SEMESTER : I

CORE COURSE : II

Inst Hour	: 6
Credit	: 5
Code	: 18KP1M02

REAL ANALYSIS

UNIT-I

Basic Topology: Finite, Countable and Uncountable sets – Metric spaces, Compact sets, Perfect sets, Connected sets.

Chapter 2 of Text Book 1

UNIT-II

Numerical Sequences and Series: Convergent Sequences – Subsequences – Cauchy Sequences Upper and Lower Limits – Some Special sequences – Series: Series of Non-negative Terms The Number e – The Root and Ratio Tests – Power Series – Summation by Parts – Absolute convergence – Addition and Multiplication of Series – Rearrangements.

Chapter 3 of Text Book 1

UNIT-III

Differentiation : The Derivative of a Real function – Mean Value Theorems – The Continuity of Derivatives – L'Hospital's Rule – Derivatives of Higher Order – Taylor's Theorem – Differentiation of Vector valued functions.

Chapter 5 of Text Book 1

UNIT-IV

Riemann Stieltje's Integral: Notation and Definition – Linear properties – Integration by Parts – Change of Variable – Reduction – Step functions as Integrators – Reduction to a finite sum – Euler's Summation Formula – Monotonically Increasing Integrators – Additive and Linearity properties of Upper and Lower Integrals – Riemann's condition – Comparison Theorem – Integrators of Bounded Variation – Sufficient and Necessary conditions for existence – Mean Value Theorem – Integral as a function of the interval – Second fundamental theorem of Integral Calculus – Change of variable – Second Mean Value Theorem – Riemann – Stieltje's Integrals depending on a parameter – Differentiation under the Integral sign – Interchanging the order of Integration – Lebesgue's criterion for existence – Complex-valued Riemann Stieltje's Integrals.

Chapter 7 of Text Book 2

UNIT-V

Functions of Several Variables: Linear Transformations – Differentiation – The Contraction Principle – The Inverse Function Theorem – The Implicit Function Theorem – The Rank Theorem – Determinants – Derivatives of Higher Order – Differentiation of Integrals.

Chapter 9 of Text Book 1

TEXT BOOK

1. W.Rudin, Principles of mathematical Analysis, IIIEd.,1976, McGrawHillBookCo
2. Apostol, Mathematical Analysis, Narosa Publishing House, New Delhi, 2nd Edition

REFERENCES

1.A.J. White, Real Analysis: An Introduction, Addison Wesley Publishing Co., Inc 1968.

2.Tom.M.Apostal Mathematical Analysis – II, Edition Narosa Publishing House – 1974.

3. Rokert G.Bartle, Donal.R.Shelbert, Introduction to Real Analysis

4. Ajith Kumar, S.Kumaresan, A Basic Course in Real Analysis

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.

Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit

Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

18KP1M02

REAL ANALYSIS

UNIT - I

Metric Spaces

Defn A set X, whose elements, we shall call points, is said to be a metric space. If with any two points p and q of X there is associated a real number $d(p,q)$, called the distance from p to q such that

(a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$

(b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$

Any function with these three properties is called a distance function or a metric.

Example The Euclidean spaces R^k especially R^1 (the real line) and R^2 (the complex plane) the distance in R^k is defined by $d(x,y) = |x-y|$ $(x, y \in R^k)$

Defn: By the segment (a,b) we mean the set of all real numbers x such that $a < x < b$. (open interval)

By the interval $[a,b]$ we mean the set of all real numbers x such that $a \leq x \leq b$. (closed interval)

Half open intervals The half open intervals $[a,b)$ and $(a,b]$; the first consists of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

K-cell If $a_i < b_i$ for $i = 1, 2 \ldots K$ the set of all points $x = (x_1, x_2 \ldots x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a k-cell

Thus a 1-cell is an interval

a 2-cell is a rectangle.

<u>Open (or) closed ball</u>: If $x \in R^k$ and $r > 0$, the open (or closed) ball B with centre at x and radius r is defined to be the set of all $y \in R^k$ such that $|y-x| < r$ (or $|y-x| \leq r$)

<u>Convex</u>: We call a set $E \subset R^k$ convex if

$$\lambda x + (1-\lambda) y \in E$$

whenever $x \in E$, $y \in E$ and $0 < \lambda < 1$

<u>Example</u>: Balls are convex

For if $|y-x| < r$, $|z-x| < r$ and $0 < \lambda < 1$ we have

$$\begin{aligned} |\lambda y + (1-\lambda) z - x| &= |\lambda y + \lambda x - \lambda x + (1-\lambda) z - x| \\ &= |\lambda (y-x) + (1-\lambda)(z-x)| \\ &\leq \lambda |y-x| + (1-\lambda) |z-x| \\ &< \lambda r + (1-\lambda) r \\ &= r \end{aligned}$$

$\therefore$ the balls are convex.

<u>Neighbourhood point</u>: A neighbourhood of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$ for some $r > 0$, The number r is called the radius of $N_r(p)$

<u>Limit point</u>: A point p is a limit point of the set E if every neighbourhood of p contains a point $q \neq p$ such that $q \in E$.

<u>Isolated point</u>: If $p \in E$ and p is not a limit of E, then p is called an isolated point of E

E is closed if every limit point of E is a point of E

Interior Point: A point is an interior point of E if there is a neighbourhood N of P such that $N \subset E$

E is open if every point of E is an interior point of E.

The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

E is perfect if E is closed and if every point of E is a limit point

E is bounded if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$

E is dense in X if every point of X is a limit point of E, or a point of E. (or both)

Theorem: Every neighborhood is an open set.

Solution: Consider a neighborhood $E = N_r(P)$. and let q be any point of E. Then there is a positive real number h such that.

$$d(p,q) = r - h.$$

For all points s such that $d(q,s) < h$ we have

$$d(p,s) \leq d(p,q) + d(q,s)$$
$$< r - h + h = r$$

So that $s \in E$. Thus q is an interior point of E.

Theorem: If p is a limit point of a set E then every neighborhood of p contains infinitely many points of E

Proof: Given q is an limit point of the set E suppose there is a neighborhood N of p which

$2a \tan x$ $\leq$ $\int_0^1 \tan x$

$x \in (\bigcup_\alpha E_\alpha)^c$

contains only a finite number of points of E.

i.e) let $q_1 q_2 \dots q_n$ be finite number of points.

i.e $N \cap E \{q_1, q_2 \dots q_n\}$

These finite number of points are distinct from p and put,

$r = \min_n d(p, q_m)$, $1 \leq m \leq n$.

[we use this notation to denote the smallest of numbers]

$d(p, q) \dots d(p, q_n)$

The minimum of a finite set of +ve numbers is clearly +ve. so that $r > 0$.

the neighborhood $N_r(p)$ contains no points q of E ~~st~~ $q \neq p$.

p is not a limit point of E.

There is a contradiction. to the fact that p is a limit point of the set E.

Hence Every neighborhood of p contains infinitely many points of E.

Hence the proof.

<u>Theorem</u>: Let $\{E_\alpha\}$ be a finite or infinite collection of sets E_α then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha)^c$

<u>proof</u>: Let x be any element of $(\bigcup_\alpha E_\alpha)^c$

i.e, $x \in (\bigcup_\alpha E_\alpha)^c$ for every α

$\Rightarrow x \notin \bigcup_\alpha E_\alpha$

$x \in E_\alpha$ for every α

$x \in E_\alpha^c$ $\quad x \in \bigcap E_\alpha^c$

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$x \in (\bigcup_\alpha E_\alpha)^c$

$x \in \bigcap_\alpha E_\alpha^c$

$(\bigcup_\alpha E_\alpha)^c \subset \bigcap E_\alpha^c$ — (1)

Let y be any element of $\bigcap_\alpha E_\alpha^c$

i, $y \in \bigcap_\alpha E_\alpha^c$

then $y \in E_\alpha^c$ for every α

$y \notin E_\alpha$ for every α

$y \notin \bigcup_\alpha E_\alpha$

$y \in (\bigcup_\alpha E_\alpha)^c$

$\therefore y \in \bigcap_\alpha E_\alpha^c \Rightarrow y \in (\bigcup_\alpha E_\alpha)^c$

$\bigcap_\alpha E_\alpha^c \subset (\bigcup_\alpha E_\alpha)^c$ — (2)

from (1) & (2)

Hence $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha E_\alpha^c$

Theorem: A set E is open iff its complement is closed.

proof: First suppose that E^c is closed.

choose $x \in E$, then $x \notin E^c$

$\Rightarrow$ x is not a limit point of E^c

Hence there exists atleast one neighborhood say $N(x)$ such that ~~$E^c \cap N(x)$~~ $E^c \cap N(x)$ is empty.

i, $N(x) \subset E$

x is an interior point of E for every $x \in E$.

Hence E is open

Conversely

suppose E is open

Let x be any limit point of E^c

Then every neighborhood of x contains atleast one a point of E^c

so that x is not an interior point of E.
since E is open this means that $x \in E^c$
It follows that E^c is closed.

~~Theorem~~ corrollary: A set E is closed iff its complement is open.

2. Theorem:

a. For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open

b. For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed

c. For any finite collection $G_1, \dots G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open.

d. For any finite collection $F_1 \dots F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed.

proof:

a. For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open.

put $G = \bigcup_\alpha G_\alpha$

if $x \in G$, then $x \in G_\alpha$ for some α

since x is an interior point of G_α,

x is also an interior point of G, and G is open.

$\therefore \bigcup_\alpha G_\alpha$ is open.

b. For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

F_α is closed set for every α

F_α^c is an open set for every α

$\bigcup_\alpha F_\alpha^c$ is an open set for every α

$\bigcup_\alpha F_\alpha^c = \left(\bigcap_\alpha F_\alpha\right)^c$ is an open set

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$\bigcap_{\alpha} F_\alpha$ is a closed set.

c. For any finite collection $G_1, \ldots G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open.

$G_1, G_2, \ldots G_n$ are open sets.

we have to prove that $\bigcap_{i=1}^{n} G_i$ is open

Suppose $A = \bigcap_{i=1}^{n} G_i$

Let $x \in A \Rightarrow x \in \bigcap_{i=1}^{n} G_i$

$x \in G_i, \; i = 1, 2 \ldots n$

G_i is an open set for every i

x is an interior point of G_i $i = 1, 2 \ldots n$.

$\therefore$ There exists a neighborhood N_i of x with radii r_i $i = 1, 2 \ldots n$

Such that $N_i \subset G_i$ $i = 1, 2 \ldots n$.

Let $r = \min \{ r_1 r_2 \ldots r_n \}$

Let N be the neighborhood of x radius r.

$N \subset G_i$ for $i = 1, 2 \ldots n$

$N \subset H$ for every $x \in H$

H is an open set.

$\bigcap_{i=1}^{n} G_i$ is open.

(d) For any collection $F_1 F_2 \ldots F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed.

$F_1 F_2 \ldots F_n$ are closed sets To prove that $\bigcup_{i=1}^{n} F_i$ is closed

F_i is closed for $i = 1, 2 \ldots n$

F_i^c is open, for $i = 1, 2 \ldots n$.

$\bigcap_{i=1}^{n} F_i^c$ is open.

But $\bigcap_{i=1}^{n} F_i^c = \left(\bigcup_{i=1}^{n} F_i \right)^c$

$\left(\bigcup_{i=1}^{n} F_i \right)^c$ is open.

$\bigcup_{i=1}^{n} F_i$ is closed.

Defn: If X is a metric space. if $E \subset X$ and if E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$

Theorem: If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed.

(b) $E = \bar{E}$ if and only if, E is closed

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

Proof: Let $p \in X$ and $p \notin \bar{E}$

$\Rightarrow$ ~~$p \notin E$~~ $p \notin \bar{E} \Rightarrow p \in X$

$\therefore p$ is neither a point of E nor a limit of E.

$\therefore p$ has neighborhood which does not intersect E.

$\therefore p$ has neighborhood which does not intersect $\bar{E}$ also.

$\therefore$ It does not intersect $\bar{E}$ also.

$\therefore$ The complement of $\bar{E}$ is open

i.e., $\bar{E}^c$ is open

$\Rightarrow \bar{E}$ is closed.

(b) If $E = \bar{E}$ then E is closed, since $\bar{E}$ is closed.

Assume that E is closed.

prove that $E = \bar{E}$

If P is any limit point of E then $P \in E$.

i.e., $P \in E' \Rightarrow P \in E$

$\Rightarrow E' \subset E$

$\bar{E} = E \cup E' = E$

Assume that $E = \bar{E}$

$E \cup E' = E \Rightarrow E' \subset E$

$P \in E' \Rightarrow P \in E$ for every $P \in E'$ $\therefore E$ is closed

(c) Given $F \subset X$, $E \subset F$, F is closed.

F is closed $\Rightarrow F' \subset F$

$E \subset F$, $E' \subset F' \subset F$

$F \subset F$ and $E' \subset F$

$\Rightarrow E \cup E' \subset F$

$\bar{E} \subset F$.

Hence the proof.

Theorem: Closed subset of compact sets are compact.

Proof: Suppose $F \subset K \subset X$

F is closed relative to X

K is compact.

Let $\{V_\alpha\}$ be an open cover of F.

F is closed.

F^c is an open subset of X

Let $\Omega = \{V_\alpha\} \cup F^c$

Then Ω is an open cover of K.

There is a finite subcover ϕ of Ω that covers K.

It covers F also.

If F^c is a member of ϕ remove F^c from ϕ.

i, $\phi - F$ is also an open cover of F.

A finite subcollection of $\{V_\alpha\}$ covers F

$\therefore F$ is compact.

Theorem. Every K-cell is compact.

Proof: Let I be a K-cell consisting of all points $x = (x_1, x_2, \ldots, x_k)$ such that $a_j \le x_j \le b_j$ $(1 \le j \le k)$

put $\delta = \left\{ \sum_{1}^{k} (b_j - a_j)^2 \right\}^{1/2}$

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then $|x-y| \leq \delta$. if $x \in I$, $y \in I$.

Suppose I is not compact. then $\exists$ an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I.

put $c_j = \frac{a_j+b_j}{2}$

The intervals $[a_j, c_j]$ $[c_j, b_j]$ then determine 2^K

K cell Q_i whose union is I

i.e, $\cup Q_i$

Atleast one of the Q_i, say $I_1 = \{\cup Q_i\}$ cannot be covered by any finite subcollection of $\{G_\alpha\}$

We next subdivided I and continue the process we obtain a sequences $\{I_n\}$ with the following properties.

(a) $I \supset I_1 \supset I_2 \ldots$

I is compact.

Also we know that "Every closed subset of a compact set is compact".

$E \subset I$, E is compact.

(b) $\Rightarrow$ (c) Assume E is compact.

If E is infinite subset of a compact set K, then E has a limit point in K.

proof: Assume that no point of K were a limit point of E. then each $q \in K$ will have a neighborhoo V_q which contains atmost one point of E.

i.e, of $q \in E$.

E is an infinite set.

$\{V_q\}$ covers E and $\{V_q\}$ covers K.

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No finite subcollection of $\{V_\alpha\}$ can cover K.

But this is not true, because K is compact.

$\therefore$ the consumption : "No point of K where a limit point of E" is false

E has a limit point in K.

Here E is compact - Replacing K by E in this proof we get,

"Every infinite subset of E has a limit point in E"

(c) $\Rightarrow$ (a)

Assume that every infinite subset of E has a limit point in E.

TPT E is closed and bounded

Assume that E is not bounded

$\exists$ points x_n in E $\ni$ $|x_n| > n$ $n = 1, 2 \ldots$

Let $S = \{x_n : |x_n| > n \quad n = 1, 2 \ldots\}$

S is an infinite subset of E, S has no limit.

This is contradiction to an assumption that

"Every infinite subset of E has a limit point in E"

E is not bounded is false.

$\therefore$ E is bounded.

TPT E is closed.

Assume that E is not closed

There is a point $x_0 \in R^k$ which is a limit point of E but not a point of E for $n = 1, 2 \ldots$

$\exists$ a point $x_n \in E \Rightarrow |x_n - x_0| < 1/n$

Let $S = \{x_n \in E ;\ |x_n - x_0| < 1/n \quad n = 1, 2 \ldots n\}$

S

S is an finite subset of E

x_0 is the limit point of S

To prove that

S has no other limit point in R^k

Let y be another limit point in R^k

$$|x_n - y| = |x_n - x_0 + x_0 - y|$$
$$= |(x_0 - y) - (x_0 - x_n)|$$
$$\geq |x_0 - y| - |x_0 - x_n|$$
$$\geq |x_0 - y| - 1/n$$
$$\geq 1/2 |x_0 - y|$$

$|x_n - y| \geq 1/2 |x_0 - y|$ for all points but finitly many n.

$\therefore$ y is not a limit of S

$\therefore$ x_0 is the only limit point of S which is not in E

But our assumption is, "Every infinite subset of E has a limit point in E"

$\therefore$ E is not closed is false.

$\therefore$ E is closed set.

WEIERSTRASS THEOREM

Every bounded infinite subset of R^k has a limit point in R^k.

Proof: Let E be a bounded infinite subset of R^k

$\therefore$ E is a subset of some k-cell I of R^k

UNIT - IV

Sequences and Series of Functions

Theorem: If γ' is continuous on $[a,b]$ then γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$

Proof: If $a_i \le x_{i-1} \le x_0 \le b_i$ then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\,dt\right|$$

$$\le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt.$$

Hence $\Lambda(P,\gamma) \le \int_a^b |\gamma'(t)|\,dt.$

for every partition P of $[a,b]$

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\,dt \quad \text{--- (1).}$$

To prove that,

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Let $\epsilon > 0$ be given,

since γ' is uniformly continuous on $[a,b]$, $\exists\ \delta > 0$,

$$|\gamma'(s) - \gamma'(t)| < \epsilon \text{ if } |s-t| < \delta.$$

Let $P = \{x_0\, x_1 \ldots x_n\}$ be a partition of $[a,b]$ with $\Delta x_i < \delta\ \forall i$.

If $x_{i-1} \le t \le x_i$ then

$$|\gamma'(t)| \le |\gamma'(x_i)| + \epsilon.$$

Hence $\displaystyle\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le |\gamma'(x_i)|\Delta x_i + \epsilon\Delta x_i$

$$= \left|\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt\right| + \int_{x_{i-1}}^{x_i} |\gamma'(x_i) - \gamma'(t)|\,dt + \epsilon\Delta x_i$$

$$\le \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\,dt\right| + \left|\int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]\,dt\right| + \epsilon\Delta x_i$$

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$\leq |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon \Delta x_i$

If we add all such inequalities we obtain

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(P,\gamma) + 2\epsilon(b-a)$$
$$\leq \Lambda(\gamma) + 2\epsilon(b-a)$$

since ϵ is arbitrary

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(\gamma)$$

ii) $\Lambda(\gamma) \geq \int_a^b |\gamma'(t)|\,dt$ — ②.

① & ②. $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

<u>Defn</u>: Suppose $\{s_n\}_{n=1,2..}$ is a sequence of function defined on a set E and suppose that the sequence of number $\{f_n(x)\}$ converges for every $x \in E$.

<u>Defn</u>: $f(x) = \lim_{n\to\infty} f_n(x) \quad x \in E$ — ①. then $\{f_n\}$ converges on E and f is the limit or the limit function of $\{f_n\}$ some times we say "$\{f_n\}$ converges to f pointwise on E" if ① holds.

<u>Example:1</u> for $m = 1, 2 \ldots$ $n = 1, 2 \ldots$

$$s_{m,n} = \frac{m}{m+n}$$

for every fixed n,

$$s_{m,n} = \frac{1}{1+n/m}$$

$$s_{m,n} = \lim_{m\to\infty} \frac{1}{1+n/m} = 1$$

so that $\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = 1$

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on the other hand, for every fixed m.

$$S_{m,n} = \frac{m/n}{m+n/n} = \frac{m/n}{m/n + 1}$$

$$\lim_{n\to\infty} S_{m,n} = \lim_{n\to\infty} \frac{m/n}{m/n+1} = 0$$

so that $\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = 0$

__Example:__ Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ x real $n = 0, 1, \dots$

__Proof:__ Consider $f(x) = \sum_{n=0}^{\infty} f_n(x)$

$$= \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

this is an Geometric Series

$$\frac{1}{1-x} = \frac{1}{1-x^2/1+x^2}$$

$$\sum_{n=0}^{\infty} x_n = \frac{1}{\frac{1+x^2-x^2}{1+x^2}} = 1+x^2$$

since $f_n(0) = 0$ we have $f(0) = 0$ for $x \neq 0$ the series is a convergent geometric series with sum $1+x^2$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1+x^2 & x \neq 0 \end{cases}$$

so that a Convergent Series of continuous function may have a discontinuous f.

__Example:__ Let $f_n(x) = n^2 x (1-x^2)^n$ — (1) $0 \leq x \leq 1$ $n = 1, 2 \dots$

for $0 \leq x \leq 1$ we have

$$\lim_{n\to\infty} f_n(x) = 0$$

(by if $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$)

since $f_n(0) = 0$ we see that.

$\lim_{n\to\infty} f_n(x) = 0 \quad 0 \le x < 1$

we calculate

$$\int_0^1 x(1-x^2)^n dx = \frac{r}{\cancel{}} = \frac{1}{2n+2}$$

Thus inspite of ②.

$$\int_0^1 f_n(x)\,dx = \frac{n^2}{2n+2} \to \infty \text{ as } n \to \infty.$$

If in ① we replace n^2 by n. ② still holds :

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2n+2} = 1/2$$

where as from ②.

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx = 0.$$

Thus the limit of the integral need not be equal to be integral of the limit even if both are finite.

Uniform Convergence.

Defn: A sequence of functions $\{f_n\}$ $n=1,2\ldots$ converges uniformly on E to a function f if for every $\epsilon>0$, there is an integer $N \ni n \ge N \Rightarrow |f_n(x) - f(x)| \le \epsilon \ \forall\ x \in E$.

Note: Every uniformly convergent sequence is pointwise convergent.

Theorem: The sequence of functions $\{f_n\}$ defined on E converges uniformly on E iff for every $\epsilon>0$, $\exists$ an integer $N \ni m \ge N$, $n \ge N$, $x \in E \Rightarrow |f_n(x) - f_m(x)| \le \epsilon$

Proof: Suppose $\{f_n\}$ converges uniformly on E and let f be the limit function, then there is an integer

$N \ni n \geq N, x \in E \Rightarrow |f_n(x) - f(x)| < \epsilon/2$

$$|f_n(x) - f_m(x)| = |f_n(x) + f(x) - f(x) - f_m(x)|$$
$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$
$$< \epsilon/2 + \epsilon/2$$
$$\leq \epsilon$$

if $n \geq N, m \geq N, x \in E$.

conversly, suppose the cauchy condition holds by a thm.

If X is a compact metric space and if $\{p_n\}$ is cauchy sequence in X then $\{p_n\}$ converges to some point of X

The sequence $\{f_n(x)\}$ converges for every x, to a limit, which we may call $f(x)$. Thus the sequence $\{f_n\}$ convergence on E to f.

we have to PT the convergence is uniform.

Let $\epsilon > 0$, be given and choose $N \ni |f_n(x) - f_m(x)| \leq \epsilon$ $\rightarrow$ (a)

fix n, and let $m \to \infty$ in (a)

since $f_m(x) \to f(a)$ as $m \to \infty$ this gives

$$|f_n(x) - f(x)| \leq \epsilon.$$

for every $n \geq N$ and every $x \in E$,

Hence the proof.

<u>Theorem</u>: suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that $\lim_{t \to x} f_n(t) = A_n$ $(n = 1, 2, \ldots)$

then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$ $\longrightarrow$ (1)

In otherwords the conclusion is that

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) \longrightarrow (2)$$

Proof: Let $\epsilon > 0$, be given By the uniform convergence $\{f_n\}$ there exists such that $n \geq N$, $m \geq N$, $t \in E$ imply

$$|f_n(t) - f_m(t)| \leq \epsilon \quad \text{—} (2).$$

Letting $t \to x$ in (2) we obtain

$$\left| \lim_{t \to x} f_n(t) - \lim_{t \to x} f_m(t) \right| \leq \epsilon$$

$$|A_n - A_m| \leq \epsilon.$$

for $n \geq N$, $m \geq N$ so that $\{A_n\}$ is a cauchy sequence and therefore converges say to A next,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \quad \text{—} (5)$$

we choose n such that

$$|f(t) - f_n(t)| < \epsilon/3 \quad \text{—} (6).$$

for all $t \in E$. [This is possible by the uniform convergence] & such that,

$$|A_n - A| \leq \epsilon/3 \quad \text{—} (7).$$

Then for this n, we choose a neighbourhood V of x, such that $|f_n(t) - A_n| < \epsilon/3$ if $t \in V \cap E$, $t \neq x$

Sub the inequalities (7), (6), (5) into (4). we get,

$$|f(t) - A| \leq \epsilon$$

provided $t \in V \cap E$, $t \neq x$.

This is equivalent to $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$

<u>Theorem</u> Let α be monotonically increasing on $[a,b]$ suppose $f_n \in R(\alpha)$ on $[a,b]$ for $n = 1, 2 \ldots$ and suppose $f_n \to f$ uniformly on $[a,b]$ Then $f \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$$

Proof: It sufficies to prove this for real f_n.

Put $\epsilon_n = \sup |f_n(x) - f(x)|$

The supremum being taken over, $a \le x \le b$. Then

$f_n - \epsilon_n \le f \le f_n + \epsilon_n$

so that the upper lower integrals of f satisfy

$$\int_a^b (f_n - \epsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le \int_a^b (f_n + \epsilon_n)\,d\alpha.$$

$$\le \int_a^b (f_n + \epsilon_n - f_n + \epsilon_n)\,d\alpha$$

$$\le 2\epsilon_n \int_a^b d\alpha.$$

Hence $0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \le 2\epsilon_n [\alpha(b) - \alpha(a)]$

Since $\epsilon_n \to 0$ as $n \to \infty$

The upper and lower integrals of f are equal thus $f \in R(\alpha)$. Another application of (1).

$$\left| \int_a^b f\,d\alpha - \int_a^b f_n\,d\alpha \right| \le \epsilon_n (\alpha(b) - \alpha(a))$$

$$\int_a^b f\,d\alpha = \lim_{n \to \infty} \int_a^b f_n\,d\alpha.$$

Uniform Convergence and Differentiation

Theorem:

Suppose $\{f_n\}$ is a sequence of f_n differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$ to a f^n f. Then $\{f_n\}$ converges uniformly on $[a,b]$

$f'(x) = \lim_{n \to \infty} f_n'(x)$ — (1)

Proof: Let $\epsilon > 0$ be given choose N such that $n \ge N$, $m \ge N$ implies,

$|f_n(x_0) - f_m(x_0)| \le \epsilon/2$ and — (2).

$|f_n'(t) - f_m'(t)| < \epsilon/2(b-a)$ $\quad a \le t \le b$ — (3).

If we apply the mean value theorem to the

$f_n \Rightarrow f_n - f_m$ — (6) such that

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x-t|}{2(b-a)} \leq \epsilon/2 \quad \text{— (A)}$$

for any x such that on $[a,b]$ if $n \geq N, m \geq N$ thus inequality,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)|$$
$$+ |f_n(x_0) - f_m(x_0)|$$
$$\leq \epsilon/2 + \epsilon/2 < \epsilon \quad \text{— (B)}$$

implies (A) & (B) $\Rightarrow$

$$|f_n(x) - f_m(x)| < \epsilon \quad (a \leq x \leq b,\ n \geq N,\ m \geq N)$$

so that $\{f_n\}$ converges uniformly on $[a,b]$

Let $f(x) = \lim_{n \to \infty} f_n(x)$ $(a \leq x \leq b)$ and define

Let us now fix a point x on $[a,b]$

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \quad \phi(t) = \frac{f(t) - f(x)}{t-x}.$$

for $a \leq t \leq b$, $t \neq x$ then,

$$\lim_{t \to x} \phi_n(t) = f_n'(x) \quad n = 1, 2 \ldots$$

The first inequality in (2) shows that

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\epsilon}{2(b-a)} \quad (n \geq N, m \geq n)$$

uniformly for $t \neq x$.

so that $\{\phi_n\}$ converges uniformly for $t \neq x$, we conclude from (3) since $\{f_n\}$ converges to f, we conclude that

$$\lim_{n \to \infty} \phi_n(t) = \phi(t)$$

uniformly for $a \leq t \leq b$ $t \neq x$. (4) & (5)

If we now apply thm 7.11 to $\{\phi_n\}$

$$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x)$$

$$\lim_{n \to \infty} f_n'(x) = f'(x).$$

Thm : The Stone - Weierstrass theorem.

If f is a continuous complex function on $[a,b]$ there exist a sequence of polynomial P_n such that

$\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$. If f is real then P_n may be taken real.

proof : we may assume without loss of generality that $[a,b]$, $[0,1]$ we may also assume that $f(0) = f(1) = 0$

$\varphi_n(x) = C_n(1-x^2)^n$

for if the theorem is proved for this case consider,

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad [0 \le x \le 1]$$

Here $g(0) = g(1) = 0$ and if g can be obtained as the limit of a uniformly convergent sequence of polynomial. It is clear that the same is true for f, since $f - g$ is polynomial. furthermore, we define $f(x)$ to be zero for x outside $[0,1]$. Then f is uniformly continuous on the whole line, we put.

$$\varphi_n(x) = C_n(1-x^2)^n \quad n = 1, 2 \ldots \quad \text{— (1)}$$

when C_n is choosen so that

$$\int_{-1}^{1} \varphi_n(x)\, dx = 1 \quad \text{— (2)} \quad n = 1, 2 \ldots$$

$\therefore$ we need some information about the order of magnitude of C_n.

$$\text{since } \int_{-1}^{1} (1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx$$

$$\ge 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx$$

$$\ge 2\left[(x)_0^{1/\sqrt{n}} - n\left(\frac{x^3}{3}\right)_0^{1/\sqrt{n}} \right]$$

$$\ge 2\left[1/\sqrt{n} - \frac{n \; 1/n\sqrt{n}}{3} \right]$$

$$\geq 2\left(\frac{1}{\sqrt{n}} - \frac{3}{3\sqrt{n}}\right]$$

$$\geq 2\left[\frac{3-1}{3\sqrt{n}}\right] > 2 \cdot \frac{2}{3\sqrt{n}}$$

$$\geq 4/3\sqrt{n} \geq \frac{1}{\sqrt{n}}$$

It follows from (2) that

$$c_n < \sqrt{n} \quad \text{—} (3)$$

the inequality $(1-x^2)^n \geq 1-nx^2$ which are we used above is easily shown to be true by considering the functions

$$(1-x^2)^n - 1 + nx^2$$

which is zero at $x=0$ and whose derivative is positive in $(0,1)$

For any $\delta > 0$ (3) implies

$$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n \quad (\delta \leq |x| \leq 1) \text{ — } (4)$$

$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ $\delta \leq |x| \leq 1$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$

now let. $Q_n(x) = \int_{-1}^{1} (x+t) Q_n(t)\,dt \quad (0 \leq x \leq 1)$ — (5)

our assumptions about f, show by a simple change of variable that

$$Q_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\,dt$$

$$= \int_0^1 f(t)\, Q_n(t-x)\,dt$$

and the last integral is clearly of polynomial ~~of~~ and the last integral is clearly of polynomial in x. Thus (P_n) is a sequence of polynomials, ~~of~~ which are real if f is real.

~~Thus~~ (P_n) ~~of~~

given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y-x| < \delta$

$$|f(y) - f(x)| < \varepsilon/2$$

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Let $M = \sup |f(x)|$ using ② & ③ and the fact that $\varphi_n(x) \geq 0$, we see that for $0 \leq x \leq 1$.

$$|\varphi_n(x) - f(x)| = \left| \int_{-1}^{1} \{f(x+t) - f(x)\} \varphi_n(t)\,dt \right|$$

$$\leq 2M \int_{-1}^{-\delta} \varphi_n(t)\,dt + \epsilon/2 \int_{-\delta}^{\delta} \varphi_n(t)\,dt$$

$$+ 2M \int_{\delta}^{1} \varphi_n(t)\,dt.$$

$$\leq 4M \int_{\delta}^{1} \varphi_n(t)\,dt + \epsilon/2$$

$$\leq 4M \sqrt{n}\,(1-\delta^2)^n + \epsilon/2$$

$$< \epsilon.$$

for all large enough n which proves the theorem.

Theorem: Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions then $\mathcal{B}$ is a uniformly closed algebra.

proof: If $f \in \mathcal{B}$ and $g \in \mathcal{B}$ there exist uniformly convergent $\{f_n\}$, $\{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathcal{A}$, $g_n \in \mathcal{A}$. since we are dealing with bounded functions

It is easily show that $f_n + g_n \to f + g$.

$$f_n g_n \to fg.$$

$$cf_n \to cf.$$

where c is any constant, the convergence being uniform in each case,

Since $f+g \in \mathcal{B}$, $fg \in \mathcal{B}$ and $cf \in \mathcal{B}$ so that $\mathcal{B}$ is an algebra by thm ①.

If X is a metric space and $E \subset X$ then

(a) $\bar{E}$ is closed.

(b) $E = \bar{E}$ iff E is closed.

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

by (a) & (b). $\bar{E}$ is the smallest closed subset of X

that contains E.

2) $\mathcal{B}$ is uniformly closed.

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UNIT – III

Differentiation

Defn: Let f be defined on $[a,b]$. for any $x \in [a,b]$ form the quotient.

$$\varphi(t) = \frac{f(t) - f(x)}{t-x} \quad (a<t<b,\ t \neq x.)$$

and define $f'(x) = \lim_{t \to x} \varphi(t)$.

Theorem: Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$ g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$ If

$$h(t) = g(f(t)) \quad a \le t \le b$$

then h is differentiable at x, and.

$$h'(x) = g'(f(x)) f'(x)$$

Proof: Let $y = f(x)$. By the definition of the derivative, we have $f(t) - f(x) = t-x\,[f'(x) + u(t)]$

$$g(s) - g(y) = (s-y)\,[g'(y) + v(s)]$$

where $t \in [a,b]$ $s \in I$, and. $u(t) \to 0$ as $t \to x$, $v(s) \to 0$ as $t \to x$, $v(s) \to 0$ as $s \to y$.

Let $s = f(t)$.

using first (5) and then (4) we obtain,

$$h(t) - h(x) = g(f(t)) - g(f(x))$$

$$= [f(t) - f(x)]\,[g'(y) + v(s)]$$

$$= (t-x)\,[f'(x) + u(t)]\,[g'(y) + v(s)]$$

(or) $t \neq x$.

$$\frac{h(t) - h(x)}{t-x} = [g'(y) + v(s)]\,[f'(x) + u(t)]$$

Letting $t \to x$ we see that $s \to y$ by the continuity of f, so that the right side of above thm. tends to

$g'(y) f'(x)$.

<u>Example</u>. Let f be defined by

$$f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Taking for granted that the derivative of $\sin x$ is $\cos x$. we can apply thm,

$$f'(x) = \sin 1/x - 1/x \cos 1/x \quad x \neq 0.$$

At $x = 0$, these theorem do not apply any longer. since $1/x$ is not defined there, we appeal directly to the definition for $t \neq 0$

$$\frac{f(t) - f(0)}{t - 0} = \sin 1/t.$$

as $t \to 0$ this does not tend to any limit. So that $f'(0)$ does not exist.

<u>Thm</u>: Let f be defined on $[a,b]$. if f has a local maximum at a point $x \in (a,b)$ and if $f'(x)$ exists then $f'(x) = 0$.

<u>Proof</u>: Choose δ in accordance with enough that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$ then $\dfrac{f(t) - f(x)}{t - x} \geq 0$.

Letting $t \to x$ we see that $f'(x) \geq 0$.

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If $x < t < x + \delta$ then

$$\frac{f(t) - f(x)}{t - x} \leq 0.$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

<u>Theorem</u>: If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b) then there is a point $x \in (a,b)$ at which,

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

<u>Proof</u>: put, $h(t) = [f(b) - f(a)]\, g(t) - [g(b) - g(a)]\, f(t)$.

Then h is continuous on $[a,b]$, h is differentiable in (a,b) and.

$$h(a) = f(a) g(a) - f(a) g(b)$$

$$= h(b)$$

we have to show that $h'(x) = 0$, for some $x \in (a,b)$.

If h is constant, this holds for every $x \in (a,b)$.

If $h(t) > h(a)$ for some $t \in (a,b)$.

Let x be a point on $[a,b]$ at which h attains its maximum.

we know that Th then, holds $h'(x) = 0$.

if $h(t) < h(a)$ for some $t \in (a,b)$. The same argument applies if we choose for x a point on $[a,b]$ where h ~~attains~~ attains its minimum.

<u>Thm</u>: suppose f is a real valued differentiable func on $[a,b]$ and suppose that $f'(a) < \lambda < f'(b)$ then the

Proof: put $g(t) = f(t) - \lambda t$.

Then $g'(a) < 0$ so that $g(t_1) < g(a)$ for some $t_1 \in (a,b)$ and $g'(b) > 0$, so that

$$g(t_2) < g(b) \text{ for some } t_2 \in (a,b).$$

Hence g attains its minimum on $[a,b]$ at some point x such that $a < x < b$.

By thm ①. $g'(x) = 0$.

Hence $f'(x) = \lambda$.

L'Hospital's rule.

Thm.: Suppose f and g are real and differentiable in (a,b) and $g'(x) \neq 0$, for all $x \in (a,b)$ where $-\infty \leq a < b \leq +\infty$

Suppose $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$.

$f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

$g(x) \to \infty$ as $x \to a$.

$\dfrac{f(x)}{g(x)} \to A$ as $x \to a$.

Proof: We first consider the case in which $-\infty \leq A < \infty$ choose a real number q such that $A < q$. and then choose r such that $A < r < q$.

There is a point $c \in (a,b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r.$$

If $a < x < y < c$, there is a point $t \in (x,y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ holds.

Letting $x \to a$ in the above eqn.

$$\frac{f(y)}{g(y)} \le r < q. \quad (a<y<c)$$

Keeping y fixed,

we can chose a point $c_1 \in (a, y)$ such that

$g(x) > g(a)$ and $g(x) > 0$ if $a<x<c_1$ multiplying.

by $[g(x) - g(a)]/g(x)$ we obtain,

$$\frac{f(x)}{g(x)} < r - r\frac{g(a)}{g(x)} + \frac{f(a)}{g(x)} \quad a<x<c_1$$

If we let $x \to a$, then is a pt $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q. \quad (a<x<c_2)$$

there is point c_2 such that $f(x)/g(x) < q$ & $a<x<c_2$

In the manner if $-\alpha < A \le \alpha$ and p is chosen.

so that $p<A$ we can find a point c_3 such that

$$p < \frac{f(x)}{g(x)} \quad (a<x<c_3)$$

<u>Taylor's thm</u>:

Suppose f is a real function on $[a,b]$, n is a positive integer, f^{n-1} is continuous on $[a,b]$, $f^{n}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$ and define,

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{k}(\alpha)}{k!}(t-\alpha)^k.$$

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Then there exist a point x between α and β such that $f(\beta) = P(\beta) + \frac{f^n(x)}{n!}(\beta-\alpha)^n$.

Proof :- Let M be the number defined by
$$f(\beta) = P(\beta) + M(\beta-\alpha)^n$$
and put $g(t) = f(t) - P(t) - M(t-\alpha)^n \quad (a \le t \le b)$

we have to show that $n!M = f^{(n)}(x)$, for some x between α and β.

$$g^n(t) = f^n(t) - n!M. \quad (a < t < b)$$

Hence the proof will be complete if we can show that $g^n(x) = 0$ for some x between α and β.

Since $P^k(\alpha) = f^k(\alpha)$ for $k = 0 \ldots n-1$
$$g(\alpha) = g'(\alpha) = \ldots\ldots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β, by the mean value thm, since $g'(\alpha) = 0$, we conclude that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps we arrive at the conclusion that $g^n(x_n) = 0$ for some x_n between α and x_{n-1}, that is between α and β.

Example:

$$f(x) = e^{ix}.$$
$$= \cos x + i \sin x.$$

$$f(2\pi) - f(0) = 1 - 1 = 0.$$

$$f'(x) = i e^{ix}.$$

so that $|f'(x)| = 1$ for all real x,

Example: on the segment $(0, 1)$ define

$$f(x) = x \text{ and.}$$
$$g(x) = x + x^2 e^{i/x^2}$$

Since $|e^{it}| = 1$ for all real t, we see that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1$$

$$g'(x) = 1 + \left\{ 2x - \frac{2i}{x} \right\} e^{i/x^2} \quad (0 < x < 1)$$

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1$$

$$\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x}.$$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

L' Hospital rule fails

$$\therefore g'(x) \neq 0.$$

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Thm: Suppose f is a continuous mapping of $[a,b]$ in to $\mathbb{R}^k$ and f is differentiable in (a,b). Then $\exists$ $x \in (a,b)$ such that

$$|f(b) - f(a)| \le (b-a)\, f'(x).$$

Proof: Put $z = f(b) - f(a)$ and define

$$\varphi(t) = z \cdot f(t) \quad a \le t \le b$$

Then φ is a real valued continuous function on $[a,b]$ which is differentiable in (a,b).

The mean value theorem shows therefore that

$$\varphi(b) - \varphi(a) = (b-a)\, \varphi'(x)$$

$$= (b-a)\, z \cdot f'(x)$$

for some $x \in (a,b)$. On the other hand,

$$\varphi(b) - \varphi(a) = z \cdot f(b) - z \cdot f(a)$$

$$= z \cdot z$$

$$= |z|^2$$

The Schwarz inequality now gives,

$$|z|^2 = (b-a)|z \cdot f'(x)| \le (b-a)|z|\,|f'(x)|$$

Hence $|z| \le (b-a)\,|f'(x)|$

which is the desired conclusion.

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UINT - IV

Riemann Stieltjes integral

Defn: Let $[a,b]$ be a given interval. A partition P of $[a,b]$ is a finite set of points $x_0, x_1, \dots x_n$ where $a = x_0 < x_1 \le \dots < x_{n-1} \le x_n = b$

we write $\Delta x_i = x_i - x_{i-1}$ $(i = 1, 2 \dots)$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\overline{\int_a^b} f\,dx = \inf U(P,f)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f)$$

$\overline{\int_a^b} f\,dx$ is called the upper Riemann integral of f over $[a,b]$.

$\underline{\int_a^b} f\,dx$ is called the lower Riemann integral of f over $[a,b]$.

$\int_a^b f\,dx$ is called Riemann integral.

Let f be any function which is bounded on $[a,b]$.

$m_i = \inf f(x)$ $\qquad U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$M_i = \sup f(x)$ $\qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

we define $\overline{\int_a^b} f\,dx : \inf U(P,f,\alpha)$

$\underline{\int_a^b} f\,dx : \sup L(P,f,\alpha)$.

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

is Riemann Stieltjes integral of f with respect to α over $[a,b]$.

Theorem: If p^* is a refinement of p then
$L(P, f, \alpha) \leq L(p^*, f, \alpha)$ and
$U(p^*, f, \alpha) \leq U(P, f, \alpha)$

Proof: Assume that p^* contains only one point more than P.

Let the point be x^* and assume $x_{i-1} \leq x^* \leq x_i$

where x_{i-1}, x_i are two consecutive point of P.

$P = \{x_1 x_2 \dots x_{i-1} \dots x_n\}$

$P^* = \{x_1 x_2 \dots x_{i-1}, x_i \dots x_n\}$

Put $w_1 = \inf f(x) (x_{i-1} \leq x \leq x^*\}$

$w_2 = \inf f(x) (x^* \leq x \leq x_i\}$

we know that,

$m_i = \inf f(x) (x_{i-1} \leq x \leq x_i)$

$w_1 \geq m_i$ and $w_2 \geq m_i$

$\therefore L(P^*, f, \alpha) - L(P, f, \alpha) =$

$w_1 [\alpha(x^*) - \alpha(x_{i-1})] + w_2 [\alpha(x_i) - \alpha(x^*)]$
$- m_i [\alpha(x_i) - \alpha(x_{i-1})]$

$= w_1 \alpha(x^*) - w_1 \alpha(x_{i-1}) + w_2 \alpha(x_i) - w_2 \alpha(x^*)$
$- m_i \alpha(x_i) + m_i \alpha(x_{i-1}) + m_i \alpha(x^*) - m_i \alpha(x^*)$

$= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)] \geq 0$

If p^* contains k points more than P then repeating the process k-times,

$L(P^*, f, \alpha) - L(P, f, \alpha) \geq 0.$

$L(P, f, \alpha) \leq L(P^*, f, \alpha)$

Put $w_1 = \sup f(x) (x_{i-1} \leq x < x^*)$

$w_2 = \sup f(x) (x^* \leq x < x_i)$

we know that $M_i = \sup f(x) (x_{i-1} \leq x \leq x_i)$, $m_i \geq w_1$ & $m_i \geq w_2$

$$U(P,f,\alpha) - U(P^*,f,\alpha) = M_i[\alpha(x_i) - \alpha(x_{i-1})] - w_1[\alpha(x^*) - \alpha(x_{i-1})] - w_2[\alpha(x_i) - \alpha(x^*)]$$

$$= M_i\alpha(x_i) - M_i\alpha(x_{i-1}) - w_1\alpha(x^*) + w_1\alpha(x_{i-1}) - w_2\alpha(x_i) + w_2\alpha(x^*) + M_i\alpha(x^*) - M_i\alpha(x^*)$$

$$= (M_i - w_1)[\alpha(x^*) - \alpha(x_{i-1})] + M_i - w_2[\alpha(x_i) - \alpha(x^*)] \geq 0.$$

$$U(P,f,\alpha) - U(P^*,f,\alpha) \geq 0.$$

$$U(P,f,\alpha) \geq U(P^*,f,\alpha)$$

$$U(P^*,f,\alpha) \leq U(P,f,\alpha).$$

<u>Theorem</u>: $\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx.$

<u>Proof</u>: f is bounded continuous function on $[a,b]$ α is a monotonically increasing function on $[a,b]$.

Let P_1 and P_2 be any partition on $[a,b]$

Let P^* be any refinement of P_1 and P_2

we know that

" If P^* is a refinement of P, then

$L(P,f,\alpha) \leq L(P^*,f,\alpha)$ and

$U(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha)$

$L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$

$\therefore$ L.H.S are bounded above by each R.H.S least upper bound $\leq$ upper bound.

$\Rightarrow \int_a^b f\,dx$ is the least upper bound of $L(P_1,f,\alpha)$

$\int_a^b f\,dx \leq U(P_2,f,\alpha)$

Each member of R.H.S is bounded below by $\underline{\int_a^b} f\,dx$

& $\overline{\int_a^b} f\,dx$ is the greatest lower bound of $U(P_2,f,\alpha)$

$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx.$

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Theorem: If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$ then
(a) $fg \in R(\alpha)$ (b) $|f| \in R(\alpha)$ & $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$.

Proof: We know that "If $f \in R(\alpha)$ on $[a,b]$ $m \le f \le M$
ϕ is continuous on $[m, M]$ and $h(x) = \phi f(x)$ on
$[a,b]$ then $h \in R(\alpha)$ on $[a,b]$"
In the proof on this theorem if $\phi(t) = t^2$
then $f^2 \in R(\alpha)$ $\longrightarrow$ (1) if $f \in R(\alpha)$
Also if $f \in R(\alpha)$, $g \in R(\alpha)$ then
$(f+g) \in R(\alpha)$
$|f+g|^2 \in R(\alpha)$ by (1).
If $g \in R(\alpha)$, $cg \in R(\alpha)$ where c is a constant.
If $c = -1$ then $-g \in R(\alpha)$
$\therefore f \in R(\alpha)$ $-g \in R(\alpha)$
$f + (-g) \in R(\alpha)$
$f - g \in R(\alpha)$
$(f-g)^2 \in R(\alpha)$ by (1)

$$|f+g|^2 - |f-g|^2 = f^2 + g^2 + 2fg - f^2 - g^2 + 2fg$$
$$= 4fg \in R(\alpha)$$
$$= fg \in R(\alpha)$$

If $\phi(t) = t$ then $|f| \in R(\alpha)$
choose $c = \pm 1$ so that $c\int f\, d\alpha \ge 0$

$$\left|\int f\, d\alpha\right| = c\int f\, d\alpha = \int cf\, d\alpha \le \int |f|\, d\alpha$$

since $cf \le |f|$

$$\left|\int f\, d\alpha\right| \le \int |f|\, d\alpha$$

Defn: The unit step function I is defined by

$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

Theorem: Suppose $c_n \geq 0$ for $1, 2, 3 \cdots \in c_n$ converges $\{s_n\}$ is a sequence of distinct points in (a, b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ Let f be continuous on $[a, b]$ then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

Proof: $c_n I(x - s_n) \leq c_n$

since $I(x-s) = 0$ or by comparison test the series $\sum_{n=1}^{\infty} c_n I(x - s_n)$ converges for every α.

Its sum $\alpha(x)$ is monotonic.

Also $\alpha(a) = 0$ and $\alpha(b) = \sum c_n$

α is monotonically increasing function.

(i) If $b > a \Rightarrow \alpha(b) > \alpha(a)$. choose N so that

$$\sum_{N+1}^{\infty} c_n < \epsilon$$

put
$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$$
$$\alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n)$$

we know that
$$\int_a^b f d\alpha = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$ and also if $a \leq s \leq b$. f is bounded on $[a, b]$ continuous at s and $\alpha(x) = I(x - s)$ then $\int_a^b f d\alpha = f(x)$

Then Let $P = \{a < s_1 < s_2 < \ldots s_n < b\}$

$$U(P, f, \alpha) = M_1(\alpha_1(s_1) - \alpha(a)] + M_2[\alpha_1(s_2) - \alpha(s_1)] + \cdots + M_{n+1}[\alpha_1(b) - \alpha_1(s_N)]$$

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$$

$$\alpha_1(s_1) = \sum_{n=1}^{N} c_n I(s_1 - s_n)$$

$$= c_1 I(s_1 - s_1) + c_2 I(s_1 - s_2) + \cdots$$

$$\alpha_1(s_1) = \sum_{n=1}^{N} c_1 I(a - s_n) = 0$$

$\alpha_1(s_2) = \sum_{n=1}^{N} c_n I(s_2 - s_n) = c_1$

$\alpha_1(s_3) = c_1 + c_2$

$\alpha_1(s_4) = c_1 + c_2 + c_3$

$\int_a^b f\,d\alpha_1 = c_1 f_1 + c_2 f_2 + \dots + c_N f_N$

$= \sum_{n=1}^{N} c_n f(s_n)$

$\int_a^b f\,d\alpha = \sum_{n=1}^{N} c_n f(s_n)$

$M = \sup |f(x)|, x \in [a,b]$

But $\alpha_2(x) = \sum_{N+1}^{\infty} c_n I(P - s_n)$

$\alpha_2(b) = \sum_{N+1}^{\infty} c_n I(b - s_n) = \sum_{N+1}^{\infty} c_n$

$\alpha_2(a) = \sum_{N+1}^{\infty} c_n I(a - s_n)$

but $\alpha_2(b) - \alpha_2(a) < M.$

$\left| \int_a^b f\,d\alpha_2 \right| \le M\epsilon$

But $\alpha = \alpha_1 + \alpha_2$

$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

$\left| \int_a^b f\,d\alpha - \int_a^b f\,d\alpha_1 \right| = \left| \int_a^b (f\,d\alpha_2) \right| \le M\epsilon.$

$\left| \int_a^b f\,d\alpha - \int_a^b f\,d\alpha_1 \right| \le M\epsilon$

$\left| \int_a^b f\,d\alpha - \sum_{n=1}^{N} c_n f(s_n) \right| \le M\epsilon.$

If $N \to \infty$ then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

Theorem: Let $f \in R$ on $[a,b]$ for $a \le x \le b$ put $F(x) = \int_a^x f(t)$ Then F is continuous on $[a,b]$. Furthermore iff it continuous at a point x_0 of closed interval $[a,b]$. F is differentiable at x_0 and $F'(x_0) = f(x_0)$

Proof: Given $f \in R(\alpha)$ on $[a,b]$

f is bounded.

Suppose $|f(t)| \le M$ for $a \le t \le b$

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If $a \le x \le y \le b$ then $|F(y) - F(x)| = \int_x^y f(t)\,dt$

$\left|\int_x^y f(t)\,dt\right| \le M(y-x)$

Given $\varepsilon > 0$ if $|y-x| < \varepsilon/M$ then $|F(y) - F(x)| < \varepsilon$

$\Rightarrow$ F is uniformly continuous on $[a,b]$

$\Rightarrow$ F is continuous on $[a,b]$

Suppose f is continuous at x_0.

Given $\varepsilon > 0$, choose $\delta > 0 \ni |f(t) - f(x_0)| < \varepsilon$. if

$|t - x_0| < \delta$ & $a \le t \le b$.

If $x_0 - \delta \le s < x_0 < t < x_0 + \delta$ and.

$a \le s < t < b$, we have

$$\left|\frac{F(t) - F(s)}{t-s} - f(x_0)\right| = \left|\frac{1}{t-s}\int_s^t f(u) - f(x_0)\right| < \varepsilon$$

$\Rightarrow |F'(x_0) - f(x_0)| < \varepsilon$

$F'(x_0) = f(x_0)$

. F is differentiable at x_0.

Theorem: The fundamental theorem of calculus if $f \in R$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: Let $\varepsilon > 0$ be given.

Choose a partition $P = \{x_0, x_1, x_2 \ldots x_n\}$ of $[a,b]$.

so that $U(P,f) - L(P,f) < \varepsilon$ by mean value thm there exists point $t_i \in [x_{i-1}, x_i]$ s.t

$F(x_i) - F(x_{i-1}) = f(t_i)\,\Delta x_i \quad i = 1,2 \ldots n$

$\sum_{i=1}^{n} f(t_i)\,\Delta x_i = F(b) - F(a)$

we know that, and

If $f \in R(\alpha)$ and if $P = \{x_0, x_1, \ldots x_n$

& s_i, t_i are arbitrary point in $[x_{i-1}, x_i]$.

with $\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i < \epsilon$ then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta x_i - \int_a^b f dx \right| < \epsilon'$$

$$\left| \{F(b) - F(a)\} - \int_a^b f(x) dx \right| < \epsilon$$

This holds for every $\epsilon > 0$,

therefore $\int_a^b f(x) dx = F(b) - F(a)$

<u>Theorem</u>: Integration by parts suppose F & G are differentiable function on $[a, b]$ $F' = f \in R$ &

$G' = g \in R$ then

$$\int_a^b F(x) g(x) dx = F(b) G(b) - F(a) G(a) - \int f(x) G(x) dx.$$

<u>Proof</u>: Put $H(x) = F(x) \cdot G(x)$

using the fundamental theorem of calculus

$$\int_a^b H'(x) dx = H(b) - H(a).$$

$$\Rightarrow \int_a^b [F(x) G'(x) + F'(x) G(x)] dx = F(b) G(b) - F(a) G(a)$$

$$\Rightarrow \int_a^b F(x) G'(x) + F'(x) G(x)] dx = F(b) G(b) - F(a) G(a)$$

$$\int_a^b F(x) g(x) = F(b) G(b) - F(a) G(a)$$

$$= \int_a^b f(x) G(x) dx.$$

<u>Theorem</u>: If f and F map $[a, b]$ into R^k. If $f \in R$ on $[a, b]$ & if $F' = f$ then $\int_a^b f(t) dt = F(b) - F(a)$

<u>Proof</u>: using the fundamental theorem of calculus we get the result.

Theorem: If f maps $[a,b]$ into R^k and if $f \in R(\alpha)$ for some monotonically increasing function α on $[a,b]$ then $|f| \in R(\alpha)$ & $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

Proof: Given f maps $[a,b]$ into R^k.

$f = f_1 \dots f_k$.

i) $f_1 \dots f_k$ are the components of f.

$|f| = |f_1^2 + \dots + f_k^2|^{1/2}$

we know that,

"If $f \in R(\alpha)$ on $[a,b]$ $m \leq f \leq M$, ϕ is continuous on $[m,M]$ & $h(x) = \phi f(x)$ on $[a,b]$ then $h \in R(\alpha)$ on $[a,b]$"

$\therefore$ Each of the function f_i^2 belongs to $R(\alpha)$.

$\therefore \sum f_i^2 \in R(\alpha)$

$\because$ x^2 is a continuous function of x.

The square root function is continuous on $[0,M]$ for every real M. for real M.

$|f| \in R(\alpha)$

Put $y = y_1 \dots y_k$ where $y_i = \int f_i\,d\alpha$

If $y = \int f\,d\alpha$ and $|y|^2 = \sum_{j=1}^{n} y_j^2 = \sum y_j \int f_j\,d\alpha$

$= \int (\sum y_j f_j)\,dx$.

By the Schwartz inequality

$|\sum y_j f_j|^2 \leq \sum |y_j|^2 < \sum |f_j|^2$

$|\sum y_j f_j| \leq |f(t)|$

$|y|^2 \leq \int |y|\,|f_t|\,d\alpha$

$|y| \leq \int |f|\,d\alpha$.

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$$\left|\int f\,d\alpha\right| \le \int |f|\,d\alpha.$$

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha.$$

Rectificable curves.

Defn:

A continuous mapping γ of an interval $[a,b]$ in to R^k is called a curve in R^k

i, γ is a curve on $[a,b]$

If γ is one - one

γ is closed an arc.

If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

UNIT - V

Functions of Several Variables.

Defn: (a) A nonempty set $X \subset R^n$ is a vector space if $x+y \in X$ and $cx \in X$, for all $x \in X, y \in X$ and for all scalars c.

(b) If $x_1 \dots x_k \in R^n$ and $c_1, c_2 \dots c_k$ are scalars, the vector $c_1x_1 + \dots c_nx_n$ is called a linear combination of $x_1 x_2 \dots x_k$.

If $S \subset R^n$ and if E is the set of all linear combinations of elements of S.

we say that S spans E. (or) E is the span of S.

Theorem: Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then $\dim X \leq r$.

Proof: If this is false, there is a vector space X which contains an independent set $Q = \{y_1, y_2 \dots y_{r+1}\}$ and which is spanned by a set S_0 consisting of r vectors.

suppose $0 \leq i < r$, and suppose a set S_i has been constructed which spans X and which consists of all y_j with $1 \leq j \leq i$ plus a certain collection of $r-i$ members of S_0, say $x_1 \dots x_{r-i}$.

Since S_i spans X, y_{i+1} is in the span of S_i.

Hence there are scalars $a_1, a_2 \dots a_{i+1}, b_1 \dots b_{r-i}$ with $a_{i+1} = 1$ such that

$$\sum_{j=1}^{r+1} a_j y_j + \sum_{k=1}^{r-i} b_k x_k = 0$$

If all b_k's were zero, the independence of Q would force all a_j's to be zero.

This is contradiction.

It follows that some $x_k \in S_i$ is a linear combination of the other members of $T_i = S_i \cup \{y_{i+1}\}$.

Remove this x_k from T_i and call the remaining set S_{i+1}. Then S_{i+1} spans the same set as T_i namely X. So that S_{i+1} has the properties postulated for S_i with $i+1$ in place of i.

Starting with S_0, we thus construct sets $S_1 S_2 \ldots S_r$. The last of these consists of $y_1 y_2 \ldots y_r$ and our construction shows that it spans X.

But Q is independent,

hence y_{r+1} is not in the span of S.

This is contradiction.

__Defn__: A mapping A of a vector space X into a vector space Y is said to be a linear transformation if $A(x_1+x_2) = Ax_1 + Ax_2$ $A(cx) = cAx$ for all $x, x_1, x_2 \in X$. and all scalars c.

Ax instead of $A(x)$ if A is linear.

__Thm__ (a) If $A \in L(R^n, R^m)$ then $\|A\| < \infty$ and A is a uniformly continuous mapping of R^n into R^m.

(b) If $A, B \in L(R^n, R^m)$ and c is a scalar, then
$$\|A+B\| \le \|A\| + \|B\|, \quad \|cA\| = |c|\,\|A\|.$$
with the distance between A and B defined as $\|A-B\|$, $L(R^n, R^m)$ is a metric space.

(c) If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$ then $\|BA\| \le \|B\|\,\|A\|$

Proof: (a) Let $\{e_1, e_2, \ldots e_n\}$ be the standard basis in R^n and suppose $x \in \sum c_i e_i$, $|x| \le 1$ so that $|c_i| \le 1$ for $i = 1, 2, \ldots n$.

Then
$$|Ax| = \left|\sum c_i A e_i\right| \le \sum |c_i|\,|Ae_i| \le \sum |Ae_i|$$

So that
$$\|A\| \le \sum_{i=1}^{n} |Ae_i| < \infty.$$

Since $|Ax - Ay| \le \|A\|\,|x-y|$ if $x, y \in R^n$, we see that A is uniformly continuous.

(b) The inequality in (b) follows from,
$$|(A+B)x| = |Ax + Bx| \le |Ax| + |Bx| \le (\|A\| + \|B\|)|x|$$

The second part of (b) is proved in the same manner if $A, B, C \in L(R^n, R^m)$

we have the triangle inequality
$$\|A - C\| = \|(A-B) + (B-C)\| \le \|A-B\| + \|B-C\|$$
and it is easily verified that $\|A-B\|$ has the other properties of a metric

(c) Finally (c) follows that.
$$|(BA)(x)| = |B(Ax)| \le \|B\|\,|Ax| \le \|B\|\,\|A\|\,|x|$$

Since we now have metrices in the spaces $L(R^n, R^m)$ the concepts of open set, continuity make sense for these spaces.

Defn. Suppose E is an open set in R^k, f maps E into R^m and $x \in E$. If there exists a linear transformation A of R^n into R^m such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0.$$

Then we say that f is differentiable at x, and we write.

$$f'(x) = A$$

Theorem: Suppose E is an open set in R^n, f maps E into R^m, f is differentiable at $x_0 \in E$. g maps an open set containing $f(E)$ into R^k and g is differentiable at $f(x_0)$. Then the mapping F of E into R^k defined by

$F(x) = g(f(x))$ is differentiable at x_0.

and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

Proof: put $y_0 = f(x_0)$, $A = f'(x_0)$, $B = g'(y_0)$

$$u(h) = f(x_0+h) - f(x_0) - Ah$$
$$v(k) = g(y_0+k) - g(y_0) - Bk.$$

for all $h \in R^n$ and $k \in R^m$ for which $f(x_0+h)$ and $g(y_0+k)$ are defined then,

$$|u(h)| = \epsilon(h)|h|. \quad |v(k)| = \eta(k)|k| \qquad (A)$$

where $\epsilon(h) \to 0$ as $h \to 0$ and $\eta(k) \to 0$ as $k \to 0$.

Given h, put $k = f(x_0+h) - f(x_0)$ then.

$|k| = |Ah + u(h)| \le [\|A\| + u(h)]\,|h|$ — (B)

$F(x_0+h) - F(x_0) - BAh = g(y_0+k) - g(y_0) - BAh$

$= B(k - Ah) + v(k)$

$= Bu(h) + v(k)$

Hence (A) & (B) imply for $h \neq 0$, that

$$\frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} \le \|B\|\, \epsilon(h) + [\|A\| + \epsilon(h)]\,\eta(k)$$

Let $h \to 0$ then $\epsilon(h) \to 0$. Also $k \to 0$ by (A) so that $\eta(k) \to 0$.

It follows that $F'(x_0) = BA$.

Inverse function theorem:

Thm! Suppose f is a C'-mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. then,

(a) there exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one-to-one on U and $f(U) = V$.

(b) If g is the inverse of f defined in V by

$g(f(x)) = x \quad x \in U$. then

$g \in C'(V)$.

Proof:

(a) Put $f'(a) = A$ and choose λ so that

$$2\lambda \|A^{-1}\| = 1$$

since f' is continuous at a, there is an open ball $U \subset E$ with centre at a,

such that $\|f'(x) - A\| < \lambda \quad (x \in U)$ — (1)

we associate to each $y \in R^n$ is a function φ defined by $\varphi(x) = x + A^{-1}(y - f(x))$, $x \in E$ — (2)

since $\varphi'(x) = 1 - A^{-1} f'(x)$

$= A^{-1}(A - f'(x))$

$\|\varphi'(x)\| < 1/2 \quad x \in U$ — ③

Hence. $|\varphi(x_1) - \varphi(x_2)| \le 1/2 |x_1 - x_2| \quad (x_1, x_2 \in U)$ — Ⓐ

It follows that φ has atmost one fixed point in U. so that $f(x) = y$. for at most one $x \in U$.

Thus f is 1-1 on U.

next put $V = f(U)$ and pick $y_0 \in V$ then $y_0 = f(x_0)$ for some $x_0 \in U$.

Let B be an open ball with centre at x_0 and radius $r > 0$, so small that its closure $\bar{B}$ lies in U.

we shall show that $y \in V$ whenever $|y - y_0| < \lambda r$.

fix $|y - y_0| < \lambda r$. with

$|\varphi(x_0) - x_0| = |A^{-1}(y - y_0)| < \|A^{-1}\| \lambda r$

$= \frac{r}{2}$.

If $x \in \bar{B}$ it therefore follows from Ⓐ

$|\varphi(x) - x_0| \le |\varphi(x) - \varphi(x_0)| + |\varphi(x_0) - x_0|$

$< 1/2 |x - x_0| + \frac{r}{2}$

$\le r$.

Hence $\varphi(x) \in \bar{B}$. eqn Ⓐ holds if $x_1 \in \bar{B}$, $x_2 \in \bar{B}$

Then φ is contraction of $\bar{B}$ into $\bar{B}$. Being a closed subset of $\mathbb{R}^n$.

$\bar{B}$ is complete.

dan (A).

Rank.

Defn: Suppose X and Y are vector spaces and $A \in L(X,Y)$ as . The null space of X, $N(A)$ is the set of all $x \in X$, at which $Ax = 0$. It is shown that $N(A)$ is a vector space in X.

The range of A, $R(A)$ is a vector space in Y.

The rank of A is defined to be the dimension of $R(A)$?

Projection: Let X be a vector space. An operator $P \in L(X)$ is said to be a projection in X if $P^2 = P$.

Theorem:

(a) If I is the identity operator on R^n then
$\det[I] = \det(e_1 e_2 \ldots e_n) = I$.

(b) det is a linear function of each of the column vectors x_j, if the others are held fixed.

(c) If $[A]_1$ is obtained from $[A]$ by interchanging two columns, then $\det[A]_1 = -\det[A]$

(d) If $[A]$ has two equal columns, then $\det[A] = 0$.

Proof: If $A = I$ then $a(i,i) = 1$ and $a(i,j) = 0$ for $i \neq j$.

Hence $\det[I] = s(1,2 \ldots n) = 1$

which proves (a).

If any two of the j's are equal. Each of the remaining $n!$ products in $\det[A] = \sum s(j_1 \ldots j_n)\, a(1,j_1)\, a(2,j_2) \ldots a(n,j_n)$ contains exactly one factor from each column.

This proves (b)

we know that, φ has a fixed point $x \in \bar{B}$, for this x, $f(x) = y$ then $y \in f(\bar{B}) \subset f(U) = V$.

(b) Pick $y \in V$, $y + k \in V$. then $\exists$ $x \in U$, $x+h \in U$, so that $y = f(x)$, $y + k = f(x+h)$ with φ as in above eqn!

$$\varphi(x+h) - \varphi(x) = h + A^{-1}(f(x) - f(x+h)]$$

$$= h - A^{-1}k.$$

By (A) :! $|h - A^{-1}k| \leq \frac{1}{2}|h|$ Hence,

$$|A^{-1}k| \geq \frac{1}{2}|h| \text{ and,}$$

$$|h| \leq 2\|A^{-1}\|\,|k|.$$

$$= \lambda^{-1}|k|.$$

By (1), (2). and we know that $f'(x)$ has an inverse, say T, since

$$g(y+k) - g(y) - Tk = h - Tk = -T(f(x+h) - f(x) - f'(x)h]$$

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \leq \frac{\|T\|}{\lambda} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|}$$

as $k \to 0$ shows that $h \to 0$.

The right side of the last inequality then tends to 0. Hence the same is true of the left. we have proved. that $g'(y) = T$

But T was chosen to be the inverse of $f'(x) = f'(g(y))$ thus,

$$g'(y) = (f'(g(y)))^{-1} \quad y \in V.$$

Part (c) is an immediate consequence of the fact that $s(j_1, \ldots j_n)$ changes sign if any two of the j's are interchanged. and d is a corollary of (c).

Theorem: If [A] and [B] are n by n matrices, then $\det([B][A]) = \det[B] \det[A]$.

Proof: If $x_1 \ldots x_n$ are the columns of [A].

define, $\Delta_B(x_1 x_2 \ldots x_n) = \Delta_B[A] = \det[B][A]$

The columns of [B][A] are the vectors $Bx_1 \ldots Bx_n$.

Thus $\Delta_B(x_1 \ldots x_n) = \det(Bx_1 \ldots Bx_n)$

Hence $\Delta_B[A] = \Delta_B(\sum_i a(i,1) e_i . x_2 \ldots x_n)$

$= \sum_i a(i,1) \Delta_B(e_i\, x_2 \ldots x_n)$

Replacing this process with $x_2 \ldots x_n$ we obtain,

$$\Delta_B[A] = \sum a_i(i_1, 1) a(i_2, 2) \ldots a(i_n, n) \Delta_B(e_{i_1} \ldots e_{i_n})$$

the sum being extended over all ordered n-tuples $(i_1 \ldots i_n)$ with $1 \leq i \leq n$.

$$\Delta_B(e_{i_1} \ldots e_{i_n}) = t(i_1 \ldots i_n) \Delta_B(e_1 \ldots e_n)$$

where $i = 1, 0$ or -1 and since $[B][I] = [B]$

$$\Delta_B(e_1 \ldots e_n) = \det[B]$$

Substituting $\det[B][A] = \sum (a(i_1, 1) \ldots a(i_n, n)\, t(i_1 \ldots i_n)) \det[B]$

for all n by n matrices [A] and [B].

Taking $B = I$ we see above sum in braces is $\det[A]$.