of strings.

Definitions!-

When a uniform string or chain hangs freely between two points not in the same vertical line, the curve in which it hangs under the action of gravity is called a catenary. If the weight per unit length of the chain (or) string is constant. the catenary is called the uniform (or) common catenary.

Equation of the Common Catenary!-

A uniform heavy inextensible string hangs freely under the action of gravity. to find the equation of the curve which it forms.

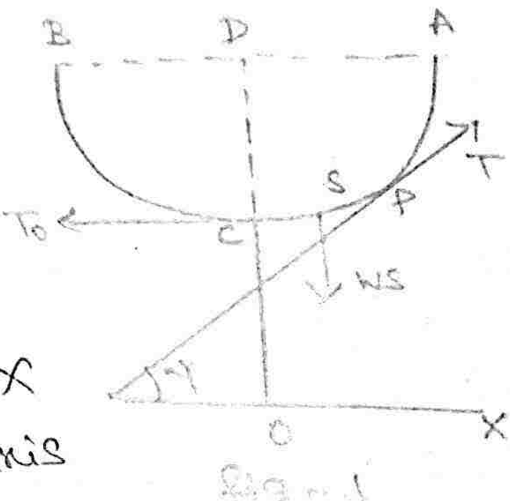


fig-1

Let ACB be a uniform heavy flexible cord attached to two points A and B at the same level, C being the lowest of the cord. Draw CO vertical, OX horizontal and take OX as x axis and OC as y axis. Let P be any point of the string so that the length of the arc CP = S.

Let w be the weight per unit length of the chain. (2)

Consider the equilibrium of the portion CP of the chain.

The forces acting on it are!

(i) Tension T_0 acting along the tangent at C and which is therefore horizontal.

(ii) tension T acting at P along the tangent at P making an angle ψ with Ox.

(iii) its weight ws acting vertically downwards through the C.G' of the arc CP.

For equilibrium, these three forces must be concurrent.

Hence the line of action of the weight ws must pass through the point of intersection of T and T_0.

Resolving horizontally and vertically we have

$$T\cos\psi = T_0 \rightarrow (1)$$

and

$$T\sin\psi = ws \rightarrow (2)$$

Dividing (2) by (1), $\tan\psi = \dfrac{ws}{T_0}$.

Now it will be convenient to write the value of T_0 the tension at the lowest point, as $T_0 = wc \rightarrow (3)$ where c is a constant. This means that we assume

T_0, to be equal to the weight of an unknown length c of the cable.

Then. $\tan\psi = \frac{ws}{wc} = \frac{s}{c}$

$\therefore s = c\tan\psi \rightarrow (4)$

Equation (4) is called the intrinsic equation of the catenary.

Problem:1

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be $\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$

Solution:

Let y_A and y_C be the y-coordinates of the highest point A and the lowest point C. Let w be the weight per unit length of the chain and c the parameter of the catenary.

Tension at $A = wy_A$.

and tension at $C = w.y_C$ Refer fig-1

since $T = wy$ at any point

Now $w.y_A = n.wy_C$.

$\therefore y_A = ny_C = nc$.

But $y_A = c\cosh\frac{x_A}{c}$

$= nc$.

$\therefore \cosh \frac{x_A}{c} = n.$

(or) $\frac{x_A}{c} = \cosh^{-1} n = \log\left(n + \sqrt{n^2-1}\right)$

$\therefore x_A = c \log\left(n + \sqrt{n^2-1}\right) \rightarrow (1)$

we have to find c.

$y_A^2 = c^2 + s_A^2$, s_A denoting the length of CA.

$= c^2 + \frac{l^2}{4}$ (as total length $= l$)

(ie) $n^2c^2 = c^2 + \frac{l^2}{4}$.

(or) $c^2 = \frac{l^2}{4(n^2-1)}$

$\therefore c = \frac{l}{2\sqrt{n^2-1}} \rightarrow (2)$

Substituting (2) in (1),

$x_A = \frac{l}{2\sqrt{n^2-1}} \cdot \log\left(n + \sqrt{n^2-1}\right)$

$\therefore$ span $AB = 2x_A = \frac{l}{\sqrt{n^2-1}} \log\left(n + \sqrt{n^2-1}\right)$

Problem : 2.

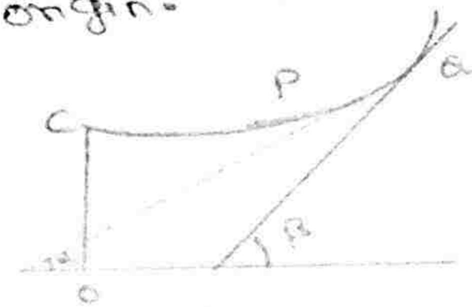
If α, β be the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary and l the length of the portion, show that the height of the one extremity above the other

is $l. \dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$, the two extremities being

on one side of the vertex of the catenary.

C is the vertex of the Catenary.

O is the origin. $CO = c$.



Let P and Q be the two extremities of a portion.

It is given that $\psi_P = \alpha$ and $\psi_Q = \beta$

since arc $PQ = l$,

$$S_Q - S_P = l \longrightarrow (1)$$

But $S = c\tan\psi$.

$\therefore S_Q = c\tan\psi_Q = c\tan\beta$ and $S_P = c\tan\alpha$.

Hence (1) becomes $c(\tan\beta - \tan\alpha) = l$.

$$(or)\ c = \frac{l}{\tan\beta - \tan\alpha} \longrightarrow (2)$$

We know that $y = c\sec\psi$.

$\therefore y_Q = c\sec\beta$ and $y_P = c\sec\alpha$

$$\therefore y_a - y_p = c(\sec\beta - \sec\alpha)$$

$$= \frac{l}{(\tan\beta - \tan\alpha)} \cdot (\sec\beta - \sec\alpha)$$

$$= \frac{l}{\left(\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha}\right)} \cdot \left(\frac{1}{\cos\beta} - \frac{1}{\cos\alpha}\right)$$

$$= \frac{l(\cos\alpha - \cos\beta)}{(\sin\beta\cos\alpha - \cos\beta\sin\alpha)}$$

$$= \frac{l(\cos\beta - \cos\alpha)}{\sin(\alpha - \beta)}$$

$$= \frac{l \cdot 2\sin\frac{\beta+\alpha}{2} \cdot \sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha-\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}}$$

$$= \frac{l\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

Problem: 3.

Show that the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with two-thirds of the circumference of the pulley is $a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right]$

Solution :-

Let CBLAC be an endless chain hanging over the circular pulley MBLA of radius a.

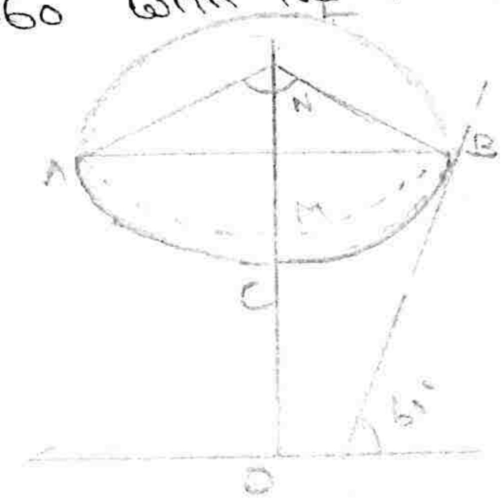
The portion ALB $= \frac{2}{3} \times$ Circumference of the pulley.

$$= \frac{2}{3} \times 2\pi a = \frac{4\pi a}{3}.$$

The remaining portion ACB will hang in the form of a Catenary with C as the lowest point.

Since $\angle AO'B = 120^\circ$, $\angle O'BA = 30^\circ$.

The tangent at B is $\perp$ to $O'B$ and so it makes 60° with the horizontal.



Let the origin, as usual, be taken at a depth c below C.

B is a point of the circle and the catenary,

x coordinate of B $= NB = O'B \cos 30^\circ$

$$= \frac{a\sqrt{3}}{2}$$

Since B is also on the catenary,

$$x = c\log(\sec\psi + \tan\psi) \longrightarrow (1)$$

Applying (1) at the point B where $\psi = 60°$,

we have $\frac{a\sqrt{3}}{2} = c\log(\sec 60° + \tan 60°)$

$$= c\log(2+\sqrt{3})$$

$$\therefore c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})} \longrightarrow (2)$$

Now $s = c\tan\psi \longrightarrow (3)$

Applying (3) for the point B, we have

arc CB $= s = c\tan 60°$

$$= \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}\cdot\sqrt{3}.$$

$$= \frac{3a}{2\log(2+\sqrt{3})} = \text{arc AC also}$$

Hence length of the chain

$$= \frac{4\pi a}{3} + 2\,\frac{3a}{2\log(2+\sqrt{3})}$$

$$= a\left[\frac{4\pi}{3} + \frac{3}{\log(2+\sqrt{3})}\right]$$

Problem: 4

A telegraph wire stretched two poles at a distance 'a' feet apart, sags n feet in the middle. Prove that the tension at the ends is approximately $w\left(\frac{a^2}{8h}+\frac{7h}{6}\right)$ where w is the weight of unit length of the wire.

Solution:- Refer to figure 6. C is the vertex and A the highest point of the catenary.

we have $y = c\cosh x/c$

At A, $x = a/2$.

$$y \text{ at } A = c\cosh\frac{a}{2c}$$

$$= c\left[e^{a/2c}+e^{-a/2c}\right]$$

$$= \frac{c}{2}\left[\left[\left(1+\frac{a}{2c}+\frac{a^2}{8c^2}+\frac{a^3}{48c^3}+\dots\right)\right]\right.$$

$$\left.+\left(1-\frac{a}{2c}+\frac{a^2}{8c^2}-\frac{a^3}{48c^3}+\dots\right)\right]$$

$$= \frac{c}{2}\cdot 2\left(1+\frac{a^2}{8c^2}+\frac{a^4}{384c^4}+\dots\right)$$

$$= c+\frac{a^2}{8c}+\frac{a^4}{384c^3}+\dots \quad (1)$$

y at A is also $= c+h \rightarrow (2)$ since h is the sag.

$$\therefore c+h = c+\frac{a^2}{8c}+\frac{a^4}{384c^3}+\dots$$

(or) $$h = \frac{a^2}{8c}+\frac{a^4}{384c^3}+\dots \quad (3)$$

we have to determine c using equation (3)

As a first approximation, keep only the first term in the right side of (3), This means that c is large and n is small.

Then $n = \frac{a^2}{8c}$ (or) $c = \frac{a^2}{8n} \rightarrow (4)$

As a second approximation in equation (3),

Let us put $c = \frac{a^2}{8n} + h \rightarrow (5)$

Here h is small and we now take the first two terms in the right side of (3).

Applying (5) in (3), we have

$$n = \frac{a^2}{8\left(\frac{a^2}{8n} + h\right)} + \frac{a^4}{384\left\{\frac{a^2}{8n}\left(1+\frac{8nh}{a^2}\right)\right\}^3}$$

$$= \frac{a^2}{8 \cdot \frac{a^2}{8n}\left(1+\frac{8nh}{a^2}\right)} + \frac{a^4}{384\left\{\frac{a^2}{8n}\left(1+\frac{8nh}{a^2}\right)\right\}^3}$$

$$= n\left(1+\frac{8nh}{a^2}\right)^{-1} + \frac{a^4(64) \times 8n^3}{384 \cdot a^6} \cdot \left(1+\frac{8nh}{a^2}\right)^{-3}$$

$$= n\left(1-\frac{8nh}{a^2}\right) + \frac{4n^3}{3a^2}\left(1 - \frac{3\times 8nh}{a^2}\right)$$

expanding by the binomial theorem and omitting h^2, h^3 etc.

(i.e) $n = n - \frac{8n^2h}{a^2} + \frac{4n^3}{3a^2} - \frac{32n^4h}{a^4}$

(i.e) $0 = n^2\left(-\frac{8h}{a^2} + \frac{4n}{3a^2} - \frac{32n^2h}{a^4}\right)$

(i.e) $-\frac{8h}{a^2} + \frac{4n}{3a^2} - \frac{32n^2h}{a^4} = 0 \rightarrow (6)$.

In equation (6), we can keep only the first power of n and omit n^2.

Also since h and n are both small, the term containing n^2h can be omitted.

Then $\frac{-8h}{a^2} + \frac{4n}{3a^2} = 0$

(or) $h = \frac{n}{6} \rightarrow (7)$.

$\therefore$ From (5), $c = \frac{a^2}{8n} + \frac{n}{6} \rightarrow (8)$

From (2),

y at $A = \frac{a^2}{8n} + \frac{n}{6} + n$

$= \frac{a^2}{8n} + \frac{7n}{6}$

Hence tension at $A = w\left(\frac{a^2}{8n} + \frac{7n}{6}\right)$ nearly.

UNIT - 5

Friction.

1. Definitions:-

If two bodies are in contact with one another the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called friction. The force exerted is called the force of friction.

2. Statical, Dynamical and Limiting Friction.

When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called statical friction.

When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called limiting friction, the equilibrium in this case is said to be limiting.

When motion ensues by one body sliding over another, the friction exerted is called dynamical friction.

3. Define the coefficient of friction.

By law 3, we know that limiting friction between two bodies bears a constant ratio to the normal reaction between them.

The ratio of the limiting friction to the normal reaction is called the Coefficient of friction. It is usually denoted by the letter μ. ②

Let F be the friction and R the normal reaction between two bodies when equilibrium is limiting.

Then $\frac{F}{R} = \mu$, ie) $F = \mu R$.

The constant μ depends on the nature of the materials in Contact. It is different for different pairs of Substances and is ordinarily less than unity.

Since friction is maximum when it is limiting, μR is the maximum value of friction. When equilibrium is nonlimiting F is less than μR and $\frac{F}{R} < \mu$.

4. Define Cone of friction.

The greatest angle which the direction of the resultant reaction can make with the normal is λ. (ie) $\tan^{-1}(\mu)$.

Now the motion of one body at O, its point of Contact with another, can take place in any direction perpendicular to the normal. Hence when two bodies are in Contact, we can Consider a Cone drawn with the point of Contact as the vertex, the Common normal as the axis and its Semi-vertical angle being equal to λ, the angle of friction.

3

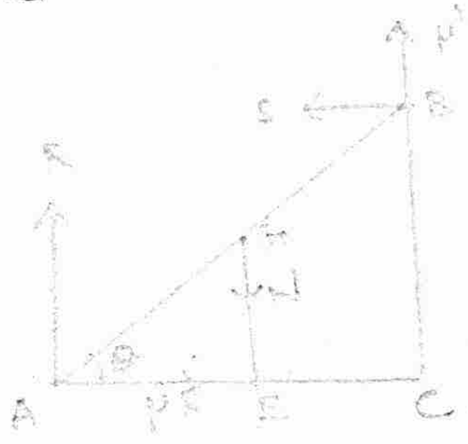
It is clear that the resultant reaction will have a direction which entirely lies within the surface (or) on the surface of that cone. It cannot fall outside the cone, such a cone is called the cone of friction.

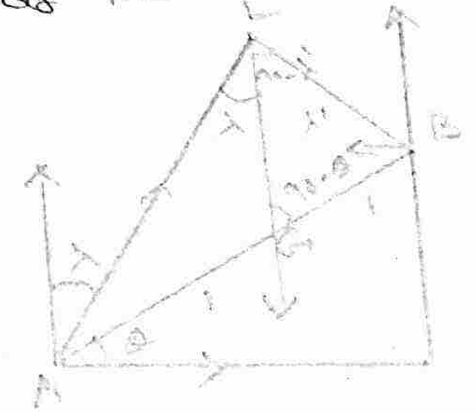
5. What is the angle of friction.

When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called the angle of friction and it is denoted by λ.

6. A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall. If the ground and wall be both rough, the coefficients of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$.

Find also the reactions at the wall and ground.





First method :- (4)

Let AB $(=2a)$ be the ladder, G its centre of gravity and W its weight. Let R and S be the normal reactions acting on the ladder at the ground and wall.

The lower end A will have a tendancy to move away from the wall and hence friction there will be acting in the direction AC. The upper end B will have a tendancy to move downwards and hence friction there will be acting upwards. Since both ends are slipping, the frictions at A and B are limiting. They are μR and $\mu' S$ as marked in the figure.

Resolving horizontally, $S = \mu R \rightarrow (1)$

Resolving vertically, $\mu' S + R = W \rightarrow (2)$

Taking moments about A,

$$S . BC + \mu' S . AC = W . AE$$

i.e) $S . 2a \sin\theta + \mu' S . 2\cos a \cos\theta = W . a\cos\theta \rightarrow (3)$

putting the value of S from (1) in (2), we have

$$\mu' \mu R + R = W.$$

$$\text{i.e)} \quad R = \frac{W}{1+\mu\mu'} \rightarrow (4)$$

Then, from (1), $S = \dfrac{\mu W}{1+\mu\mu'} \rightarrow (5)$

(4) and (5) give the reactions.

putting the value of S from (5) in (3), we get

$$\frac{\mu W}{1+\mu\mu'} 2\sin\theta + \frac{\mu' \cdot \mu W}{1+\mu\mu'} 2\cos\theta = W\cos\theta.$$

(i.e.) $2\mu\sin\theta + 2\mu\mu'\cos\theta = (1+\mu\mu')\cos\theta.$

(i.e.) $2\mu\sin\theta = (1-\mu\mu')\cos\theta.$

(or) $\frac{\sin\theta}{\cos\theta} = \frac{1-\mu\mu'}{2\mu} = \tan\theta \rightarrow (6).$

7. A Square lamina whose plane is vertical rests with the ends of a side against a rough vertical wall and a rough horizontal ground. If the coefficients of friction for the ground and the wall be μ and μ' respectively. Prove that when the lamina is on the point of motion, the inclination of the side in question to the horizontal is $\tan^{-1}\left(\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}\right)$.

Solution:-

Let ABCD be the square lamina and the side AB (= a) be in contact with the wall at A and the ground at B. The forces acting on the lamina are:

(i) normal reaction R at B perpendicular to the ground.

(ii) limiting friction μR at B as shown in the figure. ⑥

(iii) normal reaction S at A perpendicular to the wall.

(iv) limiting friction $\mu' S$ along the wall upwards and

(v) its weight W acting vertically downwards at G, its centre of gravity.

Let AB make an angle θ to the horizontal.

Resolving horizontally and vertically.

$$S = \mu R \rightarrow (1)$$

$$R + \mu' S = W \rightarrow (2),$$

Taking moments about B,

$$S \cdot OA + \mu' S \cdot OB = W \cdot BE.$$

$$S \cdot OA + \mu' S \cdot OB = W \cdot BG \cos(45^\circ + \theta).$$

(i.e) $S \cdot a\sin\theta + \mu' S \cdot a\cos\theta = W \cdot BG\cos(45^\circ+\theta)$

$$= \frac{Wa}{\sqrt{2}}\left(\cos 45^\circ \cos\theta - \sin 45^\circ \sin\theta\right)$$

$$= \frac{Wa}{2}\left(\cos\theta - \sin\theta\right).$$

(i.e) $S(\sin\theta + \mu'\cos\theta) = \frac{W}{2}(\cos\theta - \sin\theta) \rightarrow (3).$

From (2), $R = W - \mu' S$ and putting this in (1),

we have, $S = \mu(W - \mu' S)$

(i.e) $S = \dfrac{\mu W}{1+\mu\mu'} \rightarrow (4).$

Substituting (4') in (3'),

$$\frac{\mu W}{1+\mu\mu'}(\sin\theta+\mu'\cos\theta)=\frac{W}{2}(\cos\theta-\sin\theta).$$

(i) $2\mu(\sin\theta+\mu'\cos\theta) = (1+\mu\mu')(\cos\theta-\sin\theta)$

$\sin\theta(2\mu+1+\mu\mu') = \cos\theta(1+\mu\mu'-2\mu\mu')$

$$\therefore \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}$$

$$(\text{or})\quad \theta=\tan^{-1}\left(\frac{1-\mu\mu'}{1+2\mu+\mu\mu'}\right).$$

8. A solid homogeneous hemisphere rests on a rough horizontal plane and against a smooth vertical wall. Show that if the coefficient of friction be greater than 3/8, the hemisphere can rest in any position and, if it be less, the least angle that the base of the hemisphere can make with the vertical is $\cos^{-1}\left(\frac{8\mu}{3}\right)$. If the wall be rough (coefficient of friction μ') show that the angle is $\cos^{-1}\frac{8\mu(1+\mu')}{3(1+\mu\mu')}$.

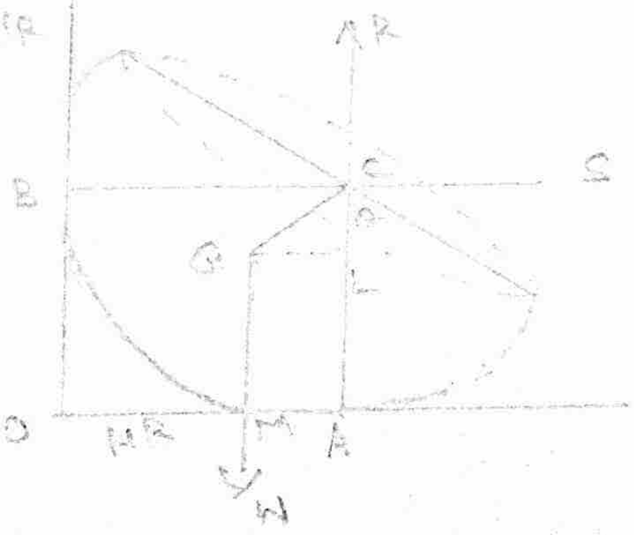


Figure represents a section of the hemisphere by the plane perpendicular to the wall. A is the point of contact with the rough horizontal plane and B with the smooth wall. LM is the base and C is the centre of the hemisphere.

The centre of gravity is at G on the radius perpendicular to LM such that $CG = \frac{3r}{8}$, r being the radius of the hemisphere.

The forces acting are:

(i) reaction R perpendicular to the ground at A.

(ii) friction F at A.

(iii) reaction S perpendicular to the smooth wall at B and

(iv) weight W of the hemisphere acting vertically downwards at G.

Let $\angle ACM = \theta$.

Resolving horizontally and vertically,

$$S = F \rightarrow (1)$$

$$R = W \rightarrow (2)$$

Taking moments about A,

$$S \cdot CA = W \cdot GD$$

$$= W \cdot CG \cos\theta.$$

i.e) $$S \cdot r = W \cdot \frac{3r}{8} \cos\theta.$$

(or) $$S = \frac{3W}{8} \cos\theta \rightarrow (3).$$

From ① and ③, $F = \frac{3W\cos\theta}{8}$ and $R = W$

From ②.

$\therefore \frac{F}{R} = \frac{3W\cos\theta}{8} = \frac{3\cos\theta}{8}$

For equilibrium,

$\frac{F}{R}$ must be $\leq \mu$, the coefficient of friction

(ie) $\frac{3\cos\theta}{8} \leq \mu$.

(or) $\mu \geq \frac{3}{8}\cos\theta \rightarrow$ (4)

If μ is $3/8$, then as $\cos\theta$ is < 1, μ will be automatically $\geq 3/8 \cos\theta$.

Hence condition (4) is satisfied and the hemisphere can rest in any position.

From (4), $\cos\theta \leq \frac{8\mu}{3}$

If $\mu < 3/8$, $\frac{8\mu}{3} < 1$.

Let α be the angle such that $\cos\alpha = \frac{8\mu}{3}$ for equilibrium.

Then $\cos\theta \leq \cos\alpha$

(ie) $\theta \geq \alpha$.

(or) $\theta \geq \cos^{-1}\left(\frac{8\mu}{3}\right)$.

Hence the least value of θ is $\cos^{-1}\left(\frac{8\mu}{3}\right)$.

When the wall is rough, limiting frictions μR and $\mu' S$ act at A and B. Refer to figure

Resolving horizontally and vertically.

$S = \mu R \rightarrow$ ⑤

$$R + \mu' S = W \rightarrow (6)$$

Taking moments about A.

$$S.CA + \mu' S.BC = W.GD$$

i.e) $Sr + \mu' S.r = W.\frac{3r}{8}\cos\theta$.

$$S(1+\mu') = \frac{3W}{8}\cos\theta \rightarrow (7)$$

putting the value of R from (6) in (5) we have

$$S = \mu(W - \mu' S).$$

i.e) $S = \frac{\mu W}{1+\mu\mu'} \rightarrow (8)$

Substituting (8) in (7), we have

$$\frac{\mu W}{1+\mu\mu'}(1+\mu') = \frac{3W}{8}\cos\theta.$$

i.e) $\cos\theta = \frac{8\mu(1+\mu')}{3(1+\mu\mu')}$ which gives the least angle θ.