K. N. GOVT. ARTS COLLEGE FOR WOMEN (AUTONOMOUS), THANJAVUR -07

DEPARTMENT OF COMPUTER SCIENCE

I M.SC., COMPUTER SCIENCE

SUBJECT NAME: OPERATIONS RESEARCH

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OPERATIONS RESEARCH (18KP1CS03)

UNIT I: Introduction – origin and development of O.R – Nature and characteristics features of O.R – Models in O.R – General solution methods for O.R models – Methodology of O.R – Scientific method in O.R – Operations research and decision-making – Applications of O.R – Uses and Limitations of O.R – Linear Programming Problem: Introduction – mathematical formulation of the problem – Graphical solution method – General Linear Programming problem – Canonical and standard forms of L.P.P.

UNIT II: The Simplex method – Introduction – The Computational procedure – Artificial variable techniques – Problem of degeneracy – Applications of simplex method.

UNIT III: The Transportation problem – Introduction - Mathematical formulation of the problem -Triangular Basis – Loops in a Transportation Table – Finding initial Basic feasible solution – Moving towards Optimality – Unbalanced Transportation Problem – Transhipment Problem.

UNIT IV: The Assignment Routing Problem s– Introduction - Mathematical formulation of A.P – Assignment Algorithm – A typical Assignment Problem – Routing Problem – Games and Strategies : Introduction – Two-Person Zero-sum Games – The Maxmin-Minmax Principle – Games without saddle points – Mixed Strategies – Solution of 2×2 rectangular games.

UNIT V : Network Scheduling by PERT/CPM – Introduction – Network and basic components - Rules of network construction – Time calculations in network- Critical path method (CPM) – PERT – PERT Calculations.

Reference : " Operations Research" – Kanti Swarup, P.K . Gupta, Man Mohan – Sultan Chand and Sons, New Delhi – 8th Edition - Reprint 1999.

Department of Computer Science
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Operations Research (18KP1CS03)

UNIT - I

Operations Research:

Operations Research is the application of Scientific methods, techniques and tools to problems involving the operations of a System So as to provide those in control of the system with optimum Solutions to the Problem.

Operations - Some action that we apply to Some problems or hypothesis.

Research - Seeking out facts about the same.

Features of operation Research:

(i) Decision Making : The mathematical models of operations. research allow people to analyze a greater number of alternatives and constraints than would usually be possible, if they were to use only an intuitive approach. Using operations research, it is easier to analyze multiple alternatives, which results in greater confidence in the optimal choice.

(b) Scientific Approach: Scientific methods for the purpose of Solving problems.

(c) Objective: To locate the best or optimal Solution to the problem under consideration.

(d) Inter-disciplinary Team Approach: Team approach to a Solution of the problem. This requires a blend of people with expertise in the areas of mathematics, Statistics, economics, management computerScience and so on.

N.SUBHA

(e) Digital computer: Digital computer has become an integral part of the operations Research approach to decision making.

Models in operations Research:

A model is a vehicle used to arrive at a well Structured view of reality. A model in operations research is a representation of an operation or a process.

Types of model:

* Account model: is a typical budget in which business accounts are referred to with the intention of providing business measurements such as rate of expense, quantity, sold, etc,
* Mathematical Model: Mathematical equations
* Quantitative Model: possibility of measuring observations.
* Physical Model: a product, a device or any tangible thing used for experimentation may represent a physical model.

Main characteristics:

* New formulations without having any significant change in its frame.
* Assumption made as small as possible.
* It should be simple and coherent. Number of variables used should be less.
* It should be open to parametric type of treatment.
* It should not take much time in its construction for any problem.

Advantages of Models:

a) Through a model the problem under consideration becomes Controllable.

b) It provides some logical and systematic approach to the problem.

c) It indicates the limitations and scope of an activity.

d) It eliminate duplication and solve any specific problem.

e) Models help in finding avenues for new research and improvement.

f) It provides economic descriptions and explanations of the operations.

Classification of Models:

Models by degree of abstraction: Based on the past data or information of the problem.

Ex: (a) language model (Book)

(b) case studies.

Models by function:

(a) Descriptive model: Describe, explain and predict facts and relationships among the various activities of the problem.

(b) Predictive model: Based on conditions.

(c) Normative or Optimization model: Decision rules or criteria for optimum solution.

Models by structure:

(a) Iconic or physical models: Pictorial representation of real systems. Ex: City maps.

(b) Analog Models: There is no 'look-alike' correspondance between these models and real life items. They are built by utilizing one set of properties to represent another set of properties.

Example: a network of pipes through which water is running could be used as a parallel of understanding the distribution of electric currents.

(c) Mathematical or Symbolic models: Most abstract in nature. They employ a set of mathematical symbols to represent the components of the real system.

Models by nature of the Environment:

(i) Deterministic models: All the parameters and functional relationship are assumed to be known.

Ex: Linear Programming problem.

(ii) Probabilistics or stochastic models: At least one parameter or decision variable is a random variable.

Models by the extent of generality:

(a) Specific models: when a model presents a system at some specific time. In these models if the time factor is not considered, then they are termed as static models.

Ex: Inventory problem.

(b) Simulation and Heuristics Models: These are general models and mainly used to explore alternative strategies. These models do not yeild any optimum solution to the problem but give a solution to a problem depending on past experience.

General Solution methods for operation Research models:

Three types of methods for deriving the solution to an O.R.

1. Analytical methods
2. Numerical methods
3. Monte-carlo methods.

Analytical or Deductive Methods: In these methods classical optimization techniques such as Calculus, Finite Difference etc., are used for solving on O.R model.

Numerical methods: Numerical methods are considered with the iterative or trial and error procedures, through the use of numerical computation at each step.

The algorithm is started with a trial and (initial) solution and continued with a set of rules for improving it towards optimality. The trial solution is then replaced by improved one, the process is repeated until either no further improvement is possible.

Monte carlo methods: These involves the use of probability and sampling concepts.

Steps:

(i) make sample observations and determine the probability distribution for the variables of interest.

(ii) Convert the probability distribution to cumulative distribution.

(iii) Select the sequence of random numbers with the help of random tables.

(iv) Determine the sequence of values of variables.

(v) Fit an appropriate standard mathematical function to the values obtained in step (iv).

Methodology of operation Research:

The systematic methodology developed for an O.R. study deals with problems involving conflicting multiple objectives, policies and alternatives.

The O.R. approach to problem solving consists of the following six steps:

1. Formulation of the problem: It involves analysis of the problem, analysis of the physical system, setting-up of objectives, decision should be adopted, alternative course of action and measurement of effectiveness.

2. Construction of a mathematical model: The next step is to express all the relevant variables (activities) of the problem into a mathematical model.

$$E = f(x_i, y_j)$$

where f represents a system of mathematical relationships between the measures of effectiveness of the objective sought (E) and the variables, both controllable (x_i) and uncontrollable (y_j).

3. Deriving the solution from the model: The next step is to determine the values of decision variables that optimize the given objective function. It is also sometimes essential to perform sensitivity analysis, ie., to determine

the behaviour of the system, changes in the system's parameters and specifications.

4. Validity of the model: To measure its accuracy. A model is valid or accurate if (a) it contain all the objectives, constraints and decision variables relevant to the problem. (b) the objectives, constraints and decision variables included in the model are all relevant to, or actually part of the problem. (c) The functional relationship are valid.

5. Estabilishing control over the solution: To establish control over the solution by proper feedback of the information of the variables.

6. Implementation of the final results: Finally, the tested results of the model are implemented to work. A careful explanation of the solution to be adopted and its relationship with the operating realities.

Scientific Method in O.R.:

The Scientific Method in operations Research consists three phases,

Judgement Phase: This phase includes, (i) Identification of the real-life problem. (ii) Selection of an appropriate goal and the values of various variables related to the goals. (iii) appropriate scale of measurement (iv) formulation of an appropriate model of the problem, abstracting the essential information so that a solution at the decision-maker's goal can be sought.

Research phase:

* better understanding of what the problem is.
* formulation of hypothesis and models.
* Observation and experimentation to test the hypothesis on the basis of additional data.
* analysis of the available information and verification of the hypothesis.
* Predictions of the various result from the hypothesis.
* Generalization of the results and consideration of alternative methods.

Action Phase: This phase consists of making recommendations for decision process by those who first posed the problem for consideration.

Operations Research and Decision-Making:

Operations Research is a tool employed to increase the effectiveness of managerial decisions as an objective supplement to the subjective feeling of the decision-maker.

The Decision maker to be objective in creating alternatives and choosing an alternative, which is the best from among these.

The essential characteristics of all decisions are,

(i) objective
(ii) alternatives at the disposal
(iii) influencing factors.

Advantages of OR Approach in Decision Making:

(i) Better Decisions: Operations Research yield actions that do improve on intuitive decision making.

(ii) Better Co-ordination:

Operations Research oriented planning model becomes a vehicle for co-ordinating marketing decisions within the limitations imposed on manufacturing capabilities.

(iii) Better control:

The managements of large organizations recognize that it is extremely costly to require continous executive supervision over routine decisions.

(iv) Better Systems: O.R. study is initiated to analyse a particular decision problem, Such as whether to open a new warehouse.

Applications of Operations Research:

O.R. has Successfully entered many different areas of research in Defense, Government Service, Organisations and Industry.

Finance, Budgeting and Investment:

(i) cash flow analysis, long range capital requirements, dividend Policies, investment portfolios.
(ii) Credit policies, credit risk and delinquent account procedures.
(iii) Claim and complaint procedure.

Marketing:

(i) Product Selection, timing, competitive actions.
(ii) Advertising media.
(iii) Number of Salesman.
(iv) Effectiveness of market research.

Physical Distribution:

(i) location and Size of warehouse
(ii) Distribution policy.

Purchasing, Procurement and Exploration:

(i) Rules for buying.

(ii) quantity and timing of purchase.

(iii) Bidding policies and vendor analysis.

(iv) Equipment replacement policies.

Personnel:

(i) manpower requirement, assignment of jobs.

(ii) selection of suitable personnel (age and skills)

(iii) optimum number of persons.

Production:

(i) Scheduling and Sequencing the production run by proper allocation of machines.

(ii) calculating the optimum product mix.

(iii) Selection, location and design of the sites for the production plant.

Research and Development:

(i) Reliability and evaluation of alternative designs.

(ii) control of developed projects.

(iii) co-ordination of multiple research projects.

(iv) Determination of time and cost requirements.

Uses and Limitations of O.R.

Advantages:

(i) optimum use of production factors.

(ii) improved quality of decision.

(iii) Preparation of future managers.

(iv) Modification of mathematical solution.

(v) Alternative solutions.

Limitations of operation Research:

(a) Magnitude of computation: O.R tries to find out the optimal solution taking all the factors into account. All these calculations cannot be handled manually and require electronic computers which bear very heavy cost.

(b) Absense of quantification: O.R provides solution only when all the elements related to a problem can be quantified. The intangible elements of the problem are excluded from the study, though these might be equally or more important than intangible factors as far as possible.

(c) Distance between managers and operations Research: O.R being specialists' job, requires a mathematician or a statistician, who might not be aware of the business problems. A manager may fail to understand the complex working operation Research. The person should have a working knowledge of each other's jobs to have better understanding of insights of the problem and its optimal solution.

Chapter 2: Linear Programming Problem.

Linear Programming Problem is a technique for determining an optimum schedule of interdependent activities in view of the available resources. Programming is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives.

2:2 Mathematical Formulation of the Problem:

Major steps:

1. Write down the decision variables of the problem.
2. Formulate the objective function to be optimised.
3. Formulate the other conditions of the problem such as resource limitations, market constraints, inter-relation between variables etc,
4. Add the 'Non-negativity' constraint.

Sample Problem:

Production Allocation Problem : A manufacturer produces two types of models M_1 and M_2. Each M_1 model requires 4 hours of grinding and 2 hours of polishing; whereas each M_2 model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinder and 3 polishers. Each grinder works for 40hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is Rs. 3.00 and on an M_2 model is Rs.4.00. whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week?

Mathematical Formulation:

Decision variables:

Let x_1 = number of units of M_1 Model

x_2 = number of units of M_2 model.

Objective Function:

The objective of the manufacturer is to determine the number of M_1 and M_2 models so as to maximize the total Profit.

$$\max z = 3x_1 + 4x_2$$

Constraints:

	Grinding	Polishing
M_1	4 Hours	2 Hours
M_2	2 Hours	5 Hours.

Grinding Hours: $4x_1 + 2x_2$

Polishing Hours: $2x_1 + 5x_2$

Max. Hours Grinding $2 \times 40 = 80$

Max. Hours Polishing $3 \times 60 = 180$.

The time constriants are, $4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

Non. Negativity: $x_1 \geq 0$ and $x_2 \geq 0$.

Manufacturer's allocation problem can be put in the following mathematical form,

Find two real numbers x_1 and x_2 Such that

(i) $4x_1 + 2x_2 \leq 80$

(ii) $2x_1 + 5x_2 \leq 180$.

(iii) $x_1, x_2 \geq 0$.

Objective function:

$$\boxed{\max z = 3x_1 + 4x_2}$$

Graphical Solution Method:

A Linear Programming problem involving two decision variables can be conveniently solved graphically.

1) Solve the following L.P.P's graphically.

$\max z = 3x_1 + 4x_2$

Subject to constraints:

$4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

A Graphical Solution of the problem:

① ⇒ $4x_1 + 2x_2 = 80$ —— ①

Put $x_2 = 0$

$4x_1 + 2(0) = 80$

$4x_1 = 80$

$x_1 = 20$

A (20, 0)

Put $x_1 = 0$

$4(0) + 2x_2 = 80$

$2x_2 = 80$

$x_2 = 40$

(0, 40)

② ⇒ $2x_1 + 5x_2 = 180$ —— ②

Put $x_1 = 0$

$2(0) + 5x_2 = 180$

$5x_2 = 180/5$

$x_2 = 36$

B = (0, 36)

put $x_2 = 0$

$2x_1 + 5(0) = 180$

$2x_1 = 180$

$x_1 = 90$

(90, 0)

① ⇒ $4x_1 + 2x_2 = 80$

② ⇒ $2x_1 + 5x_2 = 180$

$① \Rightarrow \quad 4x_1 + 2x_2 = 80$

$2 \times ② \Rightarrow \quad 4x_1 + 10x_2 = 360$

$(-) \quad (-) \quad (-)$

$-8x_2 = -280$

$\boxed{x_2 = 35}$

Put $x_2 = 35$ in equ ①

$4x_1 + 2(35) = 80$

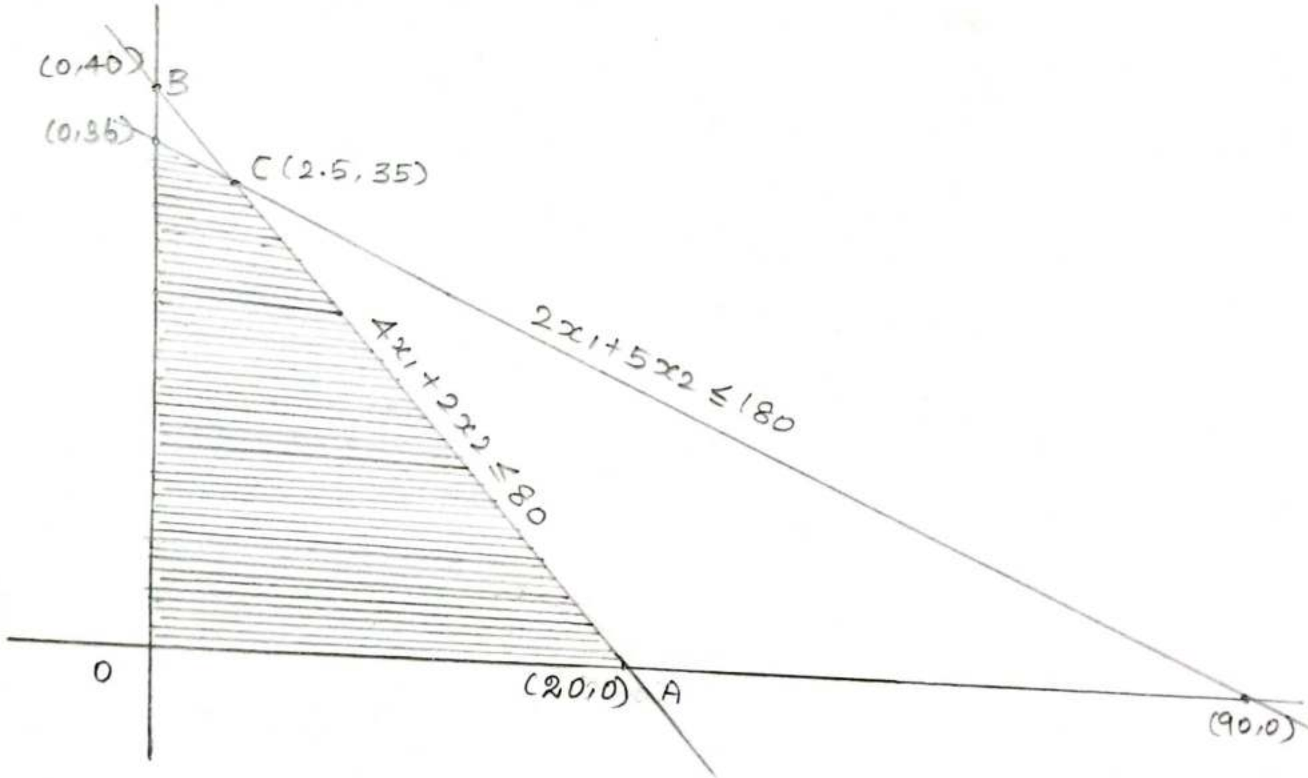
$4x_1 + 70 = 80$

$4x_1 = 10$

$\boxed{x_1 = 2.5}$ $\quad \boxed{C = (2.5, 35)}$

The points are,

$O = (0,0) \quad A = (20,0) \quad B = (0,36) \quad C = (2.5, 35)$



$\max z = 3x_1 + 4x_2$

$x_1 = 2.5 \quad x_2 = 35$

$\max z = 3(2.5) + 4(35)$

$= 7.5 + 140$

$\boxed{\max z = 147.5}$

General Linear Programming Problem:

Let z be a linear function on R^n defined by

$$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

where c_j's are constants. Let (a_{ij}) be an $m \times n$ real matrix and let $\{b_1, b_2 \dots b_m\}$ be a set of constants such that

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq \text{ or } \geq \, = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq \text{ or } \geq \, = b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq \text{ or } \geq \, = b_m.$$

and finally let

$$x_j \geq 0 \qquad (j = 1, 2 \dots n).$$

Feasible Solution:

An n-tuple $(x_1, x_2 \dots x_n)$ of real numbers which satisfies the non-negative restrictions of the problem, is called a feasible solution to the General L.P.P.

Solution: An n-tuple $(x_1, x_2 \dots x_n)$ of real numbers which satisfies the constraints of a General L.P.P is called a solution to the General L.P.P.

Optimum Solution: Any feasible solution which optimizes the objective function of a General L.P.P is called an optimum solution to the General L.P.P.

Slack variables:

Let the constraints of a General L.P.P. be

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, 2 \dots k.$$

Then, the non-negative variables x_{n+i} which satisfy,

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$$

are called slack variables.

Surplus variables

Let the constraints of a General L.P.P be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i .$$

are called Surplus variables,

Canonical and Standard forms of L.P.P

The formulation of Linear programming problem the next step is to obtain its solution. Two forms are dealt with here, the canonical form and the standard form.

The canonical Form:

maximize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to the constraints;

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq b_i;$$

$$x_1, x_2 \dots x_n \geq 0$$

This form of L.P.P is called the canonical form of L.P.P.

(i) The objective function is of the maximization type.

$$\text{Minimize } f(x) = -\text{ Maximize } \{-f(x)\}$$

(ii) All the constraints are of the "$\le$" type, except for the non-negative restrictions.

(iii) All the variables are non-negative.

The Standard Form:

The general linear Programming problem in the form

Maximize or Minimize

$$z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to constraints

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i \qquad i = 1, 2 \dots n$$

$$x_1, x_2, \dots x_n \geq 0.$$

is know as in Standard form.

The characteristics of this form are:

(i) All the constraints are expressed in the form of equations, except for the non-negative restrictions.

(ii) The right hand side of each constraint is non-negative.

The inequality constraint can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added if the constraint is of "$\le$" type and subtracted if the constraint is of "$\ge$" type.

Maximization or minimization

$$z = cx \quad \text{(object Function)}$$

Subject to constraints:

$$Ax = b \quad \text{(constraints)}$$

$$\text{and } x \geq 0 \quad \text{(non-negative constraints)}$$

Rewrite in standard form the following linear Programming Problem.

$$\min z = 2x_1 + x_2 + 4x_3$$

Subject to constraints

$$-2x_1 + 4x_2 \leq 4$$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq 2$$

$x_1, x_2 \geq 0$ and x_3 unrestricted in sign.

Solution:

we introduce slack variables $x_4 \geq 0$ and $x_5 \geq 0$ in the first and third inequalities and Surplus variable $x_6 \geq 0$.

x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$ where $x_3' \geq 0$ and $x_3'' \geq 0$.

Thus the constraints of the problem,

$$-2x_1 + 4x_2 + x_4 = 4$$

$$x_1 + 2x_2 + (x_3' - x_3'') - x_6 = 5.$$

$$2x_1 + 3x_3 + x_5 = 2$$

The non-negavite constraints,

$x_1 \geq 0$, $x_2 \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$.

Convert maximization multiplying it by (-1)

$$\max z^* = -2x_1 - x_2 - 4(x_3' - x_3'') + 0.x_4 + 0.x_5 + 0.x_6$$

where $z^* = -z$

Subject to the constraints:

$$-2x_1 + 4x_2 + x_4 = 4$$

$$x_1 + 2x_2 + x_3' - x_3'' - x_6 = 5$$

$$2x_1 + 3x_3' - 3x_3'' + x_5 = 2$$

$$x_j \geq 0 \quad j = 1, 2, 3', 3'', 4, 5, 6.$$

UNIT - II

The Simplex Method

The simplex method also called the "Simplex Technique' or 'Simplex Algorithm'. The simplex method is an iterative procedure for solving a Linear Programming Problem in a finite number of steps.

Definition : 1 (Basic Solution)

Given a System of m Simultaneous linear equations in n unknown $(m<n)$

$$Ax = b,$$

where A is an $m \times n$ matrix of rank m. Let B be any $m \times m$ submatrix, formed by m linearly independent column of A. Then a Solution obtained by setting n-m variables not associated with column of B, equal to zero, and solving the resulting System, is called a basic Solution to the given system of equations.

The m variables, which may be all different from zero, are called basic variables.

Definition: 2 Degenerate Solution:

A basic Solution to the System is called degenerate if one or more basic variables vanish.

Definition: 3 Basic Feasible Solution:

A feasible Solution to a L.P.P which is also a basic Solution to the problem is called a basic feasible Solution to the L.P.P. (non negative values)

$[5, 0, -1]$ is not a feasible solution.

$[0, 5/3, 2/3]$ is a feasible Solution.

Definition:4 Associated Cost Vectors:

Let x_B be a basic feasible Solution to the L.P.P:

Maximize $z = cx$ Subject to: $Ax = b$ and $x \geq 0$

Then the vector

$c_B = (c_{B_1}, c_{B_2} \ldots c_{Bm})$ where c_{Bi} are components of c associated with the basic variables, is called the cost vector associated with the basic feasible Solution x_B.

Definition: 5 Improved Basic Feasible Solution:

Let x_B and $\hat{x}_B$ be a two basic feasible solution to the Standard L.P.P. Then $\hat{x}_B$ is said to be an improved basic solution, as compared to x_B, if

$$\hat{c}_B \hat{x}_B \geq c_B x_B$$

where $\hat{c}_B$ is constituted of cost component corresponding to $\hat{x}_B$.

Definition: 6 Optimum Basic Feasible solution.

A basic feasible Solution x_B to the L.P.P:

Max $z = cx$ Sub. to $Ax = b$ and $x \geq 0$.

is called an optimum basic feasible Solution if

$$z_0 = c_B x_B \geq z^*$$

where z^* is the value of objective function for any feasible Solution.

01. obtain all the basic Solutions to the following system of Linear equation:

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Solution:

M equations = 2 Basic Solution n-m=0.
N variables = 3 3-2 = 1

The given system of equations can be written in the matrix form as,

$$Ax = b$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$m \times n = 2 \times 3$

Rank of A is 2, the maximum number of linear independent column of A is 2.

Thus we can take any of the following 2×2 Submatrix as basis matrix B.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix}$$

The basic solution to the given system is now obtained by setting $x_3 = 0$ and Solving the system.

Let us first take $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$x_1 + 2x_2 = 4 \quad \text{—} ①$$
$$2x_1 + x_2 = 5 \quad \text{—} ②$$

① × 2 => $2x_1 + 4x_2 = 8$

② => $\overset{(-)}{2x_1} \overset{(-)}{+} x_2 \overset{(-)}{=} 5$

$$3x_2 = 3$$

$$\boxed{x_2 = 1}$$

Put $x_2 = 1$ in equ ①

$$x_1 + 2(1) = 4$$

$$\boxed{x_1 = 2}$$

Thus the basic and non-basic solution to the given System is,

Basic $x_1 = 2$, $x_2 = 1$ Non-Basic $x_3 = 0$.

Second Submatrix:

Let take $B = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$$2x_2 + x_3 = 4 \quad — ③$$

$$x_2 + 5x_3 = 5 \quad — ④$$

$$③ \Rightarrow \quad 2x_2 + x_3 = 4$$

$$④ \times 2 \Rightarrow \quad \underset{(-)}{2x_2} \underset{(-)}{+ 10x_3} \underset{(-)}{=} 10$$

$$-9x_3 = -6$$

$$\boxed{x_3 = 2/3}$$

Put $x_3 = 2/3$ in equ ③

$$2x_2 + 2/3 = 4$$

$$2x_2 = 4 - \frac{2}{3}$$

$$2x_2 = 10/3$$

$$\boxed{x_2 = 5/3}$$

Basic $x_2 = 5/3$ $x_3 = 2/3$ Non-basic $x_1 = 0$.

Third Submatrix:

Let us take $B = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$$x_1 + x_3 = 4 \quad — ⑤$$

$$2x_1 + 5x_3 = 5 \quad — ⑥$$

(5) × 2 ⇒ $2x_1 + 2x_3 = 8$

(6) ⇒ $\underline{2x_1 + 5x_3 = 5}$

$-3x_3 = 3$

$\boxed{x_3 = -1}$

Put $x_3 = -1$ in equ (5)

$x_1 + x_3 = 4$

$x_1 + (-1) = 4$

$x_1 = 4 + 1$

$\boxed{x_1 = 5}$

Basic Solution $x_1 = 5$, $x_3 = -1$ Non Basic ⇒ $x_2 = 0$.

∴ Thus the Basic and Non-Basic solutions are,

$x_1 = 2$	$x_2 = 1$	$x_3 = 0$
$x_2 = 5/3$	$x_3 = 2/3$	$x_1 = 0$
$x_1 = 5$	$x_3 = -1$	$x_2 = 0$.

---

2) Show that the following System of Linear equation has degenarate Solution.

$2x_1 + x_2 - x_3 = 2$

$3x_1 + 2x_2 + x_3 = 3$

Solution:

The given system of equation can be written as,

$$Ax = b$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$m \times n = 2 \times 3$$

Since rank of A is 2,

2x2 submatrices of A, as basic matrix B,

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$$

Let us take $B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$2x_1 + x_2 = 2$ —— ①

$3x_1 + 2x_2 = 3$ —— ②

① x 2 ⇒ $4x_1 + 2x_2 = 4$

$\overset{(-)}{3x_1} \overset{(-)}{+ 2x_2} \overset{(-)}{=} 3$

$\boxed{x_1 = 1}$

Put $x_1 = 1$ in equ ①

$2(1) + x_2 = 2$

$x_2 = 2 - 2$

$\boxed{x_2 = 0}$

Basic Solution: $x_1 = 1$ $x_2 = 0$ Non-Basic $x_3 = 0$.

Second submatrix,

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$x_2 - x_3 = 2$ —— ③

$2x_2 + x_3 = 3$ —— ④

$3x_2 = 5$

$\boxed{x_2 = 5/3}$

Put $x_2 = 5/3$ in equ ③,

$5/3 - x_3 = 2$

$-x_3 = +2 - 5/3$

$-x_3 = 1/3$

$\boxed{x_3 = -1/3}$

Basic Solution $x_2 = 5/3$ $x_3 = -1/3$ Non-Basic $x_1 = 0$.

Third Submatrix

$$\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$-x_3 + 2x_1 = 2$ —— ⑤

$x_3 + 3x_1 = 3$ —— ⑥

$$-x_3 + 2x_1 = 2$$
$$x_3 + 3x_1 = 3$$
$$5x_1 = 5$$
$$\boxed{x_1 = 1}$$

Put $x_1 = 1$ in equ ⑤

$$-x_3 + 2(1) = 2$$
$$\boxed{x_3 = 0}$$

Basic $x_1 = 1$ $x_3 = 0$ Non Basic $x_2 = 0$.

Thus basic solution to the given problem is,

$x_1 = 1$ $x_2 = 0$ $x_3 = 0$

$x_2 = 5/3$ $x_3 = -1/3$ $x_1 = 0$

$x_1 = 1$ $x_3 = 0$ $x_2 = 0$.

In each of the two basic solution, atleast one of the basic variable is zero. Hence two of the basic solutions are degenerate solution.

The Computational Procedure:

The two fundamental conditions on which the simplex method is based are;

(i) condition of feasibility: It ensures that if the initial (starting) Solution is basic feasible then during computation only basic feasible solution will be obtained.

(ii) Condition of optimality: It guarantees that only better solutions will be encountered.

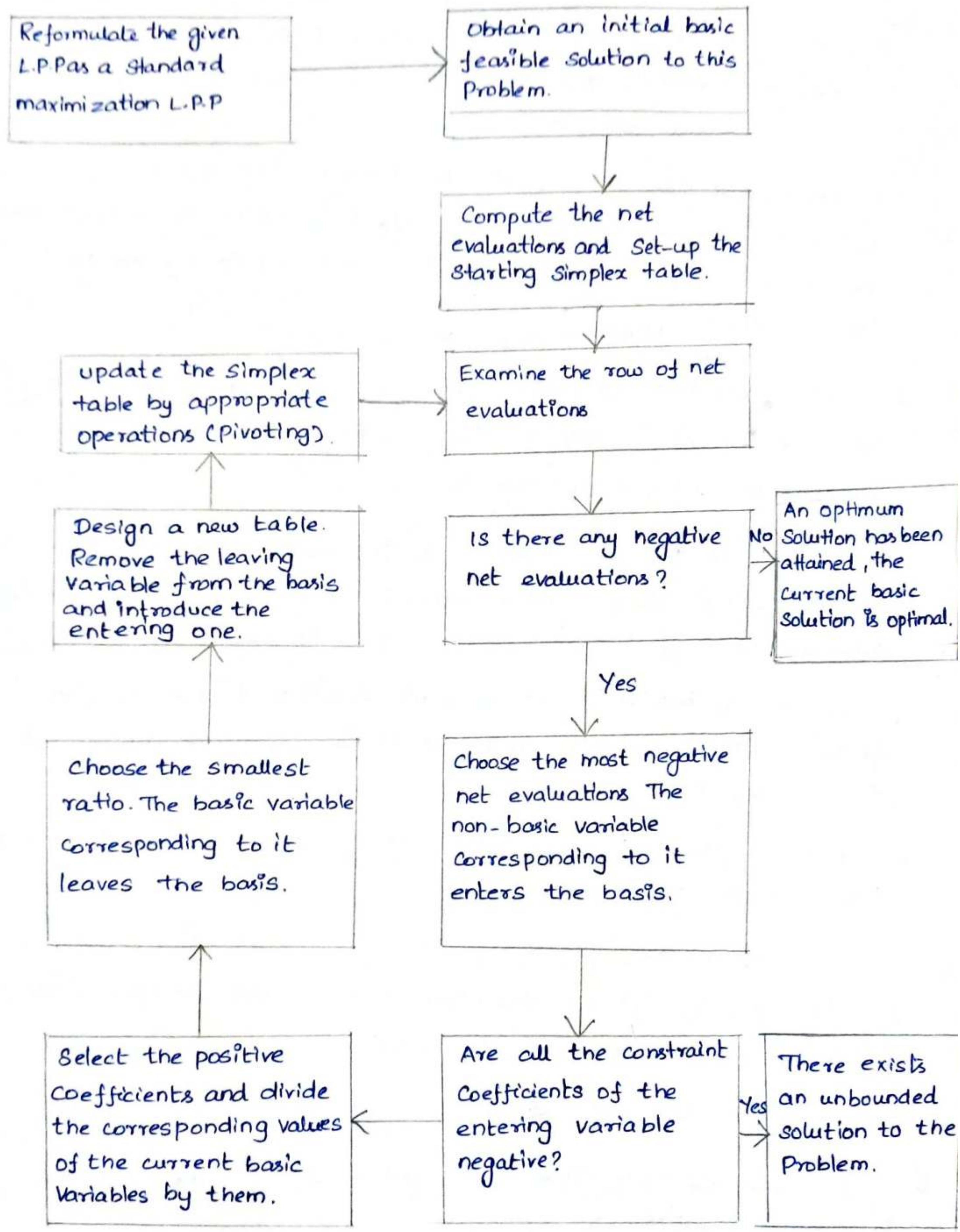
The net evalutions in a tabular form,

C_B	Y_B	X_B	Y_1	Y_2	...	Y_n
C_{B1}	Y_{B1}	X_{B1}	Y_{11}	Y_{21}	....	Y_{1n}
C_{B2}	Y_{B2}	X_{B2}	Y_{12}	Y_{22}	....	Y_{2n}
⋮	⋮	⋮	⋮	⋮		Y_{mn}
C_{Bm}	Y_{Bm}	X_{Bm}	Y_{1m}	Y_{2m}	....	
		Z_0	Z_1-C_1	Z_2-C_2	...	Z_n-C_n

The optimal solution to a General L.P.P is obtained in the following major steps:

1. Select an initial (starting) basic feasible Solution to initiate the algorithm.
2. check the objective function to see whether there is Some non-basic variable that would improve the objective function if brought in the basis. If such a Variable exists go to step 3, otherwise stop.
3. Determine how large the variable found in step 2 can be made until one of the basic variables in the current solution becomes zero. Eliminate the latter variable and let the next trial solution contain the newly found variable instead.
4. Check for optimality the current Solution.
5. Continue the iterations until either an optimum solution is attained or there is an indication that an unbounded Solution exists.

Flow chart . Simplex Algorithm for Maximization L.P.P



The Simplex Algorithm:

For the Solution of any L.P.P by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

1. Check whether the object function of the given L.P.P is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximizing it by using the result. Minimum $z = -$ Maximum(z).

2. Check whether all b_i $(i = 1, 2 \ldots m)$ are non-negative. If any one of b_i is negative then multiply the corresponding inequations of the constraints by -1.

3. Convert all the inequations of the Constraints into equations by introducing slack and/or surplus variables in the constraints. put the cost of these variables equal to zero.

4. Obtain an Initial basic feasible Solution to the problem in the form $x_B = B^{-1}b$ and put it in the first column of the simplex table.

5. Compute the net evaluations $z_j - c_j$ $(j = 1, 2, \ldots n)$ by using the relation $z_j - c_j = c_B y_j - c_j$.

Examine the sign $z_j - c_j$.

(i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible Solution x_B is an optimum basic feasible Solution.

(ii) If atleast one $(z_j - c_j) < 0$, Proceed on to the next step.

6. If there are more than one negative $z_j - c_j$, then choose the most negative of them.

(i) If all $y_{ir} \leq 0$ $(i = 1, 2 \ldots m)$ then there is an unbounded Solution to the given problem.

(ii) If at least one $y_{ir} \geq 0$ then the corresponding vector Y_r enters the basis Y_B.

7. Compute the ratios $\left\{\frac{x_{Bi}}{y_{ir}},\ y_{ir} > 0\ \ i = 1, 2 \ldots m\right\}$ and choose the minimum of them. Then the vector Y_K will level the basis Y_B. The common element Y_{kr}, which is in the kth row and the rth column is known as the leading element (or pivotal element) of the table.

8. Convert the leading element to unity by dividing its row by the leading element itself and other elements in its column to zeroes.

9. Go to step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Simplex Method:

$$\max Z = 5x_1 + 3x_2$$

Subject to constraints

$$x_1 + x_2 \leq 2$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12 \qquad x_1, x_2 \geq 0.$$

Solution:

By introducing slack variables $x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

$$x_1 + x_2 + x_3 = 2$$

$$5x_1 + 2x_2 + x_4 = 10$$

$$3x_1 + 8x_2 + x_5 = 12$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 12 \end{bmatrix}$$

The basic variables are therefore x_3, x_4, x_5 with an obvious initial basic feasible solution,

$$[x_3, x_4, x_5] = [2 \quad 10 \quad 12]$$

The initial simplex table is given below,

CB	YB	XB (C =)	5 = Y_1	3 Y_2	0 Y_3	0 Y_4	0 Y_5	x_{Bi}/y_{ij}
0	Y_3	$x_3 = 2$	1*	1	1	0	0	$2/1 = 2$
0	Y_4	$x_4 = 10$	5	2	0	1	0	$10/5 = 2$
0	Y_5	$x_5 = 12$	3	8	0	0	1	$12/3 = 4$
	$z_j =$	$C_B \cdot X_B$ 0	0	0	0	0	0	
		$z_j - c_j$	-5	-3	0	0	0	

First Iteration:

Introduce the column vector Y_1 and drop the row vector Y_3 from the basis Y_B.

CB	YB	XB (C)	5 Y_1	3 Y_2	0 Y_3	0 Y_4	0 Y_5	X_{Bi}/y_{ij}
5	Y_1	$x_1 = 2$	1	1	1	0	0	
0	Y_4	$x_4 = 0$ $(R_2 - 5R_1)$	0	-3	-5	1	0	
0	Y_5	$x_5 = 6$ $R_3 - 3R_1$	0	5	-3	0	1	
	$z_j =$	$C_B X_B$ 10	5	5	5	0	0	
	$z_j - c_j$		0	2	5	0	0	

All $z_j - c_j$ are positive and therefore an optimum solution has been obtained. Hence the optimum solution is,

$$\max z = 5x_1 + 3x_2$$

$$x_1 = 2 \qquad x_2 = 0$$

$$\max z = 5(2) + 3(0)$$

$$\boxed{\max z = 10}$$

Artificial Variable Techniques:

Introducing the slack and surplus variables, two methods,

(i) The "big M-method" or the method of penalties".

(ii) Two phase method.

The Big-M Method:

The big M-method or the method of penalties consists of the following basic steps:

1. Express the L.P.P in the standard form by introducing slack and/or surplus variables, if any

2. Introduce non-negative variables to the left handside of all constraints of ($\geq$ or $=$) type. These variables are called artifical variables.

   The purpose of introducing artificial variables is just to obtain an initial basic feasible solution.

   * Artificial variable causes violation of the corresponding constraints
   * Artificial variable would not allow them to appear in the optimum simplex table.
   * we assign a very large penalty '$-M$' to these artificial variable in the objective function, for maximization objective function.

3. Solve the modified L.P.P by simplex method.

   At any iteration of the usual simplex method there can arise any of the following 3 cases:

   (a) There is no vector corresponding to some artificial variable in the basis Y_B. In such a, case we proceed to step4.

   (b) There is atleast one vector corresponding to some artificial variable, in the basis Y_B, at zero level. Also, the co-efficient of M in each net evaluation $z_j - c_j$ $(j = 1, 2 .. n)$ is non-negative.

In Such a Case, the current basic feasible Solution is a degenerate one.

(c) At least one artificial vector is in the basis Y_B, but not at the zero level. That is, the corresponding entry in X_B is non-zero. Also co-efficient of M in each net evaluation $z_j - c_j$ is non-negative.

In this case, the given L.P.P does not process any feasible solution.

4. Application of Simplex method is continued until either an optimum basic feasible Solution is obtained or there is an indication of the existence of an unbounded Solution to the given L.P.P.

Example:

Use penalty (or Big 'M') Method to

Minimize $z = 4x_1 + 3x_2$

Subject to Constraints;

$2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6 \quad x_1 \geq 0$ and $x_2 \geq 0$.

Solution:

$\text{Min } z = 4x_1 + 3x_2$

$\text{Max } z^* = -4x_1 - 3x_2$

Subject to constraints:

$2x_1 + x_2 - x_3 + a_1 = 10$

$-3x_1 + 2x_2 + x_4 = 6$

$x_1 + x_2 - x_5 + a_2 = 6$

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 6 \end{bmatrix}$$

(columns: x_1, x_2, x_3, x_4, x_5, a_1, a_2)

The initial Simplex Table:

		c_j	-4	-3	0	0	0	-M	-M	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	A_1	A_2	x_{Bi}/y_{ij}
-M	a_1	10	2*	1	-1	0	0	1	0	$10/2 = 5$
0	x_4	6	-3	2	0	1	0	0	0	$6/-3 = -2$
-M	a_2	6	1	1	0	0	-1	0	1	$6/1 = 6$
	z_j	-16M	-3M	-2M	M	0	M	-M	-M	
	$z_j - c_j$		4-3M	3-2M	M	0	M	0	0	

First Iteration:

		c_j	-4	-3	0	0	0	-M	-M	
C_B	Y_B	x_B	Y_1	Y_2	Y_3	Y_4	Y_5	A_1	A_2	x_{Bi}/y_{ij}
-4	Y_1	5 $R_1 \div 2$	1	$1/2$	$-1/2$	0	0	$1/2$	0	$\frac{5}{1/2} = 10$
0	x_4	21 $R_2 + 3/2 R_1$	0	$7/2$	$-3/2$	1	0	$3/2$	0	$\frac{21}{7/2} = \frac{42}{7} = 6$
-M	a_2	1 $R_3 - 1/2 R_1$	0	$1/2$*	$1/2$	0	-1	$-1/2$	1	$\frac{1}{1/2} = 2$
	z_j	-20-M	-4	$-2-M/2$	$\frac{4}{2} - \frac{M}{2}$	0	M	$\frac{-4}{2} + \frac{M}{2}$	-M	
	$z_j - c_j$		0	$1 - \frac{M}{2}$	$2 - \frac{M}{2}$	0	M	$-2 + \frac{3M}{2}$	0	

Second Iteration: Introduce y_2 and drop a_2

		C_j	-4	-3	0	0	0	-M	-M	
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	a_1	a_2	x_{Bi}/y_{ij}
-4	y_1	4 $R_1 - R_2$	1	0	-1	0	1	1	-1	
0	x_4	14 $R_2 - 7R_3$	0	0	-5	1	7	5	-7	
3	y_2	2 $R_3 \times 2$	0	1	1	0	-2	-1	2	
	z_j	-22	-4	-3	1	0	2	-1	-2	
	$z_j - c_j$		0	0	1	0	2	-1+M	-2+M	

The objective Function:

$$\text{Max } z = -4x_1 + 3x_2$$

$$x_1 = 4 \qquad x_2 = 2$$

$$\text{Max } z = -4(4) + 3(2)$$

$$= -16 - 6$$

$$\text{Max } z = -22.$$

Two Phase Simplex Method:

In the first phase, the simplex method is applied to a specially constructed L.P.P. leading to a final simplex table that contain a basic feasible solution to the original problem.

Second phase, then leads from the basic feasible solution determined by first phase to an optimum basic solution, if any, by an application of simplex method.

Steps I :

1. Ensure that all b_i (constant terms) are non-negative. If some of them are negative make them non-negative by multiplying both sides of these inequations / equations by -1.

2. Add Artificial variables A_i (≥ 0) to the left-hand side of all the constraints of (= and $\geq$) type to complete the Identity Matrix I.

3. Express the given L.P.P in its Standard form and obtain an initial basic feasible solution.

4. Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function. Thus the new objective function is,

$$z^* = -A_1 - A_2 - A_3 \cdots - A_p$$

5. Write down the auxiliary L.P.P in which the new objective function is to be maximized subject to the given set of constraints.

6. Solve the auxiliary L.P.P by simplex method until either of the following three cases do arise:

(i) max $z^* < 0$ and atleast one artificial vector appears in the optimum basis at a positive level.

(ii) max $z^* = 0$ and atleast one artificial vector appears in the optimum basis at zero level.

(iii) max $z^* = 0$ and no artificial vector appears in the optimum basis.

In case (i) given L.P.P does not posses any feasible solution, whereas in case (ii) and (iii) we proceed on the phase 2.

Phase 2:

Use the optimum basic feasible solution of Phase I as a starting solution for the original L.P.P. Assign the actual costs to the original and a cost 0 to all artificial variables in the objective function. Apply simplex method to the modified simplex table obtained at the end of Phase I, till an optimum basic feasible solution is obtained.

Example:

Use two phase simplex method to

$$\max z = 5x_1 - 4x_2 + 3x_3$$

Subject to constraints:

$$2x_1 + x_2 - 6x_3 = 20$$

$$6x_1 + 5x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50, \quad x_1, x_2, x_3 \geq 0.$$

Solution:

Introducing slack variable $x_4 \geq 0$ and $x_5 \geq 0$ and an Artificial variable $A_1 \geq 0$.

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$

$$6x_1 + 5x_2 + 10x_3 + x_4 = 76$$

$$8x_1 - 3x_2 + 6x_3 + x_5 = 50.$$

$$A_1 = 20 \quad x_4 = 76 \quad x_5 = 50.$$

Phase I: Assigning a cost -1 to the artificial variable A_1 and cost 0 to all other variables, the objective function of the auxiliary L.P.P is,

$$z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0x_4 + 0.x_5 - A_1$$

Initial Iteration:

		$c_j =$	0	0	0	0	0	−1	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	A_1	x_{Bi}/y_{ij}
−1	A_1	20	2	1	−6	0	0	1	$20/2 = 10$
0	Y_4	76	6	5	10	1	0	0	$76/6 = 12.$
0	Y_5	50	8^*	−3	6	0	1	0	$50/8 = 6.$
	z_j	−20	−2	−1	6	0	0	1	
	$z_j - c_j$		−2	−1	6	0	0	1	

First Iteration: Introduce Y_1 and drop Y_5

		c_j	0	0	0	0	0	−1	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	A_1	x_{Bi}/y_{ij}
−1	A_1	$15/2$ $R_1 - 2/8 R_3$	0	$7/4^*$	$-15/2$	0	$-1/4$	1	$\frac{15}{2} \times \frac{4}{7} = \frac{30}{7} = 4.$
0	Y_4	$77/2$ $R_2 - 6/8 R_3$	0	$29/4$	$11/2$	1	$-3/4$	0	$\frac{77}{2} \times \frac{4}{29} = 5.3$
0	Y_1	$25/4$ $\div 8$	1	$-3/8$	$3/4$	0	$1/8$	0	
	z_j	$-15/2$	0	$-7/4$	$15/2$	0	$1/4$	0	
	$z_j - c_j$	0	0	$-7/4$	$15/2$	0	$1/4$	0	

		c_j	0	0	0	0	0	−1	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	A	
0	Y_2	$30/7$	0	1	$-30/7$	0	$-1/7$	$4/7$	
0	Y_4	$52/7$	0	0	$256/7$	1	$2/7$	$-29/7$	
0	Y_1	$55/7$	1	0	$-6/7$	0	$1/14$	$3/14$	
	z_j	0	0	0	0	0	0	0	
	$z_j - c_j$		0	0	0	0	0	1	

Since all $z_j - c_j \geq 0$, an optimum Solution to the auxiliary L.P.P has been reached. Furthermore, it is apparent from the table that no artificial variable appears in the basis.

phase 2:

Now we consider the actual costs associated with the original variables. The new objective function is,

$$z = 5x_1 - 4x_2 + 3x_3 + 0.x_4 + 0.x_5$$

		c_j	5	-4	3	0	0	
C_B	Y_B	x_B	Y_1	Y_2	Y_3	Y_4	Y_5	
-4	Y_2	$30/7$	0	1	$-30/7$	0	$-1/7$	
0	Y_4	$52/7$	0	0	$256/7$	1	$2/7$	
5	Y_1	$55/7$	1	0	$-6/7$	0	$1/14$	
	z_j	$\frac{-120}{7} + \frac{275}{7}$	5	-4	$\frac{120}{7} - \frac{30}{7}$	0	$\frac{4}{7} + \frac{5}{14}$	
		$= \frac{155}{7}$	5	-4	$\frac{90}{7}$	0	$\frac{13}{14}$	
	$z_j - c_j$		0	0	$\frac{90}{7} - 3$	0	$13/14$	
			0	0	$69/7$	0	$13/14$	

Since all $z_j - c_j \geq 0$ an optimum basic feasible Solution has been reached.

$x_1 = 55/7 \qquad x_2 = 30/7$

$$\text{Max } z = 5x_1 - 4x_2 + 3x_3$$

$$= 5\left(\frac{55}{7}\right) - 4\left(\frac{30}{7}\right)$$

$$= \frac{275}{7} - \frac{120}{7}$$

$$\boxed{\text{Max } z = 155/7}$$

Operations Research

UNIT - III

Chapter_6 Transportation problem.

6:1 INTRODUCTION :

The transportation problem is one of the Sub-classes of L.P.Ps. The main objective is to transport Various quantities of a Single homogeneous Commodity, that are initially Stored at Various origins, to different destinations. So that the total transportation cost is minimum.

6:2 MATHEMATICAL FORMULATION OF THE PROBLEM:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

Subject to the Constraints :

$$\sum_{j=1}^{m} x_{ij} = a_i, \quad i = 1, 2, \ldots m$$

$$\sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2, \ldots n$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j$$

Destination →

Origin ↓	1	2	3 ...	n	Supply ↓
1	x_{11} C_{11}	x_{12} C_{12}	...	x_{1m} C_{1m}	a_1
2	x_{21} C_{21}	x_{22} C_{22}	...	x_{2n} C_{2n}	a_2
...	...	...	...	...	...
m	x_{m1} C_{m1}	x_{m2} C_{m2}	...	x_{mn} C_{mn}	a_m
Demand →	b_1	b_2	...	b_n	

Theorem 6.1 (Existence of feasible Solution)

A necessary and sufficient condition for the existence of a feasible Solution to a transportation problem is that.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Theorem 6.2 (Basic feasible Solution)

The number of basic Variables in an $m \times n$ transport table are

$m+n-1$ $m \to$ no. of rows
$n \to$ no. of columns

(i) when the total capacity equals total requirement the problem is called balanced.

(ii) when number of positive allocations in any basic feasible solutions are less than $m+n-1$, the solution is said to be degenerate.

(iii) The allocated cells are called occupied cells and empty cells are called non-occupied cells.

6:3 TRIANGULAR BASIS :

A Basis for the system $Ax=b$ is said to be a triangular basis if a triangular system is obtained when all the non-basic variables are set zero in the system.

If there are m+n equations, then the total number of basic Variables will be at least 2m+2n. If n denotes the total number of basic Variables, then $N \geq 2m$ & $N \geq 2n$

6:4 LOOPS IN A TRANSPORTATION TABLE:

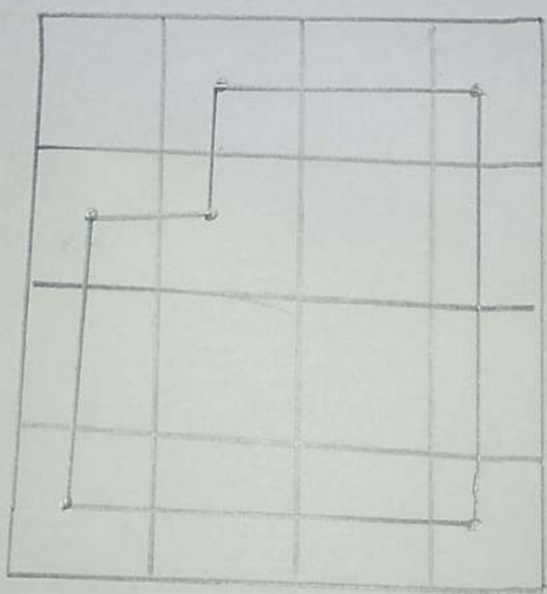
At least four cells in a transportation table is said to form a loop.

(i) any two adjacent cells of the ordered set lie either in the same row or in the same column.

(ii) no three or more adjacent cells in the ordered set lie in the same row or column. The first of the loop must follow the last in the set.

Ex:

$L = \{(2,1),(4,1),(4,4),(1,4),(1,2),(2,2)\}$



6:5 FINDING INITIAL BASIC FEASIBLE SOLUTION

Three common methods used are:

- North-West Corner Rule (NWC)
- Least Cost (or) Matrix Minima method
- Vogel's Approximation Method (VAM)

NWC - Method

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	200	225	275	250	

Solution :

[200] 11	[50] 13	17	14	250
16	[175] 18	[125] 14	10	300
21	24	[150] 13	[250] 10	400
200	225	275	250	

Since, $\Sigma a_i = 950$ & $\Sigma b_j = 950$ there exists a feasible solution.

Following North West Corner method, the first allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(250, 200) = 200$

(ie) if $(b_1 < a_1)$ move horizontally to the 2^{nd} column,

$$x_{12} = \min(a_1 - x_{11}, b_2)$$

$$\text{cell}(1,2) = \min(250 - 200, 225)$$
$$= \min(50, 225) = 50$$

(ii) if $(b_2 > a_1)$ we move down vertically to 2^{nd} row,

$x_{22} = \min(a_2, b_2 - x_{12})$ in the cell (22)

$$= \min(300, 225 - 50)$$
$$= \min(300, 175) = 175$$

(iii) if $(b_2 < a_2)$ move horizontally to the 2^{nd} column

$$x_{23} = \min(a_2 - x_{22}, b_2)$$
$$= \min(300 - 175, 225)$$
$$= \min(125, 225)$$
$$= 125$$

(iv) if $(b_3 > a_3)$ move down,

$$x_{32} = \min(a_3, b_3 - x_{23})$$
$$= \min(400, 275 - 125)$$
$$= \min(400, 150)$$
$$= 150$$

(v) if $(b_3 < a_1)$ move horizontally,

$$x_{33} = \min(a_3 - x_{32}, b_3)$$
$$= \min(400 - 150, 275)$$
$$= \min(250, 275)$$
$$= 250$$

The transport cost according to the above is

given by

$$= 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 14 + 150 \times 13 + 250 \times 10$$

$$= 12200$$

Least Cost or Matrix Minima method: (Intuitively Best Method)

Step 1: Determine the smallest cost in the cost matrix of the transportation table. Let it be c_{ij}. Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j)

Step 2: If $x_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_j by a_i. Go to Step 3.

If $x_{ij} = b_j$, cross off the j^{th} column of the transportation table and decrease a_i by b_j. Go to Step 3.

If $x_{ij} = a_i = b_j$, cross off either the i^{th} row or the j^{th} column but not both.

Step 3: Repeat Steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitary choice among the minima.

① Obtain an initial basic feasible solution to the following T.P using the matrix minima method.

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Solution:

Since $\Sigma a_i = \Sigma b_j = 24$ there exists a feasible solution to the T.P. The T.P table has 12 cells.

(i) The first allocation is made in the cell (3,1) the magnitude being $x_{31} = 4$

This satisfies the requirement at destination D_1 and thus we cross off the first column from the table,

1	2	3	4	6
4	3	2	0	8
[4] 0	2	2	1	~~10~~ 6
~~4~~ 0	6	8	6	

$x_{ij} = \min(a_i, b_j) = x_{31} (\min 4, 6) = 4$

(ii) The second allocation is made in the cell (2,4) magnitude $x_{24} = \min(6,8) = 6$. Cross off the fourth column of the table.

1	2	3	4	6
4	3	2	[6] 0	~~8~~ 2
[4] 0	2	2	1	6
0	6	8	~~6~~ 0	

(iii) There is again a tie for third allocation we choose arbitarily the cell (1,2) and allocate $x_{12} = (6, 6) = 6$, there cross off either the second column or the first row. So, we cross of the first row of the table.

1	[6] 2	3	4	~~6~~ 0
4	3	2	[6] 0	2
[4] 0	2	2	1	6
0	~~6~~ ε	8	0	

(iv) The next allocation of magnitude $x_{32} = 0$ or ε where $ε \to 0$ is made in the cell (3,2) cross of the second column.

1	[6] 2	3	4	0
4	3	2	[6] 0	2
[4] 0	[ε] 2	2	1	6
0	ε	8	0	

(v) We choose arbitarily again to make the next allocation in cell (2,3) of magnitude $x_{23} = (2,8) = 2$ cross of 2^{nd} row.

1	[6] 2	3	4	0
4	3	[2] 2	[6] 0	~~2~~ 0
[4] 0	[ε] 2	2	1	6
0	ε	~~8~~ 6	0	

(vi) the last allocation of magnitude $x_{33} = \min(6,6) = 6$ is made in the cell (3,3)

1	[6] 2	3	4
4	3	[2] 2	[6] 0
[4] 0	[ε] 2	[6] 2	1

The transportation cost accordingto the above route is given by

$$= 6\times2 + 2\times2 + 6\times0 + 4\times0 + \varepsilon\times2 + 6\times2$$

$$= 12 + 4 + 12$$

$$= 28$$

Vogel's Approximation Method (VAM or Penalty Method)

Step 1: Calculate Penalties by taking differences between the minimum and next to the minimum unit transportation Costs in each row and column.

Step 2: Circle the largest row difference or column difference. In the event of a tie choose either.

Step 3: Allocate as much as possible in the lowest cost cell of the row (or Column) having a Circled row (or column) difference.

Step 4: In Case the allocation is made fully to a row (or column), ignore that row (or column) for further consideration, by crossing it.

Step 5: Revise the differences again and cross out earlier figure Go to 2.

Step 6: Continue the procedure until all rows and Columns have been crossed out (ie) distribution is Complete.

VAM or Penalty Method.

	D	E	F	G	Available
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

11	13	17	14	250
16	18	14	10	300
21	24	13	10	400
200	225	275	250	

Step 1:

[200] 11	13	17	14	250 (2)
16	18	14	10	300 (4)
21	24	13	10	400 (3)
200 (5)	225 (5)	275 (3)	250 (0)	

Step 2:

13	17	14	50 (1)
18	14	10	300 (4)
24	13	10	400 (3)
225 (5)	275 (1)	250 (0)	

Step 3:

[175] 18	14	10	300 (4)
24	13	10	400 (3)
175 (6)	275 (1)	250 (0)	

Step 4:

14	[125] 10	125 (4)
13	10	400 (3)
275 (1)	250 (0)	

Step 5:

[275] 13	[125] 10	400
275	125	

Overall Transportation cost is

200 / 11	50 / 13	17	14
16	175 / 18	14	125 / 10
21	24	275 / 13	125 / 10

$$Z = 200 \times 11 + 13 \times 50 + 175 \times 18 + 125 \times 10 + 275 \times 13 + 125 \times 10$$
$$= 12075$$

66 Moving Towards optimality:

Determine the Net Evaluation (the UV method)

Consider the following m-origin, n-destination T.P

Determine x_{ij} So as to minimize,

$$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

Subject to Constraints,

$$a_i - \sum_{j=1}^{n} x_{ij} = 0 \qquad i = 1, 2, \ldots m$$

$$b_j - \sum_{i=1}^{m} x_{ij} = 0 \qquad j = 1, 2, \ldots n$$

$$x_{ij} \geq 0 \text{ for all } i, j$$

Let $u_1 \ldots u_m$ and $v_1, v_2 \ldots v_n$ be the dual Variables associated with above origin and destination Constraints respectively.

It Can be written in the form,

$$u_i + v_j \leq c_{ij}$$
$$u_i, v_j \text{ unrestricted.}$$

North-West Corner Rule :

[200] 11	[50] 13	17	14
16	[175] 18	[125] 14	10
21	24	[150] 13	[250] 10

$C_{ij} = u_i + v_j \quad u_1 = 0$

$C_{11} = u_1 + v_1 = C_{11} \Rightarrow 0 + v_1 = 11 \Rightarrow v_1 = 11$

$C_{12} = u_1 + v_2 = 0 + v_2 = 13 \Rightarrow v_2 = 13$

$C_{22} = u_2 + v_2 = u_2 + 13 = 18 \Rightarrow u_2 = 5$

$C_{23} = u_2 + v_3 = v_3 + 5 = 14 \Rightarrow v_3 = 9$

$C_{33} = u_3 + v_3 = u_3 + 9 = 13 \Rightarrow u_3 = 4$

$C_{34} = u_3 + v_4 = 4 + v_4 = 10 \Rightarrow v_4 = 6$

Step 1: Computing net evaluation for each non-basic cell:

$Z_{13} - C_{13} \Rightarrow u_1 + v_3 - C_{13} \Rightarrow 0 + 9 - 17 = -8$

$Z_{14} - C_{14} \Rightarrow u_1 + v_4 - C_{14} \Rightarrow 0 + 6 - 14 = -8$

$Z_{21} - C_{21} \Rightarrow u_2 + v_1 - C_{21} \Rightarrow 5 + 11 - 16 = 0$

$Z_{24} - C_{24} \Rightarrow u_2 + v_4 - C_{24} \Rightarrow 5 + 6 - 10 = 1$

$Z_{31} - C_{31} \Rightarrow u_3 + v_1 - C_{31} \Rightarrow 4 + 11 - 21 = -6$

$Z_{32} - C_{32} \Rightarrow u_3 + v_2 - C_{32} \Rightarrow 4 + 13 - 24 = -7$

				u_i
[200] 11	[50] 13	(-8) 17	(-8) 14	$u_1 = 0$
(0) 16	[175] 18	[125] 14	(1) 10	$u_2 = 5$
(-6) 21	(-7) 24	[125] 13	[250] 10	$u_3 = 4$
$v_1 = 11$	$v_2 = 13$	$v_3 = 9$	$v_4 = 6$	v_j

Step 2: Selecting the entering Variables:

The Net Evaluation $z_{rs} - c_{rs} = \max (z_{ij} - c_{ij} > 0)$

The Cell (2, 4) enter the basis in the preceding iteration since $z_{24} - c_{24}$ is most positive.

200 11	50 13	(-8) 17	(-8) 14
(0) 16	175 18	125 θ 14	(1) θ 10
(-6) 21	(-7) 24	125 θ 13	250 θ 10

Step 3: Selecting the leaving Variables.

Starting with the entering Cell (2, 4) we are able to identify a closed loop consisting of basic cells (2, 3) (3, 3) and (3, 4) ending at the starting cell (2, 4) and make ± θ adjustments in the corner cells of the loop to maintain rim requirement. The maximum Value that Can be assigned to θ is given by min [125, 250] = 125 which reduce the allocation in the basic Cell (2, 3) to Zero.

Unit - IV

The Assignment Problem, Routing problems.

Introduction:

→ An assignment Problem is a particular in which a number of operations are to be assigned to an equal number of Operators, where each operator performs only one operation.

→ The objective is to maximize Overall Profit (or) minimize overall cost for a given assignment schedule.

Assignment Algorithm:

Step 1: Check whether the Cost matrix is Square, if not, make it square by adding Suitable number of dummy rows (or columns) with 0 cost elements.

Step 2: Locate the smallest cost elements in each row of the cost matrix. Subtract this smallest element from each element in that row. As a result, there shall be at least one zero in each row of the reduced Cost matrix.

Step 3: In the reduced Cost matrix obtained, Consider each Column and locate the Smallest element in it. Subtract the Smallest value from every other entry in the Column. As a result, there would be at least one zero in each of the rows and Columns of the second reduced Cost matrix.

Step 4: In the above reduced Cost matrix search for an optimum assignment as follows

(i) Examine the rows Successively until a row with exactly one zero found Enrectangle this zero as □ and cross out all other 0's in its Column. proceed in this manner until all the rows have been examined. If there are more than one zero in any row, then do not touch that row and pass on to the next row.

(ii) Repeat the procedure for the Columns of the reduced Cost matrix. If there is no single zero in any row (or) Column of the reduced matrix, then arbitrarily choose a row or Column having the minimum number of 0's.

Arbitrally select and enrectangle any one 0 in the row or column thus chosen and cross all the other 0's in its row and column.

Repeat step (i) and (ii) until all the 0's have been either assigned or crossed.

(iii) If each row and each column of the reduced matrix has one and only assigned 0, the optimum assignment is made in the cells of enrectangled 0's. Otherwise go to next step.

Step 5:

Draw the minimum number of horizontal and/or vertical lines through all the 0's as follows.

(a) Mark(✓) the rows in which assignment has not been made.

(b) Mark (✓) columns which have zeros in the marked rows.

(c) Mark(✓) rows which have assigned in the marked columns.

(d) Repeat (b) and (c) until chain of marking is completed.

(e) Draw straight lines through all unmarked rows and marked columns.

Step 6:

If the minimum number of lines passing through all the zeros is equal to the number of row (or) columns, the optimum solution is attained. Otherwise go to next step.

Step 7: Revise the costs matrix as follows.

(i) find the smallest element not covered by any of the lines of step 4.

(ii) Subtract this from all the uncrossed elements and add the same at the point of intersection of two lines.

(iii) Other elements crossed by the lines remain unchanged.

Step 8:

Go to step 4 and repeat the procedure till an optimum solution is attained.

The above iterative method to determine an assignment schedule is known as Hungarian Assignment Method.

Problems

701.

Tasks	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

(MEN: E, F, G, H)

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Solution:

Step 1: Subtracting the smallest element of each row from every element of the Corresponding row, we get the reduced matrix:

$$\begin{bmatrix} 7 & 5 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

Step 2: Subtracting the smallest element of each Column of the reduced matrix from every element of the Corresponding Column, we get the following reduced matrix:

$$\begin{bmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

Step 3: Starting with row 1, we enrectangle □ a single zero, if any, and cross (X) all other zeros in the Column so marked. Thus we get

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \boxed{\times} & 13 \\ 23 & \boxed{0} & 2 & \boxed{\times} \\ 9 & 12 & 13 & \boxed{\times} \end{bmatrix}$$

In the above matrix, we arbitrarily enrectangled a zero in Column 1 because row 2 had two zeros.

Step 4: (i) Since row 4 does not have any assignment, we tick this row(✓).

(ii) Now there is a zero in the fourth Column of the ticked row. So we tick Column fourth [✓].

(iii) Further there is an assignment in the first row of the ticked Column. So we tick first row ( ).

(iv) Draw straight lines through all unmarked rows and marked Columns. Thus we have.

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \boxed{0} & 13 \\ 23 & \boxed{0} & 2 & \times \\ 9 & 12 & 13 & \times \end{bmatrix} \begin{matrix} \checkmark \\ \\ \\ \checkmark \end{matrix}$$

Step 5: In Step 4, we observe that the minimum number of lines so drawn is 3, which is less than the order of the cost matrix, indicating that the current assignment is not optimum.

To increase the minimum number of lines, we generate new zeros in the modified matrix.

Step 6: The smallest element not covered by the lines is 5. Subtracting this element from all the uncovered elements and adding the same to all the elements lying at the intersection of the lines.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Step 7. Repeating step 4 on the reduced matrix, we get

$$\begin{bmatrix} 2 & 6 & \boxed{0} & \times \\ \boxed{0} & 11 & \times & 18 \\ 23 & \boxed{0} & 2 & 5 \\ 4 & 7 & 8 & \boxed{0} \end{bmatrix}$$

Now, since each row and each column has one and only one assignment, an optimal solution is reached. The optimum assignment is A → G, B → E, C → F and D → H.

The minimum total time for this assignment scheduled is 17 + 13 + 19 + 10 = 59 man-hours.

102. A marketing manager has 5 Salesmen and 5 districts. Considering the capabilities of the Salesmen and the nature of districts, the marketing manager estimates that Sales per month (in hundred rupees) for each Salesman in each district would be follows.

Job	Machine A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of Salesman to districts that will result in maximum Sales.

Solution: Convert the profit matrix into opportunity loss matrix by subtracting all the entries from the highest element 41 of the given matrix.

Initial Iteration: Reduce the opportunity loss matrix so that there is at least one zero in each row and each column. Make the proper assignments in rows and columns. Also draw the minimum number of lines to cover all the zeros of the reduced matrix.

~~8~~	[0]	~~0~~	7	~~0~~	
[0]	14	12	14	4	✓
~~0~~	12	8	6	4	✓
19	1	~~0~~	[0]	5	
11	5	[0]	~~0~~	1	

$$\begin{bmatrix} 12 & \boxed{0} & \not{0} & 7 & \not{0} \\ \boxed{0} & 10 & 8 & 10 & \not{0} \\ \not{0} & 8 & 4 & 2 & \boxed{0} \\ 23 & 1 & \boxed{0} & \not{0} & 5 \\ 15 & 5 & \not{0} & \boxed{0} & 1 \end{bmatrix}$$

Final Iteration. by subtracting the element '4' from all the elements not covered by all lines and adding the same at the intersection of two lines.

The optimum assignment is

1→B 2→A 3→E, 4→C 5→D ; or

1→B 2→E 3→A, 4→C, 5→D ;

maximum profit will be RS. 191.

3. A department head has four tasks to be performed and three subordinates. the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should be allocate the tasks one to each man, so as to minimize the total man-hours?

Task	Men 1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution: Since the problem is unbalanced, we add a dummy Column with all the entries as zero and use assignment methods for optimum Solution.

Now reduce the balanced cost matrix and make assignments in rows and Columns having single zeros. Thus we have:

[0]	6	9	⊠
4	7	[0]	⊠
26	[0]	9	⊠
9	10	14	[0]

The optimum assignment is
I → 1, II → 3, and III → 2; while task IV should be assigned to a dummy man (ie) it remains to be done.

The minimum time is 35 hours.

Routing problems:

→ Network Scheduling is a technique for the planning and Scheduling of large projects.

→ A typical network problem consists of finding a route from one node to another.

→ The problem is to select the route that yields minimum cost.

→ A large variety of problem other than the routing one may be developed in connection with the construction and utilization of networks. Here we shall consider only special type of routing problem, that occurs most frequently in O.R – the travelling salesman problem.

The Travelling Salesman Problem:

The Salesman starts from his home city, he must visit every city exactly once and return to his home city.

→ The problem is to find the route shortest distance / time / cost.

→ Travelling sales man problem can be first solved as Assignment problem by using Hungarian Method to find optimum solution.

→ Then check the TSP condition.

If the Condition is satisfied then the Assignment problem solution will be optimum solution even for TSP.

If not go to next step

- The Solution can be adjusted by inspection (or)
- Form a single circuit (or)
- The Iterative procedure
  - Branch and Bound Method.

Problem

1. A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table:

To Item.

From Item	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution: Reduce the Cost matrix and make assignments in rows and Columns having Single zeros.

Initial Iteration. Draw the minimum number of lines to Cover all the zero

First Iteration: by Subtracting the lowest element from all the elements not Covered by lines and adding the Same at the intersection of two lines.

0	1	3	[0]	1	✓
1	0	2	0̸	1	✓
2	1	0	2	[0]	
0̸	[0]	3	0	4	
[0]	0̸	0	3	0	

0	0̸	2	[0]	0̸
[0]	0	1	0̸	0̸
2	1	0	3	[0]
0̸	[0]	3	0	4
0̸	0̸	[0]	4	0

→ The optimum assignment is A→D, B→A, C→E, D→B and E→C with a minimum Cost of 20

→ This assignment Schedule does not provide us the solution of travelling Salesman problem as it gives A→D, D→B, B→A, while B is not allowed to follow A unless C and E are processed.

Second Iteration: Now we try to find the next solution which satisfies this extra restriction. The next minimum (non-zero) element in the matrix is 1. So we try to bring 1 into the solution. But the element '1' occurs in two places. we shall consider all the cases seperately until the acceptable solution is reached.

→ we start with making an assignment at (2, 3) instead of zero assignment at (2, 1). The resulting feasible solution then will be

A→D, D→B, B→C, C→E, E→A

when an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting feasible solution will be

A→E, E→C, C→B, B→D, D→A

The total set-up cost in both the programmes comes out to be 21.

Game Theory

→ It is a Mathematical model which is used for decision Making.

→ In general decision Making situation can be classified into three categories

1. Deterministic situation
2. Probabilistic situation
3. uncertainty situation.

Basic Terminology of game Theory

→ players : There are two players in a game player A and player B.

Strategy:

1. Pure Startegy
2. Mixed Startegy.

→ Startegy is a Course of action taken by the player.

Example: A Company (player) will have different startegies to increase their volume of sales.

→ Among the various startegies

If the Company Select only one particular Startegy and ignore the remaining strategy then it is called pure strategy. but the Sum of probability is equal to 1.

Example: There are three startegy

P_1, P_2, P_3

→ the player will take one particular startegy ie P_2 and ignore the other.

$P_1 = 0$ $P_2 = 1$ $P_3 = 0$

ie $0+1+0=1$ so the Sum of probability is 1 ie called pure strategy.

Mixed Strategy:

→ The player will take more than one startegy. the Sum of probability is 1 but each value of probability is < 1

Ex: P_1, P_2, P_3

→ the player will take P_1 and P_2

$P_1 = 0.65$, $P_2 = 0.35$, $P_3 = 0 = 1$

Each probability startegy value will be less than 1 and sum of probability is 1.

Payoff matrix :

		Player B 1	2	3
Player A	1	20	10	22
	2	30	40	38
	3	15	18	25

→ These are the outcomes of the different combination.

→ we have two players player A and player B with three Startegy.

* If the out come value is positive means it gain to player A and loss to player B.

* If the outcome is negative means it loss to player A and gain to player B.

Game Theory has two principal

1. Maximin Principle
2. MinMax Principle.

1. Maximin :

Maximizes the minimum guaranteed gains of Player A.

2. MinMax :

Minimizes the Maximum guaranteed loss of player B.

Saddle Point:

Maximin value = Minimax value

Value of the game:

If the game has Saddle Point, then the value of the cell at the Saddle Point is called the value of the game.

Two-Person zero-Sum game:

In a game with two players if the gain of one player is equal to the loss of another Player than that game is called two-person zero-Sum game.

B

Ex

A

20	10	22
30	-40	38
15	18	25

If the outcome is posive means Player A gain and loss of player B. is called Two-Person zero-Sum game.

Problem

1. Find the optimum Strategies of the players in the following games.

Player B

		1	2	3
Player A	1	25	20	35
	2	50	45	55
	3	58	40	42

First we find row minimum and Column maximum.

	1	2	3	row min
1	25	20	35	20
2	50	45	55	45
3	58	40	42	40
Column max	58	45	55	

→ After finding Row minimum and Column Maximum we find the maximum of these minimum values is called maxmin value

	1	2	3	row min
1	25	20	35	20
2	50	45	55	[45] → maximin value
3	58	40	42	40
	58	[45]	55 → MinMax value	

→ Player A is called Maximin player
→ Player B is called Minimax Player
→ Intersecting point is $\boxed{45}$ which is called Saddle Point. (ie) Maximin value = minimax value.

→ This game has Saddle Point (45 = 45)

→ value of the game is $\boxed{V = 45}$

Hence the game has a Saddle point at the cell corresponding to Row 2 and Column 2

So optimal probability

$A[p_1 \; p_2 \; p_3] = A[0, 1, 0]$

$B[q_1 \; q_2 \; q_3] = B[0, 1, 0]$

Both the player select second strategy. The sum of the probability will be equal to 1. So this game is pure strategy.

Game with Mixed Strategies

Consider the following payoff matrix with respect to player A and solve it optimally

Player B

Player A		
	9	7
	5	11

→ The game has no saddle point the game is said to have mixed strategy.

			Row min
	9	7	7
	5	11	5
Column max	9	11	

Maximin value (7)

minimax value (9)

Here Maximin value ≠ minimax value. So this game has no saddle points.

Now we proceed mixed strategy.

			oddments
	9	7	6 (11−5)
	5	11	2 (9−7)
oddments.	4 (11−7)	4 (9−5)	

Step 1: find out the oddments in Rows as well as Columns

→ oddments is find the difference b/w the values and write it into Second row and find the difference b/w Second row and write it into first Row.

Next Step:

we need to find out the probability

9	7	6
5	11	2
4	4	

$$p_1 = \frac{6}{2+6} = \frac{6}{8} = \frac{3}{4}$$

$$p_2 = \frac{2}{2+6} = \frac{2}{8} = \frac{1}{4}$$

$$q_1 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

$$q_2 = \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}.$$

Now we need to find value of the game. take first Column value and Row oddmin

$$\frac{9\times 6 + 5\times 2}{6+2} = \frac{54+10}{6+2} = \frac{54+10}{8} = \frac{64}{8} = 8.$$

(or)

$$\frac{7\times 6 + 11\times 2}{6+2} = \frac{42+22}{8} = \frac{64}{8} = 8.$$

first Row value and Column min.

$$\frac{(9\times4)+(7\times4)}{4+4} = 8$$

(or)

$$\frac{5\times4+11\times4}{4+4} = \frac{20+44}{8} = \frac{64}{8} = 8.$$

Hence the strategy of Player A $(3/4, 1/4)$

Player B $(1/2, 1/2)$

Value = 8.

$$\left(\frac{3}{4}\ \ \frac{1}{4}\right) = \frac{3}{4}+\frac{1}{4} = \frac{4}{4} = 1$$

$$\left(\frac{1}{2}\ \ \frac{1}{2}\right) = \frac{1}{2}+\frac{1}{2} = \frac{2}{2} = 1.$$

UNIT - V
Network Scheduling by PERT / CPM

Introduction:

* Networks are diagrams, easily visualized in electrical theory, transportation systems like roads, railway lines, Pipelines, blood vessels. A large variety of problems in particular those involving sequential operations, like flow charts.

* In OR, determining an optimum solution can be looked upon as the problem of selecting the best sequence of operations out of a finite number of available alternative that can be represented as a network.

There are two basic planning and control techniques that utilize a network.

(i) PERT (Programme Evaluation and Review Technique) - Event oriented
(ii) CPM (Critical Path Method) - activity oriented.

Network and Basic Components:

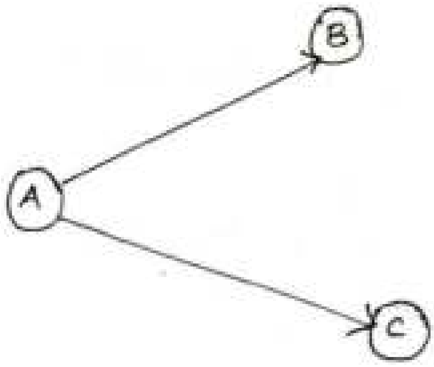
A Network is a graphic representation of a Project's operations and is composed of activities and events that must be completed to reach the end objective of a project, showing and planning sequence of their accomplishments, their dependence and inter-relationships.

Events: An Event (or node) is a specific physical or intellectual accomplishment in a programme or project plan. An Event is represented by a circle, rectangle, hexagon.

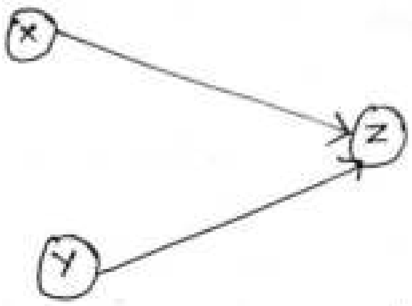
Activity :

An activity is a task, or item of work to be done, that consumes time, effort, money or other resources. It lies between two events called the 'Preceeding' and 'Succeeding' ones. An activity is represented on the network by an arrow with its head indicating the sequence in which the events are to occur.

Preceding Event Succeeding Event



Event A constraints activities AB and AC



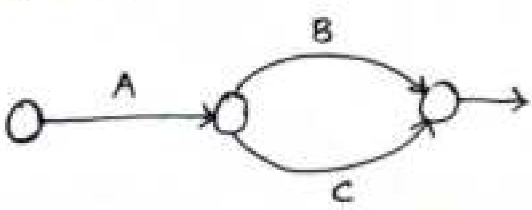
Activities XZ and YZ constrain event Z.

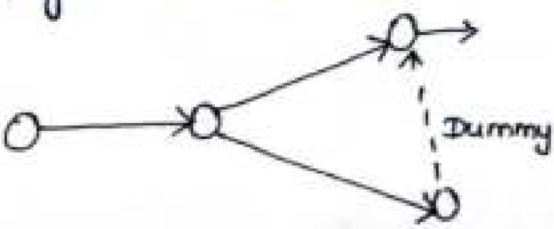
Dummy Activity :

Certain activities which neither consume time nor resources but are used simply to represent a connection between events are known as dummies. Dummy (zero time) shown by a chain of dotted arrows.

(i) to maintain uniqueness in the numbering system.

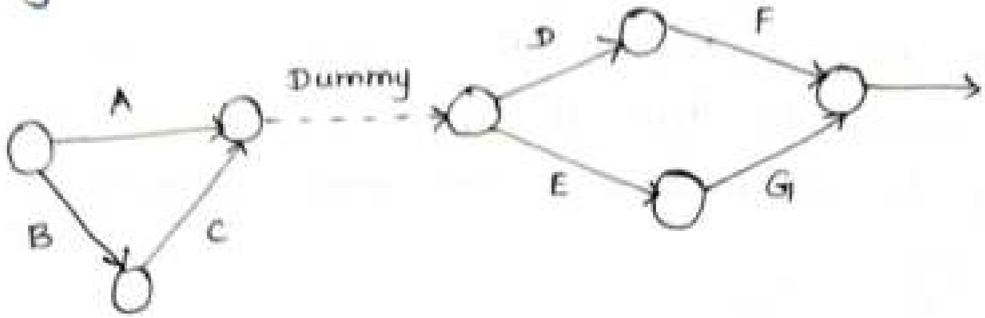
(ii) to maintain the proper logic in the network.





(c) Redundancy:

If a dummy activity is the only activity emanating from an event, it can be eliminated.



Rules of Network Construction:

(i) No event can occur until every activity preceding it has been completed.

(ii) An activity succeeding an event cannot be started until that event has occured.

(iii) An event cannot occur twice, i.e., a path of activities cannot form a loop that returns to any event previously accomplished.

(iv) Each activity must start from and terminate in an event.

(v) Time flows from left to right.

(vi) An activity must be completed in order to reach the end-event.

(vii) Dummy activities should only be introduced if absolutely necessary.

Numbering the Events:

After the network is drawn in a logical sequence every event is assigned a number. The number sequence must be such so as to reflect the flow of the network.

(i) Event numbers should be unique.

(ii) Event numbering should be carried out an a sequential basis from left to right.

(iii) The initial event which has all outgoing arrows with no incoming arrow is numbered 0 or 1.

(iv) The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.

(v) Gaps should be left in the sequence of event numbering to accommodate subsequent inclusion of activities, if necessary.

Example:

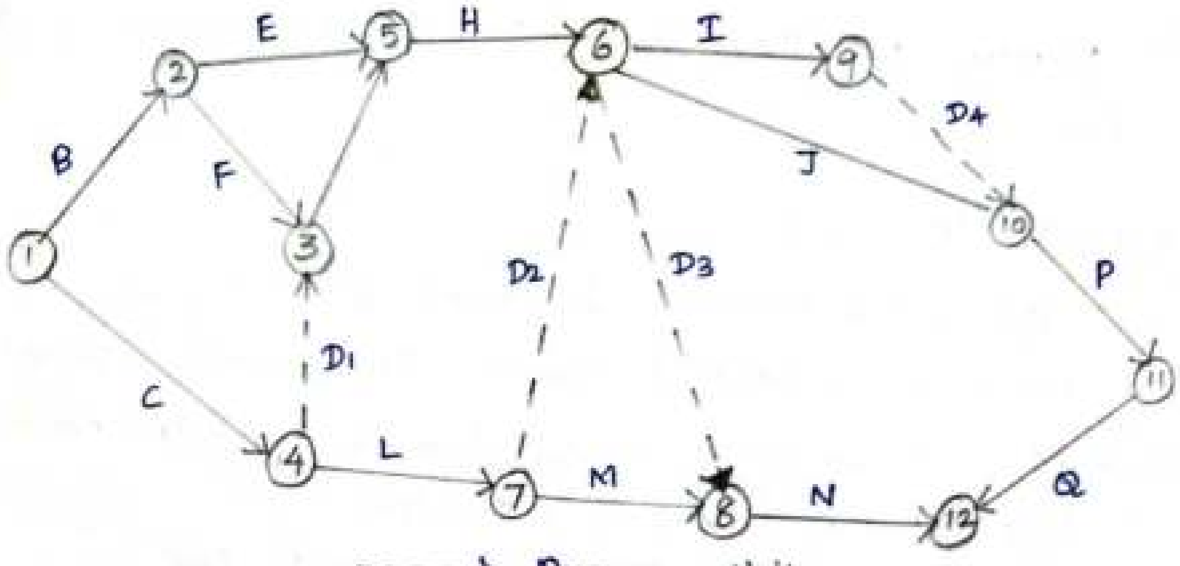
Construct the network diagram comprising activities B, C, ... Q and N such that the following constraints are satisfied.

$B < E, F$; $C < G, L$; $E, G < H$; $L, H < I$, $L < M$;

$H < N$; $H < J$; $I, J < P$; $P < Q$

The notation $X < Y$ means that the activity X must be finished before Y can begin.

Solution: The resulting network is shown in Fig. The dummy activities D_1, D_2 and D_3 are used to establish the correct precedence relationships. D_4 is used to identify the activities I and J with unique end nodes. The nodes of the project are numbered such that their ascending order indicates the direction of progress in the project.



----> Dummy activity.

Time calculations in Networks:

For each activity an estimate must be made of time that will be spent in the accomplishment of that activity. Estimates may be expressed in hours, days, weeks or any other convenient unit of time. The time estimates is the calculation of earliest time and latest time.

(a) Let zero be the starting time for the project

ES → relative to the project starting time.

EF → Earliest Finish time.

ES_i denotes the earliest start time of all the activities emanating from event i; and t_{ij} is the estimated time of activity (i, j) then

$$EF_i \text{ or } ES_j = \max \{ ES_i + t_{ij} \}$$

$ES_i = 0$ being the earliest start time.

(b) Let us suppose that we have a target time for completing the project.

LF – Latest Finish time for the final activity.

LS – Latest Start time is the latest time at which an activity can start if the target is to be maintained.

LS = LF – activity time

$LF_i = \min \{ LF_j - t_{ij} \}$ for all defined (i, j) activities.

Critical Path Calculation:

An Activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.

* non-critical activity is such that the time between its ES and LF is longer than its actual duration.

* non-critical activity is said to have a slack or float time.

* Slack - difference between the latest Finish and earliest Start time.

For the activity (i, j) to lie on the critical path following conditions must be satisfied;

(a) $ES_i = LF_i$

(b) $ES_j = LF_j$

(c) $ES_j - ES_i = LF_j - LF_i = t_{ij}$

Float or Slack values:

There are many activities where the maximum time available to finish the activity is more than the time required to complete it (i.e) its duration. The difference between the two is known as Total float available for the activity.

Three types of activity,

(a) Total Float:

(a) Determine the difference between earliest start time of tail event and the latest finish time of head for the activity.

(b) Subtract the duration time of the activity from the value obtained.

$$TF_{ij} = LF_j - ES_i - t_{ij}.$$

(ii) Free Float:

It is defined by assuming that all the activities start as early as possible. The free float for the activity (i, j) is the excess available time over its duration.

$$FF_{ij} = ES_j - ES_i - t_{ij}.$$

(iii) Interference float:

The difference between total float and free float is known as interference float.

(iv) Independent float:

The Time by which an activity can be rescheduled without affecting the preceding or the succeeding activities is known as Independent float.

IF = Free float − Tail Event Slack.

Critical Path Method:

Steps:

1. List all the jobs and then draw a network diagram. Each Job is Indicated by an arrow with the direction of the arrow showing the Sequence of the Jobs.
2. Indicate the Job's times above the arrow representing the task.
3. Calculate the earliest Start time and earliest finish time for each event and calculate the latest Start time and Latest finish time.
4. Tabulate various time (t, ES, LF)
5. Determine the total float (slack) by taking differences between ES and LF.
6. Identify the critical activities and connect them with the beginning node and the ending node in the network diagram double line arrows. This gives the critical path.
7. Calculate the total project duration.

Example:

A project consists of a series of activities / tasks labelled A, B, ... H, I with the following relationships (w < x, y means x and y cannot start until w is completed ; x, y < w means w cannot start until both x and y are completed) with this notation construct the network diagram having the following constraints ;

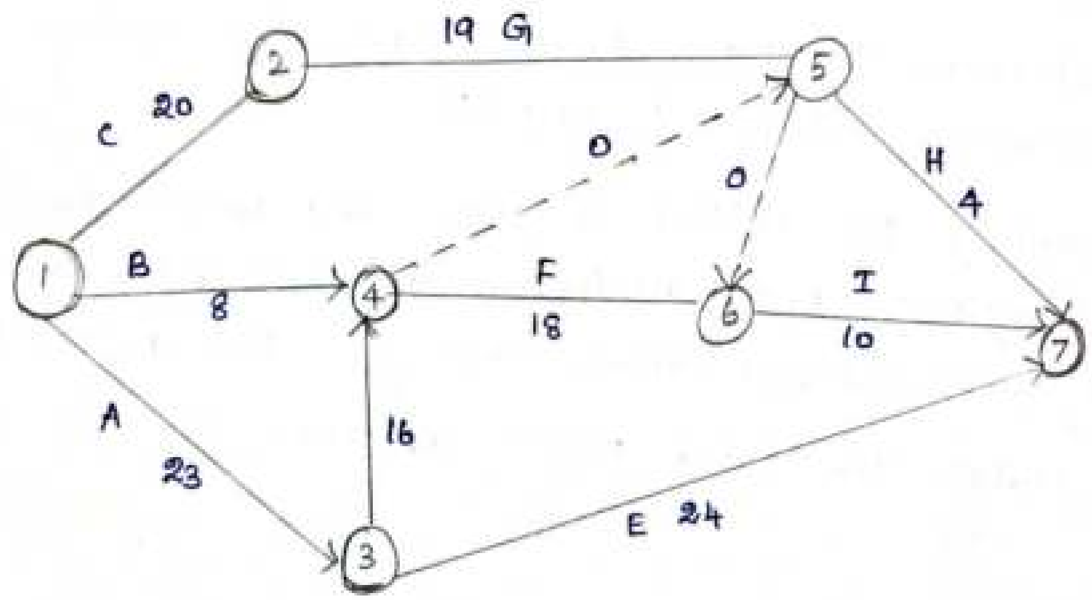
A < D, E ; B, D < F ; C < G ; B, G < H ; F, G < I

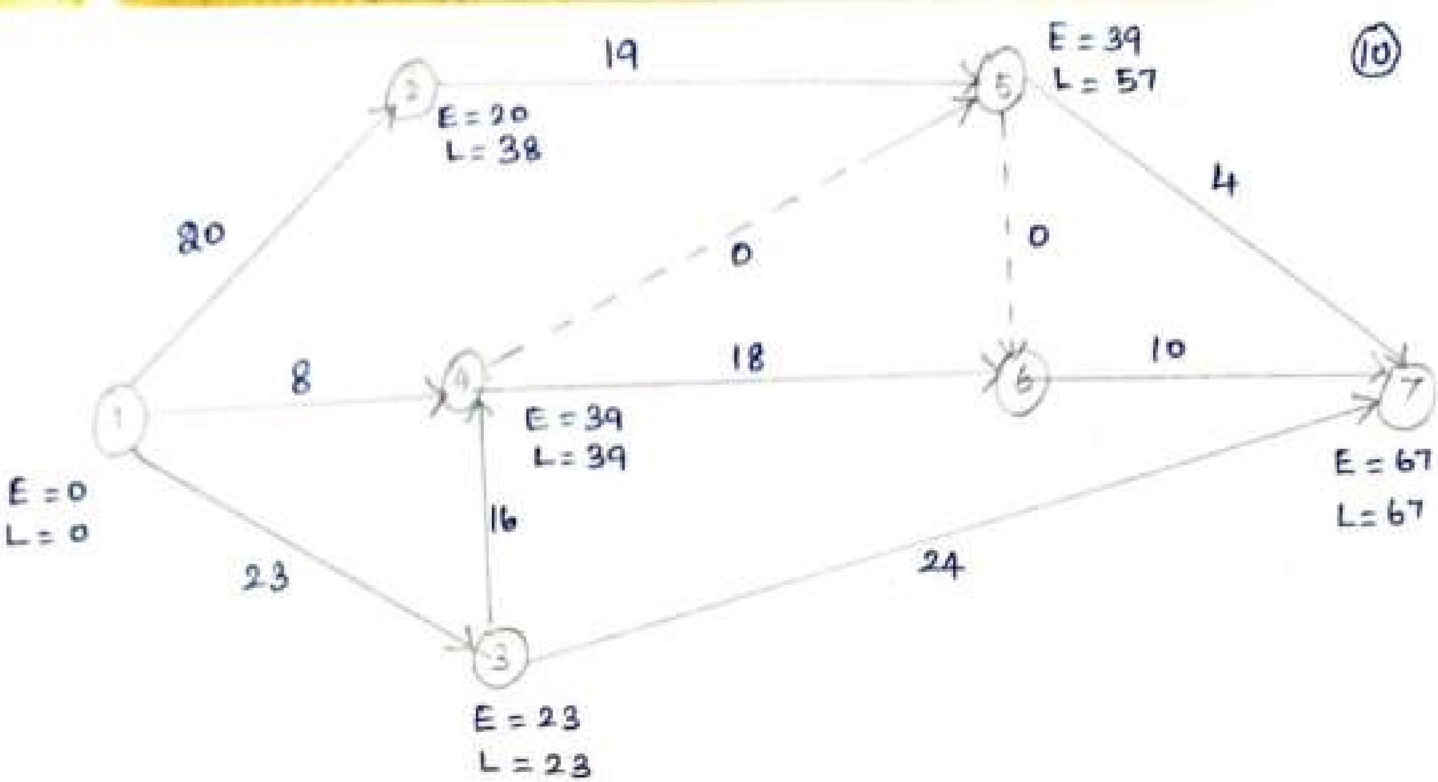
Find also the minimum time of completion of the project, when the time (in days) of completion of each task as follows ;

Task :	A	B	C	D	E	F	G	H	I
Time :	23	8	20	16	24	18	19	4	10

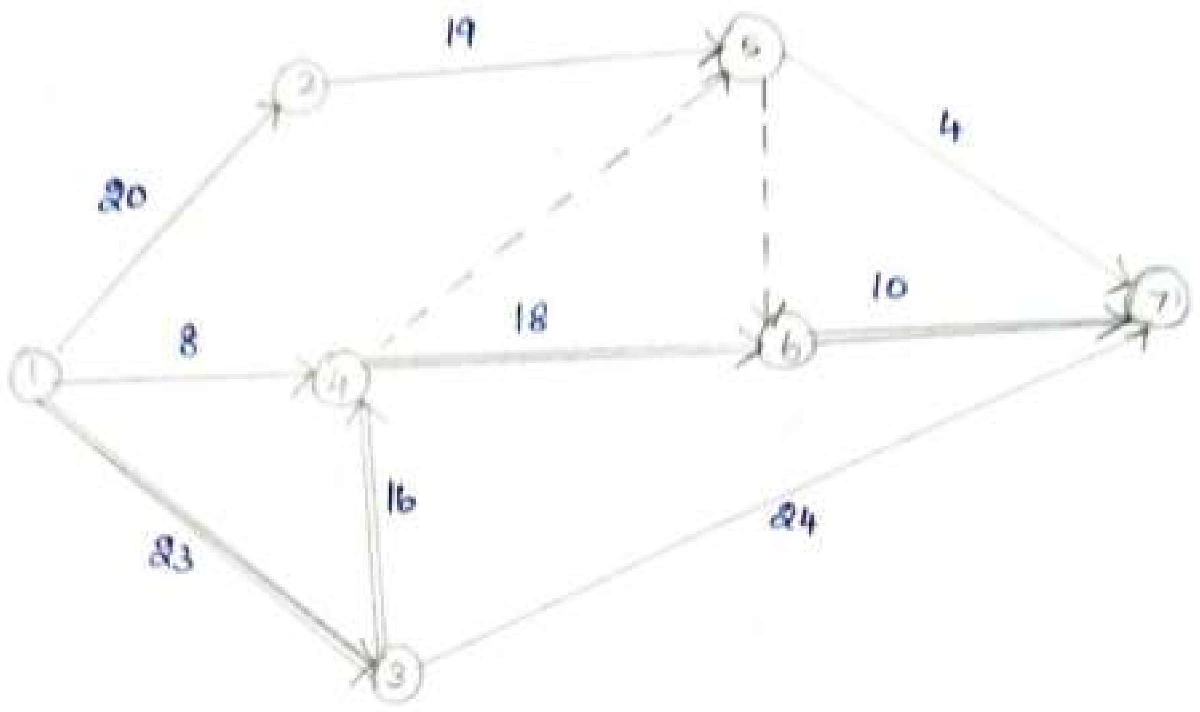
Solution:

Introduce 2 more dummy activities to establish correct precedence relationships.





Task i	j	Normal time tij	Earliest time Start ESi	Finish EFj	Latest Time Start LSi	Finish LFj	FF	TF
1	2	20	0	20	18	38	0	18
1	3	23	0	23	0	23	0	0
1	4	8	0	8	31	39	31	31
2	5	19	20	39	38	57	0	18
3	4	16	23	39	23	39	0	0
3	7	24	23	47	43	67	20	20
4	5	0	39	39	57	57	0	18
4	6	18	39	57	39	57	0	0
5	6	0	39	39	57	57	18	18
5	7	4	39	43	63	67	24	24
6	7	10	57	67	57	67	0	0



critical path = 1 → 3 → 4 → 6 → 7

Total Project Duration:

= 23 + 16 + 18 + 10

= 67 days.

PERT

Deterministic network methods assume that the expected time is the actual time take. Probabilistic methods, on the other hand, assume the reverse, more realistic situation, where activity times are represented by a probability distribution.

These are as follows;

t_o = Optimistic time

The possible time to complete the activity if all goes well.

t_p = pessimistic time

The longest time that an activity could take it everything goes wrong.

(iii) t_m = most likely time.

The estimate time of the normal time an activity would take

$$t_e = \frac{1}{3}\left[2t_m + \frac{1}{2}(t_o + t_o)\right]$$

expected time

$$t_e = \frac{1}{6}\left[4t_m + t_o + t_p\right]$$

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

PERT Same as CPM. The main difference is that instead of activity duration, we use expected time for the activity. with Each event we also associate the variance. Thus the variance of the project is the expected time.

PERT Algorithm:

1. Make a list of activities that make up the project including immediate predecessors.

2. Making use of step 1 sketch the required network.

3. Denote the most likely time by t_m the optimistic time by t_o and Pessimistic time by t_p.

4. Using beta distribution for the activity duration, the expected time t_e, for each activity is computed by using the formula,

$$t_e = (t_p + 4t_m + t_o)/6$$

5. Tabulate various times (ie) expected activity times, earliest and latest time and mark the EST and LFT on the arrow diagram.

6. Determine the total float for each activity by taking the difference between EST and LFT.

7. Identify the critical activities and connect them with the beginning node and the ending node, in the network diagram by double line arrows. This gives the critical path and the expected date of completion of the project.

8. Using the values of t_p and t_o Compute the variance (σ^2) of each activity's time estimate by using the formula,

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

9. Compute the standard normal deviate.

$$z_o = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

10. Use Standard normal tables to find the probability $P(z \le z_o)$ of completing the project with in the Scheduled time where $z \sim N(0,1)$.

Example :

A Small project is composed of Seven activities whose time estimates are listed in the table as follows,

Activity i	j	Estimated duration (weeks) Optimistic	Most likely	Pessimistic
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

a) Draw the Project work

b) Find the expected duration and variance of each activity.

c) Calculate early and late occurence times for each event. what is the expected project length.

d) Calculate the variance and Standard deviation of project length, what is the probability that the project will be completed.

(i) at least 4 weeks earlier than expected?

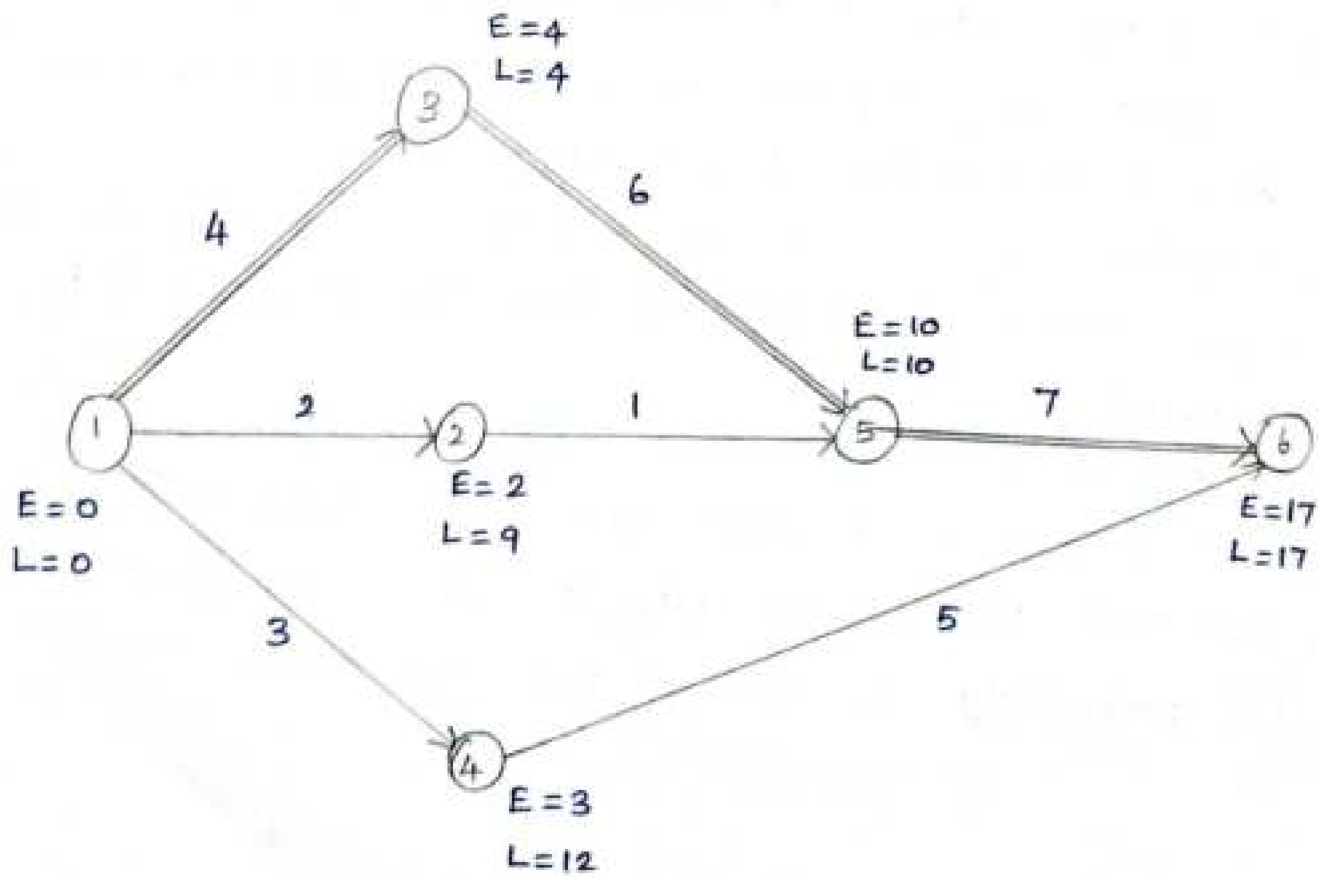
(ii) no more than 4 weeks later than expected?

e) If the project due date is 19 weeks, what is the probability of meeting the due date?

Solution:

The expected time and variance of each activity is computed in table below;

Activity $i-j$	t_o a	t_m m	t_p b	$t_e = \frac{a+4m+b}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 - 2	1	1	7	2	1
1 - 3	1	4	7	4	1
1 - 4	2	2	8	3	1
2 - 5	1	1	1	1	0
3 - 5	2	5	14	6	4
4 - 6	2	5	8	5	1
5 - 6	3	6	15	7	4



Activity i - j	t_{ij}	ES_i	ES_j	LS_i	LF_j	TT $LF_j - ES_i - T_{ij}$
1 - 2	2	0	2	7	9	7
1 - 3	4	0	4	0	4	0 ✓
1 - 4	3	0	3	9	11	8
2 - 5	1	2	3	9	10	7
3 - 5	6	4	10	4	10	0 ✓
4 - 6	5	3	8	12	17	9
5 - 6	7	10	17	10	17	0 ✓

Critical path = $1 \to 3 \to 5 \to 6$

Total Project Duration $= 4+6+7 = 17$ weeks

Variance of the project length is ;

$$\sigma^2 = 1+4+4 = 9$$

Standard Normal Deviate:

$$z = \frac{\text{Due date} - \text{Expected date of Completion}}{\sqrt{\text{Variance}}}$$

(i) $z = \frac{13-17}{3} = \frac{-4}{3} = -1.33$

$P(z < 1.33) = 0.5 - \phi(1.33)$

$= 0.5 - 0.4082$

$= 0.0918 \simeq 9.18\%$

The interpretation of the above is that if the project is performed 100 times under the same conditions, there will be 9 occasions when this job will be completed 4 weeks earlier than expected.

(ii) $z = \frac{21-17}{3} = \frac{4}{3} = 1.33$

The probability of meeting the due date

$P(z \leq 1.33) = 0.5 + \phi\, 1.33$

$= 0.5 + 0.4082$

$= 0.9082$

$= 90.82\%$

(e) when the due date is 19 weeks.

$z = \frac{19-17}{3} = \frac{2}{3} = 0.67$

$= 0.5 + \phi\,(0.67)$

$= 0.5 + 0.2486$

$= 0.7486$

$= 74.86\,\%$

Thus the probability of not meeting due date is $1 - 0.7486$ (ie) 0.2514 or $25.14\,\%$