SEMESTER : V
MAJOR BASED ELECTIVE : I

Inst Hour	**: 5**
Credit	**: 5**
Code	**: 18K5MELMIS**

PROBABILITY AND STATISTICS

UNIT 1 :
Theory of Probability : Different definitions of Probability – Sample space – Probability of an event – Independence of events – Theorems of Probability – Conditional Probability – Baye's Theorem.
(Chapter 4 : Sections 4.5 – 4.9)

UNIT 2 :
Random variables – Distribution functions – Discrete & Continuous random variables – Probability mass & density functions – Joint probability distribution functions.
(Chapter 5 : Sections 5.1 – 5.5.5)

UNIT 3 :
Expectation – Variance – Covariance – Moment generating functions – Theorems on Moment generating functions – Moments – Various measures.
(Chapter 6: Sections 6.1 to 6.10.3 & Chapter 3 : Section 3.9)

UNIT 4 :
Correlation & Regression **:** Properties of Correlation & Regression coefficients – Numerical Problems for finding the correlation & regression coefficients.
(Chapter 10 : Sections 10.1 to 10.7.4)

UNIT 5 :
Binomial, Poisson, Normal distributions – Moment generating functions of these distributions- additive properties of these distributions – Recurrence relations for the moments about origin and mean for the Binomial, Poisson and Normal distributions – Properties of normal distributions.
(Chapter 7 :Sections 7.2 to 7.2.7, 7.2.10, 7.3 to 7.3.5, 7.3.8 and Chapter 8 :Sections 8.2, 8.2.2)

Text Book :
[1]. Fundamental of Mathematical Statistics by Gupta. S.C & Kapoor, V.K. Published by Sultan Chand & Sons, New Delhi – 2000 Edition.

Book For Reference :-
1]. Practical Statistics – Thambidurai . P – Rainbow Publishers – CBE (1991)
2]. Probability and Statistics – A. Singaravelu – A.R. Publications -2002

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

Theory of Probability :

Random Experiment : If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a random experiment.

Ex: random experiments are tossing a coin, throwing a die, etc.,

Outcome : The result of a random experiment will be called an outcome.

Trial and Event : Any particular performance of a random experiment is called a trial and outcome or combination of outcomes are termed as events.

Ex: Tossing of a coin is a random experiment or trial and getting of a head or tail is an event.

Exhaustive Events : The total number of possible outcomes of a random experiment is known as exhaustive events.

Ex: In tossing of a coin there are two exhaustive events.

Favourable events: The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

Ex: In throwing of two dice, the number of cases favourable to getting the sum 5 is

(1, 4), (4, 1), (2, 3), (3, 2) (ie) 4

Mutually exclusive events: Events are said to be mutually exclusive if no two or more of them can happen simultaneously in the same trial.

Ex: In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive.

In tossing a coin the events head and tail are mutually exclusive.

Equally likely events: Outcomes of trial are said to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others.

Ex: In throwing an unbiased die, all the six faces are equally likely to come.

Independent Events: Several events are said to be independent if the happening of an event is not affected by the supplementary knowledge concerning the occurence of any number of the remaining events.

Ex: In tossing an unbiased coin, the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

- When a die is thrown twice, the result of the first throw does not affect the result of the second throw.

Probability of an event:

If a random experiment results in 'n' exhaustive mutually exclusive and equally likely events, out of which m are favourable to the occurence of an event E, then the probability 'p' of occurence of E is denoted by P(E) it is defined as

$$P(E) = \frac{\text{Number of favourable events}}{\text{Total number of exhaustive events}} = \frac{m}{n}$$

Note :

(i) $0 \leq P(E) \leq 1$

(ii) $P(E) + P(\bar{E}) = 1$

(iii) Probability 'p' of the happening of an event is known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure. (ie) $P + q = 1$.

(iv) If $P(E) = 1$, E is called a Certain event

If $P(E) = 0$, E is called an impossible event

(v) The Probability can be computed prior to obtaining any experimental data, it is also called as 'a priori' or mathematical Probability.

(vi) The total possible outcomes of a random experiment is called Sample Space

(vii) Each possible outcome in a sample space is called Sample Point.

(viii) The number of sample points in the sample spaces are denoted by $n(S)$.

(viii) Every non-empty subset A of S, which is a disjoint union of single element subsets of the sample space S of a random experiment is called an <u>event</u>

<u>Acceptable assignment of Probabilities</u>:

Let $e_1, e_2 -- e_N$ be mutually disjoint and exhaustive outcomes of a random experiment so that its sample space S is $\{e_1, e_2 -- e_N\}$

To each elementary event e_i belonging to S, let us assign a real number called the probability of the elementary event e_i it is denoted by $P(e_i)$ such that

(i) The probability of each elementary event is non-negative real number (ie) $P(e_i) \geq 0$ for $i=1,2-N$

(ii) The sum of the probabilities assigned to all elementary events of the sample space is 1.

(ie) $\sum_{i=1}^{N} P(e_i) = 1$

Such an assignment of real nos. to the elementary events of the sample space is called acceptable assignment of probabilities.

Probability function: P(A) is the Probability function defined on a σ field B of events if the following Properties or axioms hold.

(i) For each $A \in B$ P(A) is defined, is real and $P(A) \geq 0$

(ii) $P(S) = 1$

(iii) If $\{A_n\}$ is finite or infinite Sequence of disjoint events in B then

$$P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} P(A_i)$$

Independent events: An event A is Said to be independent of another event B, if the Conditional Probability of A given B is equal to the unconditional Probability of B. (ie) if $P[A/B] = P[A)$

Similarly $P[B/A] = P(B)$; $P(A) \neq 0$

Note: If A and B are independent events

then $P[A \cap B] = P(A) \cdot P(B)$

Problems: ① Find the Probability of getting a Head (9) in tossing a Coin.

② Find the Probability of getting 2 in throwing a die.

③ Find the Probability of getting a tail in tossing a Coin.

④ Find the Probability of throwing (i) 4 (ii) an odd number (iii) an even number with an ordinary die.

⑤ Find the Probability of throwing 7 with two dice.

⑥ A bag Contains 6 red and 7 black balls. Find the Probability of drawing a red ball.

⑦ Find the Probability that if a Card is drawn at random from an ordinary Pack, it is a diamond.

⑧ From a Pack of 52 Cards, one Card is drawn at random. Find the Probability of getting a Queen.

⑨ Four Persons are chosen at random from a group Containing 3 men, 2 women and 4 children. S.T the Chance that exactly two of them will be children is $10/21$.

Total no. of Persons $= 9$

Let 4 Persons are choosen at random.

(ie) $n(S) = {}^9C_4 = \frac{9\times8\times7\times6}{1\times2\times3\times4} = 126$ ways

Let A be a favourable event with exactly two of them will be children.

(ie) $n(A) = {}^4C_2 \times {}^5C_2 = 60$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{60}{126} = \frac{10}{21}$

(10) From a group of 3 Indians, 4 Pakistanis and 5 Americans, a Sub-Committee of four People is Selected by lots. Find the Probability that the Sub-Committee will Consists of

(i) 2 Indians and 2 Pakistanis

(ii) 1 Indian, 1 Pakistani and 2 Americans

(iii) 4 Americans.

Ans: (i) $n(S) = {}^{12}C_4$; $n(A) = {}^3C_2 \times {}^4C_2$; $P(A) = 18/495$

(ii) $n(S) = {}^{12}C_4$; $n(B) = {}^3C_1 \times {}^4C_1 \times {}^5C_2 = 120$

$P(B) = \frac{120}{495} = \frac{24}{99}$

(iii) $n(S) = {}^{12}C_4$; $n(C) = {}^5C_4$; $P(C) = \frac{5}{495} = \frac{1}{99}$

(11) A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.

Ans: $n(S) = 18c_2$; $n(A) = 7c_2$; $P(A) = \frac{21}{153}$

(12) What is the probability of having a king and a queen when two cards are drawn from a pack of 52 cards.

Ans: $n(S) = 52c_2$; $n(A) = 4c_1 \cdot 4c_1$; $P(A) = 8/663$

(13) What is the probability that of 6 cards taken from a full pack, 3 will be black and 3 will be red.

Ans: $n(S) = 52c_6$; $n(A) = 26c_3 \cdot 26c_3$; $P(A) = \frac{n(A)}{n(S)}$

(14) Find the probability that a hand at bridge will consist of 3 spades, 5 hearts, 2 diamonds and 3 clever.

Ans: $n(S) = 52c_{13}$; $n(A) = 13c_3 \cdot 13c_5 \cdot 13c_2 \cdot 13c_3$.

(15) What is the chance that a leap year selected at random will contain 53 Sundays?

Ans: $P(A) = 2/7$ $\left[\because \frac{366}{7}\text{ days} = 52 \text{ weeks } 2 \text{ days}\right]$

Theorems on Probability

① Probability of the impossible event is Zero

(i.e) $P(\phi)=0$

Pf/ The Certain event 'S' and the impossible event ϕ are mutually exclusive.

(i.e) $S\cup\phi = S$

$P[S\cup\phi]=P(S) \Rightarrow P(S)+P(\phi)=P(S)$

$\Rightarrow P(\phi)=0$

② Probability of the Complementary event $\bar{A}$ of A is given by $P(\bar{A})=1-P(A)$

Pf/. W.K.T A and $\bar{A}$ are disjoint

Since $A\cup\bar{A}=S$

$P(A\cup\bar{A})=P(S)=1$

$P(A)+P(\bar{A})=1 \Rightarrow P(\bar{A})=1-P(A)$

③ For any two events A and B

$P(\bar{A}\cap B) = P(B)-P(A\cap B)$

Pf/. We have $\bar{A}\cap B$ and $A\cap B$ are disjoint events.

$$(A \cap B) \cup (\bar{A} \cap B) = B$$

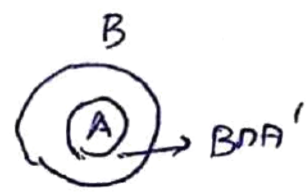
$$P[(A \cap B) \cup (\bar{A} \cap B)] = P(B)$$

$$P(A \cap B) + P(\bar{A} \cap B) = P(B)$$

$$\therefore P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

(4) If A and B are two events such that $A \subset B$

P.T $P(B \cap A') = P(B) - P(A)$

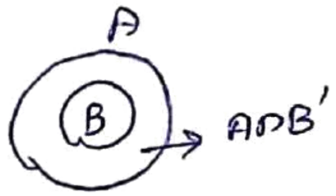


Pf/ Given $A \subset B$

$$B = A \cup (B \cap A')$$

$$P(B) = P(A) + P(B \cap A') \quad [\because A \;\&\; B \cap A' \text{ are mutually exclusive}]$$

(ie) $P(B \cap A') = P(B) - P(A)$



(5) If $B \subset A$ P.T $P(A) > P(B)$

Pf/ Given $B \subset A$

$$\therefore A = B \cup (A \cap B')$$

$$P(A) = P(B) + P(A \cap B')$$

$$P(A) - P(B) = P(A \cap B') \quad [\because P(A \cap B') \geq 0]$$

(ie) $P(A) - P(B) \geq 0$

$$\therefore P(A) \geq P(B)$$

(8) Multiplication Theorem on Probability (7)

(6) For two events A and B $P(A \cap B) = P(A) \cdot P(B/A)$ $P(A)>0$

$= P(B) \cdot P(A/B)$ $P(B)>0$

Soln. Given: $A \cap B = \phi$ Conditional Probability of occurrence Where $P(B/A)$ represents already happened and $P(A/B)$ of happened the event $A \cup (A \cap B')$ is the Conditional Probability of happening A given that B has already happened.

$= \phi \cup (A \cap B')$

$A = A \cap B'$ (i)

Pf/ W.K.T $P(A) = \frac{n(A)}{n(S)}$; $A \subseteq B'$; $P(B) = \frac{n(B)}{n(S)}$ & $P(A \cap B) = \frac{n(A \cap B)}{n(S)}$

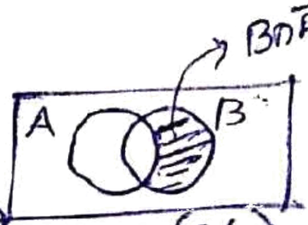
For the Conditional event A/B all the favourable out comes must be one of the sample points of B. $P(A \cap B)$ Where A and B

Thm/ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(i) $P(A/B) = \frac{n(A \cap B)}{n(B)}$

are 2 events and are not disjoint.

Pf/ $P(A/B) = \frac{P(A \cap B) \cdot n(S)}{P(B) \cdot n(S)}$ We have $A \cup B = A \cup (B \cap \bar{A})$



$\left[\because A \text{ and } B \cap \bar{A} \text{ are disjoint events}\right]$ $\Rightarrow P(A \cap B) = P(B) \cdot P(A/B)$

$\Rightarrow P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cup B) = P[A \cup (B \cap \bar{A})]$

|||ly $P(A \cap B) = P(A) \cdot P(B/A)$

$= P(A) + P(B \cap \bar{A})$

$= P(A) + P(B) - P(A \cap B)$ $\left[\because P(B \cap \bar{A}) = P(B) - P(A \cap B)\right]$

(9) If A and B are two events with Positive Probabilities $P(A) \neq 0$, $P(B) \neq 0$ then A & B are independent $\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

Pf/ W.K.T If A and B are indep. Then $P(A/B) = P(A)$ →(1); $P(B/A) = P(B)$ →(2)

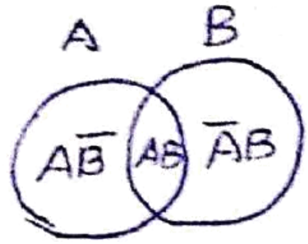
From (1) & (2) $P(A \cap B) = P(A) \cdot P(B)$

Conversely $\frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow P(A/B) = P(A)$; $\frac{P(A \cap B)}{P(A)} = P(B)$

$\Rightarrow P(B/A) = P(B)$

(9) If $P(A) = P(B) = P(A \cap B)$ Prove that

$P[A \cap \bar{B} + \bar{A} \cap B] = 0$



Soln: W.K.T

$$P[A \cup B] = P(A) + P(B) - P(A \cap B)$$

From the fig. $A \cup B = A\bar{B} + AB + \bar{A}B$

$$\therefore P(A \cup B) = P(A\bar{B}) + P(AB) + P(\bar{A}B)$$

$$P(A\bar{B}) + P(\bar{A}B) = P(A \cup B) - P(AB)$$

$$= P(A) + P(B) - P(AB) - P(AB)$$

$$= P(AB) + P(AB) - P(AB) - P(AB)$$

$$= 0$$

$$\therefore P(A\bar{B}) + P(\bar{A}B) = 0$$

(10) If the events A and B are such that $P(A) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is indep. of A.

Pf/

Since A is indep. of B (ie) $P(A/B) = P(A) \Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow \frac{P(B \cap A)}{P(A)} = P(B) \Rightarrow P[B/A] = P(B) \Rightarrow$ B is indep. of A.

Problems:

① A Person is Known to hit the target in 3 out of 4 Shots, Whereas another Person is Known to hit the target in 2 out of 3 Shots. Find the Probability of the targets being hit at all when they both Person try.

Soln. The Prob. that the 1st Person hit the target

(ie) $P(A) = 3/4$

The Prob. that the 2nd Person " " "

(ie) $P(B) = 2/3$

The two events are not mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P[A \cap B] = P(A) + P(B) - [P(A) \cdot P(B)] = \frac{17}{12} - \frac{6}{12} = \frac{11}{12}$$

② If from a Pack of Cards a Single Card is drawn What is the Probability that either a Spade or King.

Ans: $4/13$. [A and B are not mutually exclusive]

③ If $P(A) = 0.35$, $P(B) = 0.73$, $P(A \cap B) = 0.14$

Find $P(A' \cup B')$

Soln $P(A' \cup B') = 1 - P(A \cap B) = 1 - 0.14 = 0.86$

④ If A and B are independent and $P(A) = 1/3$, $P(B) = 1/4$

Sol: Since A and B are independent

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

⑤ If $P(A) = 0.65$, $P(B) = 0.4$ and $P(A \cap B) = 0.24$

Can A and B are dependent events.

Sol: W.K.T If A and B are independent then

$P(A \cap B) = P(A) \cdot P(B) \Rightarrow 0.24 = 0.65 \times 0.4$

$\Rightarrow 0.24 \neq .380$

$\therefore$ A and B are dependent events.

⑥ If A and B are independent events P.T

(i) $\bar{A}$ and B are independent

(ii) A and $\bar{B}$ are independent

(iii) $\bar{A}$ and $\bar{B}$ are independent

Sol: Since A and B are independent then

$$P(A \cap B) = P(A) \cdot P(B)$$

(i) W.K.T $B = (A \cap B) \cup (\bar{A} \cap B)$ where $A \cap B$ and $\bar{A} \cap B$ are disjoint

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) = -P(AB) + P(B) = P(B) - P(A) \cdot P(B)$$

$$= P(B)[1 - P(A)] = P(B) \cdot P(\bar{A})$$

$\therefore$ $\bar{A}$ and B are independent.

(ii) W.K.T $A = (A \cap B) \cup (A \cap \bar{B})$

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B)$$

$$= P(A)(1 - P(B)) = P(A) \cdot P(\bar{B})$$

$\therefore$ A and $\bar{B}$ are independent.

(iii) W.K.T $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= 1 - P(A) - P(B)[1 - P(A)]$$

$$= [1 - P(A)][1 - P(B)] = P(\bar{A}) \cdot P(\bar{B})$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent.

(7) The Probability that machine A will be Performing an usual function in 5 years time is $\frac{1}{4}$ while the Probability that machine B will still be operating usefully at the end of the Same Period is $\frac{1}{3}$. Find the Probability that both machines will be Performing as usual function.

<u>Sol</u> P (machine A operating usefully) $= \frac{1}{4}$

P (machine B " ") $= \frac{1}{3}$

P (Both A and B will operate usefully) $= P(A) \cdot P(B)$

$= \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$

(8) A bag Contains 8 white and 10 black balls. Two balls are drawn in Succession. What is the Probability that first is white and 2nd is black.

<u>Sol</u> Total no. of balls = 18

$P(A) = \frac{8}{18}$; $P(B) = \frac{10}{18}$

$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{8}{18} \cdot \frac{10}{18}$

(10) An article manufactured by a Company Consists of two Parts A and B. In the Process of manufacture of Part A, 9 out of 100 likely to be defective. Similarly 5 out of 100 are likely to be defective in the manufacture of Part B. Calculate the Probability that the assembled article will not be defective. (10)

Soln

$P(\text{A will be defective}) = 9/100$

$P(\text{A will not be defective}) = 1 - \frac{9}{100} = \frac{91}{100}$

$P(\text{B will be defective}) = 5/100$

$P(\text{B will not be defective}) = 1 - 5/100 = \frac{95}{100}$

$P[\text{assembled article will not be defective}]$

$= P(\text{A will not be defective}) \cdot P(\text{B will not be defe.})$

$= \frac{91}{100} \times \frac{95}{100} = 0.86.$

(11) From a bag Containing 4 white and 6 black balls two balls are drawn at random. If the balls are drawn one after the other without replacement, find the Probability that

(i) both balls are white

(ii) both balls are black

(iii) the first ball is white and 2^{nd} is black

(iv) One ball is white and other is black.

Soln.

(i) Total no. of balls = 10

P [1st ball is white] $= \frac{4}{10}$

P [2nd ball is white] $= 3/9$

P [both balls are white] $= \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15}$

(ii) P [1st ball is black] $= \frac{6}{10}$

P [2nd ball is black] $= 5/9$

P [both balls are black] $= \frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}$

(iii) P [1st ball is white] $= \frac{4}{10}$

P [2nd ball is black] $= 6/9$

P [1st white and 2nd black] $= \frac{4}{10} \cdot \frac{6}{9} = \frac{4}{15}$

(iv) P[1st ball is white and 2nd is black]

$= \frac{4}{10} \cdot \frac{6}{9} = \frac{24}{90}$

P[1st ball is black and 2nd ball is white]

$= \frac{6}{10} \times \frac{4}{9} = \frac{24}{90}$

Here both events are mutually exclusive

∴ P[one ball is white and the other is black]

$= \frac{24}{90} + \frac{24}{90} = \frac{8}{15}$

(12) Find the Probability in each of the above four cases, if the balls are drawn one after the other with-replacement.

Soln

(i) P[both balls are white] $= \frac{4}{10} \cdot \frac{4}{10} = \frac{4}{25}$

(ii) P[both balls are black] $= \frac{6}{10} \cdot \frac{6}{10} = \frac{9}{25}$

(iii) P[1st white 2nd black] $= \frac{4}{10} \cdot \frac{6}{10} = \frac{6}{25}$

(iv) P[1st ball is white and 2nd ball is black] $= \frac{4}{10} \cdot \frac{6}{10} = \frac{24}{100}$

P[1st ball is black and 2nd is white] $= \frac{6}{10} \cdot \frac{4}{10} = \frac{24}{100}$,

P[One is white & other is black] $= \frac{24}{100} + \frac{24}{100} = \frac{12}{25}$

(13) Four Cards are drawn without replacement. What is the Probability that they are all Aces?

Soln:

$P(A) = P[\text{getting 1st Ace}] = \frac{4}{52}$

$P(B) = P[\text{getting 2nd Ace}] = \frac{3}{51}$

$P(C) = P[\text{getting 3rd Ace}] = \frac{2}{50}$

$P(D) = P[\text{getting 4th Ace}] = \frac{1}{49}$

$\therefore P[\text{all four Cards are Aces}]$

$= P(A) \cdot P(B) \cdot P(C) \cdot P(D) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$.

(14) If two dice are thrown what is the Probability that the Sum is (i) greater than 8 (ii) neither 7 nor 11.

Soln (i) $P[S=9] = 4/36$; $P[S=10] = \frac{3}{36}$; $P[S=11] = \frac{2}{36}$

$P[S=12] = \frac{1}{36}$ $\therefore P(S>8) = 5/18$

(ii) $P[\text{neither 7 nor 11}] = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$= 1 - \{P(A) + P(B)\} = 1 - \frac{1}{6} - \frac{1}{18} = 7/9$

Theorem : If A and B are independent events, then (i) A and $\bar{B}$ (ii) $\bar{A}$ and B (iii) $\bar{A}$ and $\bar{B}$ are also independent

proof

Since A and B are independent,

$P(A \cap B) = P(A)P(B)$

(i) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$

$= P(A)[1 - P(B)] = P(A)P(\bar{B})$

$\Rightarrow$ A and $\bar{B}$ are independent events.

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B)$

$= P(B)[1 - P(A)] = P(\bar{A})P(B)$

$\Rightarrow$ $\bar{A}$ and B are independent events

(iii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= [1 - P(B)] - P(A)[1 - P(B)]$

$= [1 - P(A)][1 - P(B)] = P(\bar{A})P(\bar{B})$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent events.

Conditional probability:

conditional probability is the probability of one event occuring with some relationship to one or more events

$P(B/A) = P(A \text{ and } B) / P(A)$

Also can be written as,

$P(B/A) = P(A \cap B) / P(A)$

$\left.\begin{array}{l} P(\text{defective bolt manufactured} \\ \text{by machine } A_2) \end{array}\right\} = P(A_2/B)$

$$= \frac{P(A_2) \cdot P(B/A_2)}{\Sigma P(A_i) \cdot P(B/A_i)}$$

$$= \frac{0.014}{0.0345}$$

$$= 0.405.$$

$\left.\begin{array}{l} P(\text{defective bolt manufactured} \\ \text{by machine } A_3 \end{array}\right\} = \frac{P(A_3) \cdot P(B/A_3)}{\Sigma P(A_i) \cdot P(B/A_i)}$

$$= \frac{0.008}{0.0345}$$

$$= 0.231.$$

Ex:2 The 1st bag contains 3 white balls, 2 red balls and 4 black balls. second bag contains 2 white, 3 red, 5 black balls and third bag contains 3 white, 4 red and 2 black balls. one bag is chosen at random from it 3 balls are drawn. out of 3 balls 2 balls are white and one red. what are the probability that they were taken from 1st bag, 2nd bag, 3rd bag.

Soln: selection of bags are mutually exclusive events. selection of 2 white and one red ball is an independent event.

let $P(\text{selection bag}) = P(A_i) = 1/3$

$$= \frac{0.0476}{0.0746}$$

$$= 0.6380.$$

EX:3 let 5 men out of 100 and 25 women out of 1000 are colour blind. A colour blind person is chosen at random. What is the probability of his being made. (Assume that males and females are in equal proportion).

Soln: let M denotes a person is Male
let F denotes a person is female.
let C denotes a person is colour blind.

Given $P(M) = 1/2$, $P(F) = 1/2$.

$$P(C/M) = 5/100$$

$$P(C/F) = 25/1000$$

$$P(M/C) = ?$$

$$P(M/C) = \frac{P(C/M) \cdot P(M)}{P(C/M) P(M) + P(C/F) \cdot P(F)}$$

$$= \frac{5/100 \times 1/2}{5/100 \cdot 1/2 + \frac{25}{1000} \cdot 1/2}$$

$$= \frac{0.05}{0.05 + 0.025}$$

$$= 2/3$$

Ex: 4 An urn contains 10 W, 3 B balls while another urn contains 3 W, 5 B balls. two balls are drawn from the 1st urn and put into the 2nd urn Then a ball is drawn from the latter. what is the probability that it is a white ball.

soln:

The 2 balls drawn from the 1st urn may be

(i) both white (event A1)

(ii) Both black (event A2)

(iii) 1 W, 1 B (event A3)

$\therefore p(A_1) = \frac{10C_2}{13C_2} = 15/26.$

$p(A_2) = \frac{3C_2}{13C_2} = 1/26.$

$p(A_3) = \frac{10C_1 \times 3C_1}{13C_2} = 10/26.$

After the balls are transformed from 1st urn to 2nd urn will contain.

(i) 5W, 5B, (ii) 3W, 7B (iii) 4W, 6B.

let B be the event of drawing a white ball from the 2nd urn.

Now $P(B/A_1) = \frac{5C_1}{10C_1} = 5/10$

$P(B/A_2) = \frac{3C_1}{10C_1} = 3/10.$

$P(B/A_3) = \frac{4C_1}{10C_1} = 4/10$

$$\therefore P(B) = \sum_{i=1}^{3} P(B/A_i) \cdot P(A_i)$$

$$= 59/130.$$

Unit-2

Random variables

A real variable 'x' whose value is determined by the outcome of a random experiment is called Random Variables.

Ex:1 consider random experiment of throwing a dice Then 'x' the number of points on the dice is a random variable 'x' take the values 1, 2, 3, 4, 5, 6.

Sol'n: Here the random variable 'x' takes the values 1, 2, 3, 4, 5, 6 each with the probability $\frac{1}{6}$

Note that "twice the number of points on a dice" which takes the values 2, 4, 6, 8, 10, 12 is also a random variables.

The "square of number of points on a die" which takes the values 1, 4, 9, 16, 25, 36 is also a random Variable.

Ex:2

consider a random experiment of throwing a coin twice. we have the following result HH - 2, HT - 1, TH - 1, TT = 0. The number of heads" Which takes the value 2, 1, 1, 0 is a random variable.

Discrete Random Variable :-

If the random variable takes the values only on the set $\{0, 1, 2, 3 --- n\}$ is called a discrete random variable.

The number of printing mistakes in each page of a book. the number of telephone calls received by the telephones operator are examples of discrete random variables.

Clearly the above examples (1) & (2) are the examples of discrete random variables.

continuous Random Variables :

If a random variable takes on all values within a certain interval, then the random variable is called continuous Random Variable.

The height, age, weight of indivuals, the amount of rainfall on a rainy day are clear examples of Continuous random variables.

Thus to each outcome ω of a random experiment there corresponds a real number $X(\omega)$ which is defn for each point of the Sample S.

Ex:1 If a coin is tossed, then sample is,

$$S = \{H, T\}.$$

(ie) $S = \{\omega_1, \omega_2\}$, $\omega_1 = H$, $\omega_2 = T$.

$$\text{Now } X(\omega) = \begin{cases} 1, & \text{if } \omega = H \\ 0 & \text{if } \omega = T \end{cases}$$

Here $X(\omega)$ is a random variables which takes only two values.

Some Important theorems on Random Variables:

Thm-1: If X_1 and X_2 are random variables and K is constant then KX_1, $X_1 + X_2$, X_1X_2, $K_1X_1 + K_2X_2$, $X_1 - X_2$ are also random variables.

Thm-2: If 'X' is a random variables and $f(\cdot)$ is a continuous function then $f(x)$ is a random variable.

Distribution function of the random variable x.

The distribution function of a random variable x defn in $(-\infty, \infty)$ is given by

$$F(x) = P(X \leq x)$$

Note: let the random variable X take values $x_1, x_2 \ldots x_n$ with probabilities $P_1, P_2 \ldots P_n$ and let $x_1 < x_2 < x_3 \ldots <$

$= p(x=a) + p(b) - F(a)$

[using property (1)].

property-3: $p(a<x<b) = p(a<x\leq b) - p(x=b)$.

proof: $P(a<x<b) = P(a<x\leq b) - P(x=b)$

$= F(b) - F(a) - P(x=b)$

[using property (1)]

$p(a\leq x<b) = p(a<x<b) + p(x=a)$

$= F(b) - F(a) - p(x=b) + p(x=a)$.

Note-1 : If $F(x)$ is the distribution function of one dimensional random variable, then

(i) $0\leq F(x)\leq 1$

(ii) If $x<y$, then $F(x)\leq F(y)$

(iii) $F(-\infty)=0$, $F(\infty)=1$

Probability Mass function :-

let 'x' be a one - dimensional discrete random variable which takes the values $x_1, x_2, x_3 \ldots$

let each possible outcome 'x_i' we can associate a number p_i [$p(x=x_i)$] $= p(x_i) = p_i$.

The numbers $p(x_i)$ $i=1,2,3\ldots$ statisfies the following conditions.

(i) $p(x_i)\geq 0$,

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

This function 'p' statisfying the above two conditions is called the probability mass function or probability function of the random variables X and the set $\{x_i, P(x_i)\}$ is called probability distribution of the random variable X.

Ex:1 A random variable X be the following probability function.

Values of X, x	0	1	2	3	4	5	6	7	8
probability p(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Determine the value of 'a'.
(ii) Find $P(X<3)$, $P(X\geq 3)$, $P(0<X<5)$.
(iii) Find the distribution function of X

Soln: (i) We know that if p(x) is the probability mass function, then $\sum_{i=1}^{\infty} p(x_i)=1$. [Hence 'i' varies from 0 to 8]

(ie) $a+3a+5a+7a+9a+11a+13a+15 = 1$

$$81a = 1$$
$$a = 1/81$$

(ii) $P(X<3) = P(0)+P(1)+P(2)$

$= a+3a+5a$

$= \frac{1}{81}+\frac{3}{81}+\frac{5}{81}$

$= 9/81$

$= 1/9$

$p(x \geq 3) = p(3) + p(4) + p(5) + p(6) + p(7) + p(8)$

$= 1 - [p(0) + p(1) + p(2)]$

$= 1 - 1/9$

$= 8/9.$

(iii) $p(0 < x < 5)$

$= p(1) + p(2) + p(3) + p(4)$

$= 3a + 5a + 7a + 9a$

$= 24/81$

(iv) To find the distribution function $F(x)$.

x	$F(x) = P(X \leq x)$	
0	a	$p(0) = a$
1	$a + 3a = 4a$	$P(X \leq 1) = p(0) + p(1)$
2	$4a + 5a = 9a$	$p(X \leq 2) = p(0) + p(1) + p(2)$
3	$9a + 7a = 16a$	$p(X \leq 3) = p(0) + p(1) + p(2) + p(3)$.
4	$16a + 9a = 25a$	
5	$25a + 11a = 36a$	
6	$36a + 13a = 49a$	
7	$49a + 15a = 64a$	
8	$64a + 17a = 81a$	

EX:2 suppose that the random variable 'x' assumes three values 0, 1, 2 with probability $\frac{1}{3}, 1/6, 1/2$ respectively. obtain the distribution function of x.

Soln; Given

x	0	1	2
$p(x)$	$1/3$	$1/6$	$1/2$

x	$F(x) = P(X \le x)$	
0	$1/3$	
1	$1/3 + 1/6 = 1/2$	$\therefore P(X \le 1) = P(0) + P(1)$
2	$1/2 + 1/2 = 1$	$\therefore P(x \le 2) = P(0) + P(1) + P(2) = 1$

continuous Random Variable :

A random variable 'X' which takes all possible values in a given interval is called continuous random variable.

Ex: Age, height, weight, etc.. are continuous random Variables.

probability density function :-

The probability density function for the continuous random variable 'X' in the interval (a, b) is given by

$$f(x) = \begin{cases} 0, & x < a \\ \phi(x) & a \le x \le b \\ 0. & x > b. \end{cases}$$

Note:

1. $f(x) \ge 0, \; -\infty < x < \infty$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. The probability $P(E)$ is given by $P(E) = \int_E f(x)\,dx$

Cumulative Distribution Function:

If $f(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$ then $F(x)$ defn as the comulative distribution function (or) distribution function of the continuous random variable X.

<u>Note</u>: (i) $F'(x) = f(x) \geq 0$

(ii) $F(-\infty) = 0$.

(iii) $F(\infty) = 1$.

(iv) $F(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$

$$= \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= F(b) - F(a).$$

EX: 1

(i) Is the function defined as follows a density function?

$$f(x) = e^{-x}, \quad x \geq 0$$
$$= 0, \quad x < 0.$$

(ii) If so determine the probability that the variate having this density will fall in the interval $(1, 2)$

(iii) Also find the cumulative probability function $F(2) = ?$

Soln: In the interval $(1, 2)$ e^{-x} is always +ve.

(ie) $f(x) \geq 0$ in $(1, 2)$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx.$$

$$= \left[-e^{-x}\right]_0^{\infty}$$

$$= -e^{-\infty} + 1$$

$$= 1$$

Hence $f(x)$ statisfies the condition of the density function.

(ii) $$P(1 \le x \le 2) = \int_1^2 f(x)\,dx.$$

$$= \int_1^2 e^{-x}\,dx.$$

$$= \left[-e^{-x}\right]_1^2$$

$$= -e^{-2} + e^{-1}$$

$$= 0.368 - 0.135$$

$$= 0.233.$$

(iii) cummulative probability function

$$F(2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^2$$

$$= e^{-2} + 1$$

$= 1 - 0.135$

$= 0.865.$

Ex: 2 A continuous random variable 'x' has a p.d.f $f(x) = 3x^2$, $0 \leq x \leq 1$. Find 'a' and 'b' such that (2)

(i) $p(X \leq a) = p(X > a)$ and

(ii) $p(x > b) = 0.05$

soln: Given $p(X \leq a) = p(X > a)$.

Since total probability is 1, we have

$$p(X \leq a) = 1/2 \text{ and } p(X > a) = 1/2$$

When $p(x \leq a) = 1/2 \Rightarrow \int_0^a f(x)\,dx = 1/2$

$$\Rightarrow \int_0^a 3x^2\,dx = 1/2$$

$$\Rightarrow 3\left(x^3/3\right)_0^a = 1/2$$

$$\Rightarrow a^3 = 1/2$$

$$\Rightarrow a = (1/2)^{1/3}$$

When $p(x > b) = 0.05$

$$\Rightarrow \int_b^1 f(x)\,dx = 0.05$$

$$\Rightarrow \int_b^1 3x^2\,dx = 0.05$$

$$\Rightarrow 3\left(x^3/3\right)_b^1 = 0.05$$

$$\Rightarrow 1 - b^3 = 1/20$$

$$\Rightarrow b^3 = 1 - \frac{1}{20} = 19/20$$

$$b = (19/20)^{1/3}$$

Ex:3 The diameter of an electric cable say x is assumed to be a continuous random variable with p.d.f $f(x) = 6x(1-x)$ $0 \le x \le 1$.

(i) check that above is a p.d.f.

(ii) Determine a number 'b' such that

$$p(x<b) = p(x>b)$$

Soln: (i) The interval $0 \le x \le 1$, $f(x)$ is always +ve

(ie) in $0 \le x \le 1$, $f(x) > 0$

$$\int_0^1 f(x)dx = \int_0^1 6x(1-x)dx$$

$$= 6\int_0^1 (x - x^2)\, dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 6\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 1$$

$\therefore$ $f(x)$ is a p.d.f of a random variable x.

(ii) Given $p(x<b) = p(x>b)$.

(ie) $\int_0^b f(x)dx = \int_b^1 f(x)dx$.

$$6\int_0^b (x - x^2)dx = 6\int_b^1 (x - x^2)dx.$$

$$b\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^b = b\left[\frac{x^2}{2}-\frac{x^3}{3}\right]$$

$$b\left[\frac{b^2}{2}-\frac{b^3}{3}\right] = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{b^2}{2}-\frac{b^3}{3}\right)$$

$$\Rightarrow 3b^2-2b^3 = 1-3b^2+2b^3$$

$$\Rightarrow 4b^3-6b^2+1 = 0$$

$$\Rightarrow (2b-1)(2b^2-2b-1)=0$$

$$\Rightarrow 2b-1=0 \quad \text{or} \quad 2b^2-2b-1=0$$

$$\Rightarrow b=\tfrac{1}{2}\ ; \quad b=\frac{2\pm\sqrt{4-8}}{4}$$

$$=\frac{2\pm i2}{4}$$

$$=\frac{1\pm i}{2}.$$

Here $b=\frac{1}{2}$ is the real value and $b=\frac{1\pm i}{2}$ is imaginary

$\therefore b=\frac{1}{2}$ which lies in $(0,1)$

Note: If $f(x)$ is a p.d.f of a random variable 'x' which is defined in the interval (a,b) then

(i) Arithmetic mean $=\int_a^b x(f(x))\,dx$

(ii) Harmonic mean $=\int_a^b \frac{1}{x} f(x)\,dx$

(iii) Geometric mean G is given by

$$\therefore \log G = \int_a^b \log f(x)\,dx.$$

(iv) Moments about origin

$$\mu_r' = \int_a^b x^r f(x)\,dx.$$

(v) Moment about any point A

$$\mu_r' = \int_a^b (x-A)^r f(x)\,dx$$

(vi) Moment about mean

$$\mu_r = \int_a^b (x-\text{mean})^r f(x)\,dx$$

(vii) Mean derivation about the mean is

$$M.D = \int_a^b |x-\text{mean}|\, f(x)\,dx .$$

Ex: 1 A probability curve $y = f(x)$ has a range from 0 to ∞. If $f(x) = e^{-x}$ find the mean and variance and the third moment about mean.

Soln:

$$\text{Mean} = \int_0^\infty x f(x)\,dx \quad \left[\text{using mean} = \int_a^b x f(x)\,dx\right]$$

$$= \int_0^\infty x e^{-x}\,dx = \left[x(-e^{-x}) - (e^{-x})\right]_0^\infty$$

$$= 1 \qquad \boxed{\because e^{-\infty} = 0}$$

$$\text{variance } \mu_2 = \int_0^\infty (x-\text{mean})^2 f(x)\,dx$$

$$= \int_0^\infty (x-1)^2 e^{-x}\,dx$$

$$= \left[(x-1)^2(-e^{-x}) - 2(x-1)(e^{-x}) + 2(-e^{-x})\right]_0^\infty$$

[By Using Integration by part]

$$= 1 - 2 + 2$$

$$= 1$$

Third moment about mean

$$= \int_0^\infty (x-1)^3 e^{-x} dx \quad \left[\text{using } \mu_r = \int_a^b (x-\text{mean})^r f(x) dx\right.$$

$$= \left\{(x-1)^3(-e^{-x}) - 3(x-1)^2(e^{-x}) + 6(x-1)(-e^{-x}) - 6(e^{-x})\right\}_0^\infty$$

[By using Bernoulli's thm for integration]

$$= -1+3-6+6$$

$$= 2.$$

Ex: 2 The length of time (in min) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the probability density fn $f(x)$ as

$$f(x) = Ae^{-x/3}, \text{ for } x \geq 0.$$

$$= 0, \text{ otherwise}$$

Find the value of A that makes $f(x)$ a p.d.f.

Soln: (a) If $f(x)$ is p.d.f then

$$\int_0^\infty f(x) dx = 1$$

(ie) $$\int_0^\infty A e^{-x/5} dx = 1$$

$$A\left[\frac{e^{-x/5}}{-1/5}\right]_0^\infty = 1$$

$$A\left[0 - \left(\frac{5}{5}\right)\right] = 1$$

$$5A = 1 \Rightarrow A = 1/5.$$

Ex 3 Is the function defined as follows probability density function.

$$f(x) = \begin{cases} 0, & \text{if } x < 2 \\ (3+2x)/18, & \text{if } 2 \le x \le 4 \\ 0, & \text{if } x > 4. \end{cases}$$

If so find the $P(2 \le X \le 3)$.

Soln: In the interval $2 \le x \le 4$ $f(x) \ge 0$

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$

$$= 0 + \int_{2}^{4} \frac{3+2x}{18}\,dx + 0$$

$$= \frac{1}{18}\left[\frac{(3+2x)^2}{4}\right]_2^4$$

$$= \frac{1}{72}\left[121 - 49\right] = \frac{72}{72}$$

$$= 1$$

$\therefore f(x)$ is a p.d.f

Now $P(2 \le X \le 3) = \int_{2}^{3} f(x)\,dx$

$$= \int_{2}^{3} \frac{3+2x}{18}$$

$$= \frac{1}{18}\left[3x + x^2\right]_2^3$$

$$= \frac{1}{18}\left[18 - 10\right]$$

$$= 8/18.$$

Ex: 4 If a random variable 'x' has the p.d.f

$$f(x) = \begin{cases} \frac{1}{2}(x+1), & \text{if } -1<x<1 \\ 0, & \text{otherwise} \end{cases}$$

find the mean and variance of x.

Soln:

$$\text{Mean} = \int_{-1}^{1} x f(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{1}(x^2+x)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\right]$$

$$\text{Mean} = \frac{1}{3}$$

$$\text{variance} = \int_{-1}^{1}\left(x-\frac{1}{3}\right)^2\left(\frac{x+1}{2}\right)dx$$

$$\left[\mu_2 = \int_a^b (x-\text{mean})^2 f(x)\,dx\right]$$

$$= \frac{1}{6}\int_{-1}^{1}(3x-1)^2(x+1)\,dx$$

$$= \frac{1}{6}\int_{-1}^{1}(3x^2+2x-1)\,dx$$

$$= \frac{1}{6}\left[(x^3+x^2-x)\right]_{-1}^{1}$$

$$= \frac{1}{6}[1-1]$$

$$= 0$$

$$\text{Variance} = 0.$$

Ex:5 For the following density function,

$f(x) = a e^{|x|}$, $-\infty < x < \infty$, find (i) the value of 'a'

(ii) mean and variance.

Soln: Given $f(x)$ is pdf.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

(ie) $$\int_{-\infty}^{\infty} a\, e^{-|x|}\,dx = 1$$

(ie) $$a \cdot 2 \int_{0}^{\infty} e^{-|x|}\,dx = 1$$ [$e^{-|x|}$ is a even fn]

$$2a \int_{0}^{\infty} e^{-x}\,dx = 1$$ [In $(0,\infty)$ $e^{-|x|} = e^{-x}$]

$$2a\left[e^{-x}\right]_{0}^{\infty} = 1$$

$$2a[0+1] = 1$$

(ie) $$a = 1/2$$

(ii) Mean $$= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \tfrac{1}{2} e^{-|x|}\,dx.$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (x e^{-|x|})\,dx = 0$$ [$\because x e^{-|x|}$ is an odd fn].

variance $$\mu_2 = \int_{-\infty}^{\infty} (x-0)^2 \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$$

$$= 2 \times \frac{1}{2}\int_{0}^{\infty} x^2 e^{-x}\,dx$$ [$\because x^2 e^{-|x|}$ is a even fn]

$$= \int_{0}^{\infty} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty$$

$$= 3$$

Variance $\mu_2 = \int_0^\infty (x-3)^2 f(x)\,dx.$

$$= \int_0^\infty (x-3)^2 \tfrac{1}{2} x^2 e^{-x}\,dx.$$

$$= \frac{1}{2}\int_0^\infty (x^2 - 6x + 9)\, x^2 e^{-x}\,dx.$$

$$= \tfrac{1}{2}\int_0^\infty (x^4 - 6x^3 + 9x^2)\, e^{-x}\,dx$$

$$= \frac{1}{2}\Big[(x^4 - 6x^3 + 9x^2)(-e^{-x}) - (4x^3 - 18x^2 + 18x)(e^{-x})$$

$$+ (12x^2 - 36x + 18)(-e^{-x}) - (24x - 36)(e^{-x})$$

$$+ (24)(-e^{-x})\Big]_0^\infty$$

[using integration by parts]

$$= \frac{1}{2}[18 - 36 + 24]$$

$$= \frac{6}{2}$$

$$= 3.$$

Ex:7 A random variable 'x' has the p.d.f

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find (i) $P(x < 1/2)$ (ii) $P(1/4 < x < 1/2)$.

(iii) $P(x > 3/4 \mid x > 1/2)$

$= 7/16 \quad -②$

$$P(x>1/2) = \int_{1/2}^{1} f(x)\,dx$$

$$= \int_{1/2}^{1} 2x\,dx$$

$$= 2\left(x^2/2\right)_{1/2}^{1}$$

$$= 1 - 1/4$$

$$= 3/4 \quad -③$$

sub ② and ③ in (1) we get

$$P(x>3/4 \,/\, x>1/2) = \frac{P(x>3/4)}{P(x>1/2)}$$

$$= \frac{7/16}{3/4}$$

$$= 7/12.$$

Ex: 8 A continuous random variable 'x' is distributed over the interval [0,1] with p.d.f ax^2+bx where a, b are consants. If the A.M of 'x' is 0.5. Find the values of a and b

Soln: Let $f(x) = ax^2+bx$

Given $f(x)$ is a p.d.f in [0,1]

(ie) $\int_0^1 f(x)\,dx = 1$

$\int (ax^2+bx)\,dx = 1$

$$\left(\frac{ax^3}{3}+\frac{bx^2}{2}\right)_0^1=1$$

$$a\left(\frac{1}{3}\right)+b\left(\frac{1}{2}\right)=1 \quad -(1)$$

Now mean $= \int_0^1 x f(x)\,dx$ [using mean $= \int_0^b x f(x)\,dx$]

$$= \int_0^1 x(ax^2+bx)\,dx$$

$$= \int_0^1 (ax^3+bx^2)\,dx$$

$$= \left[ax^4/4 + bx^3/3\right]_0^1$$

$$= a/4 + b/3$$

Given mean $= 0.5 = 1/2$

$$\therefore 1/2 = a/4 + b/3$$

$$1/2 = \frac{3a+4b}{12}$$

$$3a+4b=6 \quad -(2)$$

$$2a+3b=6 \quad -(3)$$

Solving (2) & (3) we get.

Ex:9 Find the probability density function

$$f(x) = \frac{2(b+x)}{b(a+b)}, -b \le x < 0,$$

$$= \frac{2(a-x)}{a(a+b)}, \quad 0 \le x \le a$$. Find the

mean.

Soln: Mean $= \int_{-b}^{a} x f(x)\,dx$.

$$= \int_{-b}^{0} \frac{x\,2(b+x)}{b(a+b)}\,dx + \int_{0}^{a} \frac{x\,2(a-x)}{a(a+b)}\,dx$$

$$= \frac{2}{b(a+b)} \int_{-b}^{0} (bx+x^2)\,dx + \frac{2}{a(a+b)} \int_{0}^{a} (ax-x^2)\,dx.$$

$$= \frac{2}{b(a+b)} \left[\frac{bx^2}{2} + \frac{x^3}{3}\right]_{-b}^{0} + \frac{2}{a(a+b)} \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_{0}^{a}$$

$$= \frac{2}{b(a+b)} \left[-\frac{b^3}{2} + \frac{b^3}{3}\right] + \frac{2}{a(a+b)} \left[\frac{a^3}{2} - \frac{a^3}{3}\right]$$

$$= \frac{2}{b(a+b)} \left[\frac{-3b^3+2b^3}{6}\right] + \frac{2}{a(a+b)} \left[\frac{3a^3-2a^3}{6}\right]$$

$$= \frac{2}{3(a+b)} \left(-b^2/b\right) + \frac{a^2}{3b(a+b)}$$

Mean $= \dfrac{a-b}{3b}$.

Ex:10 Prove that the geometric mean G of the distribution $dF = 6(2-x)(x-1)dx$, $1 \le x \le 2$ is given by $6 \log(16G) = 19$.

Soln: Given $dF = 6(2-x)(x-1)dx$.

$\therefore$ pdf $f(x) = 6(2-x)(x-1)$

$$\log G = \int_{1}^{2} \log x\; f(x)\,dx$$

$$= \left[\text{using } \log G = \int_{a}^{b} \log x \cdot f(x)\,dx\right]$$

$$= 6\int \log x\,(2-x)\,(x-1)\,dx$$

$$= -6\int_2^1 (x^2-3x+2)\log x\,dx$$

$$= -6\left[\int_1^2 \log x\, d\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)\right]$$

$$= -6\left[\left\{\log x\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)\right\}_1^2 - \int_1^2\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)\frac{1}{x}dx\right]$$

$$= -6\left[\log 2\left(\frac{8}{3}-3\times 2+4\right)-\int_1^2\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)\frac{1}{x}dx\right]$$

$$= -6\left[\log 2\left(8/3-3\times 2+4\right)-\int_1^2\left(\frac{x^3}{3}-\frac{3x}{2}+2\right)dx\right]$$

$[\because \log 1 = 0]$

$$= -6\left[\log 2\times 2/3 - \left(\frac{x^3}{9}-\frac{3x^2}{4}+2x\right)_1^2\right]$$

$$= -6\left[2/3\log 2-\left(8/9-3+4-1/9+3/4-2\right)\right]$$

$$= -4\log 2+6\left(8/9-1-1/9+3/4\right)$$

$$= -4\log 2+6\left(19/36\right)$$

$\Rightarrow \log G = -4\log 2+19/6$ $\quad[\because \log m^n = n\log m]$

(ie) $\log G+\log 2^4 = 19/6$ $\quad[\because \log mn = \log m+\log n]$

$6\log(G\times 16) = 19$

Ex:1 verify that the following is distribution fn

$$F(x) = \begin{cases} 0, & x < a \\ \frac{1}{2}\left(\frac{x}{a}+1\right), & -a \le x \le a \\ 1 & x > 1 \end{cases}$$

solo: Clearly (i) $0 \le F(x) \le 1$ $\quad [\because F(x) = 0,\ x < -a,$ $\therefore F(-\infty) = 0]$

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$ $\quad [\because F(x) = 1,\ x > 1$ $\therefore F(\infty) = 1,\ x > 1]$

Hence $F(x)$ statisfies all the condition of a distribution function.

(iv) $f(x) = \frac{dF(x)}{dx} \begin{cases} 1/2a, & -a \le x \le a \\ 0, & \text{otherwise}. \end{cases}$

Also $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} \frac{1}{2a}\,dx = \frac{1}{2a}[x]_{-a}^{a} = \frac{1}{2a}(2a) = 1$

$\therefore f(x)$ is a p.d.f and $F(x)$ is a distribution fn.

Ex:2 Suppose that the amount of money that a person has saved is found to be a random variables with

$$F(x) = \begin{cases} \frac{1}{2}e^{-(x/50)^2} & x < 0 \\ 1 - \frac{1}{2}e^{-(x/50)^2} & x \ge 0. \end{cases}$$

(i) Is $F(\cdot)$ continuous? what is the pdf?

(ii) what is the probability that the amount of saving possessed by him will be (i) more than 50 (ii) equal to 50 rupees (iii) what is the conditional probability that the amount of savings will be less than Rs 100. given that it is more than 50?

soln: The given distribution function is continuous since the value $F(x)$ at $x=0$ is the Same.

probability density

function $f(x) = d/dx\, F(x)$

$$f(x) = \begin{cases} \frac{1}{2} e^{-(x/50)^2} (-2)\left(\frac{x}{50}\right)\left(\frac{1}{50}\right) \\ -\frac{1}{2} e^{-(x/50)^2} \left(-\frac{2x}{50}\right)\left(\frac{1}{50}\right) \end{cases}$$

$$= \begin{cases} \frac{-x}{2500} e^{-(x/50)^2}, & x < 0 \\ x/2500\, e^{-(x/50)^2}, & x \geq 0 \end{cases}$$

(ii) let 'x' be the random variable which represen the amount of savings.

$$p(x > 50) = 1 - P[x \leq 500]$$

$$= 1 - F(50)$$

$$= 1 - \left\{ 1 - \tfrac{1}{2} e^{-1} \right\}$$

$$= 1/2e$$

$$= \frac{1}{2.718}$$

$$= 0.1839.$$

$$p(x = 50) = 0.$$

[∵ The probability that a continuous variable takes a fixed value is zero].

(iii) let $P(A) = P(X < 100)$

$= P(X \leq 100) - P(X = 100)$

$= F(100)$ $[\because P(X = 100) = 0]$

$= 1 - \frac{1}{2}e^{-4}$

$= 1 - \frac{1}{109.19}$

$= 0.99$

$P(B) = P(X > 50) = 0.3679$ [From (ii)]

$P(A \cap B) = P\{X \leq 100 \cap X > 50\}$

$= P[50 < X < 100]$

$= F(100) - F(50)$

$= 0.99 - 0.817$

$= 0.173.$

Ex:3 The probability distribution function of a random variable X is

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & x \geq 2. \end{cases}$$

Find the cumulative distribution function of X.

Soln: we know that c.d.F $F(x) = \int_{-\infty}^{x} f(x)dx$, when x lies in $0 < x \leq 1$.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x\,dx .$$

$$= \left(\frac{x^2}{2}\right)_0^x$$

$$= x^2/2 .$$

When x lies in $1 < x \le 2$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left(x^2/2\right)_0^1 + \left(2x - x^2/2\right)_1^x$$

$$= 1/2 + 2x - x^2/2 - 3/2$$

$$= 2x - x^2/2 - 1$$

when x lies in $x \ge 2$,

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx + \int_{2}^{\infty} 0\,dx .$$

otherwise.

To find $F(x)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+x^2}\,dx.$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right]$$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1} x + \pi/2\right]$$

To find $P(X \geq 0)$

$$\therefore P(X \geq 0) = 1 - P(X \leq 0) = 1 - F(0) = 1 - 1/\pi\left[0 + \pi/2\right]$$

$$= 1 - 1/2$$

$$= 1/2$$

Ex:5 The length (in hours) x of a certain type of light bulb may be supposed to be a continuous random variable with p.d.f.

$$f(x) = a/x^3, \quad 1500 < x < 2500$$

$$= 0, \quad \text{elsewhere}.$$

Determine the constant 'a' the distribution function of x and compute the probability of the event $1,700 \leq X \leq 1,900$.

<u>Soln:</u> Given $f(x)$ is a pdf

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(ie) $a\int_{1500}^{2500} 1/x^3 \, dx = 1$

$$a\left[-\frac{1}{2x^2}\right]_{1500}^{2500} = 1$$

$$a\left[-\frac{1}{2(2500)^2} + \frac{1}{2(1500)^2}\right] = 1$$

$$a = 70,31,250.$$

To find the d.f $F(x)$.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{1500} f(x)\,dx + \int_{1500}^{x} f(x)\,dx.$$

$$= 0 + \int_{1500}^{x} a/x^3 \, dx.$$

$$= \left[-1/2x^2\right]_{1500}^{x}$$

$$= a\left[\frac{1}{-2x^2} + \frac{1}{2(1500)^2}\right]$$

$$= \frac{a}{2}\left[\frac{1}{(1500)^2} - 1/x^2\right]$$

To find $P(1700 \le X \le 1900)$

$$P(1700 \le X \le 1900) = F(1900) - F(1700)$$

$$= a/2\left[\frac{1}{2890000} - \frac{1}{3610000}\right]$$

Two dimensional Random Variables:

let S be the sample space. let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is two dimensional r.v random variables.

Two dimensional discrete Random Variables :

If the possible values of (x, y) are finite then (x, y) is called a two dimensional discrete random variable and it can be represented by $(x_i\, y_j)\; i = 1, 2 \ldots n;\; j = 1, 2 \ldots m$.

Note :

If (x, y) can take all the values in a region R in the xy - plane then (x, y) is called a two - dimensional continuous random variable

Joint probability function of the discrete random variables x and y.

For two discrete random variable x and y we write the probability that x will take the value x_i and y will take the value y_j as $P = (X = x_i, y = y_j)$ consequently $P(x = x_i, y = y_j)$ is the probability of the intersection of the events $X = x_i$ and $Y = y_j$

The function $f(x_i, y_j) = P(X = x_i, Y = y_j)$ is called the joint probability function or joint probability mass function for discrete random variable X and y.

Note: $f(x_i, y_j) = P(X = x_i, Y = y_j) = P_{ij}$ and P_{ij} should satisfies the following conditions

Marginal probability function of Y.

If the joint probability distribution of two random variable X and Y is given, then the marginal function of Y is given by

$$f(y) = p_Y(y_j) = P(Y = y_j) = p_{.j}$$

Here $p_{.j} = \sum_i p_{ij} = p_{1j} + p_{2j} + \cdots$ [Refer to above table]

Note: The set $\{y_j, p_{.j}\}$ is called the marginal distribution of Y.

Conditional probability:-

The conditional probability function of X given $Y = y_j$ is given by

$$P[X = x_i \mid Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]} = \frac{p_{ij}}{p_{.j}}$$

The conditional probability function of Y given $X = x_i$ is given by

$$P[Y = y_j \mid X = x_i] = \frac{p_{ij}}{p_{i.}} \quad (\text{or}) \quad f(y/x) = \frac{f(x,y)}{f(x)}$$

The two random variables of X and Y are said to be independent if $P[X = x_i \mid Y = y_j] = \frac{P(X=x_i)}{P(Y=y_j)}$

$$p_{ij} = p_{i.} \times p_{.j}$$

(ie) otherwise dependent.

Note: 1) The two random variables x and y are independent if each entry in the given table is the product of corresponding row and column entries using

$p_{ij} = p_{i\cdot} \times p_{\cdot j}$.

2) The two random variables x and y with joint pdf $f(x,y)$ [or $f_{XY}(x,y)$] and marginal p.d.f's $f(x)$ [or $f_X(x)$] and $g(y)$ [or $g_Y(y)$] and said to be independent if $f(x,y) = f(x) \cdot g(y)$.

3. joint probability function of continuous random variables x and y.

If x and y are continuous random variables, then we shall refer to $f(x,y)$ as the joint probability density function of these two random variables if the probability the $a_1 \le x_1 \le b_1$; $a_2 \le x_2 \le b_2$ is given by the multiple integral $\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx$.

(ie) $P[a_1 \le x \le b_1, a_2 \le y \le b_2] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx$

provided (i) $f(x,y) \ge 0$; (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$.

Note: Table - I.

To calculate marginal distribution when the random variable x takes horizontal values and y takes vertical values.

y \ x	x_1	x_2	x_3	$P_Y(y)=f(y)$
y_1	P_{11}	P_{12}	P_{31}	$P_{11}+P_{12}+P_{13}=P(Y=y_1)$
y_2	P_{21}	P_{22}	P_{32}	$P_{21}+P_{22}+P_{23}=P(Y=y_2)$
y_3	P_{31}	P_{32}	P_{33}	$P_{31}+P_{32}+P_{33}=P(Y=y_3)$
$P_X(x)=f(x)$	$P_{11}+P_{21}+P_{31}=P(X=x_1)$	$P_{12}+P_{22}+P_{32}=P(X=x_2)$	$P_{13}+P_{23}+P_{33}=P(X=x_3)$	

Table - II.

To calculate marginal distributions when the random variable x takes vertically and y takes horizontally.

x \ y	y_1	y_2	y_3	$P_X(x)=f(x)$.
x_1	P_{11}	P_{12}	P_{13}	$P_{11}+P_{12}+P_{13}=P(X=x_1)$
x_2	P_{21}	P_{22}	P_{23}	$P_{21}+P_{22}+P_{23}=P(X=x_2)$
x_3	P_{31}	P_{32}	P_{33}	$P_{21}+P_{32}+P_{33}=P(X=x_3)$
$P_Y(y)=f(x)$	$P_{11}+P_{21}+P_{31}=P(Y=y_1)$	$P_{12}+P_{22}+P_{32}=P(Y=y_2)$	$P_{13}+P_{23}+P_{33}=P(Y=y_3)$	

Ex: 4 From the following table for bivariate distribution of (X,Y) find.

(i) $P(X\leq 1)$ (ii) $P(Y<3)$

(iii) $P(X\leq 1, Y\leq 3)$ (iv) $P(X\leq 1/Y\leq 3)$

(v) $P(Y\leq 3/X\leq 1)$ (vi) $P(X+Y\leq 4)$

y / x	1	2	3	4	5	6
0	0	~~1/16~~ 0	1/32	2/32	2/32	3/32
1	1/6	~~1/8~~ 1/16	1/8	1/8	1/8	1/8
2	1/32	~~1/64~~ 1/32	1/64	1/64	0	2/64

soln:

y / x	1	2	3	4	5	6	$p_x(x) = f(x)$
0	0	0	1/32	2/32	2/32	3/32	$8/32 \rightarrow p(x=0)$
1	1/16	1/16	1/8	1/8	1/8	1/8	$10/16 \rightarrow p(x=1)$
2	1/32	1/32	1/64	1/64	0	2/64	$8/64 \rightarrow p(x=2)$
$p_y(y) = f(y)$	$3/32 \downarrow p(y=1)$	$3/32 \downarrow p(y=2)$	11/64	13/64	6/32	$16/64 \downarrow p(y=6)$	1

(i) $P(X \le 1) = P(X=0) + P(X=1)$

$= 8/32 + 10/16$

$= 28/32$

$= 7/8$

(ii) $P(Y \le 3) = P(y=1) + P(y=2) + P(y=3)$

$= 3/32 + 3/32 + 11/64$

$= 23/64$

(iii) $P(X \le 1, Y \le 3) = f(0,1) + f(0,2) + f(0,3) + f(1,1) + f(1,2) + f(1,3)$

$= 0 + 0 + 1/32 + 1/16 + 1/16 + 1/8$

$= 9/32.$

(iv) $P(X \leq 1 / Y \leq 3) = \dfrac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}$

$= \dfrac{9/32}{23/64}$

$= 9/28$

(v) $P(Y \leq 3 / X \leq 1) = \dfrac{P(X \leq 1, Y \leq 3)}{P(X \leq 1)}$

$= \dfrac{9/32}{7/8}$

$= 9/28$

(vi) $P(X+Y \leq 4) = f(0,1) + f(0,2) + f(0,3) + f(0,4)$

$= 0 + 0 + 1/3 + 2/32 + 1/16 + 1/16 + 1/8 + \frac{1}{32} + 1/32$

$= 13/32.$

EX:5 From the following joint distribution of X and Y find the marginal distribution.

y \ x	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0.

Soln: The marginal distributions are given in the below table.

y \ x	0	1	2	$P_Y(y)$
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	6/14
2	1/28	0	0	1/28
$P_X(x)$	10/28	15/28	3/28	$\sum p(x) = 1$ $\sum p(y) = 1$

The marginal distribution of x

$$P(X=0) = 10/28 \;;\; P(X=1) = 15/28 \;;\; P(X=2) = 3/28$$

The marginal distribution of y.

$$P(Y=0) = \frac{15}{28} \;,\; P(Y=1) = 6/14 \;,\; P(Y=2) = 1/28.$$

Ex: 6 let x and y have the following joint probability distribution.

y \ x	2	4
1	0.10	0.15
3	0.20	0.30
5	0.10	0.15

show that x and y are independent.

Soln:-

y \ x	2	4	$P_{i\cdot}$
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
$P_{\cdot j}$	0.40	0.60	1

$$P_{ij} = P_{i\cdot} \times P_{\cdot j}$$

For example $P_{12} = 0.15$

$P_{1\cdot} = 0.25$ — (1)

$P_{\cdot 2} = 0.60$

$\therefore P_{1\cdot} \times P_{\cdot 2} = 0.25 \times 0.60$

$= 0.15$ — (2)

From (1) and (2) we get

$$P_{12} = P_{1\cdot} \times P_{\cdot 2}$$

$\therefore$ The random variables X and Y are independent

Ex:7 The joint distribution of X and Y is given by $f(x,y) = \frac{x+y}{21}$, $x = 1,2,3$. $y = 1,2$, find the marginal distributions.

Soln: Given $f(x,y) = \frac{x+y}{21}$, $x = 1,2,3 \ldots ; y = 1,2$

The marginal distribution are given in the table.

$f(x,y)$		X			$P_Y(y) = f(y)$
		1	2	3	
Y	1	2/21	3/21	4/21	9/21
	2	3/21	4/21	5/21	10/21
$P_X(x) = f(x)$		5/21	4/21	9/21	1

The maginal distribution of X:

$$P(X=1) = 5/21 ; \; P(X=2) = \frac{7}{21} ; \; P(X=3) = \frac{9}{21}.$$

The maginal distribution of Y:

$$P(Y=1) = 9/21 ; \; P(Y=2) = 12/21$$

EX: 8 The two dimensional random variable (X, Y) has the joint density function $f(x,y) = \frac{x+2y}{27}$, $x = 0,1,2$; $y = 0,1,2$ ... Find the conditional distribution of Y for $X = x$. Also find the conditional distribution of X given $Y=1$

Soln: We know that the conditional probability distribution of Y for $X = x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)} \quad -(1)$$

where $f(x,y)$ is the joint probability distribution X and Y.

Joint probability distribution function for continuous random Variable x and y.

The joint probability distribution function of a two dimensional random variables (X, Y) is denoted by $F_{XY}(x, y)$ and is given by.

$$F_{XY}(x,y) = P[-\infty < X \leq x; -\infty < Y \leq y]$$
$$= \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f_{XY}(x,y)\, dx \right\} dy$$

Where $f_{XY}(x,y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

simply we can write as follows

$$F(x,y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f(x,y)\, dx \right\} dy$$

properties of joint distribution function :

1. $F(-\infty, y) = 0 = F(x, \infty)$ and $F(-\infty, \infty) = 1$
2. $P(a_1 < x \leq b_1 ; a_2 < y < b_2)$.

$$= F(b_1, b_2) + F(a_1, a_2) - F(a_1, b_2) - F(b_1, a_2)$$

Marginal distribution functions :

If the joint distribution function of the random variable (X, Y) is $F_{XY}(x,y)$, then the marginal distribution function of x is denoted by $F_X(x)$ is given by

$$F_X(x) = \sum_{y} P(X \leq x, Y = y) \quad \text{(For discrete random variables)}$$
$$= \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \right\} dx \quad \text{(For continuous random variables)}$$

$|||^{ly}$ the marginal distribution function of y is denoted by $F_Y(y)$ and is given by

$$F_Y(y) = \sum_x P(X=x, Y \le y) \text{ (For discrete random variable)}$$
$$= \int_{-\infty}^{y} \left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \right] dy \text{ (For continuous random variable)}$$

Joint probability density function :-

let $F_{XY}(x,y)$ be the joint probability distribution function. Then the joint probability density function of X and Y is given by

$$f_{XY}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}.$$

Marginal probability function :-

The marginal probability function of the two random variables X and Y are defined as follows

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \text{ (for continuous random variable)}$$
$$= \sum_y P_{XY}(x,y) \text{ (for discrete random variable)}$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \text{ (for continuous random variable)}$$
$$= \sum_x P_{XY}(x,y) \text{ (for discrete random variable)}$$

Marginal density functions :-

If we know the marginal probability function of X and Y as $F_X(x)$ and $F_Y(y)$ then marginal density function of X and Y can be obtained as follows.

The marginal density function of X and Y are

$$= \int_{x}^{1} 2dy$$

$$= 2(y)\Big|_{x}^{1}$$

$$= 2(1-x), \; 0 < x < 1$$

Marginal density function of y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{y} f(x,y)\,dx \quad [0 < x < y]$$

$$= \int_{0}^{y} 2dx$$

$$= 2y, \; 0 < y < 1$$

Then conditional density function of y given $X = x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}.$$

Ex:2 Find the marginal density function of x and y if $f(x,y) = \frac{2}{5}(2x+5y), \; 0 \le x \le 1, \; 0 \le y \le 1.$

Soln: Marginal density function of x is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{1} f(x,y)\,dy \quad [0 \le y \le 1]$$

$$= \int_{0}^{1} \frac{2}{5}(2x+3)\,dy = \frac{4x+3}{5}$$

Maaginal density function of y is

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 \frac{2}{5}(2x+3y)\,dx$$

$$= \frac{2+6y}{5}.$$

Ex: 3 The joint probability density function of the two dimensional random variable is

$$f(x,y) = \begin{cases} 8/9\, xy & , 1 \le x \le y \le 2 \\ 0, & \text{otherwise}. \end{cases}$$

(i) Find the maaginal density function of x and y.

(ii) Find the conditional density function of y given $X = x$.

Soln: Maaginal density function of x is given by

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} 8/9\, xy\,dy \quad \therefore [x \le y \le 2]$$

$$= 8/9\, x \left[y^2/2\right]_x^2.$$

$$= \frac{4x}{9}[4 - x^2], \quad 1 \le x \le 2.$$

III^{ly} $f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_1^y 8/9\, xy\,dx \quad [1 \le x \le y]$$

$$= \frac{8y}{9}\left[\frac{x^2}{2}\right]_1^y$$

$$= 4y/9\,[y^2 - 1], \quad 1 \le y \le 2.$$

Now $f(x) \cdot f(y) = e^{-x} \cdot e^{-y}$

$$= e^{-(x+y)}$$

But $f(x,y) = e^{-x} \cdot e^{-y}$

$$= e^{-(x+y)}$$

$$\therefore f(x,y) = f(x) \cdot f(y)$$

$\therefore$ x and y are independent.

EX : 2 The bivariate random variable x and y has the p.d.f

$$f(x,y) = \begin{cases} k x^2 (8-y), & x < y < 2 \\ & 0 \leq x \leq 2 \end{cases}.$$

Find k.

Soln: we know that if $f(x,y)$ is a p.d.f then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1 \quad [\text{condn for a p.d.f}]$$

$$\therefore \int_0^2 \int_x^{2x} k x^2 (8-y)\, dy\, dx = 1.$$

$$k \int_0^2 x^2 \left(8y - y^2/2\right)_x^{2x} dx = 1$$

$$k \int_0^2 x^2 \left(16x - \frac{4x^2}{2} - 8x + x^2/2\right) dx = 1$$

$$k \int_0^2 \left(8x^3 - 3/2\, x^4\right) dx = 1$$

$$k \left[\frac{8x^4}{4} - \frac{3}{2} \cdot \frac{x^5}{5}\right]_0^2 = 1$$

$$k \left[32 - 48/10\right] = 1$$

$$k = 5/110$$

Ex: 3 If the join p.d.f X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & , 0<x<2, 2<y<4 \\ 0 & \text{otherwise} \end{cases}$$

find (i) $P(X<1 \cap Y<3)$

(ii) $P(X<1 / Y<3)$

Soln: We know that

$$P(X<1 \cap Y<3) = \int_0^1 \int_2^3 f(x,y)\,dx\,dy$$

$$\left[\text{using } P[((a_1 < X \leq b_1) \cap (a_2 < Y < b_2)) = \int_{a_1}^{b_2} \int_{a_2}^{b_2} f(x,y)\,dx\,dy \right]$$

$$= \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx.$$

$$= \frac{1}{8} \int_0^1 \left(6y - xy - \frac{y^2}{2}\right)_2^3 dx$$

$$= \frac{1}{8} \int_1^0 \left[\left(18 - 3x - \frac{9}{2}\right) - \left(12 - 2x - \frac{4}{2}\right) \right] dx$$

$$= \frac{1}{8} \int_0^1 (6 - x - 5/2)\,dx$$

$$= \frac{1}{8} \left[6x - \frac{x^2}{2} - \frac{5x}{2} \right]_0^1$$

$$= 3/8 \quad \text{--- (1)}$$

(ii) $P(X<1 / Y<3) = \dfrac{P(X<1 \cap Y<3)}{P(Y<3)}$ --- (2)

$$P(Y<3) = \int_0^2 \int_3^{?} \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2 \left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^2 \left\{\left(18 - 3x - \frac{9}{2}\right) - (12 - 2x - 2)\right\} dx$$

$$= \frac{1}{8}\left[\int_0^2 \frac{7}{2} - x\right] dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{8}[7 - 2]$$

$$= 5/8 \quad \text{--- (3)}$$

$\therefore P(Y<3) = 5/8$

sub (1) and (3) in (2) we get

$$P(X<1/Y<3) = \frac{3/8}{5/8}$$

$$= 3/5$$

Ex: 4 the j.d.f of the random variables X and Y is given by

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

find (i) $f_X(x)$, (ii) $f_Y(y)$, (iii) $f(y/x)$.

soln: We know that

$$\text{(i) } f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

[By defn of marginal probability fn]

Given $f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(i) $f(x,y) \ge 0$ in the given interval $0 \le (x,y) \le 1$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$$= \frac{2}{5}\int_0^1 \left[x^2 + 3xy\right]_0^1 dy$$

$$= \frac{2}{5}\int_0^1 (1+3y)\,dy$$

$$= \frac{2}{5}\left[y + \frac{3y^2}{2}\right]_0^1$$

$$= \frac{2}{5}\left[1 + \frac{3}{2}\right]$$

$$= \frac{2}{5}\left(\frac{5}{2}\right)$$

since $f(x,y)$ statisfies the given two conditions, it is a j.d.f.

Ex:6 If the joint distribution function X and Y is given by

$$F(x,y) = (1-e^{-x})(1-e^{-y}) \text{ for } x>0,\ y>0.$$
$$= 0, \text{ otherwise}.$$

(i) Find the marginal densities of X and Y

(ii) Are X and Y independent.

(iii) $P(1<X<3),\ 1<Y<2)$.

Soln:1, Given $F(x,y) = (1-e^{-x})(1-e^{-y})$

$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint p.d.f is given by

$$f(x,y) = \partial^2 F(x,y) / \partial x \partial y$$

$$= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} - e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right]$$

$$f(x,y) = \begin{cases} e^{-(x+y)} & , x \geq 0, \ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = e^{-(x+y)}$$

The marginal probability function of x is

$$f(x) = f_X(x)$$

$$= \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{\infty} e^{-(x+y)}\,dy$$

$$= \left[-e^{-(x+y)}\right]_0^{\infty} \quad -(1)$$

$$= e^{-x}.$$

The marginal probability function of y is

$$f(y) = f_Y(y)$$

$$= \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{\infty} e^{-(x+y)}\,dx$$

$$= \left[-e^{-(x+y)}\right]_0^{\infty} \quad -(2)$$

$$= e^{-y}.$$

From (1) and (2) we get

$$f(x) \cdot f(y) = e^{-x} e^{-y} = e^{-(x+y)}$$

$$= f(x,y)$$

∴ x and y are independent.

3. $P(1<x<3,\ 1<y<2) = P(1<x<3)\cdot P(1<y<2)$

[since x and y are independant]

$$= \int_1^3 f(x)\,dx \times \int_1^2 f(y)\,dy$$

$$= \int_1^3 e^{-x}dx \cdot \int_1^2 e^{-y}dy \quad \text{[using (1) and (2)]}$$

$$= (-e^{-x})_1^3 \times (-e^{-y})_1^2$$

$$= (-e^{-3}+e^{-1})(-e^{-2}+e^{-1})$$

$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$

EX:7 The joint density function of two random variables x and y is

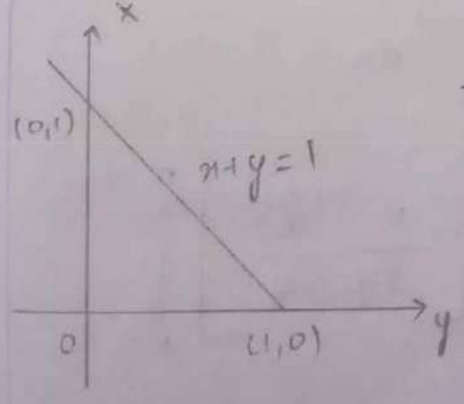
$$f(x,y) = \begin{cases} \frac{1}{3}(3x^2+xy) & 0\le x\le 1,\ 0\le y\le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P[x+y\ge 1]$.

Soln: $P[x+y\ge 1] = 1 - P[x+y<1]$ —(1)

Now $P[x+y<1] = \int_0^1\int_0^{1-y} f(x,y)\,dx\,dy$

$$= \int_0^1\int_0^{1-y} \frac{1}{3}(3x^2+xy)\,dx\,dy \quad \text{[from fig]}$$



$$= \frac{1}{3}\int_0^1 \left(x^3 + \frac{x^2y}{2}\right)_0^{1-y} dy$$

$$= \frac{1}{3}\int_0^1 \left[(1-y)^3 + \frac{(1-y)^2y}{2}\right] dy$$

$$= \frac{1}{3}\int_0^1 \left[-\frac{y^3}{2} + 2y^2 - \frac{5y}{2} + 1\right] dy$$

$$= \frac{1}{3}\left[-\frac{y^4}{8} + \frac{2y^3}{3} - \frac{5y^2}{2} + y\right]_0^1$$

$$= \frac{1}{8}\left[-\frac{1}{8}+\frac{2}{3}-\frac{5}{4}+1\right]$$

$$= 19/144$$

$$\therefore P[x+y \geq 1] = 1 - P[x+y<1]$$

$$= 1 - \frac{19}{144}$$

$$= 65/72.$$

Ex:8 Examine whether the variables x and y are independent, whose joint density is $f(x,y) = x e^{-x(y+1)}$ $0 < x, y < \infty$.

Soln: The marginal probability function of x is

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} x \cdot e^{-x(y+1)}\,dy$$

$$= x\left[\frac{e^{-x(y+1)}}{-x}\right]_0^{\infty}$$

$$= -\left[0 - e^{-x}\right]$$

$$= e^{-x}$$

$$\text{lly } f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} x e^{-x(y+1)}\,dx$$

similarly $f(x/y) \neq 1$

$$= \left\{ x\left[\frac{e^{-x(y+1)}}{-(y+1)}\right] - \left[\frac{e^{-x(y+1)}}{(y+1)^2}\right]\right\}_0^{\infty}$$

$$= \frac{1}{(y+1)^2}$$ [Integration by parts]

Here $f(x)\,f(y) = e^{-x} \times \frac{1}{(y+1)^2}$

$\therefore$ X & Y are not independent.

Ex:9 The joint probability distribution of x and y is given by

$$f(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise.} \end{cases}$$

Find $P(2<y<4 / x=2)$.

Soln:-

We know that the conditional density function of y given x $f(y/x) = \frac{f(x,y)}{f(x)}$

Now $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{2}^{4} \frac{6-x-y}{8}\,dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right]$$

$$\therefore f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{1}{8}(6-x-y)}{\frac{6-2x}{8}}$$

$$= \frac{6-x-y}{6-2x}$$

$$f(y|x=2) = \frac{4-y}{2}.$$