SEMESTER : I
CORE COURSE : III

Inst Hour	: 6
Credit	: 4
Code	: 18KP1M03

ORDINARY DIFFERENTIAL EQUATIONS

UNIT-I

Second Order Linear Equation: The general solution of the homogeneous equation – The use of a known solution to find another – The method of variation of parameters – Power Series solutions and special functions: Introduction – Series solution of first order equations – Second order Linear equations.

Chapter 3: sections 15, 16, 19, and Chapter 5: Sections 26 to 28

UNIT-II

Power series and Special functions: Regular singular points – Gauss's hypergeometric equation – The point at infinity – Some special functions of Mathematical Physics: Legendre Polynomials – Properties of Legendre Polynomials – Bessel functions – Properties of Bessel Functions.

Chapter 5: sections 29 to 32 and Chapter 8 : Sections 44, 45, 46, 47

UNIT-III

Systems of First Order Equations: Linear systems – Homogenous Linear system with constant Coefficients – The Existence and Uniqueness of Solutions: The Method of successive Approximations – Picard's Theorem.

Chapter 10: Sections 55, 56 and Chapter 13: Sections 68, 69

UNIT-IV

Qualitative Properties of Solutions: Oscillations and the Sturm separation Theorem –The Sturm Comparison Theorem – Fourier series and Orthogonal functions: Orthogonal functions – The Mean Convergence of Fourier series.

Chapter 4: Sections 24 to 25: Chapter 6: 37, 38

UNIT-V

Non Linear equations:Autonomous Systems; the phase plane and its phenomena – Types of critical points ; stability – critical points and stability for linear systems – Stability by Liapunov's direct method – Simple critical points of nonlinear systems.

Chapter 11: Sections 58 to 62

TEXT BOOK

G.F.Simmons, Differential Equations with Applications and Historical Notes, TMH, New Delhi,1984

REFERENCES

1. W.T.Reid, Ordinary Differential Equations, John Wiley & Sons, New York, 1971
2. E.A. Coddington and N.Levinson, Theory of Ordinary Differential Equations, McGraw Hil Publishing Company, New York, 1955.

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

ORDINARY DIFFERENTIAL EQUATIONS

UNIT-1

Definition :

An Equation involving one dependent Variable and its derivatives with respect to one are more independent variables is called differential equation (or) its called Ordinary differential equation.

Second order differential equation

The general form of 2^{nd} order differential Equation is;

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$ This equation is can be written as;

$$y'' + P(x)y' + Q(x)y = R(x)$$

(or)

$$y'' + Py' + Qy = R$$

Where $P(x)$, $Q(x)$ and $R(x)$ are all functions of x alone

Homogeneous and Non-Homogeneous Equations:

consider the equation

$$y'' + P(x)y' + Q(x)y = R(x)$$ ~~where k is~~

(i) If $R(x)$ is zero, $R(x)=0$, then the above equation becomes $y''+R(x)y'+Q(x)y=0$ this Equation called "homogeneous Equation".

ii If $R(x) \neq$ then the above equation becomes;

$y'' + P(x)y' + Q(x)y = R(x)$ is called non-homogeneous Equations.

Results :

(i) Let $P(x)$, $Q(x)$ and $R(x)$ be continuous functions on $[a\ b]$ if $x_0 \in [a, b]$ and if y_0 and y_0' are any numbers then equations;

$y + P(x)y' + Q(x)y = R(x)$ has one and only solution on $[a, b]$ such that: $y(x_0) = y_0$ and $y'(x_0) = y_0'$

(ii) If y_g is the general solution of, $y'' + P(x)y' + Q(x)y = 0 \rightarrow 2$ and y_p is any particular solution $y'' + P(x)y' + Q(x)y = R(x) \rightarrow 1$ then $y_g + y_p$ is the general solution of Equation 1

(iii) If $y_1(x)$ and $y_2(x)$ are any two solution of $y'' + Py' + Q(y) = 0$ then $c_1y_1(x) + c_2y_2(x)$ is also x Solution for any constants c_1 and c_2 this solution is also called a linear combination of the solution $y_1(x)$ and $y_2(x)$

Definition : {Linearly independent & dependent}

If two functions $f(x)$ & $g(x)$ are defined on $[a, b]$ and have the property that are is the constant multiple of the other ie: $f(x) = k\,g(x)$

Where k is constant then they are said to be linearly dependent on otherwise they are called linearly independent

Definition : {.

Let $y_1(x)$ and $y_2(x$ be intuo functions define on $[a,b]$ then $W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$ is called Wronskian of $y_1(x)$ & $y_2(x)$

(i) If $W(y_1, y_2) = 0$ then the functions $y_1(x)$ & $y_2(x)$ are linearly dependent.

(ii) If $W(y_1, y_2) \neq 0$ then the functions $y_1(x)$ & $y_2(x)$ are linerarly dependent.

Note :

Wronskain of n functions ; $y_1, y_2, y_3 \cdots y_n$ can be defined as follows ;

$$W(y_1, y_2, y_3 \cdots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y_1' & y_2' & \cdots\cdots & y_n' \\ y_1'' & y_2'' & \cdots\cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-2} & & y_n^{n-1} \end{vmatrix}$$

Theorem : 1

Let $y_1(x)$ and $y_2(x)$ be linearly independent Solutions of a homogenious equations, $y'' + p(x)y' + q(x)y = 0 \rightarrow 1$ on the intervel $[a,b]$ then P.T. $c_1 y_1(x) + c_2 y_2(x) \rightarrow 2$ is the general solution of equ (1) on $[a,b]$ in the same that every solution on this intervel can be obtained from (2) by the suitable choice of constant

Proof-

Let $y(x)$ be any solution of (1) on $[a,b]$ then we must show that constants c_1 and c_2 canbe found so that $y(x) = c_1 y_1(x) + c_2 y_2(x)$ $\forall x$ in $[a,b]$ But of solution of (1) on $[a,b]$ is completely determined by its value and values of the derivative at a single point.

Since, $c_1 y_1(x) + c_2 y_2(x)$ and $y(x)$ are solutions of (1) on $[a,b]$, it is enough to show that we can find c_1 and c_2 for some point x_0 in $[a,b]$ ~~such~~ such that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y'(x_0)$$

$$c_1 y_1' + c_2 y_2' (x_0) = y'(x_0)$$

Solving for c_1 and c_2, we have;

$$= \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$

$$= y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0)$$
$$\neq 0$$

Lemma:

If $y_1(x)$ and $y_2(x)$ are any two solution of the equation $y'' + P(x)y' + Q(x)y = 0 \rightarrow (1)$ on $[a,b]$ then the wronskian $W = W(y_1, y_2)$ is either identically zero (or) never be zero on $[a,b]$

Proof

We know that

$$W = y_1 y_2' - y_1' y_2 \rightarrow 2$$

then $\quad w' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$

$$W' = y_1 y_2'' - y_1'' y_2 \rightarrow 3$$

Given that y_1 and y_2 are any two solutions of (1) and therefore $y_1'' + Py_1' + Qy_1 = 0 \rightarrow 4$

$$y_2'' + Py_2' + Qy_2 = 0 \rightarrow 5$$

Now; $5 \times y_1 - 4 \times y_2$;

$y_1 \times 5 \Rightarrow y_1 y_2'' + P y_1 y_2' + Q y_1 y_2$

$4 \times y_2 \Rightarrow - y_1'' y_2 - P y_1' y_2 - Q y_1 y_2 = 0$

$$\Rightarrow (y_1 y_2'' - y_1'' y_2) + P(y_1 y_2' - y_1' y_2) = 0$$

$$\Rightarrow W' + PW = 0$$

$$\Rightarrow \frac{dw}{dx} = -Pw$$

$$\Rightarrow \frac{dw}{w} = -Pdx$$

$$\Rightarrow \log w = -\int Pdx + \log c \quad \text{where is constant}$$

$$\Rightarrow \log \frac{W}{c} = \log e^{-\int Pdx}$$

$$\Rightarrow \frac{W}{c} = e^{-\int Pdx}$$

$$\Rightarrow W = ce^{-\int Pdx}.$$

Since the exponential ~~our overall task of proving~~ factor is never zero we see that W is identically zero if the constant $c = 0$, and never zero if $c \neq 0$, and the proof is complete.[3]

Lemma :

If $y_1(x)$ & $y_2(x)$ are two solutions of the equation on; $y'' + Py' + qy = 0 \rightarrow 1$ on $[a, b]$ then they are linearly independent on this interval if the wronskians $W = (y_1, y_2) = y_1 y_2' - y_1' y_2$ is not identically zero on $[a, b]$

Proof:

Suppose that y_1 and y_2 on linearly dependent solution of (1)

claim: $y_1 y_2 - y_1' y_2 = 0 \quad \forall x \in [a,b]$

If $y_1(x) = 0$ & $y_2(x) = 0 \quad \forall x \in [a,b]$

Then there is nothing to prove because automatically

$y_1 y_2' - y_1' y_2 = 0 \quad \forall x \in [a,b]$

suppose that; $y_1(x) \neq 0$ & $y_2(x) \neq 0$

$\forall x \in [a,b]$ since, y_1 and y_2 are linearly independent, we can write

$y_2 = c y_1$ { c is constant

$y_2' = c y_1'$

Hence, $y_1 y_2' - y_1' y_2 = y_1 (c y_1') - y_1' (c y_1)$

$= c (y_1 y_1' - y_1' y_1)$

$= 0$

$\Rightarrow y_1 y_2' - y_1' y_2 = 0 \quad \forall x \in [a,b]$

suppose that $w(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0 \quad \forall x \in [a,b]$

$\Rightarrow$ y_1 and y_2 are linearly dependent

If wronskian is identically zero on $[a,b]$

dividing it by y_1^2 we have;

$$\frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0$$

$$\Rightarrow d\left(\frac{y_2}{y_1}\right) = 0$$

$$\Rightarrow \frac{y_2}{y_1} = k, \quad k \text{ is constant}$$

$\Rightarrow$ $y_2 = ky_1$, which is the for all $x \in [a,b]$

$\Rightarrow$ y_1 and y_2 are linearly dependent.

Note

* If $W \neq 0$ then the functions are linearly independent in that interval.

* If $W = 0$ then the functions are linearly dependent in that interval.

Problems:

prove that $y = c_1 \sin x + c_2 \cos x$ is the different solution of $y'' + y = 0$ on any interval and the find perticular solutions for which $y(0) = 2$ and $y'(0) = 3$.

Proof

Let $y_1 = \sin x$ and $y_2 = \cos x$

Then, $\frac{y_1}{y_2} = \frac{\sin x}{\cos x} = \tan x$; Not a constant

again; $W = y_1 y_2' - y_1' y_2$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$- \sin^2 x - \cos^2 x$$

$$W = -(\sin^2 x + \cos^2 x)$$

$$W = -1 \; (\neq 0$$

$\therefore$ Solutions $\sin x$ & $\cos x$ linearly independent and hence,

$y = c_1 \sin x + c_2 \cos x$ will be a general solution on the differential equation $y'' + y = 0$

$$\Rightarrow y(0) = 2$$

$$\Rightarrow y(0) = c_1(0) + c_2(1)$$

$$= c_2$$

$$\Rightarrow c_2 = 2$$

$$y'(x) = c_1 \cos x - c_2 \sin x$$

Hence, $y'(0) = 3$

$$y'(0) = c_1(1) + c_2(0)$$

$$= c_1$$

$$\Rightarrow c_1 = 3$$

Hence the required solution is; $y = 3 \sin x + 2 \cos x$.

2. If y_1 and y_2 are linearly independent solution of homogeneous equations $y''+p(x)y'+Q(x)y=0$, show that, $p(x) = \dfrac{-[y_1y_2''-y_2y_1'']}{W(y_1,y_2)}$ and

$$Q(x) = \frac{y_1'y_2''-y_1''y_1'}{W(y_1,y_2)}$$

Soln:- Given that
$$y''+py'+Qy=0 \rightarrow 1$$
$$y_1''+py_1'+Qy_1=0 \rightarrow 2$$
$$y_2''+py_2'+Qy_2=0 \rightarrow 3$$

$$W(y_1,y_2) = y_1y_2'-y_1'y_2 \rightarrow 4$$

$$2\times y_2 \Rightarrow y_1''y_2+py_1''y_2+Q\cancel{y_1y_2}=0$$
$$3\times y_1 \Rightarrow y_2''y_1+py_2''y_1+Q\cancel{y_2y_1}=0$$
$$\overline{y_1''y_2-y_2''y_1+py_1'y_2-py_2'y_1=0}$$

$$p(y_1'y_2-y_2'y_1) = y_2''y_1-y_1''y_2$$

$$p = \frac{y_2''y_1-y_1''y_2}{y_1'y_2-y_2'y_1}$$

$$\therefore p(x) = \frac{-[y_2''y_1-y_1''y_2]}{W(y_1,y_2)}$$

$$2\times y_2' \Rightarrow y_1''y_2'+p\cancel{y_1'y_2'}+Qy_1y_2'=0$$
$$3\times y_1' \Rightarrow y_2''y_1'+\cancel{p}y_1'y_2'+Qy_1'y_2=0$$
$$\overline{y_1''y_2'-y_2''y_1'+Qy_1y_2'-Qy_1'y_2=0}$$

$$Q(y_1y_2' - y_1'y_2) = y_2''y_1' - y_1''y_2'$$

$$Q(x) = \frac{y_2''y_1' - y_1''y_2'}{y_1y_2' - y_1'y_2}$$

$$\therefore Q(x) = \frac{y_2''y_1' - y_1''y_2'}{W(y_1, y_2)}$$

<u>To use of known solution to find another</u>

Given a solution, find the 2nd solution of homogeneous differential Equation of order 2

Consider the 2nd order homogeneous equation

$y'' + p(x)y' + Q(x)y = 0 \to (1)$

Let $y = y_1 \to (2)$ be a known solution. Let us take $y_2 = vy_1 \to (3)$ as unknown solution of equ (1)

Now, y_1 is the solution of equ (1)

$\Rightarrow y_1'' + Py_1' + Qy_1 = 0 \to (4)$

Again, y_2 is the solution of equation (1)

$\Rightarrow y_2'' + Py_2' + Qy_2 = 0 \to (5)$

Now, $y_2 = vy_1$

$\Rightarrow y_2' = vy_1' + v'y_1$ and

$y_2'' = vy_1'' + v'y_1 + v'y_1' + v''y_1$

$= vy_1'' + 2v'y_1' + v_1''y_1$

Hence $(5) \Rightarrow vy_1'' + 2v'y_1' + v''y_1 + p(vy_1' + v'y_1) + Q(vy_1) = 0$

$\Rightarrow v(y_1'' + py_1' + Qy_1) + v''y_1 + pv'y_1 + 2v'y_1' = 0$

$$\Rightarrow v''y_1 + v'(Py_1 + 2y_1') = 0$$

Divide throughout by $v'y_1$

Then, $\dfrac{v''}{v'} + \left(P + 2\dfrac{y_1'}{y_1}\right) = 0$

$$\Rightarrow d(\log v') + 2d[\log y_1] = -P$$

$$\Rightarrow \quad \log v' + 2\log y_1 = -\int P\,dx$$

$$\Rightarrow \log(v'y_1^2) = \log e^{-\int P\,dx}$$

$$v'y_1^2 = e^{-\int P\,dx}$$

$$v' = \frac{e^{-\int P\,dx}}{y_1^2}$$

$$v = \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

Hence $y_2 = vy_1$

$$= y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

To show that y_1 and y_2 are linearly independent we have to prove that, $W(y_1, y_2) \neq 0$

Now
$$W(y_1, y_2) = y_1y_2' - y_1'y_2$$
$$= y_1(vy_1' + v'y_1) - y_1'(vy_1)$$
$$= v\cancel{y_1y_1'} + v'y_1^2 - \cancel{y_1'vy_1}$$
$$= v'y_1^2$$
$$= \frac{e^{-\int P\,dx}}{\cancel{y_1^2}}\cancel{y_1^2}$$

$$W(y_1, y_2) = e^{-\int P\,dx} \neq 0 \quad \forall x$$

Example

If $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$, find the general solution.

Soln:

The given differential equation may be rewritten as,

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

Hence $P(x) = 1/x$, $Q(x) = -1/x^2$

Let $y_2 = v y_1$

$\Rightarrow y_1 = x \Rightarrow y_2 = vx$ be an unknown solution

Now, $v' = 1/y_1^2 \, e^{-\int P dx}$

$$= \frac{1}{x^2} e^{-\int \frac{1}{x} dx}$$

$$= \frac{1}{x^2} e^{-\log x} \Rightarrow \frac{1}{x^2} e^{\log 1/x}$$

$$= \frac{1}{x^2} \cdot \frac{1}{x}$$

$$v' = \frac{1}{x^3}$$

$$v = \int \frac{1}{x^3} dx$$

$$= -\frac{1}{2x^2}$$

Hence $y_2 = vx$

$$= -\frac{1}{2x^2} \cdot x$$

$$= -\frac{1}{2x}$$

$\therefore$ General solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 x + c_2 \left(-\frac{1}{2x}\right)$$

2. Find the general solution of the equation $y''+y=0$, given that, $y_1 = \sin x$ is one solution.

Soln: Here $P(x)=0$, $Q(x)=1$

$$y_2 = v y_1$$

$y_2 = v \sin x$ be an unknown solution of the differential Equation

$$v' = \frac{1}{y_1^2} e^{\int -P dx}$$

$$= \frac{1}{\sin^2 x} e^{-\int 0 dx}$$

$$v' = \operatorname{cosec}^2 x$$

$$v = \int \operatorname{cosec}^2 x \, dx$$

$$v = -\cot x$$

$$\therefore y_2 = v \sin x$$

$$= -\cot x \sin x$$

$$= -\frac{\cos x}{\sin x} \sin x$$

$$y_2 = -\cos x$$

$\therefore$ The general solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 \sin x - c_2 \cos x$$

3. Find the differential solution of $y''-y=0$ given $y_1 = e^x$ be the given solution.

Solution:

Hence $P(x)=0$, $Q(x)=1$

Let $y_2 = v y_1$

$y_2 = v e^x$ be an unknown solution of the differential Equations.

The Method of Variation

Now $v' = \frac{1}{y_1^2} e^{-\int P dx}$

$= \frac{1}{(e^{x})^2} e^{-\int 0 dx} = \frac{1}{e^{2x}}$

$v = \int \frac{1}{e^{2x}} dx = \int e^{-2x} dx$

$= -\left(\frac{e^{-2x}}{2}\right)$

$y_2 = v y_1$

$= -\frac{e^{-2x}}{2} \cdot e^{x}$

$y_2 = -\frac{e^{-x}}{2}$

∴ Hence the general solution is $y = c_1 y_1 + c_2 y_2$

$y = c_1 e^{x} - c_2 e^{-x}/2$

<u>The Method of Variation of parameters:-</u>

Consider the non-homogeneous second order differential equation $y'' + P(x)y' + Q(x)y = R(x) \rightarrow (1)$ Where $P(x)$, $Q(x)$ and $R(x)$ are all functions of x

The general solution of equ (1) is of the form $y(x) = y_g(x) + y_p(x)$ where $y_g(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of $y'' + P(x)y' + Q(x)y = 0 \rightarrow (2)$

Here $y_1(x)$ and $y_2(x)$ are linearly independent solutions of equ(2) this method describes how to find $y_p(x)$

Let $y_p = v_1 y_1 + v_2 y_2 \rightarrow 3$

Now $y_p' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$

$= (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2)$

Let us assume that $v_1' y_1 + v_2' y_2 = 0 \rightarrow 4$

Then $y_p' = v_1 y_1' + v_2 y_2'$

$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$

$\therefore y'' + p(x) y' + Q(x) y = R(x)$ becomes

$(v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2') + P(v_1 y_1' + v_2 y_2') +$

$Q(v_1 y_1 + v_2 y_2) = R(x)$

$v_1(y_1'' + P y_1' + Q y_1) + v_2(y_2'' + P y_2' + Q y_2) + v_1' y_1' + v_2' y_2' = R(x)$

Since y_1 and y_2 are solutions of (2) $y'' + p(x) y' + Q(x) y = 0$ they must satisfy that equation are

$y_1'' + P y_1' + Q y_1 = 0$ and also $y_2'' + P y_2' + Q y_2 = 0$

$\Rightarrow v_1' y_1 + v_2' y_2 = R(x) \rightarrow 5$

To find v_1' and v_2' from the equations (4) and (5), applying Grass Multiplication rule, we have

$v_1' y_1 + v_2' y_2 = 0$

$v_1' y_1' + v_2' y_2 = R(x)$

$$\frac{v_1'}{-y_2 R(x)} = \frac{v_2'}{y_1 R(x)} = \frac{1}{y_1 y_2' - y_1' y_2} \qquad \begin{vmatrix} v_1' & v_2' & 1 \\ y_1 & y_2 & 0 \\ y_1' & y_2' & R(x) \end{vmatrix}$$

$$\Rightarrow v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \text{ and } v_2' = \frac{y_1 R(x)}{W(y_1, y_2)} \rightarrow 6$$

By integrating above equations with respect to 'x', we can find v_1 and v_2. Using these v_1 and v_2, $y_p = v_1 y_1 + v_2 y_2$ and also $y = y_g + y_p$ can be obtain of this process of finding the solutions of method of variation of parameters.

Problems

1. Find the particular solution of $y'' + y = \operatorname{cosec} x$

soln: consider $y'' + y = 0$

A.E is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$m = \pm i$

$\therefore y_g = c_1 \cos x + c_2 \sin x$

Let us take $y_1 = \cos x$ and $y_2 = \sin x$

let the required particular solution be,

$y_p = v_1 y_1 + v_2 y_2$

$y_p = v_1 \cos x + v_2 \sin x \rightarrow 1$

we know that, $v_1' = \dfrac{-y_2 R(x)}{W(y_1, y_2)} \rightarrow 2$ and

$$v_2' = \frac{y_1 R(x)}{W(y_1, y_2)} \rightarrow 3$$

Here $R(x) = \operatorname{cosec} x$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$
$$= \cos x (\cos x) - (-\sin x) \sin x$$
$$= \cos^2 x + \sin^2 x$$
$$= 1$$

$$v_1' = -\frac{\sin x \operatorname{cosec} x}{1} = -\sin x \frac{1}{\sin x}$$
$$= -1$$
$$v_1 = -\int dx = -x$$

and $$v_2' = \frac{\cos x \operatorname{cosec} x}{1} = \cos x \frac{1}{\sin x}$$
$$v_2' = \cot x$$
$$v_2 = \int \cot x \, dx$$
$$v_2 = \log \sin x$$

Hence the required particular solution is

$$y_p = -x \cos x + \sin x \log(\sin x)$$

2) Find the general solution of the equation $(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2$, given that $y_1 = x$ and $y_2 = x^2+1$

Soln:-

The given differential equation may be written as,

$$y'' - \frac{2x}{x^2-1} y' + \frac{2}{x^2-1} y = (x^2-1)$$

Hence, $y_1(x) = x$, $y_2(x) = (x^2+1)$ and $R(x) = x^2-1$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$
$$= x(2x) - 1(x^2+1)$$
$$= 2x^2 - x^2 - 1$$
$$W(y_1, y_2) = x^2-1$$

Let $y_p = v_1 y_1 + v_2 y_2$

$y_p = x v_1 + (x^2+1) v_2$ be the particular solutions

Where v_1 and v_2 are functions of x

We know that,

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} = -\frac{(x^2+1)(x^2-1)}{(x^2-1)}$$

$$v_1' = -(x^2+1)$$

$$v_1 = -\int (x^2+1)\, dx$$

$$= -(x^3/3 + x)$$

$$v_2' = \frac{y_1 R(x)}{W(y_1, y_2)} = \frac{x(x^2-1)}{(x^2-1)}$$

$$v_2' = x$$

$$v_2 = \int x\, dx$$

$$= x^2/2$$

$$y_p = -x(x^3/3 + x) + \frac{x^2}{2}(x^2+1)$$

$$= -x^4/3 - x^2 + \frac{x^4}{2} + \frac{x^2}{2}$$

$$= \frac{-2x^4+3x^4}{6} + \frac{x^2-2x^2}{2}$$

$$= \frac{x^4}{6} - \frac{x^2}{2}$$

$$= \frac{x^2}{6}(x^2-3)$$

Hence the general solution is,

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$y = c_1 x + c_2 [x^2+1] + \frac{\cancel{x}}{6}(x^2-3)$$

Using variation of parameter, find the particular solution of the following equation

$y'' \cancel{+} 2y' + y = 2x$

Soln:

consider $y'' - 2y' + y = 2x$

$$A.E \Rightarrow m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$\therefore m = 1 \text{ (twice)}$$

$$y_g = (c_1 x + c_2) e^x$$

$$y_g = c_1 x e^x + c_2 e^x \longrightarrow A$$

Let us take $y_1 = xe^x$ and $y_2 = e^x$

Let the required particular solution be,

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p = v_1 x e^x + v_2 e^x \rightarrow 1$$

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \text{ and } \rightarrow 2$$

$V_2' = \dfrac{y_1 R(x)}{W(y_1,y_2)} \rightarrow 3$

Here $R(x) = 2x$

$W(y_1,y_2) = y_1 y_2' - y_2 y_1'$

$= x e^{x}(e^{x}) - e^{x}(xe^{x} + e^{x})$

$= x e^{2x} - x e^{2x} - e^{2x}$

$\therefore W(y_1,y_2) = -e^{2x}$

$V_1' = \dfrac{-y_2 R(x)}{W(y_1,y_2)} = \dfrac{e^{x}\, 2x}{e^{2x}}$

$= \dfrac{2x}{e^{x}}$

$v_1' = 2x e^{-x}$

$V_1 = \int 2x e^{-x}$ $\qquad u = x \quad dv = e^{-x}$

$\qquad\qquad du = dx \quad V = -e^{-x}$

$= 2\{-x e^{-x} + \int e^{-x} dx\}$

$= -2x e^{-x} + 2\int e^{-x} dx$

$= -2x e^{-x} - 2 e^{-x}$

$V_2' = \dfrac{y_1 R(x)}{W(y_1,y_2)} = \dfrac{x e^{x} 2x}{-e^{2x}} = \dfrac{2x^2}{e^{x}}$

$V_2 = 2\int x^2 e^{-x} dx$ $\qquad \boxed{\begin{array}{ll} u = x^2 & dv = e^{-x} \\ du = 2x\,dx & v = -e^{-x} \end{array}}$

$= 2[x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx$

$= 2(-x^2 e^{-x} + 2\int x e^{-x} dx)$

$V_2 = -2x^2 e^{-x} + 4x e^{-x} + 4e^{-x}$

Hence the required particular solution is

$y_p = v_1 y_1 + v_2 y_2$

$$= \frac{x^2}{6}(x^2-3)$$

Hence the general solution is,

$$y = c_1 y_1 + c_2 y_2 + y_p$$

$$y = c_1 x + c_2 (x^2+1) + \frac{x^2}{6}(x^2-3)$$

Using variation of parameter, find the particular solution of the following equation

$$y'' - 2y' + y = 2x$$

Soln:

consider $y'' - 2y' + y = 2x$

$$A.E \Rightarrow m^2 - 2m + 1 = 0$$

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$$y_g = (c_1 x + c_2) e^x$$

$$y_g = c_1 x e^x + c_2 e^x \longrightarrow A$$

Let us take $y_1 = x e^x$ and $y_2 = e^x$

Let the required particular solution be,

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p = v_1 x e^x + v_2 e^x \rightarrow 1$$

$$v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \text{ and } \rightarrow 2$$

$$V_2' = \frac{y_1 R(x)}{W(y_1, y_2)} \rightarrow 3$$

Here $R(x) = 2x$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= x e^{x}(e^{x}) - e^{x}(x e^{x} + e^{x})$$

$$= x e^{2x} - x e^{2x} - e^{2x}$$

$$\therefore W(y_1, y_2) = -e^{2x}$$

$$V_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} = \frac{\cancel{-} e^{x}\, 2x}{\cancel{-} e^{2x}}$$

$$= \frac{2x}{e^{x}}$$

$$V_1' = 2x e^{-x}$$

$$V_1 = \int 2x e^{-x}$$

$u = x \quad dv = e^{-x}$

$du = dx \quad v = -e^{-x}$

$$= 2\{-x e^{-x} + \int e^{-x} dx\}$$

$$= -2x e^{-x} + 2\int e^{-x} dx$$

$$= -2x e^{-x} - 2 e^{-x}$$

$$V_2' = \frac{y_1 R(x)}{W(y_1, y_2)} = \frac{x e^{x}\, 2x}{-e^{2x}} = -\frac{2x^2}{e^{x}}$$

$$V_2 = -2\int x^2 e^{-x} dx$$

$u = x^2 \quad dv = e^{-x}$

$du = 2x\,dx \quad v = -e^{-x}$

$$= -2\left[x^2(-e^{-x}) - \int -e^{-x}(2x)\, dx\right]$$

$$= -2\left(-x^2 e^{-x} + 2\int x e^{-x} dx\right)$$

$$V_2 = +2x^2 e^{-x} + 4x e^{-x} + 4e^{-x} = 2e^{-x}(x^2 + 2x + 2)$$

Hence the required particular solution is

$$y_p = V_1 y_1 + V_2 y_2$$

$y_p = v_1 y_1 + v_2 y_2$

$= xe^{x}(-2e^{-x}x - 2e^{-x}) + 2e^{-x}(x^2+2x+2)e^{x}$

$= -2x^2 \cancel{e^{x}} e^{-x} - 2xe^{x}e^{-x} + 2x^2 \cancel{e^{-x}} \cancel{e^{x}} + 4xe^{-x}e^{x} + 4e^{x}e^{-x}$

$= 2x + 4$

$y_p = 2(x+2)$

Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \rightarrow (1)$$

is called power series in x.

The series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1(x - x_0) + a_2(x-x_0)^2 + \cdots \rightarrow 2$

is a power series in $(x - x_0)$.

The series (1) converge at a point x

$\lim_{n\to\infty} \sum_{m=0}^{m} a_n x^n$ exists.

consider the equation $y' = y \rightarrow (1)$ this equation has a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \rightarrow 1$$

which converge for $|x| < R$ with $R > 0$.

is called Radius of convergent of a series.

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots$

$\therefore y' = y$

Equating the corresponding terms.

UNIT - II

Regular singular points

If a point x_0 is a singular point of the differential equation $y'' + p(x)y' + Q(x)y = 0 \rightarrow ①$ if one or the other of the coefficient functions $p(x)$ and $Q(x)$ fails to be analytic at x_0.

A singular point x_0 of equation ① is said to be regular if the functions $(x-x_0)\,p(x)$ and $(x-x_0)^2 Q(x)$ are analytic and irregular otherwise.

Ex: 1

Consider Legendre's equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

This equation can be written as,

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{p(p+1)}{(1-x^2)}y = 0$$

It is clear that $x=1$, and $x=-1$ are singular points. The first is regular because

$$(x-1)\,p(x) = \frac{2x}{x+1} \text{ and } (x-1)^2 Q(x) = -\frac{(x-1)\,p(p+1)}{x+1}$$

are analytic at $x=1$, and the second is also regular.

$\therefore x = 1, -1$ are regular singular point.

Ex: 2

consider the Bessel's equation

$x^2y'' + xy' + (x^2 - p^2)y = 0$.

If this is written in the form,

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

$P(x) = 1/x$, $Q(x) = \frac{x^2 - p^2}{x^2}$ are analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular point

Ex: 3

consider the equation $2x^2y'' + x(2x+1)y' - y = 0$

p(x) This equation can be written as,

$$y'' + \frac{x(2x+1)}{2x^2}y' - \frac{y}{2x^2} = 0$$

$$P(x) = \frac{x(2x+1)}{2x^2}, \quad Q(x) = -\frac{1}{2x^2}$$

$$= \frac{(2x+1)}{2x}$$

$P(x)$ & $Q(x)$ are a not analytic at $x=0$

$\therefore$ $x=0$ is a singular point.

$$(x-x_0)P(x) = \frac{x(2x+1)}{2x}, \quad (x-x_0)^2Q(x) = \frac{-x^2}{2x^2}$$

$$= \frac{2x+1}{2}, \quad (x-x_0)^2Q(x) = -1/2$$ are

analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular point.

Frobenius series

If $x=0$ is a regular singular point then we can express the solution as of the form,

$$y = x^m \sum_{m=0}^{\infty} a_m x^m,$$ where m is zero or positive or negative integers.

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots)$$

which is called Frobenius series.

Ex: Solve $2x^2y'' + x(2x+1)y' - y = 0$

$$y'' + \frac{(2x+1)}{2x}y' - \frac{y}{2x^2} = 0$$

$$y'' + \frac{(x+1/2)}{x}y' - \frac{y/2}{x^2} = 0 \rightarrow (1)$$

$$P(x) = \frac{x+1/2}{x}; \quad Q(x) = -1/x^2$$

$P(x)$ and $Q(x)$ are not analytic at $x=0$,

$\therefore x=0$ is a regular singular point

Let $y = x^m \sum_{m=0}^{\infty} a_m x^m$ be the solution

$$y = a_0 x^m + a_1 x^{m+1} + \cdots \rightarrow (2)$$

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + \cdots \rightarrow (3)$$

$$y'' = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + \cdots \rightarrow (4)$$

Sub these values in (1),

$$[m(m-1)a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + \dots] + \frac{x+1/2}{x}$$

$$[m a_0 x^{m-1} + (m+1) a_1 x^m + \dots] - \frac{1}{2x^2}(a_0 x^m + a_1 x^{m+1} + \dots) = 0$$

$$a_0 m(m-1) + a_1 (m+1)m x + a_2 (m+2)(m+1) x^2 + \dots$$

$$+ (1/2 + x)\left(a_0 m + a_1 (m+1) x + a_2 (m+2) x^2 + \dots\right]$$

$$-1/2 (a_0 + a_1 x + \dots) = 0$$

Equating constant term to zero,

$$\{m(m-1) + 1/2 (m-1)\} a_0 = 0$$

since $a_0 \neq 0$

$$m(m-1) + 1/2 m - 1/2 = 0$$

$$2m^2 - 2m + m - 1 = 0$$

$$2m(m-1) + 1(m-1) = 0$$

$$(2m+1)(m-1) = 0$$

$$m = 1, \text{ or } m = -1/2$$

Equating coefficient of x,

$$m(m+1) + \left(\frac{m+1}{2} - 1/2\right) a_1 + a_0 m = 0 \rightarrow (5)$$

coefficient of x^2,

$$\left((m+1)(m+2) + \frac{m+2}{2} - 1/2\right) a_2 + a_1 (m+1) = 0 \rightarrow (6)$$

coefficient of x^3,

$$\left[(m+3)(m+2) + \frac{m+3}{2} - 1/2\right] a_3 + a_2 (m+2) = 0 \rightarrow (7)$$

put $m = 1$ in (5),

$(2+1-\frac{1}{2})a_1 + a_0 = 0$

$\frac{5}{2} a_1 = -a_0$

$a_1 = -\frac{2}{5} a_0$

put $m=1$ in (6),

$(6+\frac{3}{2}-\frac{1}{2})a_2 + 2a_1 = 0$

$7a_2 - 2\times\frac{2}{5} a_0 = 0$

$7a_2 = \frac{4}{5} a_0$

$a_2 = \frac{4}{35} a_0$

put $m=1$ in (7)

$(12+2-\frac{1}{2})a_3 + 3a_2 = 0$

$\frac{27}{2} a_3 = -3a_2$

$a_3 = \frac{-6}{27} \times \frac{4}{35} a_0$

$= \frac{-8}{315} a_0$

$y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots)$

put $m=1$,

$y = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$

$= a_0 x (1 - \frac{2}{5} x + \frac{4}{35} x^2 - \frac{8}{315} x^3 + \cdots)$

put $a_0 = 1$

$y_1 = x(1 - \frac{2}{5} x + \frac{4}{35} x^2 - \frac{8}{315} x^3 + \cdots)$

For $m = -\frac{1}{2}$ in (5)

$$-\tfrac{1}{2}(-\tfrac{1}{2}+1)\left[\frac{-\tfrac{1}{2}+1}{2}-\tfrac{1}{2}\right]a_1-\frac{a_0}{2}=0$$

$$(-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{2})\,a_1=\frac{a_0}{2}$$

$$-\frac{a_1}{2}=\frac{a_0}{2}$$

$$\therefore \boxed{a_1=-a_0}$$

$$⑥ \Rightarrow \left[\tfrac{1}{2}\cdot\tfrac{3}{2}+\tfrac{3}{4}-\tfrac{1}{2}\right]a_2+\frac{a_1}{2}=0$$

$$a_2=-\frac{a_1}{2}$$

$$a_2=\frac{a_1}{2}=\frac{-a_0}{2}$$

$$⑦ \Rightarrow \left[\frac{5}{2}\cdot\frac{3}{2}+\frac{5}{4}-\frac{1}{2}\right]a_3+a_2(\tfrac{3}{2})=0$$

$$\left(\frac{15}{4}+\frac{5}{4}-\tfrac{1}{2}\right)a_3=-\frac{3a_2}{2}$$

$$\frac{9}{2}\,a_3=-\frac{3a_2}{2}$$

$$a_3=-\tfrac{1}{6}a_0$$

$$y_2=a_0\,x^{-1/2}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)$$

put $a_0=1$,

$$y_2=x^{-1/2}(1-x+x^2/2-x^3/6+\cdots)$$

The general solution is,

$$y=c_1y_1+c_2y_2$$

$$y=c_1\left[x(1-2/5\,x+4/35\,x^2-\cdots)\right]+c_2\left[x^{-1/2}(1-x+x^2/2-x^3/6+\cdots)\right]$$

Gauss Hypergeometric Equation

Solve the differential Equation,

$$x(1-x)y''+[c-(a+b+1)x]y'-aby=0 \rightarrow (1)$$

$$y''+\frac{c-(a+b+1)}{x(1-x)}y'-\frac{aby}{x(1-x)}=0$$

Here $P(x)=\frac{c-(a+b+1)x}{x(1-x)}$; $Q(x)=\frac{-ab}{x(1-x)}$

$P(x)$ and $Q(x)$ are not analytic at $x=0$.

$$xP(x)=[c-(a+b+1)x](1-x)^{-1}$$

$$=[c-(a+b+1)x](1+x+x^2+\cdots)$$

$$=[c-(a+b+1)x]+[c-(a+b+1)x]x+\cdots$$

$$x^2Q(x)=-abx(1-x)^{-1}$$

$$=-abx(1+x+x^2+\cdots)$$

clearly both $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$.

$\therefore$ $x=0$ is a regular singular point.

$|||^{ly}$ we can show $x=1$ is also a regular singular point.

Let $y=\sum_{n=0}^{\infty} a_n x^{m+n} \rightarrow (2)$

be a solution of equation (1)

$$y'=\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1}$$

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$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1)\, a_n\, x^{m+n-2}$$

Sub these values in ①,

$$x(1-x)\sum_{n=0}^{\infty} (m+n)(m+n-1)a_n\, x^{m+n-2} + [c-(a+b+1)x]$$

$$\sum_{n=0}^{\infty} a_n (m+n)\, x^{m+n-1} - ab\sum_{n=0}^{\infty} a_n\, x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (m+n)(m+n-1)\, x^{m+n-1} - \sum_{n=0}^{\infty} a_n (m+n)(m+n-1)\, x^{m+n}$$

$$+ c\sum_{n=0}^{\infty} a_n (m+n)\, x^{m+n-1} - \sum_{n=0}^{\infty} (a+b+1)\, a_n (m+n)\, x^{m+n}$$

$$- ab\sum_{n=0}^{\infty} a_n\, x^{m+n} = 0$$

$$a_0 m[(m-1)+c]x^{m-1} + \sum_{n=1}^{\infty} a_n (m+n)[m+n-1+c]\, x^{m+n-1}$$

$$- \sum_{n=0}^{\infty} a_n\{[(m+n)(m+n+a+b)] + ab\}\, x^{m+n} = 0$$

Indical equation is

$$m[(m-1)+c] = 0$$

$$m-1+c = 0$$

$$m = 0,\ m = 1-c$$

Recurrence formula is

$$a_{n+1}(m+n+1)(m+n+c) = a_n\{(m+n)(m+n+a+b)+ab\}$$

$$\therefore a_{n+1} = \frac{(m+n)(m+n+a+b)+ab}{(m+n+1)(m+n+c)} a_n$$

put $m=0$,

$$a_{n+1} = \frac{n(n+a+b)+ab}{(n+1)(n+c)} a_n$$

$$= \frac{n^2+(a+b)n+ab}{(n+1)(n+c)} a_n$$

$$a_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)} a_n$$

Take $a_0 = 1$, $n=0$

$$a_1 = \frac{ab}{c}$$

put $n=1$, $$a_2 = \frac{(a+1)(b+1)}{2(c+1)} a_1$$

$$= \frac{ab(a+1)(b+1)}{2c(c+1)}$$

put $n=2$, $$a_3 = \frac{(a+2)(b+2)}{3(c+2)} a_2$$

$$= \frac{ab(a+1)(a+2)(b+1)(b+2)}{2\cdot 3\, c(c+1)(c+2)}$$

Equation (2) $\Rightarrow$ $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

$m = 0 \Rightarrow y = \sum_{n=0}^{\infty} a_n x^n$

$y = a_0 + a_1 x + a_2 x^2 + \cdots$

$y = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2\, c(c+1)} x^2 +$

$\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\, c(c+1)(c+2)} x^3 + \cdots$

$$y = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)(c+2)\cdots(c+n-1)} x^n.$$

This is known as Gauss hypergeometric series and is denoted by $F(a, b, c, x)$.

Ex: 1

P.T $F(1, b, b, x) = \frac{1}{1-x}$

soln:-

$$F(a, b, c, x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)\cdots(c+n-1)} x^n$$

$$F(1, b, b, x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdots n\, (b)(b+1)\cdots(b+n-1)}{n!\, b(b+1)\cdots(b+n-1)} x^n$$

$$= 1 + \sum_{n=1}^{\infty} x^n$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$= (1-x)^{-1} = \frac{1}{1-x}.$$

$\therefore F(1, b, b, x) = \frac{1}{1-x}$

Ex:2

P.T $\log(1+x) = x F(1,1,2,-x)$

$$F(1,1,2,-x) = 1+\sum_{n=1}^{\infty} \frac{1\cdot 2 \cdots n \cdot 1\cdot 2\cdots n}{n!\, 2\cdot 3 \cdots (n+1)} (-x)^n$$

$$= 1+\sum_{n=1}^{\infty} \frac{n!}{n(n+1)!} (-x)^n$$

$$= 1+\sum_{n=1}^{\infty} \frac{(-x)^n}{n+1}$$

$$= 1+\left(-x/_2 + x^2/_3 - \frac{x^3}{4} + \cdots\right)$$

$$F(1,1,2,x) = 1 - x/_2 + x^2/_3 - x^3/_4 + \cdots$$

$$x[F(1,1,2,x)] = x - x^2/_2 + x^3/_3 - x^4/_4 + \cdots$$

$$xF(1,1,2,x) = \log(1+x)$$

Ex: 3

p.T $\sin^{-1} x = x F(1/2, 1/2, 3/2, x^2)$

$$F(1/2, 1/2, 3/2, x^2) = 1+\sum_{n=1}^{\infty} \frac{1/2 \cdot 3/2 \cdots (n-1/2) \cdot 1/2 \cdot 3/2 \cdots (n-1/2)}{n!\; 3/2 \cdot 5/2 \cdots (n+1/2)} x^{2n}$$

$$= 1+ \frac{1/2 \cdot 1/2}{3/2} x^2 + \frac{(1/2 \cdot 3/2)^2}{2!\, 3/2 \cdot 5/2} x^4 + \cdots$$

$$= x + x^3/_6 + \frac{3}{40} x^5 + \cdots$$

$$F(1/2, 1/2, 3/2, x^2) = \sin^{-1} x$$

The point at infinity

Consider the equation $y'' + p(x)y' + Q(x)y = 0 \rightarrow (1)$

put $t = 1/x \Rightarrow x = 1/t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt}(-1/x^2) = -t^2 \frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(-t^2 \frac{dy}{dt}\right)(-1/x^2)$$

$$= -t^2\left[-t^2 \frac{d^2y}{dt^2} - \frac{dy}{dt}(2t)\right]$$

$$y'' = t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}$$

Hence (1) $\Rightarrow$

$$\left[t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt}\right] + p(1/t)(-t^2)\frac{dy}{dt} + Q(1/t)y = 0$$

$$t^4 \frac{d^2y}{dt^2} + (2t^3 - p(1/t)t^2)\frac{dy}{dt} + Q(1/t)y = 0$$

$$\frac{d^2y}{dt^2} + \left[2/t - 1/t^2\, p(1/t)\right]\frac{dy}{dt} + 1/t^4\, Q(1/t)y = 0$$

This is a 2nd order homogeneous equations and obvious $t = 0$ is a singular point.

Legendre polynomials :-

consider the equation,

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow (1)$$

put $t = \frac{1}{2}(1-x)$

This makes $x=1$ correspond to $t=0$. and transforms (1) into,

$$t(1-t)y'' + (1-2t)y' + n(n+1)y = 0 \rightarrow (2)$$

This is hypergeometric equation with $a = -n$, $b = n+1$ and $c = 1$.

$\therefore$ Equation (2) has a polynomial solution near $t=0$.

Hence $y_1 = F(-n, n+1, 1, t)$ is its 1^{st} solution.

To find the 2^{nd} solution :.

Let to use of known solution to find another

Let $y_2 = vy_1$

where $v' = \frac{1}{y_1^2} e^{-\int p\,dx} = \frac{1}{y_1^2} e^{-\int \frac{1-2t}{t(1-t)} dt} \rightarrow (3)$

Hence $v' = \frac{1}{y_1^2} e^{-\log t(1-t)}$

$\int \frac{f'(x)}{f(x)} dx = \log f(x)$

$$= \frac{1}{y_1^2} e^{\log[t(1-t)]^{-1}}$$

$$= \frac{1}{y_1^2} \frac{1}{t(1-t)} = \frac{1}{t[y_1^2(1-t)]}$$

we know that y_1^2 is polynomial with constant term 1 and

$\therefore \frac{1}{y_1^2(1-t)}$ is an analytic function of the form

$$1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$$

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Hence $v' = \frac{1}{t}(1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)$

$= \frac{1}{t} + a_1 + a_2 t + a_3 t^2 + \cdots$

$v = \log t + a_1 t + \frac{a_2}{2} t^2 + \frac{a_3}{3} t^3 + \cdots$

$y_2 = v y_1$

$= y_1 [\log t + a_1 t + \frac{a_2}{2} t^2 + \cdots]$

∴ The general solution equation (2)

$y = c_1 y_1 + c_2 y_2 \rightarrow 4$

Equation (4) has $\log t$ in y_2 and its bounded near $t=0$ iff the constant term $c_2 = 0$

The solution (1) bounded near $x=1$.

If $y = F[-n, n+1, 1, \frac{1}{2}(1-x)]$.

Rodrigue's formula:

To derive the rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$\left[\text{Here } P_n(x) = F[-n, n+1, 1, \tfrac{1}{2}(1-x)]\right]$

Proof:

The n^{th} degree Legendre polynomial as,

$$P_n(x) = F[-n, n+1, 1, \tfrac{1}{2}(1-x)]$$

we know that,

$$F[a, b, c, x] = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\; c(c+1)\cdots(c+n-1)} x^n$$

$$P_n(x) = 1 + \frac{(-n)(n+1)}{(1!)^2}\left(\frac{1-x}{2}\right) + \frac{(-n)(-n+1)(n+1)(n+2)}{(2!)^2}\left(\frac{1-x}{2}\right)^2 + \cdots$$

$$\frac{(-n)(-n+1)(-n+2)\cdots(-n+(n-1))(n+1)(n+2)\cdots n+1+(n-1)}{(n!)^2}\left(\frac{1-x}{2}\right)^n$$

$$= 1 + \frac{n(n+1)}{(1!)^2}\left(\frac{x-1}{2}\right) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2}\left(\frac{x-1}{2}\right)^2 + \cdots,$$

$$\frac{n(n-1)\cdots(n-n-1)(n+1)(n+2)\cdots 2n}{(n!)^2}\left(\frac{x-1}{2}\right)^n$$

$$P_n(x) = 1 + \frac{n(n+1)}{(1!)^2 2}(x-1) + \frac{n(n-1)(n+1)(n+2)}{(2!)^2 \cdot 2^2}(x-1)^2 + \cdots$$

$$+ \frac{n(n-1)(n-2)\cdots(n+1)(n+2)\cdots 2n}{(n!)^2 \cdot 2^n}(x-1)^n \rightarrow (1)$$

(1) Implies that $P_n(x)$ is a polynomial of degree n, and this polynomial contains only even (or) only odd powers of x. According as n is even or odd.

$$\therefore P_n(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \cdots \rightarrow (2)$$

where equation (2) ends with a_0 when n is even and $a_1 x$ when n is odd.

Now the recursion formula is,

$$a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n$$

$$a_k = \frac{-(n-k+2)(n+k-1)}{k(k-1)} a_{k-2}$$

$$\Rightarrow a_{k-2} = \frac{-k(k-1)}{(n-k+2)(n+k-1)} a_k$$

put $k = n, n-2, n-4, \ldots$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$= \frac{-(n-2)(n-3)}{4(2n-3)} \left[\frac{-n(n-1)}{2(2n-1)} a_n \right]$$

$$= \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}$$

$$a_{n-6} = \frac{-(n-4)(n-5)}{6\cdot(2n-5)} a_{n-4}$$

$$= \frac{-(n-4)(n-5)}{6(2n-5)} \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} a_n.$$

$$= -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)} a_n$$

$\therefore$ Equation (2) becomes,

$$P_n(x) = a_n \, x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} x^{n-4} + \ldots$$

$$+ \frac{(-1)^k n(n-1)(n-2)\ldots(n-\overline{2k-1})}{2\cdot 4\ldots 2k(2n-1)(2n-3)\ldots(2n-\overline{2k-1})} x^{n-2k} + \ldots$$

$$= \frac{(2n)!}{(n!)^2 2^n}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdots(2n-1)(2n-1)}x^{n-4} + \cdots\right.$$

$$\left. + \frac{(-1)^k n(n-1)\cdots(n-2k+1)}{2^k k!(2n-1)(2n-3)\cdots(2n-2k+1)}x^{n-2k} + \cdots\right\} \quad \rightarrow (3)$$

since $n(n-1)\cdots\cdots(n-2k+1) = \frac{n!}{(n-2k)!}$

and,

$(2n-1)(2n-3)\cdots\cdots(2n-2k+1)$

$$= \frac{(2n-2k+1)(2n-2k+2)\cdots(2n-3)(2n-2)(2n-1)2n}{(2n-2k+2)\cdots\cdots(2n-2)\cdot 2n}$$

$$= \frac{(2n)!}{(2n-2k)!}\,\frac{1}{2^k(n-k+1)\cdots(n-1)n}$$

$$= \frac{(2n)!\,(n-k)!}{(2n-2k)!\,2^k n!}$$

The coefficient of x^{n-2k} in (3) is

$$(-1)^k \frac{n!}{2^k k!(n-2k)!}\cdot\frac{(2n-2k)!\,2^k n!}{(2n)!\,(n-k)!} = \frac{(-1)^k (n!)^2 (2n-2k)!}{k!\,(2n)!\,(n-k)!\,(n-2k)!}$$

$\therefore$ The equation (3) can be written as,

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!}x^{n-2k}$$

where $[n/2]$ is the greatest integer $\leq n/2$

$$= \sum_{K=0}^{n/2} \cdot \frac{(-1)^K}{2^n K!\,(n-K)!} \left\{ \frac{(2n-2K)!}{(n-2K)!} x^{n-2K} \right\}$$

$$= \sum_{K=0}^{n/2} \frac{(-1)^K}{2^n K!\,(n-K)!} \left\{ \frac{d^n}{dx^n} (x^{2n-2K}) \right\}$$

Since $\frac{d^n}{dx^n}(x^{2n-2k}) = (2n-2K)(2n-2K-1)(2n-2K-2)\ldots\ldots (2n-2K-n+1)\, x^{(2n-2K-1)}$

$$= \frac{(2n-2K)\ldots(2n-2K-(n-1))\,(2n-2k-n)\ldots 3\cdot 2\cdot 1}{(2n-2K-n)\ldots\ldots 3\cdot 2\cdot 1} x^{n-2K}$$

$$= \frac{(2n-2K)!}{(n-2K)!} x^{n-2K}$$

$$\therefore P_n(x) = \frac{1}{2^n n!} \sum_{K=0}^{n/2} \frac{n!\,(-1)^K}{K!\,(n-K)} \frac{d^n}{dx^n} (x^{2n-2K})$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{K=0}^{n/2} (-1)^K \frac{n!}{K!\,(n-K)!} (x^2)^{n-K}$$

$$nC_K = \frac{n!}{K!\,(n-K)!}$$

Hence, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{K=0}^{n/2} (-1)^K nC_K (x^2)^{n-K}$

If we Extended the range of the sum by letting K various from 0 to n, nothing will be change in the Σ, because the new terms are of and also. the n^{th} derivatives are zero.

$$\therefore P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{K=0}^{n} (-1)^K \, {}^nC_K \, (x^2)^{n-K}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

This is called Rodrigue's formula.

Note:

By Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)$$

Hence, $P_1(x) = \frac{1}{2!\,1!} \frac{d}{dx}(x^2-1) \Rightarrow \frac{1}{2}(2x) \Rightarrow x$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2-1) \Rightarrow \frac{1}{8}\frac{d}{dx}\left(2(x^2-1)2x\right)$$

$$= \frac{1}{2} \frac{d}{dx} (x^3-x)$$

$$P_2(x) = \frac{1}{2} (3x^2-1)$$

III^{ly} $P_3(x) = \frac{1}{2} (5x^3-3x)$

Properties of legender's polynomial:

Orthogonal property of legender's polynomial

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0\ ; & \text{If } m \neq n \\ \dfrac{2}{2n+1}\ ; & \text{If } m = n \end{cases}$$

Here $P_0(x), P_1(x), \ldots P_n(x)$ is the sequence of orthogonal functions. In the interval $-1 \leq x \leq 1$.

proof!

part - I $m \neq n$

Let $f(x)$ be a function with atleast 'n' continuous derivatives on $-1 \leq x \leq 1$

Consider,

$$I = \int_{-1}^{1} f(x)\, P_n(x)\, dx$$

$$= \int_{-1}^{1} f(x) \cdot \left\{ \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \right\} dx$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2-1)^n\, dx$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} f(x)\, d\left\{ \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right\}$$

$$= \frac{1}{2^n n!} \left\{ \left[f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right]_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n f'(x)\, dx \right\}$$

$$= -\frac{1}{2^n n!} \int_{-1}^{1} f'(x)\, d\left\{ \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n \right\}$$

$$= -\frac{1}{2^n n!}\left\{\left[f'(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n f''(x)\,dx\right\}$$

$$= \frac{(-1)^2}{2^n n!}\int_{-1}^{1} f''(x)\, d\left\{\frac{d^{n-3}}{dx^{n-3}}(x^2-1)^n\right\}$$

If we continue this process,

$$I = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^n(x)\,(x^2-1)^n\,dx.$$

If $f(x) = P_m(x)$ with m is $<n$, then $f^n(x) = 0$ $m<n$

$\therefore I = 0$

$$\Rightarrow \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

Part-II: If $m = n$

put $f(x) = P_n(x)$

$f^n(x) = P_n^n(x)$

$$f^n(x) = \frac{(2n!)}{(n!)\,2^n}$$

$$\text{Hence } I = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^n(x)\,(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\frac{(2n)!}{(n!)\,2^n}(x^2-1)^n\,dx$$

$$= \frac{(2n)!\,(-1)^{2n}}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx$$

$$= \frac{(2n!)}{2^{2n}(n!)^2} \, 2\int_0^1 (1-x^2)^n dx$$

put $x = \sin t \Rightarrow dx = \cos\theta \, d\theta$

when $x = 0 \Rightarrow \theta = 0$

When $x = 1 \Rightarrow \theta = \pi/2$

$$\therefore I = \frac{2(2n!)}{2^{2n}(n!)^2} \int_0^{\pi/2} (1-\sin^2\theta)^n \cos\theta \, d\theta$$

$$= \frac{2(2n!)}{2^{2n}(n!)^2} \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$= \frac{2(2n!)}{2^{2n}(n!)^2} \frac{2n(2n-2)(2n-4)\cdots 2}{(2n+1)(2n-1)(2n-3)\cdots 3 \cdot 1}$$

$$= \frac{2(2n!)}{2^{2n}(n!)^{\cancel{2}}} \frac{2^n \cancel{n!}}{(2n+1)(2n-1)\cdots 3\cdot 1}$$

$$= \frac{2(2n!)}{2^n n!} \frac{2\cdot 4\cdots(2n-2)\cdot 2n}{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)(2n+1)}$$

$$= \frac{2(2n!)}{\cancel{2^n}(\cancel{n!})^2} \cdot \frac{\cancel{2^n}\cancel{n!}}{(2n+1)!(2n+1)\,2n!}$$

$$= \frac{2}{(2n+1)}$$

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$$\therefore \int_{-1}^{1} P_n(x) P_n(x)\, dx = \frac{2}{2n+1}, \text{ If } m=n$$

Hence,

$$\int_{-1}^{1} P_m(x) P_n(x) = \begin{cases} 0, & \text{If } m \neq n \\ \frac{2}{2n+1}, & \text{If } m=n \end{cases}$$

Bessel Function:

Def:

The differential equation, $x^2y''+xy'+(x^2-p^2)y=0 \rightarrow (1)$ called bessel equation of order p. where p is a non-negative constant. Its solutions are called bessel function.

Solution of Bessel Equation

Given that, $x^2y''+xy'+(x^2-p^2)y=0 \rightarrow (1)$

$$\Rightarrow y''+\frac{1}{x}y'+\frac{x^2-p^2}{x^2}y=0$$

Hence, $p(x)=1/x$ and, $Q(x)=\frac{x^2-p^2}{x^2}$

Hence $x\,p(x)=1$ and $x^2Q(x)=x^2-p^2$

$xp(x)$ and $x^2Q(x)$ are analytic at $x=0$.

The indicial equation is, $m(m-1)+m-p^2=0$

$\Rightarrow m^2-p^2=0$

$m=\pm p$ [$\because p$ is non-negative]

Let us take $m_1=p$ and $m_2=-p$

$\therefore$ Equation (1) has a solution of the form,

$$y = x^p \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+p} \rightarrow (2) \text{ where } a_0 \neq 0$$

This power series converges for all values of x

Now, $y' = \sum_{n=0}^{\infty} a_n (n+p) x^{n+p-1}$

$$y'' = \sum_{n=0}^{\infty} a_n (n+p)(n+p-1) x^{n+p-2}$$

$$+x^2 y'' = \sum_{n=0}^{\infty} a_n (n+p)(n+p-1) x^{n+p}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+p) x^{n+p}$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+p+2} = \sum_{n=0}^{\infty} a_{n-2} x^{n+p}$$

Equation (1) $\Rightarrow x^2 y'' + xy' + (x^2 - p^2) y = 0$

$$\sum_{n=0}^{\infty} a_n [(n+p)(n+p-1) x^{n+p}] + \sum_{n=0}^{\infty} a_n (n+p) x^{n+p} + \sum_{n=0}^{\infty} a_{n-2} x^{n+p}$$

$$- \sum_{n=0}^{\infty} p^2 a_n x^{n+p} = 0$$

$$\sum_{n=0}^{\infty} \{ a_n [(n+p)(n+p-1) + (n+p) - p^2] + a_{n-2} \} x^{n+p} = 0$$

$$= \sum_{n=0}^{\infty} \{ a_n (n^2 + np - \cancel{n} + np + \cancel{p^2} - \cancel{p} + \cancel{n} + \cancel{p} - \cancel{p^2}) + a_{n-2} \} x^{n+p} = 0$$

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$$= \sum_{n=0}^{\infty} \{a_n (n^2+2np) + a_{n-2}\} x^{n+p} = 0$$

$$= \sum_{n=0}^{\infty} \{a_n (n+2p) n + a_{n-2}\} x^{n+p} = 0$$

Hence the recursion formula is,

$$a_n = \frac{-a_{n-2}}{n(n+2p)}$$

Put $n=1 \Rightarrow a_1 = -\frac{a_{-1}}{1(1+2p)} = 0$

$n=2$, $a_2 = \frac{-a_0}{2(2+2p)} \neq 0$

$n=3$, $a_3 = \frac{-a_1}{3(3+2p)} \Rightarrow 0$

$n=4$, $a_4 = \frac{-a_2}{4(4+2p)} \Rightarrow \frac{a_0}{2\cdot 4(2+2p)(4+2p)}$

$|||^{ly}$ $a_5 = a_7 = a_9 = \dots = 0$

$$(2) \Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+p} = a_0 x^p + a_1 x^{p+1} + a_2 x^{p+2} + \dots$$

$$y = a_0 x^p - \frac{a_0 x^{p+2}}{2(2+2p)} + \frac{a_0 x^{p+4}}{2\cdot 4(2+2p)(4+2p)} + \frac{a_0 x^{p+6}}{2\cdot 4\cdot 6(2+2p)(4+2p)(6+2p)}$$

$$= a_0 x^p \left\{ 1 - \frac{x^2}{2^2(1+p)} + \frac{x^4}{2^4\cdot 1\cdot 2\cdot (1+p)(2+p)} - \frac{x^6}{2^6\cdot 1\cdot 2\cdot 3(1+p)(2+p)(3+p)} \right\}$$

$$= a_0 x^p \left\{ 1 - \frac{x^2}{2^2(1+p)} + \frac{x^4}{2^4 \cdot 2!(1+p)(2+p)} - \frac{x^6}{2^6 \cdot 3!(1+p)(2+p)(3+p)} + \cdots \right\}$$

$$y = a_0 x^p \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (p+1)(p+2) \cdots (p+n)} \right\} \rightarrow (4)$$

The Bessel Function of 1st kind of order p

The Bessel function of 1st kind of order p is dended by $J_p(x)$, is defined by putting a_0,

$a_0 = \frac{1}{2^p p!}$ in equation (4),

$$\therefore J_p(x) = \frac{x^p}{2^p p!} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (p+1)(p+2) \cdots (p+n)} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n+p} n! (p+n)!}$$

$$\therefore J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n! (p+n)!}$$

UNIT - III

Linear Systems:

Systems of only two first order equations in two unknown functions of the form

$$\left.\begin{aligned} \frac{dx}{dt} &= F(t, x, y) \\ \frac{dy}{dt} &= G(t, x, y) \end{aligned}\right\} \quad \text{--- (1)}$$

The equations are linked together

t - independent variable

x and y - dependent variables

Linear systems of the form:

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t) \end{aligned}\right\} \quad \text{--- (2)}$$

The functions $a_i(t)$, $b_i(t)$ and $f_i(t)$ where $i = 1, 2$ are continuous on a certain closed interval $[a, b]$ of the t-axis.

If $f_1(t)$ and $f_2(t)$ are identically zero, then the system is called homogeneous otherwise it is said to be non-homogeneous.

A Solution is a path of functions $x(t)$ and $y(t)$ on $[a,b]$.

Example

Solve the homogeneous linear system

2m

$$\left.\begin{aligned} \frac{dx}{dt} &= 4x - y \\ \frac{dy}{dt} &= 2x + y \end{aligned}\right\} \quad \text{--- (3)}$$

soln:

$Dx - 4x + y = 0$ $\qquad Dx = \frac{dx}{dt}$

$-2x + Dy - y = 0$ $\qquad Dy = \frac{dy}{dt}$

$(D-4)x + y = 0$ — (i)

$-2x + (D-1)y = 0$ — (ii)

(i) $\times 2 \Rightarrow 2(D-4)x + 2y = 0$ — (iii)

(ii) $\times (D-4) \Rightarrow -2x(D-4) + (D-1)(D-4)y = 0$ — (iv)

$(D-1)(D-4)y + 2y = 0$

$(D^2-5D+4)y + 2y = 0$

$(D^2 - 5D + 6)y = 0$

$(D-1)(D-4) = 0$

$D^2 - 4D - D + 4 = 0$

$D^2 - 5D + 4 = 0$

$(D-3)(D-2)y = 0$

$m = 3, 2$

m_1 $\qquad m_2 = 2$

$y = e^{3t}$ $\qquad y_2 = e^{2t}$

3

(2)

$y = e^{3t}$ — (i)

$2x = Dy - y$

(2) in (i) $2x = 3e^{3t} - e^{3t}$

$2x = 2e^{3t}$

$\boxed{x = e^{3t}}$

$y = e^{2t}$ — (ii)

(2) in (ii) $2x = 2e^{2t} - e^{2t}$

$2x = e^{2t}$

$\boxed{x = e^{2t}/2}$

The system has

$x = e^{3t}$
$y = e^{3t}$

$\left.\begin{array}{l} x = e^{2t} \\ y = 2e^{2t} \end{array}\right\}$ —(4)

as solutions on any closed interval

Theorem A

If t_0 is any point of the interval $[a, b]$ and if x_0 and y_0 are any number whatever, then

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$ has one and only one solution.

$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\}$ valid throughout $[a, b]$.

Such that $x(t_0) = x_0$ and $y(t_0) = y_0$.

Remark:

Homogeneous system is

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y\end{aligned}\right\} \quad (5)$$

It is obvious that (5) is satisfied by trivial solution, in which $x(t)$ and $y(t)$ are both identically zero.

Theorem B

If the homogeneous system (1) has two solutions $x = x_1(t)$, $y = y_1(t)$ and $\left.\begin{aligned}x &= x_2(t)\\ y &= y_2(t)\end{aligned}\right\}$ (6) on $[a,b]$

then
$$\left.\begin{aligned}x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t)\end{aligned}\right\} \quad (7)$$

is also a solution on $[a,b]$ for any constant c_1 and c_2.

Proof:

The solution (7) is obtained from the pair of solutions (6) by multiplying the first by c_1 the second by c_2, and adding (7) is called a linear combination of the solution (6).

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We can restate that (any linear combinatio of two solutions of the homogeneous system (5) is also a solution)

Accordingly (3) has

$$\left.\begin{aligned} x &= c_1 e^{3t} + c_2 e^{2t} \\ y &= c_1 e^{3t} + 2c_2 e^{2t} \end{aligned}\right\} - (8)$$

as a solution for every choice of the constants c_1 and c_2.

(i.e) Eqamation (7) is the general Solution of (5) on $[a,b]$.

Theorem C

If the two solutions (6) of the homogeneous systom (5) have a wronskian $w(t)$ that does not vanish on $[a,b]$ the (7) is the general solution of (5) on this interval.

Proof:

By Theorem A, (7) will be general solution if the constants c_1 and c_2 can be chosen so as to satisfy arbitrary conditions $x(t_0)=x_0$ and $y(t_0)=y_0$ at an arbitrary point t_0 in $[a,b]$.

or equivalently

if the system of linear algebraic equation

$$c_1 x_1(t_0) + c_2 x_2(t_0) = x_0$$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$

in the unknowns c_1 and c_2 can be solved for each t_0 in $[a,b]$ and every pair of numbers x_0 and y_0.

By the elementary theory of determinants. This is possible whenever the determinants of the coefficients

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$

does not vanish on the interval $[a,b]$

This determinant is called the wronskian of the two solutions (6).

Example:

The wronskian of two solutions (4) is

$$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix} = 2e^{5t} - e^{5t} = e^{5t}$$

Which never vanishes.

Note:

The vanishing of non vanishing of the wronskian $w(t)$ of two solutions does not depend on the choice of t.

Theorem D:

If $w(t)$ is the wronskian of the two solutions (6) of the homogeneous system (5), then $w(t)$ is either identically zero as no where zero on $[a,b]$.

Proof:

$w(t)$ satisfies the first order D.E

$$\frac{dW}{dt} = [a_1(t) + a_2(t)] W$$

$$\frac{dW}{W} = [a_1(t) + a_2(t)] dt$$

$$W = C e^{\int (a_1(t) + a_2(t)) dt}$$ for some constant C.

(i.e) The exponential factor never vanishes on $[a,b]$.

Note:

The two solutions (6) are called linearly dependent on $[a,b]$ if one is a constant multiple of the other

(i.e) $x_1(t) = k x_2(t)$ or $x_2(t) = k x_1(t)$

$y_1(t) = k y_2(t)$ $\quad$ $y_2(t) = k y_1(t)$

for some constant k and all t in $[a,b]$

It is clear that linear dependence is equivalent to the condition that there exist two constants c_1 and c_2 atleast one of which is not zero, such that

$$\left.\begin{aligned} c_1 x_1(t) + c_2 x_2(t) &= 0 \\ c_1 y_1(t) + c_2 y_2(t) &= 0 \end{aligned}\right\} \text{— (11)} \quad \text{for all } t \text{ in } [a,b]$$

And linearly independent if neither is a constant multiple of the other.

Theorem E:

If the two solutions (6) of the homogeneous system (5) are linearly independent on $[a,b]$. Then (7) is the general solution of (5) on this interval.

Proof:

In view of theorems C and D. It suffices to show that the solutions (6) are linearly dependent iff their wronskian $W(t)$ is identically zero.

Assuming that they are linearly dependent. So $x_1(t) = k\,x_2(t)$

$y_1(t) = k\,y_2(t)$

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}.$$

$$= \begin{vmatrix} k\,x_2(t) & x_2(t) \\ k\,y_2(t) & y_2(t) \end{vmatrix}.$$

$$= k\,x_2(t)\,y_2(t) - k\,x_2(t)\,y_2(t)$$

$$= 0 \quad \text{for all } t \text{ in } [a,b].$$

Now assume that $W(t)$ is identically zero and show that the solutions (6) are linearly dependent in the sense of equation (11).

Let t_0 be a fixed point in $[a,b]$. Since $W(t_0)=0$ the system of linear algebraic equations

$$c_1 x_1(t_0) + c_2 x_2(t_0) = 0$$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = 0$$ has a

solution both zero.

Thus the solution of (5) given by

$$\left.\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}\right\} \quad \text{--- (13)}$$

equals the trivial solution at t_0.

It now follows from the uniqueness part of theorem A that (13) must equal the trivial solution throughout the interval $[a, b]$. So (11) holds and the proof is complete.

Theorem: F

If the two solutions $\left.\begin{aligned} x &= x_1(t) \\ y &= y_1(t) \end{aligned}\right\}$

and $\left.\begin{aligned} x &= x_2(t) \\ y &= y_2(t) \end{aligned}\right\}$ of the homogeneous system (6)

are linearly independent on $[a, b]$ and if $\left.\begin{aligned} x &= x_p(t) \\ y &= y_p(t) \end{aligned}\right\}$ is any particular soln of (2) on $[a, b]$ interval

Then $$\left.\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{aligned}\right\} \quad \text{--- (14)}$$

is the general solution of (2) on $[a, b]$

Proof:

It suffices to show that if $\left.\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\}$

is an arbitrary solution of (2), then

$\left.\begin{aligned} x &= x(t) - x_p(t) \\ y &= y(t) - y_p(t) \end{aligned}\right\}$ is a solution of (5) non-homogeneous eqn

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$$\left.\begin{array}{l} x = x_p(t) \\ y = y_p(t) \end{array}\right\} \text{ is a particular soln of (2)}$$

Then $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$.

$$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

is a general solution of (2).

non-homogeneous equation

Homogeneous linear Systems with constant co-efficients.

$$\left.\begin{array}{l} \dfrac{dx}{dt} = a_1 x + b_1 y \\[2ex] \dfrac{dy}{dt} = a_2 x + b_2 y \end{array}\right\} \text{— (1)}$$

Solution of (1) having the form

$$\left.\begin{array}{l} x = A e^{mt} \\ y = B e^{mt} \end{array}\right\} \text{— (2)}$$

If we sub (2) in (1)

$$A m e^{mt} = a_1 A e^{mt} + b_1 B e^{mt}$$

$$B m \cdot e^{mt} = a_2 A \cdot e^{mt} + b_2 B e^{mt}$$

Dividing by e^{mt} yields,

The linear algebraic system

$$\left.\begin{array}{l} (a_1 - m) A + b_1 B = 0 \\ a_2 A + (b_2 - m) B = 0 \end{array}\right\} \text{— (3)}$$

(3) has the trivial Solution if $A = B = 0$.

We should make eqn (2) is a trivial soln of

However we know that eqn (3) has non-trivial solution.

Whenever the determinant of the coefficients vanishes.

$$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$$

We get the algebraic equation.

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - b_1 a_2) = 0 \quad \text{—} (4)$$

Let m_1 and m_2 be the roots of eqn (4). If replace m by m_1 then the resulting equations have a non-trivial eqn A_1, B_1.

$$\left.\begin{aligned} x &= A_1 e^{m_1 t} \\ y &= B_1 e^{m_1 t} \end{aligned}\right\}$$

is a non-trivial soln of the system eqn (1).

Similarly $\left.\begin{aligned} x &= A_2 e^{m_2 t} \\ y &= B_2 e^{m_2 t} \end{aligned}\right\}$ be a another non-trivial soln.

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Example

Solve $\left.\begin{array}{l}\dfrac{dx}{dt} = x+y \\[2ex] \dfrac{dy}{dt} = 4x-2y\end{array}\right\} \text{— (i)}$

The linear algebraic system is

$(a_1-m)A+b_1B=0$
$a_2A+(b_2-m)B=0$

$$\left.\begin{array}{l}(1-m)A+B=0 \\ 4A+(2-m)B=0\end{array}\right\} \text{—— (ii)}$$

$a_1=1, b_1=1$
$a_2=4 \quad b_2=\cancel{2}$

The axiliary equation is

$$m^2-(a_1+b_2)m+(a_1b_2-a_2b_1)=0$$
$$m^2-(1-2)m+(-2-4)$$
$$m^2+m-6=0$$
$$(m+3)(m-2)=0$$
$$m_1=-3 \text{ and } m_2=2$$

$m_1=-3$ Substitute in (ii)

$(1+3)A+B=0$

$$4A+B=0$$
$$4A=-B=K=4$$
$$A=1, \quad B=-4$$

$x=Ae^{m_1t}$
$y=Be^{m_1t}$

Step 1: write Alg...
Step 2: Auxiliary equ
Step 3: Solve

So $\left.\begin{array}{l}x=1\cdot e^{-3t} \\ y=-4\cdot e^{-3t}\end{array}\right\}$ — (iii) as a non trivial solution of (i)

Step 4: use m_1, m_2 and get x, y values
Step 5: ... x, y values

$m_2=2$ substituting in (ii)

$(1-2)A+B$

$$-A+B=0$$
$$\cancel{4A+4B=0}$$
$$A=B=1$$

This yields $\left.\begin{aligned} x &= e^{2t} \\ y &= e^{2t} \end{aligned}\right\}$ — (iv)

as another solution of (i)

Since (iii) and (iv) are linearly independent

$$x = c_1 e^{-3t} + c_2 e^{2t}$$
$$y = -4c_1 e^{-3t} + c_2 e^{2t}$$ is the general solution of (i)

Distinct complex roots:

If m_1 and m_2 are distinct complex numbers then they can be written $a \pm ib$ where a and b are real numbers.

A's and B's are obtained from (3). Two linearly independent solutions.

$$x = A_1^* e^{(a+ib)t} \quad \& \quad x = A_2^* e^{(a-ib)t}$$

(*) denote complex number.

$$y = B_1^* e^{(a+ib)t} \quad \& \quad y = B_2^* e^{(a-ib)t}$$

If $A_1^* = A_1 + iA_2$ and $B_1^* = B_1 + iB_2$ and use euler's formula then

$$x = (A_1 + iA_2)\, e^{at} (\cos bt + i \sin bt)$$
$$y = (B_1 + iB_2)\, e^{at} (\cos bt + i \sin bt)$$

$$x = e^{at} \{(A_1 \cos bt - A_2 \sin bt) + i (A_1 \sin bt + A_2 \cos bt)\}$$

$$y = e^{at} \{ B_1 \cos bt - B_2 \sin bt) + i \, (B_1 \sin bt + B_2 \cos bt)\}$$

15 if a pair of complex valued function is a Solution of (1) in which the co-efficients are real, constants then their two real parts and their two imaginary parts are real valued Solution. So (4) yields the two real-valued Solution

Real valued Solution
$$\begin{cases} x = e^{at}\left[A_1 \cos bt - A_2 \sin bt\right] \\ y = e^{at}\left[B_1 \cos bt - B_2 \sin bt\right] \end{cases}$$ and

Imaginary valued solution
$$\begin{cases} x = e^{at}\left[A_1 \sin bt + A_2 \cos bt\right] \\ y = e^{at}\left[B_1 \sin bt + B_2 \cos bt\right] \end{cases}$$

These Solutions are linearly independent. So the general solution in this case is

$$x = e^{at}\left[c_1 \underbrace{(A_1 \cos bt - A_2 \sin bt)}_{\text{real } x \text{ value}} + c_2 \underbrace{\left[A_1 \sin bt + A_2 \cos bt\right]}_{\text{imag } x \text{ value}}\right]$$

$$y = e^{at}\left[c_1 \underbrace{(B_1 \cos bt - B_2 \sin bt)}_{\text{Real } y \text{ value}} + c_2 \underbrace{(A_1 \sin bt + A_2 \cos bt)}_{\text{imag } y \text{ value}}\right]$$

Equal real roots:

When m_1 and m_2 have the same value m. then

$$x = A_1 e^{m_1 t} \quad \text{and} \quad x = A_2 e^{m_2 t}$$

$$y = B_1 e^{m_1 t} \quad \text{and} \quad y = B_2 e^{m_2 t}$$

are not linearly independent and we have only one Solution $x = Ae^{mt}$ and $y = Be^{mt}$

16 A second linearly independent solution of the form $x = Ate^{mt}$

$y = Bte^{mt}$

A second solution of the form

$$x = (A_1 + A_2 t) \cdot e^{mt}$$

$$y = (B_1 + B_2 t) \cdot e^{mt} \quad \text{--- (i)}$$

So that the general solution of is

$$x = C_1 A e^{mt} + C_2 (A_1 + A_2 t) e^{mt}$$

$$y = C_1 B e^{mt} + C_2 (B_1 + B_2 t) e^{mt}$$

The constants A_1, A_2, B_1, B_2 are found by substituting (i) into the system

$$\frac{dx}{dt} = a_1 x + b_1 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

Example: 2

$$\frac{dx}{dt} = 3x - 4y$$

$$\frac{dy}{dt} = x - y$$

$$(3-m)A - 4B = 0$$

$$A + (-1-m)B = 0 \quad \text{--- (ii)}$$

The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

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Which has equal real roots 1 and 1 with $m=1$ (ii) becomes

$$2A-4B=0$$

$$A-2B=0$$

A simple non-trivial solution of (i).

A second linearly independent solution of the form

$$\left.\begin{aligned} x &= (A_1+A_2t)\cdot e^t \\ y &= (B_1+B_2t)\cdot e^t \end{aligned}\right\} \text{— (iv)} \quad \text{Sub in (i)}$$

$$A_1e^t + A_2e^t\cdot t + A_2e^t = 3(A_1+A_2t)e^t - 4(B_1+B_2t)e^t$$

$$B_1e^t + B_2te^t + B_2e^t = (A_1+A_2t)e^t - (B_1+B_2t)e^t$$

$$(A_1+A_2-3A_1+4B_1) + (A_2-3A_2+4B_2)t = 0$$

$$(B_1+B_2-A_1+B_1) + (B_2-A_2+B_2)t = 0$$

$$(-2A_2+4B_2)t + (-2A_1+A_2+4B_1) = 0$$

$$(-A_2+2B_2)t + (-A_1+2B_1+B_2) = 0.$$

$$-A_2+2B_2=0$$

$$A_2=2B_2=2$$

$$A_2=2, \quad B_2=1$$

$$-2A_1+2+4B_1=0 \qquad -A_1+2B_1+1=0$$

$$-2A_1+4B_1=-2 \qquad -A_1+2B_1=-1$$

We may take $A_1=-1$, $B_1=0$

Substitute in (iv)

$$x = (1+2t)e^t$$

$y = te^t$ is a second solution. It is obvious that (iii) and (iv) are linearly dependent. So,

$$x = 2c_1 e^t + c_2(1+2t)e^t$$

$y = c_1 e^t + c_2 te^t$ is the general solution of the system (i).

Example: 3

Find the general solution of the system

$$\frac{dx}{dt} = x - 2y. \quad \frac{dy}{dt} = 4x + 5y$$

$$(1-m)A - 2B = 0$$

$$4A + (5-m)B = 0$$

The auxiliary equation is

$$m^2 - 6m + 13 = 0$$

$$m = \frac{6 \pm \sqrt{36-52}}{2} \Rightarrow \frac{6 \pm \sqrt{-16}}{2}$$

$$\Rightarrow \frac{6 \pm 4i}{2}$$

$$\Rightarrow 3 \pm 2i$$

Two linearly independent solution

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$$x = A_1^* e^{(3+i2)t} \quad \text{and} \quad x = A_2^* e^{(3-i2)t}$$

$$y = B_1^* e^{(3+i2)t} \quad \text{and} \quad y = B_2^* e^{(3-i2)t} \quad \text{--- (ii)}$$

The 1st of the solution (ii) can be written as

$$x = (A_1 + iA_2) \cdot e^{3t} (\cos 2t + i \sin 2t)$$

$$y = (B_1 + iB_2) \cdot e^{3t} (\cos 2t + i \sin 2t)$$

$$x = e^{3t} \left[(A_1 \cos 2t - A_2 \sin 2t) + i (A_1 \sin 2t + A_2 \cos 2t) \right]$$

$$y = e^{3t} \left[(B_1 \cos 2t - B_2 \sin 2t) + i (B_1 \sin 2t + B_2 \cos 2t) \right]$$

The two real valued solution

$$x = e^{3t} [A_1 \cos 2t - A_2 \sin 2t]$$

$$y = e^{3t} [B_1 \cos 2t - B_2 \sin 2t] \text{ and}$$

$$x = e^{3t} [A_1 \sin 2t + A_2 \cos 2t]$$

$$y = e^{3t} [B_1 \sin 2t + B_2 \cos 2t]$$

These solutions are linearly independent so the general solution is

$$x = e^{3t} \left[C_1 (A_1 \cos 2t - A_2 \sin 2t) + C_2 (A_1 \sin 2t + A_2 \cos 2t) \right]$$

$$y = e^{3t} \left[C_1 (B_1 \cos 2t + B_2 \sin 2t) + C_2 (B_1 \sin 2t + B_2 \cos 2t) \right]$$

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The existence and uniqueness of solutions

Picard's method of successive approximation

Picard's iteration method is used for finding an approximate solution of the initial value problem of the form

$$\frac{dy}{dx} = y'(x) = f(x,y) , \; y(x_0) = y_0$$

The condition $y(x_0)=y_0$ is called the initial condition.

An iteration method is a method which consists of repeated application of repeated application of exactly the same type of steps where in each step we use the result of the previous step.

Picard's Method of successive approximation.

Consider an initial value problem of the form

$$y' = \frac{dy}{dx} = f(x,y), \; y(x_0) = y_0 \quad \text{--- (1)}$$

By integrating over the interval (x_0, x)

$$\int_{y_0}^{y} \frac{dy}{dx} \cdot dx = \int_{x_0}^{x} f(x,y)\, dy$$

$$y(x) - y_0 = \int_{x_0}^{x} f(x,y)\,dx \quad \therefore$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad \text{——} \; (2)$$

To see that (1) and (2) are indeed equivalent. Suppose that $y(x)$ is a solution of (1).

$$y'(x) = f\,[x, y(x)]$$

Integrate from x_0 to x and $y(x_0) = y_0$

$$y(x) - y_0 = \int_{x_0}^{x} f\,[x, y(x)]\,dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f\,[x, y(x)]\,dx$$

[The result i (2)]

Conversely by differentiating (2), we get

$$\frac{dy}{dx} = f(xy)$$

and putting $x = x_0$ in (2)

$$y(x_0) = y_0 \qquad \left[\because \int_{x_0}^{x} f(x,y)\,dx = 0\right]$$

(i.e) (1) and (2) are equivalent.

Since the expression of y in terms of x is absent. Hence the exact value of y cannot be obtained

$\therefore$ Determine a sequence of approximation to the solution (2).

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

The next step is to use $y_1(x)$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[x, y(x)]\, dx$$

At the nth stage of the process

$$y_n(x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1}(x)]\, dx \quad -(3)$$

Example: 4 Solve differential equation

Initial value problem

$$y' = y, \quad y(0) = 1$$

$y(x_0) = y_0$

Soln

The equivalent integral equation is

$$y(x) = y_0 + \int_{x_0}^{x} y\, dx$$

$$= 1 + \int_{0}^{x} y\, dx$$

Equation (3) becomes

$$y_n(x) = 1 + \int_{0}^{x} f[x, y_{n-1}(x)]\, dx \quad [\because y' = f(x,y)]$$

$$y_n(x) = 1 + \int_{0}^{x} y_{n-1}(x)\, dx$$

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$$y_1(x) = 1 + \int_0^x y_0(x)\,dx$$

$$= 1 + \int_0^x dx \quad = 1 + (x)_0^x$$

$$\boxed{y_1(x) = 1 + x}$$

$$y_2(x) = 1 + \int_0^x (1+x)\,dx$$

$$y_2(x) = 1 + \int_0^x y_1(x)\,dx$$

$$= 1 + \int_0^x (1+x)\,dx$$

$$y_2(x) = 1 + \left[x + \frac{x^2}{2}\right]_0^x$$

$$\boxed{y_2(x) = 1 + x + \frac{x^2}{2}}$$

$$y_3(x) = 1 + \int_0^x y_2(x)\,dx$$

$$y_3(x) = 1 + \int_0^x \left[1 + x + \frac{x^2}{2}\right] dx$$

$$y_3(x) = 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3}\right]_0^x$$

$$\boxed{y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$

and in general

$$\boxed{y_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}$$

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$$P.I = -x-1$$

$\boxed{y = Ae^{x} - x - 1}$, Apply initial condition $y(0)=1$

$$1 = A - 1$$

$$\boxed{A = 2}$$

$\boxed{y = 2e^{x} - x - 1}$ is an exact solution.

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Picard's Theorem

Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axis. [fig.1) If (x_0, y_0) is any interior point of R, then there exists a number $h>0$ with the property that intial value problem

$$y' = f(x,y), \; y(x_0) = y_0 \quad \text{—(1)}$$

has one and only solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

Proof:

We know that every solution of (1) is also continuous solution of the integral equation

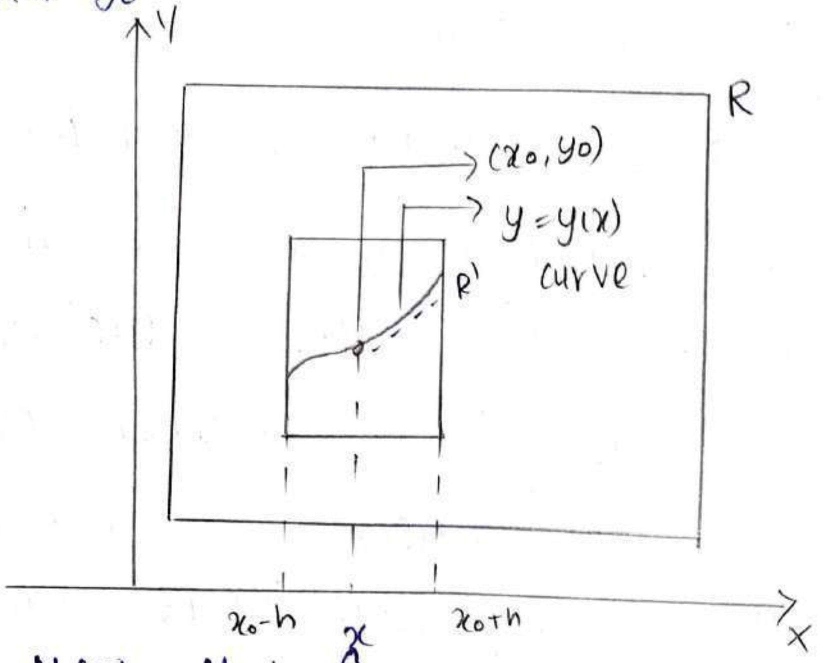
$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt \quad \text{—(2)}$$

and conversely

To conclude that (1) has a unique soln on an interval $|x-x_0| \le h$ iff (2) has a unique continuous solution on the same interval.

The sequence of functions $y_n(x)$ (successive approximation) defined by

$y_0(x) = y_0$



$$y_1(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\, dt \quad \text{— (3)}$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt$$

converges to a solution (2).

$y_n(x)$ is the n^{th} partial sum of the series of the functions.

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$$y_0(x) + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$

$$= y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots \quad \text{---} (4)$$

So the convergence of the sequence (3) is equivalent to the convergence of this series (4).

In order to complete the proof, we produce a number $h>0$ that defines the interval.

$\boxed{|x-x_0| \leq h}$ and then we show that on this interval the following statements are true (i) the series (4) converges to a function $y(x)$. $y_n(x) \to y(x)$ Sum of series $\to y(x)$

(ii) $y(x)$ is a continuous solution of (2)

(iii) $y(x)$ is the only continuous solution of (2)

Proof of Statement (i).

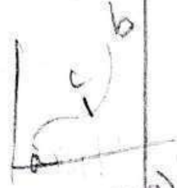
Assumed that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on the rectangle R. But R is closed and bounded, so each of these functions is necessarily bounded on R. This means that there exists constants M and K such that

$$|f(x,y)| \leq M \quad \text{—— (5) and}$$

$$\left|\frac{\partial}{\partial y} f(x,y)\right| \leq K \quad \text{—— (6)}$$

for all points (x,y) in R.

If (x,y_1) and (x,y_2) are distinct points in R with the same x coordinate then the mean value theorem says that



$$|f(b)-f(a)| = |f'(c)|\;|b-a|$$

$$|f(x,y_1)-f(x,y_2)| = \left|\frac{\partial}{\partial y}(f(x,y))\right|\;|y_1-y_2| \text{ —(7)}$$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

for some number y^* b/w y_1 and y_2.

$y^* \to c$, $y_1 \to a$, $y_2 \to b$

From (6) and (7)

$$|f(x,y_1)-f(x,y_0)| \leq K\,|y_1-y_2| \text{ —— (8)}$$

for any point (x,y_1) and (x,y_2) in R that lie on the same vertical line.

Choose h to be any positive number such that

$$Kh < 1 \text{ —— (9) and}$$

the rectangle R defined, by

$|x-x_0| \leq h$ and $|y-y_0| \leq Mh$ is contained in R.

In order to prove (ii) it suffices to show that the series

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_3(x)| + \cdots$$
$$+ |y_n(x) - y_{n-1}(x)| + \cdots \quad \text{——— (10)}$$

converges.

It is necessary that each of the functions $y_n(x)$ has a graph that lies in R' and hence in R.

The points $[t, y_0(t)]$ are in R'.

P. method $y_1(x) = y_0 + |\int_{x_0}^{x} f(t, y_0(t)) dt$

(5) $\Rightarrow$ $|f[t, y_0(t)]| \leq M$ and, $M = f(t, y$

$$|y_1(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_0(t)) dt \right| \leq M(t)\Big|_{x_0}^{x}$$

$$\leq M|x - x_0|$$

$$\leq Mh \qquad [\because |f(t, y(t))| \leq M]$$

Which proves the statement for $y_1(x)$

$$|y_2(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_1(t)] dt \right|$$

$$\leq Mh$$

$|||^{ly}$

$$|y_3(x) - y_0| \leq \left| \int_{x_0}^{x} f[t, y_2(t)] dt \right|$$

$$\leq Mh$$

32 and so on

Since a continuous function on a closed interval has a maximum and $y_1(x)$ is continuous. Define a constant 'a' by

$$a = \max |y_1(x) - y_0|$$

$$|y_1(x) - y_0(x)| \leq a$$

Next, the points $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R', so ⑧ yields

$$|f[t, y_1(t)] - f[t, y_0(t)]| \leq k |y_1(t) - y_0(t)|$$
$$\leq ka$$

We have

$$|y_2(x) - y_1(x)| = \left| y_0(x) - y_0(x) + \int_{x_0}^{x} \left(f[t, y(t)] - f[t, y_0(t)] \right) dt \right|$$

$$\leq ka |x - x_0| = a(kh)$$

$|||^{ly}$

$$|f(t, y_2(t)] - f[t, y_1(t)]| \leq k |y_2(t) - y_1(t)|$$

$a(kh)$

$$|f(t, y_2(t) - f[t, y_1(t)]| \leq k^2 ah$$
$$\leq ak^2h$$

So, $|y_3(x) - y_2(x)| = \left| \int_{x_0}^{x} (f[t, y_2(t)] - f(t, y_1(t)) dt \right|$

$$\leq (k^2ah)h$$
$$= a(kh)^2$$

33 $|y_n(x) - y_{n-1}(x)| \leq a(kh)^{n-1}$ for every $n = 1, 2 \cdots$

$\therefore$ (10) yields

$$|y_0| + |y_1(x) - y_2(x)| + \cdots + |y_n(x) - y_{n-1}(x)| + \cdots$$

$kh < 1$ - (a) $\quad \leq |y_0| + a + a(kh) + a(kh)^2 + \cdots + a(kh)^{n-1}$

1/5 But (9) gives that this series converges. So, (10) converges (4) converges to a sum which not by $y(x)$ and $y_n(x) \to y(x)$ since the graph of each $y_n(x)$ lies in R'.

It is evident that the graph $y(x)$ also has this property

The proof of Statement (ii)

$y_n(x)$ converges to $y(x)$ is uniform. This means by choosing 'n' to be sufficiently large. To make $y_n(x)$ as close as $y(x)$ for all x in the interval (or) if $\varepsilon > 0$, then there exists a positive integer n_0 such that if $n \geq n_0$.

We how $|y(x) - y_n(x)| < \varepsilon$ for all x in the interval. Since each $y_n(x)$ is clearly continuous this uniformly of the convergence implies that the limit function $y(x)$ is also continuous.

$$\left[\because \lim_{n\to\infty} y_n(x) = y(x)\right]$$

To prove that $y(x)$ is Solution of (2)

To show that $y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\, dt = 0$ —(3)

But we know that

$$y_n(x) - y_0(x) - \int_{x_0}^{x} f[t, y_{n-1}(t)]\, dt = 0 \quad \text{—— (12)}$$

So, $y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\, dt$

$$= y(x) - y_0(x) + \int_{x_0}^{x} f[t, y(t)] - f[t - y(t)]\, dt$$

$[\because 0 = 0-$

$$\left| y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\, dt \right| \le | y(x) - y_n(x) | +$$

$$\left| \int_{x_0}^{x} \{ f[t, y_{n-1}(t)] - f[t, y(t)] \}\, d \right.$$

Since the graph of $y(x)$ lies in R' and valu in R. (8) yields

$$| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t)) dt | \le | y(x) - y_n(x) | +$$

$$kh \max | y_{n-1}(x) - y(x) | -($$

The uniformity of the convergence of $y_n(x)$ to $y(x)$ implies that the right Side of (13) can be made as small as by taking n large enough. The left side of (13) must equal zero and the proof of (ii) is complete.

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The proof of statement (iii)

To prove that the solution $y=y(x)$ is the only solution for which $y(x_0)=y_0$

Assume that $\bar{y}(x)$ is also a continuous solution of (2) on the interval $|x-x_0|\leq h$.

To show that $\bar{y}(x)=y(x)$ for every x in the interval. It is necessary to prove that the graph of $\bar{y}(x)$ lies in R' and hence in R.

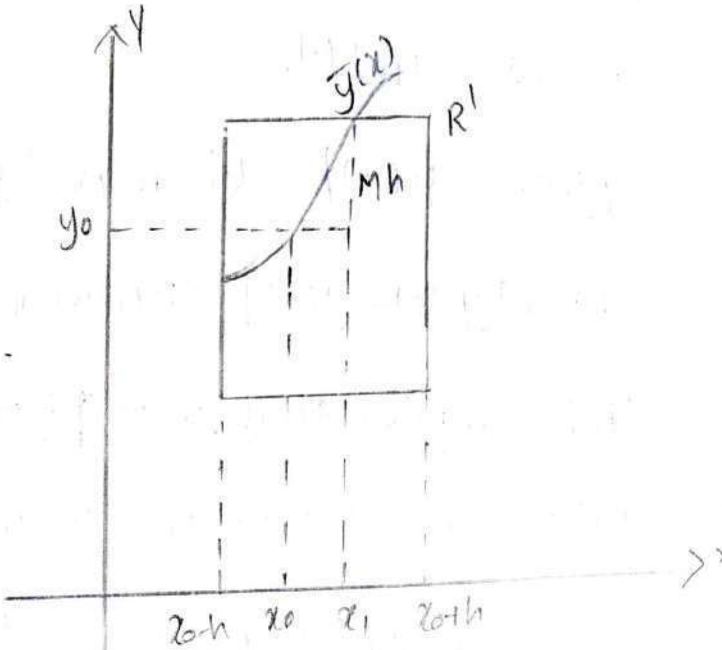
Suppose the graph of $\bar{y}(x)$ leaves R' (fig

Then the properties of this function imply that there exists an x_1 such that

$$|x_1-x_0|<h,$$

$$|\bar{y}(x_0)-y_0| = Mh \text{ and}$$

$$|\bar{y}(x)-y_0|<Mh \quad \text{if } |x-x_0|<|x_1-x_0|$$



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$$\frac{|\bar{y}(x_1)-y_0|}{|x_1-x_0|} = \frac{Mh}{|x_1-x_0|} > \frac{Mh}{h} = M$$

However, by the mean value theorem there exists a number x^* b/w x_0 and x_1 such that

$$\frac{|\bar{y}(x_1)-y_0|}{|x_1-x_0|} = |y^{-1}(x^*)|$$

$$= |f[x^*, \bar{y}(x^*)]| \le M$$

Since the point $[x^*, \bar{y}(x^*)]$ lies in R'.

This is a contradiction.

So the graph of $\bar{y}(x)$ lies in R'.

To complete the proof of (iii)

$\bar{y}(x)$ and $y(x)$ are both solutions of (2)

To write

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} \{ f[t, \bar{y}(t)] - f[t, y(t)] \} \right.$$

Since the graph of $\bar{y}(x)$ and $y(x)$ both lie in R' (8) yields.

$$|\bar{y}(x) - y(x)| \le kh \max |\bar{y}(x) - y(x)|$$

$$\max |\bar{y}(x) - y(x)| \le kh \max [\bar{y}(x) - y(x)]$$

This implies that $\max |\bar{y}(x) - y(x)| = 0$

otherwise $1 \le kh$ in contradiction to (9)

$\therefore$ $\bar{y}(x) = y(x)$ for every x in the interval $|x - x_0| \le h$. Picard's theorem is fully proved.

LIPSCHITZ CONDITION

A function $f(x,y)$ is said to satisfy a Lipschitz condition in a region R in xy-plane if there exists a positive constant K such that

$$|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$$

Whenever the points (x,y_1) and (x,y_2) both lies in R. The constant K is called the Lipschitz constant for the function $f(x,y)$.

Remark:

Picard's theorem is called a local existence and uniquences theorem because the existence of a unique solution only on some interval $|x-x_0| \le h$ where 'h' may be very small.

Existence and uniqueness theorem:

Let $f(x,y)$ be a continuous function that satisfies a Lipschitz condition.

$$|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2|$$

On a strip defined by $a \le x \le b$ and $-\infty < y < \infty$. If $f(x_0,y_0)$ is any point on the strip, then the initial value problem.

$$y' = f(x, y), \; y(x_0) = y_0 \quad \text{——} (15)$$

has one and only one solution $y = y(x)$ on the interval $a \le x \le b$.

Proof:

Write the proof of Picard's theorem upto equation (16) converges. After

Define M_0, M_1 and M by

$M_0 = |y_0|$, $M_1 = \max |y_1(x)|$, $M = M_0 + M_1$

We notice that $|y_0(x)| \le M$ and

$$|y_1(x) - y_0(x)| \le M$$

Next, if $x_0 \le x \le b$ it follows that

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} \{ f[t, y_1(t)] - f[t, y_0(t)] \} dt \right|$$

$$\le k \int_{x_3}^{x} |y_2(t) - y_1(t)| dt$$

$$\le k^2 M \frac{(x - x_0)^2}{2}$$

and in general

$$|y_n(x) - y_{n-1}(x)| \le k^{n-1} M \frac{(x - x_0)^{n-1}}{(n-1)!}$$

The some argument is also valid for $a \le x \le x_0$, provided only that $x - x_0$ is replaced by $|x-x_0|$

$$|y_n(x)-y_{n-1}(x)| \le k^{n-1} M \frac{|x-x_0|^{n-1}}{(n-1)!}$$

$$\le k^{n-1} M \frac{(b-a)^{n-1}}{(n-1)!}$$

in the interval and $n=1,2\ldots$

We conclude that

$$|y_0(x)| + |y_1(x)-y_0(x)| + \cdots + |y_n(x)-y_{n-1}(x)| + \cdots$$

$$\le M + M + Mk(b-a) + k^2 M \frac{(b-a)^2}{2!} +$$

$$k^3 M \frac{(b-a)^3}{3!} + \cdots$$

$$\le M + M\left[1 + k(b-a) + k^2 \frac{(b-a)^2}{2!} + \cdots\right]$$

So, (3) converges uniformly on the interval $a \le x \le b$ to a limit function $y(x)$

The uniformly of the convergence implies that $y(x)$ is a solution of (5) on the whole interval.

Next proof of Statement (ii) from Picard's theorem proof of Statement (ii)

Unit - IV

Qualitative properties of solution:

Oscilations and the sturm separation.

Oscillations and the S.

The familiar equation $y'' + y = 0$ — ①

w.r.t. $y_1(x) = \sin x$, $y_2(x) = \cos x$, are two linearly independent solution of ①

and they are determined by the initial conditions, $y_1(0)$, $y_1'(0)$ and $y_2(0) = 1$

$y_2'(0) = 0$ and the general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Let $y = s(x)$ be defined as the solution of ① determined by the initial conditions $s(0) = 0$, $s'(0) = 1$

Theorem: A :- Sturm separation.

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$ then the zeros of these functions are distinct and occur alternatively - in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and conversly

Proof:

Since y_1 and y_2 are linearly independent their wronskian $W(y_1, y_2) = y_1(x) y_2'(x) - y_2(x) y_1'(x)$ does not vanish and ∴ since it is continuous must have constant sign. First it is easy to see that y_1 and y_2 cannot have a common zero

If they have a common zero, then the wronskian will vanish at that point which is impossible.

we now assume that x_1 and x_2 are successive zeros of y_2 and show that y_1 vanishes between these points. the wronskian reduces to $y_1(x) \, y_2'(x)$ at x_1 and x_2. So both factors $y_1(x)$ and $y_2'(x)$ are $\neq 0$ at each of these points.

Furthermore $y_2'(x_1)$ and $y_2'(x_2)$ must have opposite signs. because if y_2 is increasing at x_1, it must be decreasing at x_2, and vice versa since the wronskian has constant sign. $y_1(x_1)$ and $y_1(x_2)$ must also have opposite signs and therefore by continuity $y_1(x)$ must vanish at some point between x_1 and x_2.

Note that y_1 cannot vanish more than once between x_1 and x_2 for if it does, then the same argument shows that y_2 must vanish between these zeros of y_1, which contradicts the original assumption that x_1 and x_2 are successive zeros of y_2.

Note :-

Any equation of the form

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—①}$$

can be written as $u'' + q(x)u = 0$ —②

by a simple change ~~bo~~ ~~rate~~ of the dependent variable.

Theorem B :-

If $q(x) < 0$ and if $u(x)$ is a nontrivial solution of $u'' + q(x)u = 0$, then $u(x)$ has atmost one zero

Proof:-

Let x_0 be a zero of $u(x)$, so that $u(x_0)=0$, since $u(x)$ is nontrivial $u'(x_0)\neq 0$ assume that $u'(x_0)>0$, so that $u(x)$ is positive over some interval to the right of x_0.

Since $q(x)<0$ $u''(x)=-q(x)u(x)$ is a +ve function on the same interval this implies that the slope $u'(x)$ is an increasing function, so $u(x)$ cannot have a zero to the right of x_0, and in the same way it has none to the left of x_0.

A similar argument holds when $u'(x_0)<0$, so $u(x)$ has either no zeros at all or only one, and the proof is complete.

Remark:-

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ with $q(x) > 0$

If we consider a portion of the graph above the x-axis then $u''(x) = -q(x)u(x)$ is negative.

So the graph is concave down and the slope $u'(x)$ is decreasing.

If this slope ever becomes -ve the curve plainly crosses the x-axis somewhere to the right and we get a zero for $u(x)$.

W.K.T this happens when $q(x)$ is constant the alternative is that although $u'(x)$ decreases it never reaches zero and the curve continues to raise.

It is clear from these remarks that $u(x)$ will have zeros as x increases whenever $q(x)$ does not decrease too rapidly.

<u>Theorem:-</u>

Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$ where $q(x) > 0$ $\forall x > 0$

If $\int_1^\infty q\,dx = \infty$ — ① then $u(x)$ has infinitely many zero's on the +ve x-axis.

Proof: Assume that contrary namely that $u(x)$ vanishes at most a finite no of times for $0 < x < \infty$. So that the point $x_0 > 1$ exists,

With the property that $u(x) \neq 0$, $\forall (x) \geq x_0$. We may clearly suppose without loss of generality, that $u(x) > 0$ $\forall x \geq x_0$. Since $u(x)$ can be replaced by its -ve if necessary.

Our purpose is to contradict the assumption by showing that $u'(x)$ is -ve somewhere to the right of x_0, for by the above remarks:-

This will imply that $u(x)$ has a zero to the right of x_0, If we put

$$v(x) = \frac{-u'(x)}{u(x)} \text{ for } x \geq x_0.$$

$$v'(x) = \frac{-u(x)\,u''(x) + u'(x)\,u'(x)}{[u(x)]^2}$$

since $u''(x) = -q(x)\,u(x)$

$$v'(x) = \frac{[u(x)]^2 q(x) + [u'(x)]^2}{[u(x)]^2}$$

$$= q(x) + \left[\frac{u'(x)}{u(x)}\right]^2$$

$$= q(x) + [v(x)]^2.$$

and $\int$ ing this (x_0 to x) where $x > x_0$

we get $v(x) - v(x_0) = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} (v(x))^2 dx$

we now use eqn (1) to conclude that $v(x)$ is +ve if x is taken large enough.

This shows that $u(x)$ and $u'(x)$ have opposite sign if x is sufficiently large, so $u'(x)$ is -ve.

The proof is complete.

Theorem: D

Sturm comparison thm

Let $y(x)$ be a non-trivial solution of eqn $y'' + q(x)y = 0$ on a closed interval $[a, b]$. Then $y(x)$ has atmost a finite no of zeros in this interval.

Proof:

Given $y'' + q(x)(y) = 0$ —(1)

we may assume the contray that $y(x)$ has an infinite number of zeros in $[a, b]$

It ~~arises~~ follows from this that if a pt x_0 in $[a,b]$ and a sequence of zeros, $x_n \neq x_0$ s.t x_n converges to x_0.

Since $y(x)$ is continuous and differenti-able at x_0 we have

$$y(x_0) = \lim_{x_n \to x_0} y(x_n) = 0 \text{ and}$$

$$y'(x_0) = \lim_{x_n \to x_0} \frac{y(x_n) - y(x_0)}{x_n - x_0} = 0.$$

W.K.T

Let $P(x), Q(x), R(x)$ be continuous on $[a,b]$ if x_0 is any point in $[a,b]$ and these statements imply that $y(x)$ is the trivial solution of (1)

which is a contradiction.

Hence the proof is complete.

Theorem: C:

Let $y(x)$ and $z(x)$ be non-trivial solutions ~~solutions~~ of $y'' + q(x)y = 0$ and $z'' + r(x)z = 0$ where $q(x)$ & $r(x)$ are +ve fn s.t $q(x) > r(x)$. Then $y(x)$ vanishes at least once between any two successive zeros of $z(x)$.

Proof:

Let x_1 and x_2 be successive zeros of $z(x)$. so that $z(x_1) = z(x_2) = 0$ and $z(x)$ does not vanish on the open interval (x_1, x_2) we assume that $y(x)$ does not vanish on (x_1, x_2) and prove the thm by deducing a contradiction. It is clear that no loss of generality is involved in supposing that both $y(x)$ and $z(x)$ are positive on (x_1, x_2) for either function can be replaced by its negative if necessary.

If we know that the Wronskian $W(y,z) = y(x)z'(x) - z(x)y'(x)$ is a function of x by writing it $W(x)$, then

$$\frac{d\,W(x)}{dx} = yz'' - zy''$$

$$= y(-rz) - z(-qy) \quad (\text{by hyp})$$

$$= (q-r)yz > 0.$$

on (x_1, x_2). we now integrate both

sides of the inequality from x_1 to x_2 and obtain

$$W(x_2) - W(x_1) > 0 \quad (or) \quad W(x_2) > W(x_1)$$

However the wronskian reduces to $y(x) z^1(x)$ at x_1 and x_2, so

$$W(x_1) \geq 0 \quad and \quad W(x_2) \leq 0.$$

which is the contradiction and hence the proof is complete.

1

UNIT - V

NON- LINEAR EQUATION

(i) The equation of motoin is

$$\boxed{\frac{d^2x}{dt^2} + \frac{g}{a} \sin x = 0}$$

Because of the presence of $\sin x$, this equati[on] is non- linear.

(ii) The vander pol equation is (used in the theorem of the vaccum tube)

$$\boxed{\frac{d^2x}{dt^2} + \mu(x^2-1) + \frac{dx}{dt} + x = 0}$$

This time the non- linearly come from the presence of the form x^2 (which is multiplie[d] by $\frac{dx}{dt}$)

Phase plane:

In equation $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$ the value of x (Position) and $\frac{dx}{dt}$ (velocity) that is the quantities that at each instant characteri[z]e the state of the system, the phases of the system, the plane determined by those

2M

(X)

two variables is called phase plane.

Autonomous Systems:

If we introduce the Substitution $y = \frac{dx}{dt}$ the equation

$$\boxed{\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)}$$ can be written as

$$\frac{dy}{dt} = f(x, y)$$

To Study systems of the form

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

Hence F and G have continuous 1^{st} partial derivatives in the entire plane. A system of this kind in which the independent variable t does not appear in the function on the right is called autonomous System.

The path of the system (or) trajectory (or) characterstic of the system. ~~(or)~~ ~~trajectory~~

~~(Or) Ch~~

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The path of the system (or trajectory):

3 Existence theorem for system. Let $[a, b]$ be an interval and $t_0 \in [a, b]$.

Suppose that a_i, b_j, f_t are continuous function on $[a, b]$ for $i = 1, 2$.

Let x_0 and y_0 be arbitrary numbers. Then there is one and only one solution to the system.

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

Satisfying $x(t_0) = x_0$, $y(t_0) = y_0$

that if t_0 is any number and (x_0, y_0) is any point in the phase plane then there is a unique solution.

$x = x(t)$, $y = y(t)$ of equation.

$$\frac{dx}{dt} = F(x, y) \text{ such that } x(t_0) = x_0$$

$$\frac{dy}{dt} = G(x, y) \text{ Such that } y(t_0) = y_0$$

If the resulting $x(t)$ and $y(t)$ are not both constant function then equation $x = x(t)$, $y = y(t)$ defines a curves the phase plane which is known as path of the system

2M Critical points.