Kunthavai Nachiaar Govt. Arts College (W) Autonomous, Thanjavur

B.Stat Major

Major Based Elective – 1(Numerical Analysis)

18K5SELS1
Hrs:5
Credit:5

Unit – I

Finite Differences – Forward and Backward differences, operators Δ, ∇ & E, and their basic properties – Interpolation with equal intervals – Newton's Forward & Backward Difference formula – Simple problem.

Unit – II

Interpolation with unequal intervals – Divided differences and their properties – Newton's divided difference formula – Lagrange's formula – Simple problems.

Unit – III

Central difference interpolation formula – Gauss Forward and Backward difference formula – Stirling's, Bessel's Central difference formula – Simple problems.

Unit – IV

Inverse interpolation : Lagrange's method – Interaction of successive approximation method – simple problems.

Unit – V

Numerical Integration : Trapezoidal Rule – Simpson's $\frac{1}{3}^{rd}$ & $\frac{3}{8}^{th}$ rules – Weddle's Rule – Euler's summation formula – Simple problems.

Books for Study :

1. Scarborough, B.Numerical Mathematical Analysis, OUP.
2. Sastry, S.S. Introductory method of numerical Analysis, P.H.I.
3. Balasubramanian : Numerical Mathematics, Vol I & II.

(Data can be taken from online)

UNIT - I

Finite DIFFERENCES:

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \dots n$ of any function $y = F(x)$ the value of x being equally spaced. i.e, $x_i = x_0 + ih$, $i = 0, 1, 2, \dots n$. Suppose that we are required to recover the values of $F(x)$ for some intermediate values of x, or to be obtain the derivative of $F(x)$. For some x in the range $x_0 \le x \le x_n$. The methods for the solution of these problems are based on the concept of the differences of a function which we now proceed to define.

FORWARD DIFFERENCES:

If $y_0, y_1, \dots, y_n$ denote a set of values of y. then $y_1 - y_0$, $y_2 - y_1$, ..., $y_n - y_{n-1}$ are called the differences of y. Denoting these differences by Δy_0, Δy_1, ..., Δy_{n-1} respectively, we have

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \dots,$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the forward difference operator and Δy_0

called first forward differences.

The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$ Similarly, one can define third forward differences, fourth forward differences, etc. thus.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0;$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 \Rightarrow y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

and

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

It is therefore clear that any higher order differences can easily be expressed in terms of the ordinates, since the co-efficients occuring on the right side are the binomial coefficients.

The following tables shows how the forward differences of all orders can be formed.

FORWARD DIFFERENCE TABLE

x	y	1st order differences Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

BACKWARD DIFFERENCES:

The differences $y_1 - y_0$, $y_2 - y_1$, ..., $y_n - y_{n-1}$ are called first backward differences if they are denoted by Δy_1 ∇y_1, ∇y_2, ... ∇y_n respectively. So that $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, ..., $\nabla y_n = y_n - y_{n-1}$ where ∇ is called the backward difference operator. In a similar way. one can define backward difference operator. In a similar way, one can define backward difference of higher orders. Thus we obtain

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \Delta^2 y_3 - \Delta^2 y_2 = y_3$$

with the since values of x and y as in, a backward difference table can be formed.

BACKWARD DIFFERENCE TABLE:

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_4		$\nabla^3 y_5$		$\nabla^6 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^5 y_6$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

Central differences:-

The central differences operation δ is detained by the relations

$$y_1 - y_0 = \delta y_{1/2}, \; y_2 - y_1 = \delta y_{3/2}, \ldots, \delta y_n - y_{n-1}$$
$$= \delta y_n - y_2$$

Similarly, higher order central difference can be defined with the values of x and y as in the preceding two values tables, a central difference table can be formed thus.

<u>Central difference table</u>

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

It is clear from the three tables that in a definite numerical case the same numbers occur in the same position whether we use forward, backward or central differences. Thus we obtain.

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}$$

$$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2}$$

etc..

<u>Symbolic Relations and separation of Symbols:</u>

Difference formulate can easily be established by symbolic methods using the shift operator E and the average or mean operator μ, in addited to the operators Δ, ∇ and δ.

The averaging operator μ is defined by the equation

$$\mu y_r = \tfrac{1}{2}(y_{r+1/2} + y_{r-1/2})$$

The shift operator E is defined by the equation

$$E y_r = y_{r+1}$$

Which shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives

$$E^2 y_r = E(E y_r)$$
$$= E y_{r+1} = y_{r+2}$$

and in general,

$$E^n y_r = y_{r+n}$$

It is now easy to derive a relationship between Δ and E, for we have

$$\Delta y_0 = y_1 - y_0$$
$$= (E-1) y_0$$

and hence $\Delta = E-1$

or $E = 1+\Delta$

The student should note that does not mean that.

existence as separate entities. It merely implies that the effects of the operator E on y_0 is the same as that of the operator $(1+\Delta)$ on y_0.

We can now express any higher order forward difference in terms of the given function values.

For example,

$$\Delta^3 y_0 = (E-1)^3 y_0$$
$$= (E^3 - 3E^2 + 3E - 1) y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

From the definitions, the following relation can easily be established.

$$\nabla = 1 - E^{-1}$$
$$\delta = E^{1/2} - E^{-1/2}; \quad \mu = 1/2\,(E^{1/2} + E^{-1/2}),$$
$$\mu^2 = 1 + 1/4\,\delta^2$$
$$\Delta = \nabla E = \delta E^{1/2}$$

As an example,

We prove the relation $\mu^2 = 1 + 1/4\,\delta^2$

We have, by definition

$$\mu y_n = 1/2\,(y_{n+1/2} + y_{n-1/2})$$
$$= 1/2\,(E^{1/2} y_n + E^{-1/2} y_n)$$
$$= 1/2\,(E^{1/2} + E^{-1/2})\, y_n$$

Hence $\mu = 1/2\,[E^{1/2} + E^{-1/2}]$

and

$$\mu^2 = \frac{1}{4}\left[E^{1/2} + E^{-1/2}\right]^2 = \frac{1}{4}\left[E + E^{-1} + 2\right]$$

$$= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right]$$

$$= \frac{1}{4}\left(\delta^2 + 4\right)$$

We have therefore

$$\mu = \sqrt{1 + \frac{1}{4}\delta^2}$$

Finally we define the operator D such that

$$D\,y(x) = \frac{d}{dx}\,y(x).$$

To relate D to E, we start with the Taylor's series

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \cdots$$

This can be written in the symbolic form

$$E(y)(x) = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots\right] y(x)$$

Since the series in the brackets is the expansion of e^{hD}, we obtain the interesting result

$$E = e^{hD}$$

Using the relation a Number of useful identities can be derived. This relation is used to separate the effect of E in to then of the powers of Δ and this method of separation is called the method of separation of symbols.

The following examples demonstrate the use of this method.

Introduction:

The difference derived from the sequences of values uptained from a given function say $F(x)$ when the variable x. changes in Arithmetic progression say $x=a$, $a+h$, $a+2h$, ... $a+nh$

The function takes the values $F(x)$
$F(x) = F(a)$, $F(a+h)$ $(a+2h)$... $f(a+nh)$

Here h is know as interval of differences. the values of the independent variable x is know as arguments and that of dependent variable say $y=F(x)$ as entry

Definition:

The calculas of finite differenies deal with the change in the values of the function [dependent variable] due to change in the independent variable.

Let $y=F(x)$ the values of the independent variable x is called arguements and the coresspording value of the dependent variable is called entry

x [Arguement] : a $a+h$ $a+2h$ ... $a+nh$

$F(x)$ [entry] : $F(a)$ $F(a+h)$ $F(a+2h)$ +...$F(a+nh)$

$\Delta F(a) = F(a+h) - F(a)$

Δ is called forward are deciding difference operators.

Forward difference Table :-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
a	$f(a)$	$f(a+h) - f(a) = \Delta f(a)$	$\Delta f(a+h) - \Delta f(a) = \Delta^2 f(a)$
$a+h$	$f(a+h)$	$f(a+2h) - f(a+h) = \Delta f(a+h)$	
$a+2h$	$f(a+2h)$	$f(a+3h) - f(a+2h) = \Delta f(a+2h)$	$\Delta f(a+nh) - \Delta f[a+(n-1)h] = \Delta^2 f(a+(n-1)h)$
$\vdots$	$\vdots$	$f(a+4h) - f(a+3h) = \Delta f(a+3h)$	
$a+nh$	$f(a+nh)$	$f(a+nh) - f(a(n-1)h] = \Delta f(a(n-1)h$	

x :	0	1	2	3	4
$f(x)$:	1	0	1	10	12

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		-1			
1	0		2		
		1		6	
2	1		8		-21
		9		-15	
3	10		-7		
		2			
4	12				

Note :

(i) $\Delta f(a)$, $\Delta f(a+h)$, $\Delta f(a+2h)$, $\Delta f[a+(n-1)h]$ are called first differences.

(ii) The higher differences $\Delta^2 f(a)$, $\Delta^3 f(a)$ ... $\Delta^n f(a)$ are defined as follows.

Second order differences :-

$$\Delta^2 f(a) = \Delta . \Delta f(a)$$
$$= \Delta [f(a+h) - f(a)]$$
$$= \Delta f(a+h) - \Delta f(a)$$
$$= [f(a+2h) - f(a+h)] - [f(a+h) - f(a)]$$
$$\Delta^2 f(a) = f(a+2h) - 2f(a+h) + f(a)$$

$$\Delta^3 f(a) = \Delta . \Delta^2 f(a)$$
$$= \Delta f(a+2h) - 2f(a+h) + f(a)$$
$$= \Delta f(a+2h) - 2\Delta f(a+h) + \Delta f(a)$$
$$= f(a+3h) - f(a+2h) - 2f(a+2h) - f(a+h)] \quad f(a+h) - f(a)$$
$$\Delta^3 f(a) = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$$

$$\Delta^n f(a) = \Delta^{n-1} \Delta f(a)$$
$$= \Delta^{n-1} [f(a+h) - f(a)]$$

Note:

(iii) $f(a)$ the first entry is termed as the leading term and the differences $\Delta f(a)$, $\Delta^2 f(a)$ are known as leading differences.

iv) $\Delta^n f(a) = f(a+nh) - {}^nC_1 f(a+(n-1)h) + {}^nC_2 f(a+(n-2)h) + \dots + (-1)^n f(a)$

Backward difference Table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$
a	$f(a)$	$f(a+h)-f(a) = \nabla f(a+h)$	
$a+h$	$f(a+h)$	$f(a+2h)-f(a+h) = \nabla f(a+2h)$	$\nabla f(a+2h)-\nabla f(a+h) = \nabla^2 f(a+2h)$
$a+2h$	$f(a+2h)$		
$\vdots$	$\vdots$		
$a+nh$	$f(a+nh)$	$f(a+nh)-f(a+(n-1)h) = \nabla f(a+nh)$	$\nabla f(a+nh) - \nabla f(a+(n-1)h) = \nabla^2 f(a+nh)$

Note:

(i) $\nabla f(a+h) = f(a+h) - f(a)$

(ii) $\nabla f(a) = f(a) - f(a-h)$

Here ∇ is called backward differences operator. The higher order differences $\nabla^2 f(x)$, $\nabla^3 f(x)$, $\nabla^n f(x)$ as define as follows.

$f(a-nh)$

$f(a-3h) = f(a-2h) - f(a-3h) = \nabla f(a-2h)$

$f(a-2h) = f(a-h) - f(a-2h) = \nabla f(a-h)$

$f(a-h)$

$f(a) \cdot f(a) - f(a-h) = \nabla f(a)$.

$$\nabla^2 f(a) = \nabla \cdot \nabla f(a)$$
$$= \nabla \cdot [f(a) - f(a-h)]$$
$$= f(a) - f(a-h) - [f(a-h) - f(a-2h)]$$
$$\nabla^2 f(a) = f(a) - 2f(a-h) + f(a-2h)$$

$$\nabla^3 f(a) = \nabla \cdot \nabla^2 f(a)$$
$$= \nabla[f(a) - 2f(a-h) + f(a-2h)]$$
$$= \nabla f(a) - 2\nabla f(a-h) + \nabla f(a-2h)$$
$$= [f(a) - f(a-h)] - [2[f(a-h) - f(a-2h)]$$
$$+ f(a-2h) - f(a-3h)]$$
$$= f(a) - f(a-h) - 2f(a-h) + 2f(a-2h) + f(a-2h) - f(a-3h)$$

$$\nabla^3 f(a) = f(a) - 3f(a-h) + 3f(a-2h) - f(a-3h)$$

Similarly the n^{th} difference is (b)

$$\nabla^n f(a) = f(a) - nC_1 f(a-h) + nC_2 f(a-2h) \ldots (-1)^n nC_n f(a-nh)$$

<u>note</u>:

1.) $\nabla^n f(a) = \nabla^{n-1} \nabla f(a)$

$$= \nabla^{n-1}[f(a) - f(a-h)]$$
$$= \nabla^{n-1} f(a) - \nabla^{n-1} f(a-h)$$

Problem 1:

Given the following table construct a difference table and from it estimate y when $\bar{x} = 0.35$ by using newton backward Interpolation formula.

x:	0	0.1	0.2	0.3	0.4
y:	1	1.095	1.179	1.251	1.310

Solution:

x	y	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1				
		0.095			
			-0.011		
0.1	1.095			-0.001	
		0.084			0
0.2	1.179		-0.012		
		0.072		-0.001	
0.3	1.251		-0.013		
		0.059			
0.4	1.310				

$$\bar{x} = \frac{\bar{x} - x_0}{x_1} = \frac{0.35 - 0}{0.1} = 3.5$$

$$y_x = y_0 + x\,\Delta f(x) + \frac{x(x-1)}{2!}\Delta^2 f(x) + \frac{x(x-1)(x-2)}{3!}$$

$$\Delta^3 f(x) + \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(x)$$

$$y_x = 1 + 3.5 \times 0.095 + \frac{3.5(3.5-1)}{1 \times 2}(-0.011) +$$

$$\frac{3.5(3.5-1)(3.5-2)}{1 \times 2 \times 3}\, 0.001 + \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{1 \times 2 \times 3 \times 4}$$

$= 1 + 0.3325 - 0.048125 + 0.0021875$

$y_x = 1.2821875.$

Problem 2:

From the following table estimate by interpolation the number of units of a commodity supplied when the price is Rs. 4

Price Rupees:	1	3	5	7	9
No. of units supplied:	256	625	935	1201	133

$$\bar{x} = \frac{\bar{x} - x_0}{x_1} = \frac{4-1}{2} = 1.5$$

$$x = 1.5$$

degree $(n-2)$ Thus by contintuing the process. We will get a polynomial of zero degree 1 for the n^{th} difference is $\Delta^n f(x) = n(n-1)(n-2)\ldots 1\, n^n\, a_n\, x^{n-1} = n!\, h^n a_n$

Thus n^{th} difference is a constant and so all higher difference are are zero is $(n+1)^{+n}$ and higher difference of a polynomial of n^{th} degree and zero.

Note:

1) $\Delta^n f(x) = n!\, h^n\, a_n x^{n-n} = n!\, h^n a_n$

2) $\Delta^m f(x) = n!\, h^n\, a_n\, x^{n-m}.$

Theorem:

To express any value of the function in terms of leading term and the leading difference of a difference table.

$$f(a+nh) = f(a) + nc_1 \Delta f(a) + nc_2 \Delta^2 f(n) + \dots$$
$$nc_n \Delta^n f(a)$$

Proof:

$$\Delta f(a) = f(a+h) - f(a)$$

$$f(a+h) = \Delta f(a) + f(a) \rightarrow ①$$

$$\Delta f(a+h) = f(a+2h) - f(a+h)$$

$$f(a+2h) = \Delta f(a+h) + f(a+h)$$

$$f(a+2h) = \Delta f(a+h) + \Delta [f(a) + f(a)]$$

$$= f(a) + \Delta f(a) + \Delta f(a+h)$$

$$= f(a) + \Delta f(a) + \Delta [\Delta f(a) + f(a)]$$

$$= f(a) + \Delta f(a) + \Delta^2 f(a) + \Delta f(a)$$

$$f(a+2h) = f(a) + 2\Delta f(a) + \Delta^2 f(a) \rightarrow ②$$

$$\Delta f(a+2h) = f(a+3h) - f(a+2h)$$

$$[\Delta f(a+3h) = f(a+2h) - \Delta f(a+2h)]$$

$$\Delta f(a+3h) = f(a+2h) + \Delta f(a+2h)$$

$$= f(a) + 2\Delta f(a) + \Delta^2 f(a) + $$

$$\Delta[f(a) + 2\Delta f(a) + \Delta^2 f(a)] = \Delta f(a) + 2\Delta^2 f(a) + \Delta^3 f(a)$$

$$= f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a)$$

Similarly proceeding line this we get

~~f(a+nk~~

$$f(a+nh) = f(a) + nc_1 \Delta f(a) + nc_2 \Delta^2 f(a) + \dots + nc_n \Delta^n f(a)$$

This result is known as newton Gregory formula or "forward interpolation."

Problem:

1) Interpolate the population for the year 1966 from the following data.

year :	1960	1970	1980	1991
population of a town :	25494	29003	32528	3

@ $F(x) = 4x^5 + 2x$ find the values of $\Delta^5 f(x)$

Solution:

assuming,

$h = 1 \quad a_n = 4$

$h = 1 \quad n = 5$

$m = 5$

$$\Delta^n f(x) = n!\, h^n\, a_n$$

$$= 5!\,(1)^5\, 4$$

$$= 120 \times 4$$

$$\Delta^n f(x) = 480$$

b) $f(x) = 4x^5 + 2x$ find the value of $\Delta^3 f(x)$

Solution:

assuming,

$h = 1$ $an = 4$

$h = 1$ $n = 5$

$m = 3$

$$\Delta^m f(x) = n!\, h^n\, an\, x^{n-m}$$

$$= 5!\,(1)^5\, 4\, x^{5-3}$$

$$= 120 \times 4x^2$$

$$\Delta^3 f(x) = 480x^2$$

3) Find the value of $\Delta^3 (1-x)(1-2x)(1-3x)$

Solution:

$$f(x) = (1-x)(1-2x)(1-3x)$$

$$= (1-2x-x+2x^2)(1-3x)$$

$$= (1-3x+2x^2)(1-3x)$$

$$= 1-3x-3x+9x^2+2x^2-6x^3$$

$$f(x) = 1-6x+11x^2-6x^3$$

$an = -6$, $n = m = 3$; $h = 1$

$$\Delta^n f(x) = n!\, an\, h^n$$

$$\Delta^3 f(x) = 3!\,(-6)\,(1)^3$$

$$\Delta^3 f(x) = -36.$$

Difference formula [or] characteristics of operator Δ

1) $\Delta c = 0$ where c is constant as, also $\Delta c = c\Delta = 0$

2) $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$

3) If 'a' 'b' are constant then $\Delta[a f(x) + b\, g(x)] = a\,\Delta f(x) + b\,\Delta g(x)$.

4) $\Delta^n \Delta^m f(x) = \Delta^{n+m} f(x)$

5) $\Delta f(x)\, g(x) = [f(x+h) \cdot \Delta g(x)] + [g(x) \cdot \Delta f(x)]$

(or)

$[g(x+h) \cdot \Delta f(x)] + [f(x) \cdot \Delta g(x)]$

5) a) $\Delta f(x)\, g(x) = f(x+h) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$ or

$g(x+h) \cdot \Delta f(x) + f(x) \cdot \Delta g(x)$

Proof:

$\Delta f(x) = f(x+h) - f(x)$

$\Delta[f(x) \cdot g(x)] = f(x+h)\, g(x+h) - f(x)\, g(x) \rightarrow ①$

Add & sub, $f(x+h) \cdot g(x)$

$\Rightarrow f(x+h)\, g(x+h) - [f(x+h)\, g(x)] + [f(x+h) \cdot g(x)] - f(x)\, g(x)$

$= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$

$\Delta f(x) = f(x+h)\, \Delta g(x) + g(x) \cdot \Delta f(x)$.

5) (b) $g(x+h) \cdot \Delta f(x) + f(x) \cdot \Delta g(x)$

$$[f(x)\, g(x) = f(x+h)\, g(x)]$$

Proof:

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta [f(x)\, g(x)] = f(x+h)\, g(x+h) - f(x)\, g(x) \rightarrow (1)$$

Add & sub $g(x+h)\, f(x)$

$$\Rightarrow f(x+h) \cdot g(x+h) - [g(x+h)\, f(x)] + [g(x+h)\, f(x)] - f(x)\, g(x)$$

$$= g(x+h)\, [f(x+h) - f(x)] + f(x)\, [g(x+h) - g(x)]$$

$$= g(x+h)\, \Delta f(x) + f(x)\, \Delta g(x)$$

---

(6) $\Delta \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\, \Delta f(x) - f(x)\, \Delta g(x)}{g(x+h)\, g(x)}$

Proof:

$$\Delta \left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)\, g(x) - f(x)\, g(x+h)}{g(x+h)\, g(x)}$$

Add & sub $f(x)\, g(x)$

$$= \frac{f(x+h)\, g(x) - [f(x)\, g(x)] + [f(x)\, g(x)] - f(x)\, g(x+h)}{g(x+h)\, g(x)}$$

$$= \frac{g(x)\, [f(x+h) - f(x)] - f(x)\, [g(x+h) - g(x)]}{g(x+h)\, g(x)}$$

$$= \frac{g(x) \cdot \Delta f(x) - f(x)\, \Delta g(x)}{g(x+h) \cdot g(x)}.$$

⑦ $$\Delta c^x = \left[c^{x+h} - c^x\right]$$
$$= c^x \left[c^h - 1\right]$$

⑧ $$\Delta \log x = \log(x+h) - \log x$$
$$= \log\left(\frac{x+h}{x}\right)$$
$$= \log\left[1 + h/x\right]$$

The shifting operators E :-

The operation of shifting $f(x)$ to $f(x+h)$ is denoted by $E f(x)$.

i.e) $f(x+h) = E f(x)$

$f(x+2h) = E f(x+h)$

$= E\left(E f(x)\right)$

$f(x+2h) = E^2 f(x)$

$f(x+3h) = E f(x+2h)$

$= E \cdot E^2 f(x)$

$f(x+3h) = E^3 f(x)$

$f(x+nh) = E^n f(x)$.

Note: Here it should be noted carefully that E^n do not mean n as an ~~exposed~~ Exponent power, but simply the number of operator.

Relation between Δ and E :-

We know that $\Delta f(x) = f(x+h) - f(x)$

$$= E f(x) - f(x)$$

$$\Delta f(x) = f(x)[E-1] \rightarrow (1)$$

Since $f(x)$ is orbitary so omiting it from (1)

we get $\Delta = E - 1$

$$\therefore E = 1 + \Delta$$

Relation between ∇ and E

$$\nabla f(x+h) = f(x+h) - f(x)$$

$$f(x) = f(x+h) - \nabla f(x+h)$$

$$f(x) = E f(x) - \nabla[E f(x)]$$

$$f(x) = E f(x)[1 - \nabla] \rightarrow (2)$$

Since $f(x)$ is orbitary so omitted it from (2)

$$1 = E(1 - \nabla)$$

$$E = 1/_{1-\nabla} \Rightarrow 1 - \nabla^{-1}$$

Properties of the operator E and Δ

(i) operator E and Δ are intributioned to consider a function $U(x)$, which is the sum of the function $f(x), g(x), h(x)$, that is, then

$$U(x) = f(x) + g(x) + h(x) + \ldots$$

$$E\,U(x) = E[f(x) + g(x) + h(x) + \ldots]$$

$$= E[f(x+h) + g(x+h) + h(x+h) \ldots]$$

$$= E f(x) + E g(x) + E h(x) +$$

$\Delta\ U(x) = \Delta[f(x) + g(x) + h(x) + \dots]$

$= [f(x+h) + g(x+h) + h(x+h)] - [f(x) + g(x) + h(x)]$

$[\because \Delta f(x) = f(x+h) - f(x)]$

$\Rightarrow \cancel{f(x+h} \; f(x+h) - f(x) + g(x+h) - g(x) + h(x+h) - h(x)$

$= \Delta f(x) + \Delta g(x) + \Delta h(x).$

i.e, Δ is distributive

(ii) E and Δ are cumulative with ~~degreed~~ (degrad) to constant.

i.e) a) $E[c\, f(x)] = c \cdot E\, f(x)$

where c is constant

$E[c\, f(x)] = c \cdot f(x+h)$

$= c \cdot E\, f(x)$

c is the constant and h being the interval of differening

(iii) h intervals kurikum

b) $\Delta[c\, f(x) = c \cdot \Delta f(x)]$

$\Delta[c\, f(x) = c\, f(x+h) - c\, f(x)]$

$= c[f(x+h) - f(x)]$

$= c\, \Delta f(x)$

(iii) E and Δ obey the law of indices

a) $E^m[E^n f(x)] = E^{m+n} \cdot f(x)$

$E^m[E^n f(x)] = E^m[f(x) + nh]$

$= f(x + nh + mh)$

$E^m[E^n f(x)] = E^{m+n} \cdot f(x)$

b) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

$\Delta^m [\Delta^n f(x)] = \Delta^m [\Delta \cdot \Delta \ldots \Delta^n \text{ times}]$

$= \Delta \cdot \Delta \cdot \Delta \ldots \Delta \ (m+n) \text{ times}$

$= \Delta^{m+n} f(x)$

(iv) $E\nabla = \nabla E = \Delta$

$\therefore \quad E[\nabla f(x)] = E[f(x) - f(x+h)]$

$= E\{f(x) - E f(x+h)$

$= f(x+h) - f(x)$

$\therefore E[\nabla f(x)] = \Delta f(x) \rightarrow ①$

From ① and ② $F(x)$ is orbitory then, we get

$E\nabla E\nabla E = \Delta$

v) $E[a f(x) + b g(x)] = aE f(x) + bE g(x)$

$E[a f(x) + b g(x)] = a f(x+h) + b g(x+h)$

$= aE[f(x) + bE g(x)]$

vi) $E^0 f(x) = f(x)$

Proof: $E^n f(x) = f(x + nh)$.

Put $n=0$ we get

$E^0 f(x) = f(x+0)$

vii) $E^{-n} f(x) = f(x+nh)$

Put $n = -n$

we get, $E^{-n} f(x) = f(x-nh)$

viii) $\Delta^n f(x) = f(x+nh) - nc_1 f[x+(n-1)h] + nc_2 f(x+(n-2)h + \ldots + (-1)^n f(x)$

$\therefore \Delta = E-1$

$(E-1)^n = [E^n - nc_1 E^{n-1} + nc_2 E^{n-2} + \ldots + (-1)^n nc_n !]$

$\Delta^n f(x) = (E-1)^n f(x)$

$= E^n f(x) - nc_1 E^{n-1} f(x) + nc_2 E^{n-2} f(x) + \ldots + (-1) nc_n f(x)]$

$\Delta^n f(x) = f(x+nh) - nc_1 f(x) + (n-1)h] + nc_2 [f(x)(n-2)h] + \ldots + (-1)^n f(x)$

Missing terms [equal Intervals]

Some times we may be given a set of equal distance terms unit some terms missing The problem of estimating such terms missing can be easily taken by use of the operator E and Δ.

Let us suppose that we are given (n+1) equal distance arguments [$x = 0, 1, 2, \ldots n$] but the n entry $f(x)$ corresponding to the $(n+1)^{th}$ arguments is not given and we want to estimate that since we are given n entries. The data can be represented by the polynomial of $(n+1)^{th}$ degree.

2) Evaluate $\Delta^n (e^{ax+b})$

$$\Delta[e^{ax+b}] = e^{a(x+1)+b} - e^{ax+b}$$

$$= e^{a(x+1)+b} - e^{ax+b}$$

$$= e^{ax+a+b} - e^{ax+b}$$

$$= e^{ax+b}(e^a - 1)$$

$$\Delta^2[e^{ax+b}] = \Delta[\Delta e^{ax+b}]$$

$$= \Delta[e^{ax+b} \cdot (e^a - 1)]$$

$$= (e^{a-1}) \Delta e^{ax+b}$$

$= (e^{a}-1) \cdot e^{ax+b}(e^{a-1})$

$= (e^{a-1})^2 \cdot e^{ax+b}$

similarly,

$\Delta^n [e^{ax+b}] = (e^{a}-1)^n (e^{ax+b})$

Interpolate the population for the year 45 from the following data

x	40	45	50	55	60	65	70	75
y	212	296	368	429	460	481	490	492

Solution:

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$	$\Delta^7 f(x)$
40	212							
		84						
45	296		-12					
		72		1				
50	368		-11		-20			
		61		-19		59		
55	429		-30		39		-120	
		31		-20		-61		210
60	460		-10		-22		90	
		21		-2		29		
65	481		-12		7			
		9		5				
70	490		-7					
		2						
75	492							

$$x = \frac{x - x_0}{x_1} = \frac{42 - 40}{5} = 0.4$$

$$y_x = y_0 + x \cdot \Delta f(x) + \frac{x(x-1)}{2!} \Delta^2 f(x) + \frac{x(x-1)(x-2)}{3!}$$

$$\Delta^3 f(x) + \frac{x(x-1)(x-2)(x-3)}{4!} \Delta^4 f(x) +$$

$$\frac{x(x-1)(x-2)(x-3)(x-4)}{5!} \Delta^5 f(x) + \frac{x(x-1)(x-2)(x-3)(x-5)(x-4)}{6!}$$

$$\Delta^6 f(x) + \frac{x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)}{7!} \Delta^7 f(x)$$

$$= 212 + 0.4(84) + \frac{0.4(0.4-1)}{2!}(-12) + \frac{0.4(0.4-1)(0.4-2)}{3!}$$

$$(1) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-20) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!}$$

$$(59) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)(0.4-5)}{6!}(-120) +$$

$$\frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)(0.4-5)(0.4-6)}{7!}(210)$$

$$= 212 + 33.6 + 1.44 + 0.064 + 0.832 + 1.767168 + 2.755584 + 3.8578 .$$

3) Prove that $e^x = \left[\frac{\Delta^2}{E}\right] e^x \cdot \frac{E^{-1} e^x}{\Delta^2 e^x}$

Interval the differencing h

Proof:

$$\Delta e^x = e^{x+h} - e^x$$

$$\Delta e^x = e^x (e^h - 1)$$

$$\Delta^2 e^x = \Delta \cdot \Delta e^x$$

$$= \Delta e^x (e^h-1)$$

$$= (e^h-1) e^x (e^h-1)$$

$$\Delta^2 e^x = (e^h-1)^2 e^x \longrightarrow (1)$$

$$\therefore \left[\frac{\Delta^2}{E}\right] e^x = \Delta^2 E^{-1} e^x$$

$$= \Delta^2 e^{x-h}$$

$$= \Delta^2 . e^x . e^{-h}$$

$$= e^{-h} \Delta^2 e^x \longrightarrow (2)$$

sub equ (1) and (2)

we get,

$$\left(\frac{\Delta^2}{E}\right) e^x = e^{-h}(e^h-1)^2 e^x$$

$$\therefore \left(\frac{\Delta^2}{E}\right) e^x \frac{E^1 e^x}{\Delta^2 e^x} = e^{-h}\cancel{(e^h-1)^2} e^x \cdot \frac{e^{x+h}}{\cancel{e^x}\cancel{(e^h-1)^2}}$$

$$= e^{\cancel{-h}+x+\cancel{h}}$$

$$= e^x$$

---

4) Evaluate $\Delta \tan^{-1} x$

Proof:

$$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$$

$$= \tan^{-1}\left[\frac{(x+h)-x}{1+(x+h)x}\right]$$

$$= \tan^{-1}\left[\frac{h}{1+x^2+xh}\right]$$

5) Evaluate $\left[\frac{\Delta^2}{E}\right] x^3$

$(a^3 - 3ab^2 + 3ab$

Proof: $\left[\frac{\Delta^2}{E}\right] x^3 = \Delta^2 (x-1)^3$

$= \Delta^2 (x^3 - 3x^2 + 3x - 1)$

$\Delta^m f(x) = n! \, h^n \, a^n \, x^{n-m}$

$\Delta^2 (x^3 - 3x^2 + 3x - 1) = 3! \, (1)^3 . 1 \quad x^{3-2}$

$m = 2 \ , \ n = 3$

$\Delta^2 f(x) = 6x .$

Interpolation:

Definition:

Suppose we are given the values of a function $f(x)$, i.e) the entries for a set of values of the independent variable x (i.e) the arguments. The interpolation is defined as the technique interpolation is defined as the of estimating the values of $f(x)$ for any intermediate values of the arguments.

USES:

1) The need for interpolating missing observations (or) making forecasts (or) projects arise in a number of discipling like economics, business,

social science, acturial work population studies etc.

2) The interpolation technique has been used to derive the formula for the computation of median and mode in case of continuous frequency distribution.

3) Interpolation technique are used to fill in the gap in the statistical data for the state of continuity of information

There are two types, they are.,

* Interpolation with equal interval
* Interpolation with unequal interval

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* <u>Interpolation with equal interval</u>.

Assumption:

1) The values of the function should be either in increasing order (or) decreasing order.

2) The raise (or) fall in the values should be uniform.

<u>Newton's forward formula</u>:

Let $y = f(x)$ represent a function which assumes the values $f(a)$, $f(a+h)$, $f(a+nh)$ for $(h+1)$ equal distance values $a, a+h, a+2h, \ldots a+nh$ of the independent variable x

$$f(a+xh) = E^x f(a) = (1+\Delta)^x f(a)$$

$$= \left[1 + xc_1\Delta + xc_2\Delta^2 + \dots + xc_n\Delta^n + \dots + \Delta^n\right] f(a)$$

$$= f(a) + xc_1\Delta f(a) + xc_2\Delta^2 f(a) + \dots + xc_n\Delta^n f(a) + \dots + \Delta^n f(a)$$

Hence a is first arguments and 'h' is the common interval of differencing the last term depending on the degree of the Polynomial F(x).

$$x = \frac{\text{Period of interpolation} - a\,(\text{first argument})}{h}$$

<u>Newton Backward Formula:-</u>

Let $y = F(x)$ represent a function which assumes the values

$F(a), f(a+h) \dots f(a+nh)$ for $(n+1)$ equal distants $a, a+h, a+2h, \dots, a+nh$ of the independent variable x.

$$f(a+nh+xh) = E^x f(a+nh)$$

$$= (1-\nabla)^{-x} f(a+nh)$$

$$= \left[1 + xc_1\nabla + (x+1)c_2\nabla^2 + (x+2)c_3\nabla^3 + \dots\right] f(a+nh)$$

$$= f(a+nh) + xc_1\nabla f(a+nh) + (x+1)c_2\nabla^2 f(a+nh) + \dots$$

the last term is depending on the degree of the polynomial $f(x)$, 'h' is the common interval

$$x = \frac{\text{Period of interpolation} - \text{last argument } (a+nh)}{h}$$

1. Estimate the premium's for policies, maturing the age of 46 yrs

Age x :	45	50	55	60	65
Premium d(x)	2.571	2.404	2.083	1.862	1.712

2) Estimate the premium for policies maturing the age of 62 years for the above data. forward:

solution.

x	$U(x)$	$\Delta U(x)$	$\Delta^2 U(x)$	$\Delta^3 U(x)$	$\Delta^4 U(x)$
45	2.871				
		-0.467			
50	2.404		0.146		
		-0.321		-0.046	
55	2.083		0.1[illegible]		0.017
		-0.221		-0.029	
60	1.862		0.071		
		-0.15			
65	1.712				

$$y_x = y_0 + x\Delta f(x) + \frac{x(x-1)}{2!}\Delta^2 f(x) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(x)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 f(x)$$

$$x = \frac{x - x_0}{x_1} = \frac{46-45}{5} = 0.2$$

$$y_x = 2.871 + (0.2)(-0.467) + \frac{(0.2)(0.2-1)}{2!}(0.146)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!} (-0.046) +$$

$$\frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!} \times 0.017$$

$$= 2.871 - 0.0934 + (-0.01165) + (-0.002208) + (-0.0005712)$$

$$y_x = 2.7631$$

2) Backward:

Solution.

x	$U(x)$	$\nabla U(x)$	$\nabla^2 U(x)$	$\nabla^3 U(x)$	$\nabla^4 U(x)$
45	2.871				
		-0.467			
50	2.404		0.146		
		-0.321		-0.046	
55	2.083		0.1		0.017
		-0.221		-0.029	
60	1.862		0.071		
		-0.15			
65	1.712				

$$x = \frac{x - x_0}{x_1}$$

$$= \frac{62 - 65}{5} = -0.6$$

$$y_x = y_0 + x \nabla f(x) + \frac{x(x-1)}{2!} \nabla^2 f(x) + \frac{x(x-1)(x-2)}{3!}$$

$$\nabla^3 f(x) + \frac{x(x-1)(x-2)(x-3)}{4!} \nabla^4 f(x)$$

$= 1.712 + (0.6)(-0.15) + \frac{(-0.6)(-0.6-1)}{2!}(0.071) +$

$\frac{(0.6)(-0.6)(-0.6-1)(0.6-2)}{3!}(-0.029) +$

$\frac{(-0.6)(-0.6-1)(-0.6-2)(-0.6-3)}{4!}(0.017)$

$= 1.712 + 0.09 + 0.03408 + 0.012064 + 0.0063648$

$y_x = 1.8545 \simeq 1.7913$

3) Given $u_0 = 3, u_1 = 12, u_2 = 81, u_3 = 200, u_4 = 100$

$u_5 = 8$. Find $\Delta^5 u_0$

Solution:

$\Delta^5 u_0 = (E-1)^5 u_0$

$= E^5 - 5C_1 E^4 + 5C_2 E^3 - 5C_3 E^2 + 5C_4 E - 5C_5] u_0$

$= [E^5 u_0 - 5C_1 E^4 u_0 + 5C_2 E^3 u_0 - 5C_3 E^2 u_0 + 5C_4 E u_0 - 5C_5 u$

$= E^5 u_0 - 5E^4 u_0 + 10E^3 u_0 - 10E^2 u_0 + 5E^1 u_0 - u_0$

$= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0$

$= 8 - 5 \times 100 + 10 \times 200 - 10 \times 81 + 5 \times 12 - 3$

$\Delta^5 u_0 = 755$

4) Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

Proof: By the definition of $\Delta f(x) = f(x+h) - f(x)$

$\therefore \Delta f_0 = f_1 - f_0$

$\Delta^2 f_0 = \Delta \cdot \Delta f_0 = \Delta[f_1 - f_0] = \Delta f_1 - \Delta f_0$

$$\Delta^2 f_1 = \Delta \cdot \Delta f_1 = \Delta[f_2 - f_1] = \Delta f_2 - \Delta f_1$$

$$\Delta^2 f_2 = \Delta \cdot \Delta f_2 = \Delta[f_3 - f_2] = \Delta f_3 - \Delta f_2$$

$$\vdots \qquad \vdots$$

$$\Delta^n f_{n-2} = \Delta \cdot \Delta f_{n-2} = \Delta[f_{n-1} - f_{n-2}]$$

$$= \Delta f_{n-1} - \Delta f_{n-2}$$

$$\frac{\Delta^2 f_{n-1}}{\sum_{k=0}^{n-1} \Delta^2 f_k} = \Delta \cdot \Delta f_{n-1} = \Delta[f_n - f_{n-1}]$$

$$= \frac{\Delta f_n - \Delta f_{n-1}}{\Delta f_n - \Delta f_0}$$

$$\therefore \sum_{k=0}^{n-1} \Delta^2 f_k = \Delta^2 f_0 + \Delta^2 f_1 + \dots + \Delta^2 f_{n-1}$$

$$= \Delta f_1 - \Delta f_0 + \Delta f_2 - \Delta f_2 - \Delta f_1 + \dots + \Delta f_n - \Delta f_{n-1}$$

$$\therefore \sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$$

5) $\Delta^n (1/x) = \dfrac{(-1)^n \, n!}{(x+1) \dots (x+n)}$

Proof:

Now,

$$\Delta^1 (1/x) = \frac{1}{x+1} - 1/x$$

$$= \frac{(-1)!}{x(x+1)}$$

again

$$\Delta^2 (1/x) = \frac{(-1)^2 \, 2!}{x(x+1)(x+2)}$$

$$\Delta^2(1/x) = \frac{(-1)^n \, n!}{x(x+1)(x+2)\cdots(x+n)}$$

Common formula:

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{B-A}{2}\right)$$

$$\cos A + \cos B = 2\sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) \cdot \cancel{\sin}\cos\left(\frac{A+B}{2}\right)$$

6) Evaluate $\Delta(x+\cos x)$ the interval of $h = a$

Proof:

$$\Delta(x+\cos x) = \Delta x + \Delta \cos x$$

$$= (E-1)x + (E-1)\cos x$$

$$= E(x) - x + E\cos x - \cos x$$

$$\therefore E(x) = x+h$$

$$h = d$$

$$= (x+d) - \cancel{x} + \cos(x+d) - \cos x$$

$$= d + 2\sin\left(\frac{x+d+x}{2}\right) \cdot \sin\left(\frac{x-x-d}{2}\right)$$

$$= d + 2\sin\left(\frac{2x+2}{2}\right) \cdot \sin\left(\frac{-d}{2}\right)$$

$$= d + \left[-2\sin\left(x+d/2\right) \cdot \sin(d/2)\right]$$

$$= d - 2\sin\left(x+d/2\right) \cdot \sin(d/2)$$

7) $\Delta \dfrac{2^x}{(x+1)!} = \dfrac{2^{x+1}}{(x+2)!} - \dfrac{2^x}{(x+1)!}$

Proof:

$$= \frac{2^{x+1} - (x+2)\,2^x}{(x+2)!}$$

$$= \frac{2^x\left[2^1 - (x+2)\right]}{(x+2)!}$$

$$= \frac{2^x\left[\cancel{2} - x \cancel{- 2}\right]}{(x+2)!}$$

$$= \frac{-x \cdot 2^x}{(x+2)!}$$

Relation between $\Delta^n 0^m$, $\Delta^{n-1} 0^{m-1}$ and $\Delta^n 0^{m-1}$

Proof:

W.k.T,

$$\Delta^n 0^m = n^m - nC_1 (n-1)^m + nC_2 (n-2)^m + \dots + (-1)^{n-1}\, nC_{n-1}$$

$$= n^m - nC_1 (n-1)^m + nC_2 (n-2)^m + \dots + n(n-1)(-1)^{n-1}$$

$$= n^m - n(n-1)^m + \frac{n(n-1)}{2!}(n-2)^m + \dots + n(-1)^{n-1}$$

$$= n\left[n^{m-1} - (n-1)^m + \frac{(n-1)}{2!}(n-2)^m + \dots + (-1)^{n-1}\right]$$

$$= n\left[n^{m-1} - (n-1)(n-1)^{m-1} + \frac{(n-1)(n-2)}{2!}(n-2)^{m-1} + \dots + (-1)^{n-1}\right]$$

$$= n\left[n^{m-1} - (n-1)C_1 (n-1)^{m-1} + (n-1)C_2 (n-2)^{m-1} + \dots + (-1)^{n-1}\right]$$

$$= n\left[c_1 + (n-1)^{n-1} - (n-1)c_1\,(1+n-2)^{m-1} + (n-1)c_2\,[1+n-3]^{m-1} + \dots + (-1)^{n-1}\right]$$

$$= n\left[c_1 + (n-1)^{n-1} - (n-1)c_1\,(1+n-2)^{m-1} + (n-1)c_2\,[1+n-3]^{m-1} + \dots + (-1)^{n-1}\right.$$

$$= n\left[E^{n-1}(1)^{m-1} - (n-1)c_1\,E^{n-2}(1)^{m-1} + (n-1)c_2\,E^{n-3}(1)^{m-1} + \dots + (-1)^{n-1}(1)^{m-1}\right]$$

$$= n\left[E-1\right]^{n-1}(1)^{m-1}$$

$$= n\,\Delta^{n-1}\,.(1)^{m-1}$$

$$= n\,\Delta^{n-1}\,.\,E(0)^{m-1}$$

$$= n\,\Delta^{n-1}.(1+\Delta)\,0^{m-1}$$

$$\therefore \Delta^n 0^m = n\left[\Delta^{n-1}0^{m-1} + \Delta^n 0^{m-1}\right]$$

Show that $u_x = u_{x-1} + \Delta u_{x-2} + \dots + \Delta^{n-1}u_{x-n} + \Delta^n u_{x-n}$

Proof: $u_x - \Delta^n u_{x-n} = u_x - \Delta^n E^{-n} u_x$ $\qquad E^{-n}F(x) = f(x-nh)$

$$= \left[1 - \Delta^n E^{-n}\right]u_x \qquad E^{-n}u(x) = u(x-n)$$

$$= \left[1 - \left(\frac{\Delta}{E}\right)^n\right]u_x$$

$$= \left[\frac{E^n - \Delta^n}{E^n}\right]u_x$$

$$= \frac{1}{E^n}\left[E^n - \Delta^n\right]u_x \qquad \therefore 1 = E - \Delta$$

$$= E^{-n}\left[\frac{E^n - \Delta^n}{1}\right]u_x$$

$$= E^{-n}\left[\frac{E^n-\Delta^n}{E-\Delta}\right]u_x$$

$$\therefore \left[\frac{A^n-B^n}{A-B} = A^{n-1}+BA^{n-2}+B^2A^{n-3}+\dots+B^{n-1}\right]$$

$$= E^{-n}\left[E^{n-1}+\Delta E^{n-2}+\Delta^2E^{n-3}+\dots+\Delta^{n-1}\right]u_x$$

$$= \left[E^{-1}+\Delta E^{-2}+\Delta^2E^{-3}+\dots+\Delta^{n-1}E^{-n}\right]u_x$$

$$u_x-\Delta^n u_{x-1} = u_{x-1}+\Delta u_{x-2}+\Delta^2 u_{x-3}+\dots+\Delta^{n-1}u_{x-n}$$

$$u_x = u_{x-1}+\Delta u_{x-2}+\Delta^2 u_{x-3}+\dots+\Delta^{n-1}u_{x-n}+\Delta^n u_{x-n}$$

Show that,

$$u_{x+n} = u_n + xC_1\Delta u_{n-1} + (x+1)C_2\,\Delta^2 u_{n-2} + (x+2)C_3\,\Delta^3 u_{n-3}+\dots$$

Proof:

R.H.S =

$$u_n + xC_1\Delta u_{n-1} + (x+1)C_2\,\Delta^2 u_{n-2} + (x+2)C_3\,\Delta^3 u_{n-3}+\dots$$

$$= u_n + xC_1\Delta E^{-1}u_n + (x+1)C_2\,\Delta^2E^{-2}u_n + (x+2)C_3\,\Delta^3E^{-3}u_n+\dots$$

$$= \left[1+xC_1\,\Delta E^{-1}+(x+1)C_2\,\Delta^2E^{-2}+(x+2)C_3\,\Delta^3E^{-3}+\dots\right]u_n$$

$$\left[\because (1-a)^{-x} = 1+xC_1a+(x+1)C_2a^2+(x+2)C_3a^3+\dots\right]$$

$$= \left[1-\Delta E^{-1}\right]^{-x}u_n$$

$$= \left[1-\frac{\Delta}{E}\right]^{-x}u_n$$

$$= \left[\frac{E-\Delta}{E}\right]^{-x} U_n \Rightarrow \left[\frac{1+\Delta-\Delta}{E}\right]^{-x} U_x$$

$\therefore\ 1 - E - \Delta$

$$= \left[\frac{1}{E}\right]^{-x} U_n$$

$$= E^{x} U_n$$

$$= U_{n+1}$$

UNIT-II

DIVIDED DIFFERENCES

Introduction:

The differences used and defined for are the ordinary differences where only the difference between successive values of the entry are taken in to account and not the change in the argument. If the change (or) differences into the values of the argument are taken into account then we call the difference divided differences.

Definition:

The differences defined by taking into consideration we change in the value of argument are called differences. Let $f(x_0)\ f(x_1) \dots f(x_n)$ be the value of the function $y = f(x)$ corresponding to the value of the $x_0, x_1, \dots x_n$ of the arguments x. where $x_1 - x_0,\ x_2 - x_1, \dots x_n - x_{n-1}$ are not necessarily equal.

(i.e) when there is a case of unequal intervals. Thus we can define a divided difference as the difference between the two successive values of the entry divided by the difference between the corresponding values of the arguments.

x	$F(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	$F(x_0)$				
		$\frac{f(x_1)-F(x_0)}{x_1-x_0} = \Delta f(x_0)(x_1) = f(x_0, x_1)$			
x_1	$F(x_1)$		$\frac{\Delta_{x_2} f(x_1) - \Delta_{x_1} f(x_0)}{x_2-x_0} = \Delta^2_{x_1x_2} f(x_0) = f(x_0, x_1, x_2)$		
		$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \Delta f(x_1)(x_2) = f(x_1, x_2)$		$\frac{\Delta^2_{x_2x_3} f(x_1) - \Delta^2_{x_1x_2} f(x_0)}{x_3-x_0} = \Delta^3_{x_1x_2x_3} f(x_0) = f(x_0, x_1, x_2, x_3)$	
x_2	$F(x_2)$		$\frac{\Delta_{x_3} f(x_2) - \Delta_{x_2} f(x_1)}{x_3-x_1} = \Delta^2_{x_2,x_3} f(x_1) = f(x_1, x_2, x_3)$		$\frac{\Delta^3_{x_2,x_3,x_4} f(x_1) - \Delta^3_{x_1,x_2,x_3} f(x_0)}{x_4-x_0} = \Delta^4_{x_1,x_2,x_3,x_4} f(x_0) = f(x_0, x_1, x_2, x_3, x_4)$
		$\frac{f(x_3)-f(x_2)}{x_3-x_2} = \Delta f(x_2)(x_3) = f(x_2, x_3)$		$\frac{\Delta^2_{x_3x_4} f(x_2) - \Delta^2_{x_2x_3} f(x_1)}{x_4-x_1} = \Delta^3_{x_2x_3x_4} f(x_1) = f(x_1, x_2, x_3, x_4)$	
x_3	$F(x_3)$		$\frac{\Delta_{x_4} f(x_3) - \Delta_{x_3} f(x_2)}{x_4-x_3} = \Delta^2_{x_3,x_4} f(x_2)$		
		$\frac{f(x_4)-f(x_3)}{x_4-x_3} = \Delta f(x_3)(x_4) = f(x_3, x_4)$			
x_4	$F(x_4)$				

Properties of divided difference

Property 1:

Divided difference are symmetric function of their argument (i.e) the value of any divided difference in independent of the order of the argument.

Proof:

$$\underset{x}{\Delta} f(x_0) = f(x_0, x_1) = \frac{F(x_1) - F(x_0)}{(x_1 - x_0)}$$

multiple by (-1) we get,

$$= \frac{f(x_0) - f(x_1)}{(x_0 - x_1)} \Rightarrow f(x_1 - x_2)$$

$$= \underset{x_0}{\overset{1}{\Delta}} f(x_1)$$

Also,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{f(x_1)}{(x_1 - x_0)} - \frac{f(x_0)}{(x_1 - x_0)}$$

multiple by (-1)

$$= \frac{-f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$$= \frac{\leq f(x_0)}{x_0 - x_1}$$

$$\underset{x_1 x_2}{\overset{2}{\Delta}} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right]$$

$$= \left[\frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)(x_1 - x_0)} + \right.$$

$$+\frac{f(x)}{(x_2-x_0)(x_1-x_0)}\Bigg]$$

$$=\Bigg[\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+\frac{f(x_1)}{(x_1-x_2)(x_2-x_0)}-\frac{f(x_1)}{(x_2-x_0)(x_1-x_0)}$$

$$+\frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}\Bigg]$$

$$=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+f(x_1)\left[\frac{(x_1-x_0)-(x_1-x_2)}{(x_1-x_2)(x_2-x_0)(x_1-x_0)}\right]+$$

$$-\frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}$$

$$=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+f(x_1)\left[\frac{x_1-x_0-\cancel{x_1}+x_2}{(x_1-x_2)(x_2-x_0)(x_1-x_0)}\right]+$$

$$\frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}$$

$$=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+f(x_1)\left[\frac{\cancel{x_2-x_0}}{(x_1-x_2)\cancel{(x_2-x_0)}(x_1-x_0)}\right]+$$

$$\frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}$$

$$= \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + f(x)\left[\frac{1}{(x_1-x_2)(x_1-x_0)}\right] + \frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}$$

$$= \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_1)}{(x_1-x_2)(x_1-x_0)} + \frac{f(x_0)}{(x_2-x_0)(x_1-x_0)}$$

$$= \frac{\Sigma f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

which is a symmetric function $f(x)$ of the arguments x_0, x_1, x_2 thus we have,

$$f(x_0, x_1, x_2) = f(x_1, x_2, x_0) = f(x_2, x_0, x_1)$$

(i.e) we can choose any of the permutations of the arguments (x_0, x_1, x_2) for the notation,

By the mathematical induction we can prove that

$$f(x_0, x_1 \dots x_n) = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

$$\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\dots(x_2-x_n)}$$

$$\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

Thus, we observed that the value of the divided difference depends only on the values of the arguments involved and not on the order in ~~whose~~ which they involve.

Note:

We also conclude that the divided difference of the n^{th} order can be expressed in terms of $(n+1)$ partial fraction.

Property 2:

The n^{th} divided difference of a polynomial of the n^{th} degree are constant.

Proof:

Consider al the n^{th} degree polynomial is

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

If $a_0 = a_1 = \dots a_{n-1} = 0$ and $a_n = 1$ then the function $f(x) = x^n$

The first divided difference of $f(x)$ for the arguments x_n, x_{n+1} is

$$f(x_n, x_{n+1}) = \frac{f(x_{n+1}) - f(x_n)}{(x_{n+1}) - x_n}$$

$$= \frac{x_{n+1}^n - x_n^n}{x_{n+1} - x_n}$$

$$= f(x_{n+1}, x_n)$$

$$\left[\frac{a^n b^n}{a-b}\right] = a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-2}a^1 + b^{n-1}$$

$$\left[\frac{x_{n+1}^n - x_n^n}{x_{n+1} - x_n}\right] = x_{n+1}^{n-1} + x_n x_{n+1}^{n-2} + x_n^2 x_{n+1}^{n-3} + \cdots x_n$$

doubt

This is a symmetric function of the $(n-1)^n$ degree in x_n, x_{n+1}

The second divided difference of $f(x)$ for the arguments x_n, x_{n+1}, x_{n+2} is given by,

$$f(x_n, x_{n+1}, x_{n+2}) = \frac{f(x_{n+1}, x_{n+2}) - f(x_n, x_{n+1})}{x_{n+2} - x_n}$$

$$= \frac{1}{(x_{n+2}) - x_n}\left[\frac{f(x_{n+2}) - f(x_{n+1})}{x_{n+2} - x_{n+1}} - \frac{f(x_n, x_{n+1})}{x_{n+1} - x_n}\right]$$

$$= \frac{1}{x_{n+2} - x_n}\left[\left(\frac{x_{n+2}^n - x_{n+1}^n}{x_{n+2} - x_{n+1}}\right) - \left(\frac{x_{n+1}^n - x_n^n}{x_{n+1} - x_n}\right)\right]$$

$$= \frac{1}{(x_{n+2} - x_n)}\left[\left(x_{n+2}^{n-1} + x_{n+1} x_{n+2}^{n-2} + x_n^2 x_{n+1}^{n-3} + \cdots x_{n+1}^{n-1}\right) - \right.$$

$$\left.\left[x_{n+1}^{n-1} + x_n x_{n+1}^{n-2} + x_n^2 x_{n-1}^{n-3} + \cdots + x_n^{n-1}\right]\right]$$

$$= \frac{1}{(x_{n+2} - x_n)} \Big[x_{n+2}^{n-1} + x_{n+1} x_{n+2}^{n-2} + x_{n+1}^{2} x_{n+2}^{n-3} + \dots x_{n+1} \dots$$

$$x_{n+2} + x_{n+1}^{n-1} - x_{n+1}^{n-1} - x_n x_{n+1}^{n-2} -$$

$$x_n^2 x_{n+1}^{n-3} \dots x_n^{n-2}(x_{n+1}) - x_n^{(n-1)} \Big]$$

$$= \frac{1}{(x_{n+2} - x_n)} \Big[\left(x_{n+2}^{n-1} - x_n^{n-1} \right) + x_{n+1} \left(x_{n+2}^{n-2} \right) x_n^{n}$$

$$x_{n+1}^{n-2} \left(x_{n+2} - x_n \right)$$

$$= \frac{x_{n+2}^{n-1} - x_n^{n-1}}{x_{n+2} - x_n} + x_{n+1} \frac{x_{n+2}^{n-2} - x_n^{n-2}}{x_{n+2} - x_n} + \dots +$$

$$\frac{x_{n+1}^{n-2} (x_{n+2} - x_n)}{x_{n+2} - x_n}$$

$$\left[\frac{a^{n-1} - b^{n-1}}{a - b} \right] = a^{n-2} + ba^{n-3} + \dots + b^{n-3} a + b^{n-2}$$

$$= \Big[x_{n+2}^{n-2} + x_n x_{n+2}^{n-3} + \dots + x_n^{n-3} x_{n+2} + x_n^{n-2} \Big] + (x_{n+1})$$

$$x_{n+2}^{n-3} + x_n x_{n+2}^{n-4} + \dots + x_n^{n-4} x_{n+2} + x_n^{n-3} + \dots x_{n+1}^{n-2} \Big]$$

Thus is a symmetric function of $(n-2)^{th}$ degree in x_n, x_{n+1} and x_{n+2} by induction it can be shown that if $(x_n, x_{n+1} \dots x_{n+n})$ is a symmetric function of degree $(n-m)$ is particular then n^{th} divided difference.

of $f(x) = x^n$ is a symmetric function of degree $(n-n) = 0$ Thus the n^{th} divided difference of $f(x) = x^n$ is constant.

Properties of Divided Difference.

1) Divided differences are symmetric function of their arguments.

2) The n^{th} divided difference of a polynomial of the n^{th} degree are constant.

3) The n^{th} divided difference of the can be express as the quotient of determinans, each of order $n+1$.

4) The n^{th} divided difference can be expressed as the Product of multiple integrals.

5) A function $f(x)$ is known is 3 points x_1, x_2, x_3 in the vaicinity of an extremal point x_0 then their exit an approximation.

Newton's formula for divided difference [unequal intervals]

Let $f(x_0), f(x_1) \dots f(x_n)$ be the value of $f(x)$ corresponding to the arguments $x_0, x_1, x_2, \dots x_n$ not necessarily equally spaced from the definition of divided difference

$$f(x_1, x_0) = \frac{f(x) - f(x_0)}{(x - x_0)} = \underset{x}{\Delta} f(x_0)$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0) \underset{x}{\Delta} f(x_0) \rightarrow (1)$$

Again

$$f(x, x_0, x_1) = \underset{x, x_1}{\Delta^2} f(x_0)$$

$$= \frac{\underset{x}{\Delta} f(x_0) - \underset{x_1}{\Delta} f(x_0)}{x - x_1}$$

$$\Rightarrow \underset{x}{\Delta} f(x_0) = \underset{x_1}{\Delta} f(x_0) + (x-x_1) \underset{x x_1}{\Delta^2} f(x_0)$$

Sub this in (1) we have,

$$f(x) = f(x_0) + (x-x_0)\left[\underset{x_1}{\Delta} f(x_0) + (x-x_1) \underset{x x_1}{\Delta^2} f(x_0)\right]$$

$$f(x) = f(x_0) + (x-x_0) \underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-1) \underset{x x_1}{\Delta^2} f(x_0) \rightarrow (2)$$

Similarly,

$$f(x, x_0, x_1, x_2) = \underset{x x_1 x_2}{\Delta^3} f(x_0)$$

$$= \frac{\underset{x x_1}{\Delta^2} f(x_0) - \underset{x_1 x_2}{\Delta^2} f(x_0)}{(x - x_2)}$$

$$\underset{x x_1}{\Delta^2} f(x_0) = \underset{x_1 x_2}{\Delta^2} f(x_0) + (x-x_2) \underset{x x_1 x_2}{\Delta^3} f(x_0)$$

Sub in (2) we have.

$$f(x) = f(x_0) + (x-x_0) \underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1)$$

$$\left[\underset{x_1 x_2}{\overset{2}{\Delta}} f(x_0) + (x_1 - x_2) \underset{x x_1 x_2}{\overset{3}{\Delta}} f(x_0)\right]$$

$$f(x) = f(x_0) + (x - x_0) \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\overset{2}{\Delta}} f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2) \underset{x x_1 x_2}{\overset{3}{\Delta}} f(x_0)$$

similarly proceducing we shall get

$$f(x) = f(x_0) + x - x_0 \underset{x_1}{\Delta} f(x_0)\, f(x - x_0)(x - x_1) \underset{x_1 x_2}{\overset{2}{\Delta}} f(x_0)$$
$$+ (x - x_0)(x_1 - x)(x - x_2) \underset{x x_1 x_2}{\overset{3}{\Delta}} f(x_0)$$

The last term depending on the degree of the polynomial $f(x)$. If $f(x)$ is a polynomial of n^{th} degree then there will be $(n+1)$ terms.

Relation between divided differences and ordinary difference.

Let the arguments are $x_0, x_1, \ldots x_n$ are equally spaced is $h = x_1 - x_0 = x_2 - x_1 \ldots = x_n - x_{n-1}$

Then by defenition.

$$\underset{x}{\overset{1}{\Delta}} f(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{\Delta f(x_0)}{h} \rightarrow ①$$

$$\underset{x_1}{\Delta^1} f(x_0) = \frac{1}{1!\,h!} \Delta f(x_0)$$

$$\underset{x_2}{\Delta^1} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1+h) - f(x_1)}{x_1 + h - x_1} = \frac{\Delta f(x_1)}{h} \rightarrow (2)$$

Consider,

$$\underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{\underset{x_2}{\Delta} f(x_1) - \underset{x_0}{\Delta} f(x_1)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}\right]$$

$$= \frac{1}{\cancel{x_0} + 2h - \cancel{x_0}}\left[\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}\right]$$

$$= \frac{1}{2h^2}\left[\Delta f(x_0+h) - \Delta f(x_0)\right]$$

$$= \frac{1}{2!\,h^2} \Delta^2 f(x_0)$$

In general we have,

$$\underset{x_1, x_2, \ldots x_n}{\Delta^n} f(x_0) = \frac{1}{n!\,h^n} \Delta^n f(x_0)$$

Newton's forward formula as a particular case of Newton's divided difference formula :-

$$F(x) = f(x_0) + [x - x_0] \underset{x_1}{\Delta} f(x_0) + (x - x_0)(x - x_1) \underset{x_1 x_2}{\overset{2}{\Delta}} f(x_0) + (x - x_0)(x - x_1)(x - x_2) \underset{x_1 x_2 x_3}{\overset{3}{\Delta}} f(x_0) + \dots \longrightarrow ①$$

Put $x = x_0 + nh$ and we

$$\underset{x_1 x_2 \dots x_n}{\overset{n}{\Delta}} f(x_0) = \frac{1}{n!\,h^n} \Delta^n f(x_0)$$

in (1) we get,

$$f(x_0 + nh) = f(x_0) + (x_0 + nh - x_0) \frac{\Delta^1 f(x_0)}{1!\,h!} + [x_0 + nh - x_0][x - (x_0 + h)] \frac{\Delta^2 f(x_0)}{2!\,h^2} + [x_0 + nh - x_0] \frac{[x - (x_0 + h)(x - (x_0 + 2h))]}{3!\,h^3}$$

Δ^3.

$$= f(x_0) + \frac{nh \Delta f(x_0)}{h} + nh \frac{h(n-1)\,\Delta f(x)}{2!\,h^2} + \frac{nh\ h(n-1)\ h(n-2)\ \Delta^3 f(x_0)}{3!\,h^3} + \dots$$

$$f(x_0) + n \Delta f(x_0) + \frac{n(n-1)\,\Delta^2 f(x_0)}{2!} + \frac{n(n-1)(n-2)\,\Delta^3 f(x_0)}{3!} \dots$$

$$f(x_0 + nh) = f(x_0) + nc_1 \Delta f(x_0) + nc_2 \Delta^2 f(x_0) + nc_3\, f(x_0) + \dots$$

which is Newton's forward formula.

Lagranges interpolation for unequal intervals :-

Let $f(x_0), f(x_1) \dots f(x_n)$ be the $(n+1)$ values of the function $y = f(x)$ corresponding to the argument $x_0, x_1, \dots x_n$ not necessary equally space. It is assumed that function $f(x)$ is a polynomial in x and since $(n+1)$ values of $f(x)$ are given so $(n+1)^{th}$ difference are zero. Thus $f(x)$ is supposed to be polynomial in x of degree n writing $f(x)$ as

$$f(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n) + A_1(x-x_0)(x-x_2)\dots(x-x_n) + \dots + A_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

where A's constant the relation equation① is true for all values of x to determine A_0,

Put $x = x_0$ is equation ①

$$f(x_0) = A_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

Put $x = x_1$ in equation(1) we get,

$$f(x_1) = A_1(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

Similarly,

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}$$

Putting these values of n's in equations we get

$$F(x) = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\left[(x-x_1)(x-x_2)\cdots(x-x_n)\right]$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\left[(x-x_0)(x-x_2)\cdots(x-x_{n-1})\right]$$

$$+ \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\left[(x-x_0)(x-x_1)\cdots x-x_{n-1}\right]$$

2m $\left[f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \right.$

$$\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) + \cdots +$$

$\left. \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} + f(x_n) \right]$ 2m

Lagrange's formula in another form :- X

Let $\phi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

$$\phi'(x) = uv' + vu'$$

$$= (x-x_0)v' + v(1)$$

$$\phi(x_0) = (x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$$

$$\therefore f(x) = \frac{\phi(x) f(x_0)}{(x-x_0)\phi(x_0)} + \frac{\phi(x) f(x_1)}{(x-x_1)\phi'(x)} + \cdots$$

Since, $f(x) = \sum_{i=1}^{n} \left[\frac{\phi(x)}{(x-x_1)\phi(x_1)} \right] f(x_i)$

2. If $f(x) = 1/x^2$ find the divided difference $f(a,b,c)$, $f(a,b,c,d)$

x	$f(x) = 1/x^2$	$\Delta f(x) = f(a,b)$	$\Delta^2 f(x) = f(a,b,c)$
a	$1/a^2$	$\frac{1/b^2 - 1/a^2}{b-a} = \Delta f(a) = \frac{-a+b}{a^2b^2}$	$\frac{-(c+b)}{c^2b^2} + \frac{(a+b)}{a^2b^2} = \Delta^2 f(a) = \frac{ab+bc+ca}{a^2b^2c^2}$
b	$1/b^2$	$\frac{1/c^2 - 1/b^2}{c-b} = \Delta f(b) = \frac{-(c+b)}{c^2b^2}$	$\frac{-c+d}{c^2d^2} + \frac{(c+b)}{c^2b^2} = \Delta^2 f(b) = \frac{bc+cd+db}{b^2c^2d^2}$
c	$1/c^2$	$\frac{1/d^2 - 1/c^2}{d-c} = \Delta f(c) = \frac{-(c+d)}{c^2d^2}$	
d	$1/d^2$		

$$\Delta^3 f(x) = f(a,b,c,d)$$

$$\frac{\frac{bc+cd+db}{b^2c^2d^2} - \frac{ab+bc+ca}{a^2b^2c^2}}{d-a} = \frac{a^2(bc+cd+bd) - d^2(ab+bc+ca)}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{a^2bc + a^2cd + a^2bd - d^2ab - d^2bc - d^2ca}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{bc(a^2-d^2) + abd(a-a) + acd(a-d)}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{-[bc(a+d)(a-d) + acd(d-a) + abd(d-a)]}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{-[abc+bcd+acd+abd]}{a^2b^2c^2d^2}$$

Sum:1

The function $y = f(x)$ is given in the points (7,3) (8,1) (9,1) and (10,9) Find the value of y for x = 9.5 using lagranges interpolation formula the data in tabulated from are,

x :	7^{x_0}	8^{x_1}	9^{x_2}	10^{x_3}
$f(x)$:	$3_{F(x_0)}$	$1_{F(x_1)}$	$1_{F(x_2)}$	$9_{F(x_3)}$

Solution. using lagranges formula.

$$f(9.5) = \frac{\overset{x-x_1}{(9.5-8)}\overset{x-x_2}{(9.5-9)}\overset{x-x_3}{(9.5-10)}}{\underset{x_0-x_1}{(7-8)}\underset{x_0-x_2}{(7-9)}\underset{x_0-x_3}{(7-10)}} \times \overset{F(x_0)}{3} + \frac{\underset{x-x_0}{(9.5-7)}\underset{x-x_2}{(9.5-9)}\underset{x-x_3}{(9.5-10)}}{\underset{x_1-x_0}{(8-7)}\underset{x_1-x_2}{(8-9)}\underset{x_1-x_3}{(8-10)}} \times \overset{F(x_1)}{1} +$$

$$\frac{\overset{x-x_0}{(9.5-7)}\overset{x-x_1}{(9.5-8)}\overset{x}{(9.5-9)}}{(9-7)(9-8)(9-10)} \times \underset{F(x_2)}{1} + \frac{\overset{x-x_0}{(9.5-7)}\overset{x-x_1}{(9.5-8)}\overset{x-x_2}{(9.5-9)}}{\underset{x_3-x_0}{(10-7)}\underset{x_3-x_1}{(10-8)}\underset{x_3-x_2}{(10-9)}} \times \underset{F(x_3)}{9}$$

$$= 0.1875 - 0.3125 + 0.9375 + 2.8125$$

$$= 3.625.$$

Sum:2

If $y(1) = -3$ $y(3) = 9$, $y(4) = 30$, $y(6) = 132$.

Find the four point lagrange interpolation polynomial that takes the same values as the function y at the given points.

x :	1	3	4	6
y :	3	9	30	132.

Solution:

$$f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times 3 + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 13$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 13$$

$$= 1/10 \left[x^3 - 13x^2 + 54x - 72\right] + 3/2 \left[x^3 - 11x^2 + 34x \ldots\right.$$

$$-5\left(x^3 - 10x^2 + 27x - 18\right) + 22/5 \left(x^3 - 8x^2 + 19 \ldots\right.$$

$$= x^3 - 3x^2 + 5x - 6.$$

Sum:3

Find the form of the function given by.

x	3	2	1	-1
$F(x)$	3	12	15	-21

Solution:

using Lagranges formula,

$$f(x) = \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(1+1)} \times 3 + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)} \times 12$$

$$+ \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)} \times 15 + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)} \times -21$$

$$= x^3 - 9x^2 + 17x + 6.$$

Sum:4

Determine by lagrange's formula percentage of criminals under 35 years.

Age [under years]	25	30	40	50
Y. number of animals	52.0	67.3	84.1	94.4

Solution:

$$y_{35} = \frac{(35-30)(35-40)(35-50)}{(25-30)(25-40)(25-50)} \times 52.0 +$$

$$\frac{(35-25)(35-40)(35-50)}{(30-25)(30-40)(30-50)} \times 61.3 + \cancel{(35-35}$$

$$\frac{(35-25)(35-30)(35-50)}{(40-25)(40-30)(40-50)} \times 84.1 +$$

$$\frac{(35-25)(35-40)(35-50)}{(50-25)(50-30)(50-40)} \times 94.4$$

$$= -1 \times 52.0 + \frac{3}{4} \times 67.3 + \frac{1}{2} \times 34.1 - \frac{1}{20} \times 94.4$$

$$= 77.49$$