SEMESTER : I
CORE COURSE : IV

Inst Hour	: 6
Credit	: 5
Code	: 18KP1M04

GRAPH THEORY

UNIT -I

Graphs and Simple Graphs – Graph Isomorphism – Incidence and Adjacency Matrices – Subgraphs – Vertex Degrees – Paths and Connections – Cycles – Shortest Path Problem – Sperner's Lemma. Trees – Cut Edges and Bonds – Cut Vertices - Cayley's Formula - Connector Problem.

Chapter 1 and 2.

UNIT -II

Connectivity – Blocks – Construction of Reliable Communication Networks. Euler Tours Hamilton Cycles – Chinese Postman Problem – Travelling Salesman Problem. Matchings – Matchings and Coverings in Bipartite Graphs – Perfect Matchings – Personnel Assignment Problem – Optimal Assignment Problem.

Chapter 3, 4 and 5.

UNIT- III

Edge Chromatic Number – Vizing's Theorem – Timetabling Problem. Independent Sets – Ramsey's Theorem - Turan's Theorem – Schur's Theorem – A Geometry Problem. Chromatic Number - Brook's Theorem – Hajo's Conjecture – Chromatic Polynomials.

Chapter 6, 7 and 8 (upto Section 8.4)

UNIT- IV

Plane and Planar Graphs – Dual Graphs – Euler's Formula – Bridges – Kuratowski's Theorem – Five - Colour Theorem and the Four – Colour Conjecture – Nonhamiltonian Planar Graphs – Planarity Algorithm.

Chapter 9

UNIT -V

Directed Graphs – Directed Paths – Directed Cycles – Job Sequencing Problem – Designing an Efficient Computer Drum – Making a Road Sysem One – Way – Ranking the Participants in a Tournament.

Chapter 10.

TEXT BOOK

A.Bondy and U.S.R Murty, Graph Theory with Applications, Macmillan, 1976.

REFERENCES

1. S.A.Choudum, Afirst Course in Graph Theory, Mac Millan India Limited, 1987
2. R.J.Wilson &b J.J.WATKINS, Graphs: An Introduction Approach, John Wiley & Sons,1989
3. Kobayashi S and Nomizu K Foundations of Differential Geometry,Interscience Publishers, 1963
4. WIHELM Klingenberg: A Course in Differential Geometry, Graduate Texts in Mathematics, Springer Verlag, 1978
5. T.J.Willmore, An Introduction to Differential Geometry, Oxford University Press,(17th Impression) New Delhi 2002.(Indian Print).

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.

Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.

Section C : 3 x 10 = 30 Marks, 3 out of 5; One Question from each Unit.

UNIT-I

1. Graph

A graph G is an ordered triplet $(V(G), E(G), \psi(G))$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function $\psi(G)$ that associates with each edge of G an unordered pair of vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v; the vertices u and v are called the ends of e.

2. Graphs that have a diagram whose edges intersect only at their ends are called planar, since such graphs can be represented in the plane in a simple manner.

3. Two vertices which are incident with a common edge are adjacent

4. An edge with identical ends is called a loop, and an edge with distinct ends is link.

5. A graph is finite if both its vertex set and edge set are finite. A graph with just one vertex is trivial, all other graphs are non-trivial.

6. A graph is simple if it has no loops and two of its links join the same pair of vertices.

Note:-

V → Set of vertices, E → Set of edges

$\nu(G)$ → Number of vertices, $\varepsilon(G)$ → Number of edges.

7. Two graphs G and H are identical if $V(G) = V(H)$, $E(G) = E(H)$ and $\psi(G) = \psi(H)$. If two graphs are identical then they can be represented by identical diagrams

8. Two graphs G and H are said to be isomorphic $(G \cong H)$ if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ iff $\psi_H(\phi(e)) = \theta(u)\theta(v)$ such a pair

(θ, ϕ) of mappings is called an isomorphism between G and H.

9) A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. It is denoted by K_n, where n denotes the number of vertices.

10) An empty graph, that has no edges.

11) A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y, such a partition (X, Y) is called a bipartition of a graph.

12) A complete bipartite graph is a simple bipartite graph with partition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ & $|Y| = n$, then it is denoted by $K_{m,n}$.

13) To any graph G there corresponds a $\nu \times \varepsilon$ matrix called the incident matrix of G. The vertices of G are denoted by $v_1, v_2, \ldots v_\nu$, and the edges are by $e_1, e_2, \ldots e_\varepsilon$. Then the incident matrix of G is the matrix $M(G) = [m_{ij}]$ where m_{ij} is the number of times that v_i and e_j are incident. Another matrix associated with G is the adjacency matrix, this is $\nu \times \nu$ matrix $A(G) = [a_{ij}]$ in which a_{ij} is the number of edges joining v_i and v_j.

14) A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. When $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and say H a proper subgraph of G.

15) A spanning subgraph of G is a subgraph H with $V(H) = V(G)$.

16) A simple graph obtained deleting the self loops [illegible] links if having more than link between adjacent

vertices keeping one between them is called the underlying simple graph.

17) Let V' be a nonempty subset of V. The subgraph of G whose vertex set V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V', is denoted by $G(V')$. $G(V')$ is an induced subgraph of G

18) Let E' be a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G induced by E', denoted by $G[E']$. $G[E']$ is an edge-induced subgraph of G.

19) Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are disjoint if they have no vertex in common and edge-disjoint if the have no edge in common.

20) The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint, we denote their union by $G_1 + G_2$.

21) The intersection $G_1 \cap G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. G_1 and G_2 must have at least one vertex is common.

22) The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges. We denote $\delta(G)$ as minimum degree and $\Delta(G)$ as the maximum degree.

23) Prove that $\sum_{v \in V} d(v) = 2\varepsilon$

Proof:-

Consider the incidence matrix M. The sum of the entries in the row corresponding to vertex v is precisely $d(v)$, and therefore $\sum_{v\in V} d(v)$ is just the sum of entries in M. But this sum is also $2E$, since each of the ε column sums of M is 2.

24) Prove that in any graph, the number of vertices of odd degree is even.

Proof:

Let V_1 and V_2 be the sets of vertices of odd and even degree in G, respectively then

$$\sum_{v\in V_1} d(v) + \sum_{v\in V_2} d(v) = \sum_{v\in V} d(v) \quad \text{--- (1)}$$

The right side value is even, also $\sum_{v\in V_2} d(v)$ is also even. Therefore $\sum_{v\in V_1} d(v_1)$ is also even. Thus $|V_1|$ is even.

25) A walk in G is a finite non-null sequence $W = v_0 e_1 v_1 e_2 \ldots e_k v_k$ whose terms are alternatively vertices and edges, such that $1 \le i \le k$, the ends of e_i are v_{i-1} and v_i.

26) If the edges $e_1, e_2, \ldots e_k$ of the walk W are distinct, W is called a trail. In addition, if the vertices $v_0, v_1, \ldots v_k$ are distinct, W is called a path.

27) Two vertices u and v of G are said to be connected if there is a (u,v)-path in G

28) A walk is closed if it has positive length and its origin and terminus are the same. A closed trail whose whose origin and internal vertices are distinct is a cycle. A cycle of length k is called k-cycle.

29) A graph is bipartite iff it contains no odd cycle.

Proof:

Suppose that G is bipartite with bipartition (X,Y) and let $C = v_0 v_1 v_2 \ldots v_k v_0$ be a cycle of G. Without loss of generality we may assume that $v_0 \in X$. Then, since $v_0 v_1 \in E$ and G is bipartite, $v_1 \in Y$. Similarly $v_2 \in X$ and in general $v_{2i} \in X$ and $v_{2i+1} \in Y$. Since $x_0 \in X$, $v_k \in Y$.

Thus $k = 2i+1$ for some i, and it follows that C is even conversely. Let G be a connected graph that contains no odd cycles. We choose an arbitrary vertex u and define partition (X, Y) of V by setting

$$X = \{x \in V : d(u,x) \text{ is even}\}$$

$$Y = \{y \in V : d(u,y) \text{ is odd}\}$$

We claim that (X, Y) is a bipartition of G. Suppose that v and w are two vertices of X. Let P be a shortest (u,v)-path and Q be the shortest (u,w)-path. Denote by u_1 the last vertex common to P and Q. Since P and Q are shortest paths, (u,u_1)-sections of both P and Q are shortest (u,u_1) paths and therefore have the same length. Now, since the lengths of both P and Q is even, the lengths of the (u_1,v)-sections P_1 of P and the (u_1,w)-section Q_1 of Q must have the same parity. It follows that the (v,w)-path $P_1^{-1}Q_1$ is of even length. If v were joined w, $P_1^{-1}Q_1wv$ would be a cycle of odd length, contrary to hypothesis. Therefore no two vertices in X are adjacent; similarly no two vertices in Y are adjacent.

30) Dijkstra's algorithm

① Set $l(u_0) = 0$, $l(v) = \infty$ for $v \neq u_0$, $S_0 = \{u_0\}$ and $i = 0$

② For each $v \in \bar{S}_i$, replace $l(v)$ by $\min\{l(v), l(u_i) + w(u_i v)\}$. Compute $\min_{v \in \bar{S}_i}\{l(v)\}$ and let u_{i+1} denote a vertex for which this minimum is attained. Set $S_{i+1} = S_i \cup \{u_{i+1}\}$.

③ If $i = \nu - 1$, stop. If $i < \nu - 1$, replace i by $i+1$ and go to Step ②.

31) Let T be a closed triangle in the plane. A subdivision of T into a finite number of smaller triangles is said to be simplical if any two intersecting triangles have either a vertex or a whole side in common.

32) Let T be a simplical subdivisioned. Then a labelling of the vertices of triangles in the subdivision in three symbols 0, 1 and 2 is said to be proper if
i) the three vertices of T are labelled 0, 1 and 2 (in any order)
ii) for $0 \leq i \leq j \leq 2$, each vertex on the side of T joining vertices labelled i and j is labelled either i or j.

33) A triangle in a subdivision whose vertices receive all three labels is a distinguished triangle.

34) Sperner's lemma:-
Every properly labelled simplical subdivision of a triangle has no odd number of distinguished triangles

Trees

35) An acyclic graph is one that contains no cycles.

36) A tree is a connected acyclic graph.

37) In a tree, any two vertices are connected by a unique path

Proof:-
By contradiction, Let us assume G be a tree and there are two distinct (u,v)-paths P_1 and P_2 in G. Since $P_1 \neq P_2$ there is an edge $e = xy$ of P_1 that is not an edge of P_2. Clearly the graph $(P_1 \cup P_2 - e)$ is connected. It therefore contains an (x,y)-path P. But the $P+e$ is a cycle in the acyclic graph G, a contradiction.

38) Prove that if G is a tree, then $\varepsilon = \nu - 1$

Proof:-
By induction on ν. When $\nu = 1$, $G \cong K_1$ and $\varepsilon = 0 = \nu - 1$
When $\nu = 2$, $G \cong K_2$ and $\varepsilon = 1 = \nu - 1$
Suppose the theorem is true for all trees on fewer than ν vertices and let G be a tree on $\nu \geq 2$ vertices. Let $uv \in E$. Then $G - uv$ contains no (u,v)-path. Since uv is the unique (u,v)-path in G. Thus $G - uv$

is disconnected and so $\omega(G-uv)=2$. The components G_1 and G_2 of $G-uv$ being acyclic are trees. Moreover each has fewer than ν vertices. Therefore by the induction hypothesis

$$\varepsilon(G_i) = \nu(G_i)-1 \quad \text{for } i=1,2$$

Thus

$$\varepsilon(G) = \varepsilon(G_1)+\varepsilon(G_2)+1 = \nu(G_1)+\nu(G_2)-1 = \nu(G)-1.$$

39) A cut edge of G is an edge e such that $\omega(G-e) > \omega(G)$.

40) An edge e of G is a cut edge of G iff e is contained in no cycle of G.

Proof:

Let e be a cut edge of G. Since $\omega(G-e) > \omega(G)$; there exist; u and v of G that are connected in G but not in $G-e$. There is therefore some (u,v)-path P in G which necessarily, traverses e. Suppose that x and y are the ends of e, and that x preceeds y on P. In $G-e$, u is connected to x by a section of P and y is connected to v by a section of P. If e were in a cycle C, x and y would be connected in $G-e$ by the path $C-e$. Thus u an v would be connected in $G-e$, a contradiction.

Conversely, suppose that $e=xy$ is not a cut edge of G; thus $\omega(G-e)=\omega(G)$. Since there is an (x,y)-path in G, x and y are in the same component of G. It follows that x and y are in the same component of $G-e$, and hence that there is an (x,y) path P in $G-e$. But then e is in the cycle $P+e$ of G.

41) Prove that a connected graph is a tree iff every edge is a cut edge.

Proof:

Let G be a tree and let e be an edge of G. Since G is acyclic, e is contained in no cycle of G

and is therefore a cut edge of G. Conversely, suppose that G is connected but is not a tree. Then G contains a cycle C. But no edge of C can be a cut edge of G.

42) A spanning tree of G is a spanning subgraph of G that is a tree.

43) Every connected graph contains a spanning tree.

Proof:

Let G be connected and let T be a minimal connected spanning subgraph of G. By the definition of $\omega(T) = 1$ and $\omega(T-e) > 1$ for each edge e of T. It follows that each edge of T is a cut edge and therefore, T is a a tree as it is connected too.

44) Let T be a spanning tree of a connected graph G and let e be an edge of G not in T. Then $T+e$ contains a unique cycle.

Proof:

Since T is acyclic, each cycle of $T+e$ contains e. Moreover, C is a cycle of $T+e$ iff $C-e$ is a path in T connecting the ends of e. As T has a unique such path; therefore $T+e$ contains a unique cycle.

45) An edge cut of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of V and $\bar{S} = V \setminus S$. A minimal edgecut of G is called a bond.

46) If G is a given connected graph, T be its spanning tree the $\bar{T}$ is the subgraph of G, called as cotree of G.

47) Let T be a spanning tree of a connected graph G, and let e be any edge of T. Then

i) the cotree $\bar{T}$ contains no bond of G;

ii) $\bar{T}+e$ contains a unique bond of G.

Proof: i) Let B be a bond of G. Then $G-B$ is disconnected and so cannot contain the spanning tree T. Therefore B is not contained in $\bar{T}$.

ii) Denote by S the vertex set of one of the two components of $T-e$. The edge cut $B = [S, \bar{S}]$ is clearly a bond of G, and is contained in $\bar{T}+e$. Now for any $b \in B$, $T-e+b$ is a spanning tree of G. Therefore every bond of G contained in $\bar{T}+e$ must include every such element b. It follows that B is the only bond of G contained in $\bar{T}+e$.

48) A vertex v of G is a cut vertex if E can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ has just a vertex v in common.

49) A vertex v of a tree G is a cut vertex of G iff $d(v) > 1$.

Proof:

If $d(v) = 0$, $G \cong K_1$ and so v cannot be a cut vertex.

If $d(v) = 1$ $G-v$ is an acyclic graph with $\nu(G-v)-1$ edges and thus a tree. Hence $\omega(G-v) = 1 = \omega(G)$ and v is not a cut vertex of G.

If $d(v) > 1$, there are distinct vertices u and w adjacent to v. The path uvw is a (u,w)-path in G. Since G is a tree, uvw is the unique path in G. It follows that there is no (u,w)-path in $G-v$, and therefore that $\omega(G-v) > \omega(G)$. Thus v is a cut vertex of G.

50) Every nontrivial loopless connected graph has atleast two vertices that are not cut vertices.

Proof:

Let G be a nontrivial loopless connected graph. G contains a spanning tree T. As T containg atleast two pendant vertices and pendant vertex cannot cannot be a cut vertex. Therefore T has at least two vertices that are not cut vertices. Let v be any such vertices Then $\omega(T-v) = 1$

Since T is a spanning subgraph of G, $T-v$ is a is a spanning subgraph of $G-v$ and therefore

$$\omega(G-v) \leq \omega(T-v)$$

It follows that $\omega(G-v)=1$, hence that v is not a cut vertex of G. Since there are at least two such vertices v.

51) An edge e of G is said to be contracted if it is ~~closed~~ deleted and its ends are identified, the resulting graph is denoted by $G \cdot e$.

52) If e is a link of G, the $\tau(G) = \tau(G-e) + \tau(G \cdot e)$

Proof:-

Since every spanning tree of G that does not contain e is also a spanning tree of $G-e$ and conversely, $\tau(G-e)$ is the number of spanning trees of G that do not contain e. Now to each spanning tree T of G that contains e, there corresponds a spanning tree $T \cdot e$ of $G \cdot e$. This correspondence is clearly a bijection. Therefore $\tau(G \cdot e)$ is precisely the number of spanning trees of G that contain e. It follows that $\tau(G) = \tau(G-e) + \tau(G \cdot e)$.

53) A minimum-weight spanning tree of a weighted graph will be called an optimal tree.

54) Kruskal's Algorithm:-

i) Choose a link e_1 such that $w(e_1)$ is as small as possible.

ii) If edges $e_1, e_2, \ldots, e_i$ have been chosen, then choose an edge e_{i+1} from $E \setminus \{e_1, e_2, \ldots e_i\}$ in such a way that

a) $G[\{e_1, e_2, \ldots, e_{i+1}\}]$ is acyclic;

b) $w(e_{i+1})$ is as small as possible subjet to (a).

iii) Stop when step cannot be implemented further.

—x— x—.

UNIT- II

1) A vertex cut of G is a subset V' of V such that G-V' is disconnected. A k-vertex cut is a vertex cut of k elements. The connectivity $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut.

2) A k-edge cut is an edge cut of k elements. The edge connectivity $\kappa'(G)$ of G is the minimum k for which G has a k-edge cut.

3) Prove $\kappa \le \kappa' \le \delta$.

Proof:

If G is trivial, then $\kappa' = 0 \le \delta$. otherwise, the set of links elements incident with a vertex of degree δ constitute a δ-edge cut of G. It follows that $\kappa' \le \delta$.

We prove that $\kappa \le \kappa'$ by induction on κ'. The result is true if $\kappa' = 0$, since then G must be either trivial or disconnected. Suppose that it holds for all graphs with edge connectivity less than k, let G be a graph with $\kappa'(G) = k > 0$, and let e be an edge in a k-edge cut of G. Setting $H = G - e$, we have $\kappa'(H) = k-1$ and so, by the induction hypothesis, $\kappa(H) \le k-1$.

If H contains a complete graph as a spanning subgraph, then so does G and $\kappa(G) = \kappa(H) \le k-1$. Otherwise, let S be a vertex cut of H with $\kappa(H)$ elements. Since H-S is disconnected, either G-S is disconnected, and then

$$\kappa(G) \le \kappa(H) \le k-1$$

orelse G-S is connected and e is a cut edge of G-S. In this latter case, either $\nu(G-S) = 2$ and

$$\kappa(G) \le \nu(G) - 1 = \kappa(H) + 1 \le k$$

or G-S has a 1-vertex cut $\{v\}$, implying that $S \cup \{v\}$ is a vertex cut of G and $\kappa(G) \le \kappa(H) + 1 \le k$.

Thus in each case we hat $\kappa(G) \le k = \kappa'(G)$.

4) A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property.

5) A graph G with $\nu \geq 3$ is 2-connected if and only if any two vertices of G are connected by at least two internally-disjoint paths.

Proof: If any two vertices of G are connected by at least two internally disjoint paths then, clearly, G is connected and has no 1-vertex cut. Hence G is 2-connected.

Conversely, let G be a 2-connected graph, we shall prove, by induction on the distance $d(u,v)$ between u and v, that any two vertices u and v are connected by at least two internally-disjoint paths.

Suppose, first that $d(u,v)=1$. Then, since G is 2-connected the edge uv is not a cut edge and therefore, by theorem 2.3, it is contained in a cycle. It follows that u and v are connected by two internally-disjoint paths in G.

Now assume that the theorem holds for any two vertices at distance less than k, and let $d(u,v)=k \geq 2$. Consider a (u,v)-path of length k, and let w be the vertex that precedes v on this path. Since $d(u,w)=k-1$, it follows from the induction hypothesis that there are two internally disjoint (u,w)-paths P and Q in G. Also since G is 2-connected, $G-w$ is connected and so contains a (u,v) path P'. Let x be the last vertex of P' that is also in $P \cup Q$. Since u is in $P \cup Q$ there is such an x; we do not exclude the possibility that $x=v$.

We may assume, without loss of generality that x is in P. Then G has two internally-disjoint (u,v)-paths. One composed of the section of P from u to x together with the section of P' from x to v, and the other composed of Q together with the path wv.

6) If G is a block with $\nu \geq 3$, then any two edges of G lie on a common cycle.

Proof:

Let G be a block with $\nu \geq 3$, and let e_1 and e_2 be two edges of G. Form a new graph G' by subdividing e_1 and e_2 and denote the new vertices by v_1 and v_2. Clearly G' is a block with atleast five vertices, and hence is

2-connected. It follows that v_1 and v_2 lie on a common cycle of G'. Thus e_1 and e_2 lie on a common cycle of G.

7) A trail that traverses every edge of G is called an Euler trail of G. A tour of G is a closed walk that traverses each edge of G at least once. An Euler tour is a tour which traverses each edge exactly once. A graph is eulerian if it contains an Euler tour.

8) A nonempty connected graph is eulerian iff it has no vertices of odd degree.

Proof-

Let G be eulerian, at let C be an Euler tour of G with origin u. Each time a vertex v occurs as an internal vertex of C, two of the edges incident with v are accounted for. Since an Euler tour contains every edge of G, $d(v)$ is even for all $v \neq u$. Similarly, since C starts and ends at u, $d(u)$ is also even. Thus G has no vertices of odd degree.

Conversely, suppose that G is a noneulerian connected graph with atleast one edge and no vertices of odd degree. Choose such a graph G with as few edges as possible. Since each vertex of G has degree atleast two contains a closed trail. Let C be a closed trail of maximum possible length in G. By assumption, C is not an Euler tour of G and $G-E(C)$ has some component G' with $\varepsilon(G')>0$. Since C is itself eulerian it has no vertices of odd degree; thus the connected graph G' also has vertices of odd degree. Since $\varepsilon(G') < \varepsilon(G)$, it follows from the choice of that G' has an Euler tour C'. Now because G is connected, there is a vertex v in $V(C) \cap V(C')$ and we may assume, without loss of generality, that v is the origin and terminus of both C and C'. But then CC' is a closed trail of G with $\varepsilon(CC') > \varepsilon(C)$, ~~contradicting~~ contradicting the choice of C.

9) A connected graph has an Euler trail iff it has atmost two vertices of odd degree.

Proof:-

If G has an Euler trail then each vertex other than the origin and terminus of this trail has even degree.

Conversely, suppose that G is a nontrivial connected graph with atmost two vertices of odd degree. If G is a nontrivial connected graph. If G has no such vertices then, G has a closed trail otherwise G has exactly two vertices, u and v of odd degree. In this case, let G+e denote the graph obtained from the addition of a new edge e joining u and v. clearly each vertex of v has even degree and so G+e has an Euler tour $v_0 e_1 v_1 \dots e_{\varepsilon+1} v_{\varepsilon+1}$ where $e_1 = e$. The trail $v_1 e_2 v_2 \dots e_{\varepsilon+1} v_{\varepsilon+1}$ is an Euler trail of G.

10) A path that contains every vertex of G is called a Hamilton path of G. A Hamilton cycle of G is a cycle that contains every vertex of G. A graph is hamiltonian if it contains a Hamilton cycle.

11) If G is hamiltonian then, for every nonempty proper subset S of V $\omega(G-S) \leq |S|$.

Proof:

Let C be Hamiltonion cycle of G. Then for every nonempty proper subset S of V

$$\omega(C-S) \leq |S|$$

Also C-S is a spanning subgraph of G-S and so

$$\omega(G-S) \leq \omega(C-S)$$

Thus we get the theorem.

12) If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, then G is hamiltonian.

Proof:-

By contradiction. Suppose that the theorem is false, and let G be a maximal nonhamiltonian simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$. Since $\nu \geq 3$ G cannot be complete. Let u and v be nonadjacent vertices in G.

By the choice of G, $G+uv$ is hamiltonian. Moreover, since G is nonhamiltonian each Hamilton cycle $G+uv$ must contain the edge uv. Thus there is a Hamilton path $v_1 v_2 \cdots v_\nu$ in G with origin $u = v_1$ and terminus $v = v_\nu$. Set $S = \{v_i : uv_{i+1} \in E\}$ and $T = \{v_i : v_i v \in E\}$

Since $v_\nu \notin S \cup T$ we have $|S \cup T| < \nu$. — ①

Furthermore $|S \cap T| = 0$. — ②

Since if $S \cap T$ contained some vertex v_i, then G would have the Hamilton cycle $v_1 v_2 \cdots v_i v_\nu v_{\nu-1} \cdots v_{i+1} v_1$ contrary to assumption. Using ① & ② we obtain

$d(u) + d(v) = |S| + |T| = |S \cup T| + |S \cap T| < \nu$ — ③

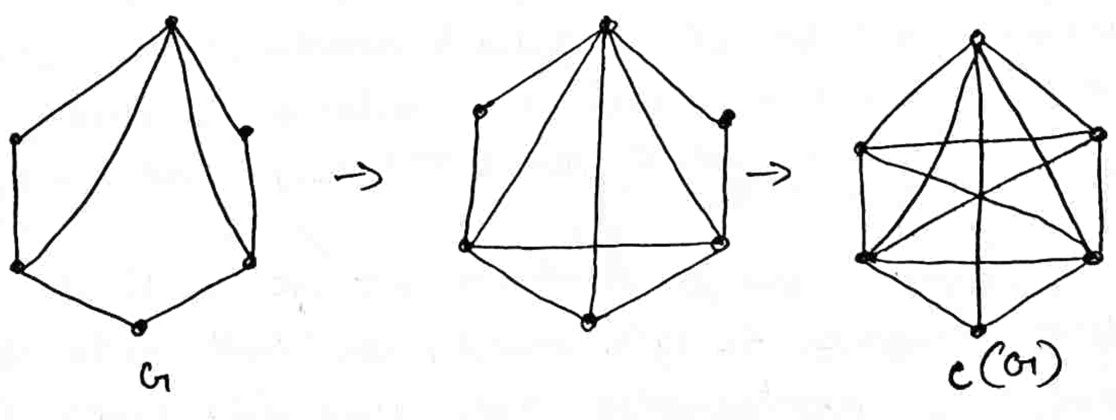
But this contradicts the hypothesis that $\delta \geq \nu/2$.

13) Let G be a simple graph and let u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq \nu$. Then G is Hamiltonion iff $G+uv$ is hamiltonian

Proof:

If G is hamiltonian then trivially, so too is $G+uv$. Conversely suppose that $G+uv$ is hamiltonian but G is not. Then as by the above proof at theorem (12) we obtain $d(u) + d(v) < \nu$. But this contradicies $d(u) + d(v) \geq \nu$.

14) The closure of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is atleast ν until no such pair remains. We denote the closure of G by $c(G)$.



The closure of a graph.

15) Prove that $c(G)$ is well defined.

Proof:-

Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is atleast ν until if such pairs remains. Denote by $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ the sequence of edges added to G in obtaining G_1 and G_2 respectively. We shall show that each e_i is an edge of G_2 and f_j is an edge of G_1.

If possible, let $e_{k+1} = uv$ be the first edge in the sequence $e_1, e_2, \ldots, e_n$ that is not an edge of G_2. set $H = G + \{e_1, e_2, \ldots, e_k\}$. It follows from the definition of G_1 that $d_H(u) + d_H(v) \geq \nu$.

By the choice of e_{k+1}, H is a subgraph of G_2. Therefore

$$d_{G_2}(u) + d_{G_2}(v) \geq \nu.$$

This is a contradiction, since u and v are nonadjacent in G_2. Therefore each e_i is an edge of G_2 and similarly each f_j is an edge of G_1. Hence $G_1 = G_2$ and $c(G)$ is well defined.

16) A simple graph is Hamiltonion iff its closure is hamiltonion.

17) Let G be a simple graph with $\nu \geq 3$. If $c(G)$ is complete, then G is hamiltonion.

18) Let G be a simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$, where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$. Suppose that there is no value of m less than $\nu/2$ for which $d_m \leq m$ and $d_{\nu-m} < \nu - m$. Then G is hamiltonian.

Proof:-

Let G satisfy the hypothesis of the theorem. We shall show that its closure $c(G)$ is complete and the conclusion will then follow from (17) we denote the degree of a vertex v in $c(G)$ by $d'(v)$.

Assume that $c(G)$ is not complete, and let u and v be two nonadjacent vertices in $c(G)$ with

$$d'(u) \leq d'(v) \quad \text{——} \ (1)$$

and $d'(u) + d'(v)$ as large as possible; since no two nonadjacent vertices in $c(G)$ can have degree sum ν or more, we have

$$d'(u) + d'(v) < \nu \quad \text{——} \ (2)$$

Now denote by S the set of vertices in $V \setminus \{v\}$ which are nonadjacent to v in $c(G)$, and by T the set of vertices in $V \setminus \{u\}$ which are nonadjacent to u in $c(G)$. clearly

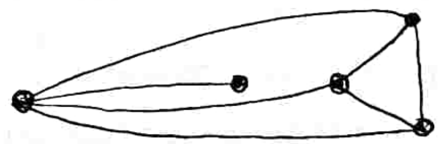
$$|S| = \nu - 1 - d'(v) \text{ and } |T| = \nu - 1 - d'(u) \quad \text{——} \ (3)$$

furthermore, by the choice of u and v, each vertex in S has degree atmost $d'(u)$ and each vertex in $T \cup \{u\}$ has degree atmost $d'(v)$. Setting $d'(u) = m$ and using (2) & (3), we find that $c(G)$ has at least m vertices of degree atmost m and atleast $\nu - m$ vertices of degree less than $\nu - m$. Because G is a spanning subgraph of $c(G)$, the same is true is true of G; therefore $d_m \leq m$ and $d_{\nu-m} < \nu - m$. But is contrary to hypothesis since, by (1) and (2) $m < \nu/2$. We conclude $c(G)$ is indeed complete and hence that G is hamiltonian.

19) The join $G \vee H$ of disjoint graphs G and H is the graph obtained from $G + H$ by joining each vertex of G to each vertex of H. Let $C_{m,n}$ denote $K_m \vee (K_m^c + K_{n-2m})$.

eg $C_{1,5}$ is $K_1 \vee (K_1^c + K_{5-2})$.

$m = 1, n = 5$



$C_{1,5}$

20) <u>Chvatal Theorem</u> : If G is a nonhamiltonian simple graph with $\nu \geq 3$, then G is degree-majorised by some $C_{m,\nu}$.

Proof:

Let G be a nonhamiltonian simple graph with degree sequence $(d_1, d_2, \ldots, d_\nu)$ where $d_1 \le d_2 \le \ldots \le d_\nu$ and $\nu \ge 3$. Then by the previous theorem (18) there exists $m < \nu/2$ such that $d_m \le m$ and $d_{\nu-m} < \nu - m$. Therefore $(d_1, d_2, \ldots, d_\nu)$ is majorised by the sequence

$(m, \ldots m, \nu-m-1, \ldots, \nu-m-1, \nu-1, \ldots \nu-1)$

with m terms equal to m, $\nu-2m$ terms equal to $\nu-m-1$ and m terms equal to $\nu-1$, and this sequence is the degree sequence of $C_{m,\nu}$.

21) A subset M of E is called a matching in G if its elements are links and no two are adjacent in G; the two ends of an edge in M are said to be matching under M.

22) A matching M saturates a vertex v, and v is said to be M-saturated if some edge of M is incident with v; otherwise v is M-unsaturated.

23) If every vertex of G is M-saturated, the matching M is perfect. M is a maximum matching if G has no matching M' with $|M'| > |M|$.

24) Let M be a matching in G. An M-alternating path in G is a path whose edges are alternatively in $E \setminus M$ and M.

25) An M-augumenting path is an M-alternating path whose origin and terminus are M-unsaturated.

26) Berge: A matching M in G is a maximum matching iff G contains no M-augmenting path.

Proof:

Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0 v_1 \ldots v_{2m+1}$. Define $M' \subseteq E$ by $M' = (M \setminus \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2m-1}v_{2m}\}) \cup \{v_0v_1, v_2v_3, v_4v_5, \ldots, v_{2m}v_{2m+1}\}$

Then M' is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.

conversely, Suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then $|M'| > |M|$ set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of M and M'.

Each vertex of H has degree either one or two in H, since it can be incident with atmost one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternatively in M and M' or else a path with edges altornatively in M and M'. H containing more edges of M' than of M and therefore some path component P of H must start and end with edges of M'. The origin and terminus of P, being M'-saturated in H, are M-unsaturated in G. Thus P is an M-angmenting path in G.

27) The neighbour set of S in G is defined to be the set of all vertices adjacent to vertices in S, this is denoted by $N_G(S)$.

28) Let G be a bipartite graph with partition (X, Y). Then G contains a matching that saturates every vertex in X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof: Suppose that G contains a matching M which saturates every vertex in X, and let S be a subset of X. Since the vertices in S are matching under M with distinct vertices in $N(S)$, we clearly have $|N(S)| \geq |S|$.

conversely suppose that G is a bipartite graph satisfying the given condition, but that contains no matching saturating all the vertices in X. We shall obtain a contradiction. Let M^* be a maximum matching in G. By our supposition M^* does not saturate all vertices in X. Let u be an M^*-unsaturated vertex in X, and let Z denote the set of all vertices connected to u by M alternating paths. Since M^* is a maximum matching, it follows that u is the only M^*-unsaturated vertex in Z. Set $S = Z \cap X$ and $T = Z \cap Y$.

Clearly, the vertices in $S \setminus \{u\}$ are matched under M^* with the vertices in T. Therefore $|T| = |S| - 1$ —①, and $N(S) \supseteq T$. In fact, we have $N(S) = T$ —②

Since every vertex in $N(S)$ is connected to u by an M^*-alternating path. But ① & ② imply that $|N(S)| = |S| - 1 < |S|$ contradicting assumption.

29) If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof:

Let G be a k-regular bipartite graph with bipartition (X, Y). Since G is k-regular, $k|X| = |E| = k|Y|$ and so, since $k > 0$, $|X| = |Y|$ Now let S be a subset of X and denote by E_1 and E_2 the sets of edges incident with vertices in S and $N(S)$, respectively. By definition of $N(S)$, $E_1 \subseteq E_2$ and therefore $k|N(S)| = |E_2| \geq |E_1| = k|S|$.

It follows that $|N(S)| \geq |S|$ and hence that G has a matching M saturating every vertex in X. Since $|X| = |Y|$, M is a perfect matching.

30) A covering of a graph G is a subset K of V such that every edge of G has atleast one end in K. A covering K is a minimum covering if G has no covering K' with $|K'| < |K|$.

31) In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering

Proof:

Let G be a bipartite graph with partition (X, Y) and let M^* be a maximum matching of G. Denote by U the set of M^*-unsaturated vertices in X, and by Z the set of all vertices connected by M^*-alternating paths to vertices of U. Set $S = Z \cap X$ and $T = Z \cap Y$. Then as by theorem (28) we have that every vertex in T is M^*-saturated and $N(S) = T$. Define $\bar{K} = (X \setminus S) \cup T$. Every edge of G must have at least one of its ends in $\bar{K}$. For otherwise, there would be an edge with one end in S and one end in $Y \setminus T$, contradicting $N(S)=T$. Thus $\bar{K}$ is a covering of G and clearly $|M^*|=|\bar{K}|$. As by the lemma that "if $|M|=|K|$, then M is maximum matching K is minimum covering", $\bar{K}$ is a minimum covering. Hence the theorem.

32) Every 3-regular graph without cut edges has a perfect matching.

Proof:-

Let G be a 3-regular graph without cut edges, and let S be a proper subset of V. Denote by $G_1, G_2, \ldots, G_n$ be the odd components of $G-S$, and let m_i be the number of edges with one end in G_i and one end in S, $1 \le i \le n$. Since G is 3-regular

$$\sum_{v \in V(G_i)} d(v) = 3\,\nu(G_i) \text{ for } 1 \le i \le n \quad \text{——} (1)$$

and

$$\sum_{v \in S} d(v) = 3|S| \quad \text{——} (2)$$

By (1) $m_i = \sum_{v \in V(G_i)} d(v) - 2\varepsilon(G_i)$ is odd. Now $m_i \neq 1$ since G has no cut edge. Thus $m_i \geq 3$ for $1 \le i \le n$ —— (3)

It follows from (3) & (2) that

$$o(G-S) = n \le \frac{1}{3}\sum_{i=1}^{n} m_i \le \frac{1}{3}\sum_{v \in S} d(v) = |S|$$

Therefore, by theorem "G has a perfect matching iff $o(G-S) \le |S|$ for all $S \subset V$", G has a perfect matching.

——x——x——

UNIT-3

1) A k-edge colouring $\mathcal{C}$ of a loopless graph G is an assignment of k colours, 1, 2, ..., k, to the edges of G. The colouring $\mathcal{C}$ is proper if no two adjacent edges have the same colour.

2) G is k-edge-colourable if G has a proper k-edge-colouring.

3) The edge chromatic number $\chi'(G)$, of a loopless graph G, is the minimum k for which G is k-edge-colourable. G is k-edge-chromatic if $\chi'(G) = k$.

4) Let G be a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree atleast two.

Proof:-

We may clearly assume that G is nontrivial. Suppose, first, that G is eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. otherwise, G has a vertex v_0 of degree atleast four, Let $v_0 e_1 v_1 \ldots e_\varepsilon v_0$ be an Euler tour of G, and set

$E_1 = \{e_i : i \text{ odd}\}$ and $E_2 = \{e_i : i \text{ even}\}$ —— (1)

Then the 2-edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex $v_0 e_1 v_1 \ldots e_\varepsilon v_0$.

It is not eulerian, construct a new graph G^* by adding a new vertex u and joining it to each vertex of odd degree in G. clearly G^* is eulerian. Let $v_0 e_1 v_1 \ldots e_{\varepsilon^*} v_0$ be an Euler tour of G^* and define E_1 and E_2 as in (1). It is then easily verified that the 2-edge colouring $(E_1 \cap E, E_2 \cap E)$ of G has the required property.

5) Let $\mathcal{C} = (E_1, E_2, \ldots, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is represented atleast twice at u, then the components of $G[E_i \cap E_j]$ that contains u is an odd cycle.

Proof:-

Let u be a vertex that satisfies the hypothesis and denote by H the component of $G[E_i \cup E_j]$ containing u. Suppose that H is not an odd cycle. Then H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colours i and j in this way, we obtain a new k-edge colouring $\mathscr{C}' = [E_1', E_2', \ldots, E_k')$ of G. Denoting by $c'(v)$ the number of distinct colours at v in the colouring $\mathscr{C}'$, we have $c'(u) = c(u)+1$.

Since now both i and j are represented at u, and also $c'(v) \geq c(v)$ for $v \neq u$. Thus $\sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$, contradicting the choice of $\mathscr{C}$. It follows that H is indeed an odd cycle.

6) If G is bipartite, then $\chi' = \Delta$.

Proof:-

Let G be a graph with $\chi' > \Delta$, let $\mathscr{C} = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of G, and let u be a vertex such that $c(u) < d(u)$. clearly, u satisfies the hypothesis of above (5) one. Therefore G contains an odd cycle and so is not bipartite. It follows that G is bipartite, then $\chi' = \Delta$

7) If G is simple then either $\chi' = \Delta$ or $\chi' = \Delta+1$

Proof:-

Let G be a simple graph. By virtue $\chi' \geq \Delta$, we need only show that $\chi' \leq \Delta+1$. Suppose then, that $\chi' > \Delta+1$. Let $\mathscr{C} = (E_1, E_2, \ldots, E_{\Delta+1})$ be an optimal $(\Delta+1)$-edge colouring of G and let u be a vertex such that $c(u) < d(u)$. There exist colours i_0 and i_1 such that i_0 is represented at u and i_1 is represented atleast twice at u. Let uv_1 have colour i_1.

Since $d(v_1) < \Delta+1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u since otherwise, by recolouring uv_1 with i_2. We would obtain an improvement on $\mathscr{C}$. Thus some edge uv_2 has colour i_2. Again since $d(v_2) < \Delta+1$, some colour i_3 is not represented at v_2; and i_3

must be represented at u since otherwise, by recolouring with uv_1 with i_2 and uv_2 with i_3. we would obtain an improved $(\Delta+1)$-edge colouring. Thus some edge uv has colour i_3. Continuing this procedure we construct a sequence $v_1, v_2, \ldots v_n$ of vertices and a sequence $i_1, i_2, \ldots$ of colours, such that

i) uv_j has colour i_j and

ii) i_{j+1} is not represented at v_j

Since the degree of u finite, there exists a smallest integer l such that, for some $k<l$,

iii) $i_{l+1} = i_k$.

We now recolour G as follows. For $1 \le j \le k-1$, recolour uv_j with colour i_{j+1}, yielding a new $(\Delta+1)$-edge colouring $\mathscr{C}' = (E_1', E_2', \ldots E_{\Delta+1}')$. Clearly $c'(v) \geq c(v)$ for all $v \in V$ and therefore $\mathscr{C}'$ is also an optimal $(\Delta+1)$-edge colouring of G. The component H' of $G[E_{i_0}' \cup E_{i_k}']$ that contains u is an odd cycle.

Now in addition, recolour uv_j with colour i_{j+1}, $k \le j \le l-1$, and uv_l with colour i_k, to obtain a $(\Delta+1)$-edge colouring $\mathscr{C}'' = (E_1'', E_2'', \ldots E_{\Delta+1}'')$. As above $c''(v) \geq c(v)$ for all $v \in V$. $G[E_{i_0}'' \cup E_{i_k}'']$ that contains u and the component H'' of $G[E_{i_0}'' \cup E_{i_k}'']$ is odd cycle. But, since v_k has degree two in H', v_k clearly has degree one in H''. This contradiction establishes the theorem.

8) A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. An independent set is maximum if G has no independent set S' with $|S'| > |S|$.

9) A set $S \subseteq V$ is an independent set of G iff $V \setminus S$ is a covering of G.

Proof:

By the above definition, S is an independent set of G iff no edge of G has both ends in S, or, equivalently iff each edge has at least one end in $V \setminus S$. But this is so iff $V \setminus S$ is covering of G.

10) The subset K of V such that every edge of G has atleast one end in K is called covering of G.

11) The number of vertices in a maximum independent set G is called the independence number of G and is denoted by $\alpha(G)$,

12) The number of vertices in minimum covering of G is the covering number of G and is denoted by $\beta(G)$.

13) $\alpha+\beta=\nu$. Prove.

Proof:

Let S be a maximum independent set of G, and let K be a minimum covering of G. Then $V\setminus K$ is an independent set and $V\setminus S$ is a covering. Therefore

$\nu-\beta=|V\setminus K|\leq\alpha$ —— ① and

$\nu-\alpha=|V\setminus S|\leq\beta$ —— ②

combining ① & ②, we get $\alpha+\beta=\nu$.

14) The edge analogue of an independent set is a set of links no two of which are adjacent that is a matching. The edge analogue of covering is called an edge covering.

15) An edge covering of G is a subset L of E such that each vertex of G is an end of some edge in L.

16) We denote the number of edges in a maximum matching of G by $\alpha'(G)$, and the number of edges in a minimum edge covering of G by $\beta'(G)$; the number $\alpha'(G)$ and $\beta'(G)$ are the edge independence number and edge covering number of G, respectively.

17) Gallai: If $\delta>0$, then $\alpha'+\beta'=\nu$.

Proof:

Let M be a maximum matching in G and let U be the set of M-unsaturated vertices. Since $\delta>0$ and M is maximum, there exists a set E of |U| edges, one incident with each vertex in U. Clearly, $M\cup E'$ is an edge covering of G, also $\beta'\leq|M\cup E'|=\alpha'+(\nu-2\alpha')=\nu-\alpha'$

or $\alpha'+\beta'\leq\nu$ —— ①

Now let L be a minimum edge covering of G, set $H = G[L]$ and let M be a maximum matching in H. Denote the set of M-unsaturated vertices in H by U. Since M is maximum, $H[U]$ has no links and therefore

$|L| - |M| = |L \setminus M| \geq |U| = \nu - 2|M|$.

Because H is a subgraph of G, M is a matching in G and so

$$\alpha' + \beta' \geq |M| + |L| \geq \nu \quad \text{——} \ (2)$$

combining (1) & (2) we get $\alpha' + \beta' = \nu$.

18) A clique of a simple graph G is a subset S of V such that $G[S]$ is complete.

19) Ramsey showed that given any positive integers k and l, there exists a smallest integer $r(k,l)$ such that every graph on $r(k,l)$ vertices contains either a clique of k-vertices or an independent set of l vertices.

For ~~eg~~ $r(1,l) = r(k,1) = 1$.

20) for any two integes $k \geq 2$ and $l \geq 2$

$$r(k,l) \leq r(k, l-1) + r(k-1, l). \quad \text{——} \ (1)$$

Furthermore, if $r(k, l-1)$ and $r(k-1, l)$ are both even, then strict inequality holds in (1).

Proof:

Let G be a graph or $r(k, l-1) + r(k-1, l)$ vertices, and let $v \in V$. We distinguish two cases:

i) v is nonadjacent to a set S of atleast $r(k, l-1)$ vertices

or

ii) v is adjacent to a set T of atleast $r(k-1, l)$ vertices.

Note that either case (i) or case (ii) must hold because the number of vertices to which v is nonadjacent plus the number of vertices to which v is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$.

In case (i), $G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of vertices or an independent set of l vertices. Similarly, in case (ii) $G[T \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices. Since one of case (i) and case (ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices. This proves ①.

Now suppose that $r(k,l-1)$ and $r(k-1,l)$ are both even, and let G be graph on $r(k,l-1)+r(k-1,l)-1$ vertices. Since G has an odd number of vertices, it follows that some vertex v is of even degree in particular, v cannot be adjacent to precisely $r(k-1,l)-1$ vertices. Consequently either case (i) or case (ii) above holds, and therefore contains either a clique of k vertices or an independent set of l vertices. Thus $r(k,l) \leq r(k,l-1)+r(k-1,l)-1$, as stated.

21) If a simple graph G contains no K_{m+1}, then G is degree majorised by some complete m-partite graph H. Moreover, if G has the same degree sequence as H, then $G \cong H$.

Proof:

By induction on m. The theorem is trivial for $m=1$. Assume that it holds for all $m<n$, and let G be a simple graph which contains no K_{n+1}. Choose a vertex u of degree Δ in G, and set $G_1 = G[N(u)]$. Since G contains no K_{n+1}, G_1 contains no K_n and therefore by the induction hypothesis, is degree-majorised by some complete $(n-1)$-partite graph H_1.

Next, set $V_1 = N(u)$ and $V_2 = V \setminus V_1$ and denote by G_2 the graph whose vertex set is V_2 and whose edge set is empty. Consider the join $G_1 \vee G_2$ of G_1 and G_2. Since

$$N_G(v) \subseteq N_{G_1 \vee G_2}(v) \text{ for } v \in V_1 \quad \text{—①}$$

and since each vertex of V_2 has degree Δ in $G_1 \vee G_2$, G is degree-majorised by the complete n-partite graph $H = H_1 \vee G_2$.

Suppose, now, that G has the same degree sequence as H. Then G has the same degree sequence as $G_1 \vee G_2$ and hence equality must hold in ①. Thus, in G, every vertex of V_1 must be joined to every vertex of V_2. It follows that $G = G_1 \vee G_2$. Since $G = G_1 \vee G_2$ has the same degree sequence as $H = H_1 \vee G_2$, the graphs G_1 and H_1 must have the same degree sequence and therefore, by induction hypothesis be isomorphic. We conclude that $G \cong H$.

22) A k-vertex colouring of G is an ~~important~~ assignment of k-colours 1, 2, ..., k to the vertices of G; the colouring is proper if no two distinct adjacent vertices have the same colour.

23) The chromatic number, $\chi(G)$, of G is the minimum k for which G is k-colourable; if $\chi(G) = k$, G is said to be k-chromatic.

24) If G is k-critical, then $\delta \geq k-1$.

Proof:-

By contradiction. If possible, let G be a k-critical graph with $\delta < k-1$, and let v be a vertex of degree δ in G. Since G is k-critical, $G-v$ is (k-1)-colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a (k-1)-colouring of $G-v$. By that, v is adjacent in G to $\delta < k-1$ vertices, and therefore v must be nonadjacent in G to every vertex of some V_j. But then $(V_1, V_2, \ldots V_j \cup \{v\}, \ldots, V_{k-1})$ is a (k-1)-colouring of G, a contradiction. Thus $\delta \geq k-1$.

25) Every k-chromatic graph has atleast k-vertices of degree atleast k-1.

Proof:-

Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. By the above theorem (24) each vertex of H has degree atleast k-1 in H, and hence also in G. Thus the given statement follows since H, being k-chromatic clearly has atleast k-vertices.

26) In a critical graph, no vertex cut is clique.

Proof:-

By contradiction. Let G be a k-critical graph and suppose that G has a vertex cut S that is a clique.

Denote the S-components of G by $G_1, G_2, \ldots, G_n$. Since G is k-critical, each G_i is (k-1)-colourable. Furthermore, because S is a clique, the vertices S must receive distinct colours in any (k-1)-colouring of G_i. It follows that there are (k-1)-colourings of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a (k-1)-colouring of G, a contradiction.

27) Dirac:- Let G be a k-critical graph with a 2-vertex cut {u,v}. Then (i) $G = G_1 \cup G_2$, where G_i is a {u,v}-component of type i (i=1,2), and (ii) both G_1+uv and $G_2 \cdot uv$ are k-critical.

Proof:-

(i) Since G is critical, each {u,v}-component of G is (k-1) colourable. Now there cannot exist (k-1)-colourings of these {u,v} components all of which agree on {u,v}, Since such colourings work together yield a (k-1)-colouring of G. Therefore there are two {u,v} components G_1 and G_2 such that no (k-1)-colouring of G_1 agrees with a (k-1)-colouring of G_2. Clearly one, say G_1, must be of type 1 and the other G_2 of type 2. Since G_1 and G_2 are of different types, the Subgraph $G_1 \cup G_2$ of G is not (k-1)-colourable. Therefore, because G is critical, it must have $G = G_1 \cup G_2$.

ii) Set $H_1 = G_1+uv$. Since G_1 is of type 1, H_1 is k-chromatic. We shall prove that H_1 is critical by showing that, for every edge e of H_1, H_1-e (k-1)-colourable. This is clearly so if $e=uv$, since then $H_1-e = G_1$. Let be some other edge of H_1. In any (k-1)-colouring, since G_2 is a subgraph of $G-e$. The restriction of such a colouring to the vertices of G_1 is a (k-1)-colouring H_1-e. Thus G_1+uv is k-critical. An analogous argument shows that $G_2 \cdot e$ is k-critical.

28) If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi \le \Delta$.

Proof:-

Let G be a k-chromatic graph which satisfies the hypothesis of the theorem. Without loss of generality, we may assume that G is k-critical. G is a block. Also

Since 1-critical and 2-critical graphs and complete and 3-critical graphs are odd cycles. We have $k \geq 1$

If G has a 2-vertex cut $\{u,v\}$ gives

$$2\Delta \geq d(u) + d(v) \geq 3k-5 \geq 2k-1.$$

This implies that $\chi = k \leq \Delta$, since 2Δ is even.

Assume, then, that G is 3-connected. Since G is not complete, there are three vertices u, v and w in G such that $uv, vw \in E$ and $uw \notin E$. Set $u = v_1$ and $w = v_2$ and let $v_3, v_4, \ldots, v_\nu = v$ be any ordering of the vertices of $G - \{u,w\}$ such that each v_i is adjacent to some v_j with $j > i$. We can now describe a Δ colouring of G: assign colour 1 to $v_1 = u$ and $v_2 = w$; then successively colour $v_3, v_4, \ldots, v_\nu$ each with the first available colour in the list $1, 2, \ldots, \Delta$. By the construction of the sequence $v_1, v_2, \ldots, v_\nu$ each vertex v_i $1 \leq i \leq \nu-1$, adjacent to some vertex v_j, with $j > i$, and therefore to atmost $\Delta - 1$ colours, and thus that one of the colours $1, 2, \ldots, \Delta$ will be available. Finally since v_ν is adjacent to two vertices of colour 1, it is adjacent to atmost $\Delta - 2$ other colours and can be assigned one of the colours $2, 3, \ldots, \Delta$.

29) If G is k-chromatic, then G contains a subdivision of K_k. This is known as Hajo's Conjecture.

30) If G is 4-chromatic, then G contains a subdivision of K_4.

<u>Proof:</u>

Let G be a 4-chromatic graph. Note that if some subgraph of G contains a subdivision of K_4, then so, too does G. Without loss of generality, therefore, we may assume that G is critical, and hence that G is a block with $\delta \geq 3$. If $\nu = 4$, then G is K_4 and the statement holds trivially. We proceed by induction on ν. Assume that the theorem true for all 4-chromatic graphs with fewer than n vertices and let $\nu(G) = n > 4$.

Suppose, first, that G has a 2-vertex cut $\{u,v\}$. G has two $\{u,v\}$-components G_1 and G_2 where $G_1 + uv$ is 4-critical. Since $\nu(G_1 + uv) < \nu(G)$, we can apply the induction hypothesis and deduce that G_1+uv contains a subdivision of K_4. It follows that, if P is a (u,v)-path in G_2, the $G_1 \cup P$ contains a subdivision of K_4. Hence so, too, does G, since $G_1 \cup P \subseteq G$.

Now suppose that G is 3-connected. Since $\delta \geq 3$, G has a cycle C of length atleast four. Let u, v be non consecutive vertices on C. Since $G-\{u,v\}$ is connected, there is a path P in $G-\{u,v\}$ connecting the two components of $C-\{u,v\}$; we may assume that the origin x and the terminus y are the only vertices of P on C. Similarly, there is a path Q in $G-\{x,y\}$.

If P and Q have no vertex in common, then $C \cup P \cup Q$ is a subdivision of K_4. otherwise, let w be the first vertex of P on Q, and let P' denote the (x,w)-section of P. Then $C \cup P' \cup Q$ is a subdivision of K_4. Hence, in both cases, G contains a subdivision of K_4.

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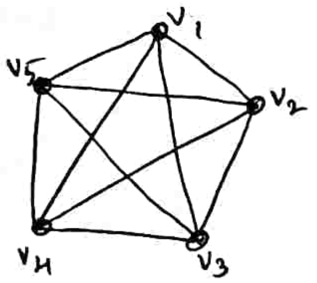
UNIT-4

1) A graph is said to be embeddable in the plane, or planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called planar embedding of G.

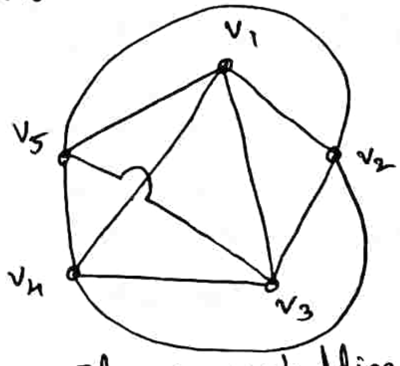
2) K_5 is nonplanar.

Proof:-

By contradiction. If possible let G be a plane graph corresponding to K_5. Denote the vertices of G by v_1, v_2, v_3, v_4 and v_5. Since G is complete, any two of its vertices are joined by an edge. Now the cycle $C = v_1v_2v_3v_1$ is a Jordan curve in the plane, and the point v_4 must lie either in int C or ext C.



Graph K_5



Planar embedding.

We shall suppose that $v_4 \in$ int C. Then the edges v_4v_1, v_4v_2 and v_4v_3 divide int C into the three regions int C_1, int C_2 and int C_3, where $C_1 = v_1v_4v_2v_1$, $C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$.

Now v_5 must lie in one of the four regions ext C, int C_1, int C_2 and int C_3. If $v_5 \in$ ext C then, since $v_4 \in$ int C, it follows from the Jordan curve theorem that the edge v_4v_5 must meet C in some point. But this contradicts the assumption that G is a plane graph. The cases $v_5 \in$ int C_i, $i = 1, 2, 3$, can be disposed of in like manner.

3) Consider a sphere S resting on a plane P, and denote by z the point of S that is diagonally opposite the

point of contact of S and P. The mapping $\pi : S \setminus \{z\} \to P$, defined by $\pi(s) = p$ iff the points z, s and p are collinear, is called stereographic projection from z.

4) A graph G is embeddable in the plane iff it is embeddable on the sphere.

Proof:

Suppose G has an embedding $\tilde{G}$ on the sphere. choose a point u of the sphere not in $\tilde{G}$ under stereographic projection from z is an embedding of G in the plane. Similarly the converse part can be proved.

5) A plane graph G partitions the rest of the plane into a number of connected regions; the closures of these regions are called the faces of G. Each plane graph has exactly one unbounded face, called the exterior face.

6) Let v be a vertex of a planar graph G. Then G can be embedded in the plane in such a way that v is on the exterior face of the embedding.

Proof:

Consider an embedding $\tilde{G}$ of G on the sphere, such an embedding exists by virtue of (4). Let z be a point in the interior of some face containing v, and let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from z. Clearly $\pi(\tilde{G})$ is a planar embedding of G of the desired type.

7) Euler's formula

If G is a connected plane graph, then $\nu - \varepsilon + \phi = 2$.

Proof:

By induction on ϕ, the number of faces of G. If $\phi = 1$. then each edge of G is a cut edge and so G, being connected, is a tree. In this case $\varepsilon = \nu - 1$, so the theorem clearly holds. suppose that it is true for all connected plane graphs with fewer than n faces, and let G be a connected plane graph with $n \geq 2$ faces. choose an edge e of G that is not a cut edge. Then $G-e$ is a connected plane graph and has $n-1$ faces, since the two faces of G separated by e combine to form one face of $G-e$. By the induction hypothesis

$$\nu(G-e) - \varepsilon(G-e) + \phi(G-e) = 2$$

and using the relations

$\nu(G-e) = \nu(G)$ $\quad \varepsilon(G-e) = \varepsilon(G)-1$ $\quad \phi(G-e) = \phi(G)-1$

we obtain $\quad \nu(G) - \varepsilon(G) + \phi(G) = 2.$

Hence the proof.

8) All planar embeddings of a given connected planar graph have the same number of faces

9) If G is a simple planar graph with $\nu \geq 3$ then $\varepsilon \leq 3\nu - 6$.

10) Let H be a given subgraph of a graph G. We define a relation $\sim$ on $E(G) \setminus E(H)$ by the condition that $e_1 \sim e_2$ if there exists a walk W such that

i) the first and last edges of W are e_1 and e_2, respectively and

ii) W is internally-disjoint from H. (that is, no internal vertex of W is a vertex of H).

11) A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$ is called a bridge of H in G.

12) In a connected graph every bridge has atleast one vertex of attachement and in a block every bridge has atleast two vertices of attachement. A bridge with k-vertices of attachement is called a k-bridge. Two k-bridges with the same vertices of attachement are equivalent k-bridges.

13) The vertices of attachement of a k-bridge B with $k \geq 2$ effect a partition of C into edge-disjoint paths, called the segments of B.

14) Two bridges avoid one another if all the vertices of attachement of one bridge lie in a single segment of the other bridge; otherwise they overlap.

15) If two bridges overlap, then either they are skew or else they are equivalent 3-bridges.

Proof:-

Suppose that the bridges B and B' overlap. Clearly each must have atleast two vertices of attachement. Now if either B or B' is a 2-bridge, it is easily verified that they must be skew. We may therefore assume that both B and B' have atleast three vertices of attachement. There are two cases

Case 1:-

B and B' are not equivalent bridges. Then B' has a vertex of attachement u' between two consecutive vertices of attachement u and v of B. Since B and B' overlap, some vertex of attachement v' of B' does not lie in the segment of B connecting u and v. It now follows that B and B' are skew.

Case 2:-

B and B' are equivalent k-bridges, $k \geq 3$. If $k \geq 4$, then B and B' are clearly skew; if $k=3$, they are equivalent 3-bridges.

16) If a bridge B has three vertices of attachement v_1, v_2 and v_3, then there exists a vertex v_0 in $V(B) \setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3 respectively, such that for $i \neq j$, P_i and P_j have only the vertex v_0 in common.

Proof:- Let P be a (v_1, v_2)-path in B, internally-disjoint from C. P must have an internal vertex v, since otherwise the bridges B would be just P, and would not contain a third vertex v_3. Let Q be a (v_3, v)-path in B, internally-disjoint from C, and let v_0 be the first vertex of Q on P. Denote by P_1 the (v_0, v_1)-section of P^{-1}, by P_2 the (v_0, v_2)-section of P, and by P_3 the (v_0, v_3)-section of Q^{-1}. Clearly P_1, P_2 and P_3 satisfy the required conditions.

17) Suppose that G is a planar graph and that C is a cycle in G. Then C is a Jordan curve in the plane, and each edge of $E(G)\setminus E(C)$ is contained in one of the two regions Int C or Ext C. It follows that a bridge of C is contained entirely in Int C or Ext C. A bridge contained in Int C is called a inner bridge, and a bridge contained in Ext C, an outer bridge.

18) Prove that Inner (outer) bridges avoid one another.

Proof:

By contradiction. Let B and B' be two inner bridges that overlap. Then they must be either skew or equivalent 3-bridges.

Case 1: B and B' are skew. By definition, there exist distinct vertices u and v in B and u' and v' in B', appearing in the cyclic order u, u', v, v' of C. Let P be a (u,v)-path in B and P' a (u', v')-path in B', both internally disjoint from C. The two paths P and P' cannot have an internal vertex in common because they belong to different bridges. At the same time, both P and P' must be contained in Int C because B and B' are inner bridges. By the Jordan curve theorem, G cannot be a plane graph, contrary to the hypothesis.

Case 2: B and B' are equivalent 3-bridges. Let the common set of vertices of attachement be $\{v_1, v_2, v_3\}$. Then there exist in B, a vertex v_0 and three paths P_1, P_2 and P_3 joining v_0 to v_1, v_2 and v_3 respectively, such that for $i \neq j$, P_i and P_j have only the vertex v_0 in common. Similarly v_2 and v_3 respectively, such that for $i \neq j$, P_i' and P_j' have only the vertex v in common.

Now the paths P_1, P_2 and P_3 divide Int C into three regions, and v_0' must be in the interior of one of these regions. Since only two of the vertices v_1, v_2 and v_3 can lie on the boundary of the region containing v_0', we may assume by symmetry, that v_3 is not on the boundary of this region. By the Jordan curve theorem, the path P_3' must cross either P_1, P_2 or C. But since B and B' are distinct inner bridges, this is clearly impossible.

Thus we conclude inner bridges avoid one another.

Similarly, outer bridges avoid one another.

19) An inner bridge that avoids every outer bridge is transferable.

<u>Proof:-</u>

Let B be an inner bridge that avoids every outer bridge. Then the vertices of attachement of B to C all lie on the boundary of some face of G contained in Ext C. B can be drawn in this face.

20) If G is nonplanar, then every subdivision of G is non planar.

21) If G is planar, then every subgraph of G is planar.

22) If G is nonplanar, then atleast one of H_1 and H_2 is also nonplanar.

<u>Proof:</u>

By contradiction, suppose that both H_1 and H_2 are planar. Let $\tilde{H}_1$ be a planar embedding of H_1 and let f be a face of $\tilde{H}_1$ incident with e. If $\tilde{H}_2$ is the planar embedding of H_2 in f such that $\tilde{H}_1$ and $\tilde{H}_2$ have only the vertices u and v and the edge e in common, then $(\tilde{H}_1 \cup \tilde{H}_2) - e$ is a planar embedding of G. This contradicts the hypothesis that G is nonplanar

23) Let G be a nonplanar connected graph that contains a subdivision of K_5 or $K_{3,3}$ and has a few edges as possible. Then G is simple and 3-connected.

Proof:

By contradiction. Let G satisfy the hypothesis. Then G is clearly a minimal nonplanar graph, and therefore must be a simple block. If G is not 3-connected, let {u,v} be a 2-vertex cut of G and let H_1 and H_2 be the graphs obtained from this cut as described above. At least one of H_1 and H_2, say H_1 is non planar. Since $\varepsilon(H_1) < \varepsilon(G)$, H_1 must contain a subgraph K which is subdivision of K_5 or $K_{3,3}$; moreover $K \not\subseteq G$, and so the edge e is in K. Let P be a (u,v)-path in H-e. Then G contains the subgraph $(K \cup P) - e$, which is a subdivision of K and hence a subdivision of K_5 or $K_{3,3}$. This contradiction establishes the lemma given.

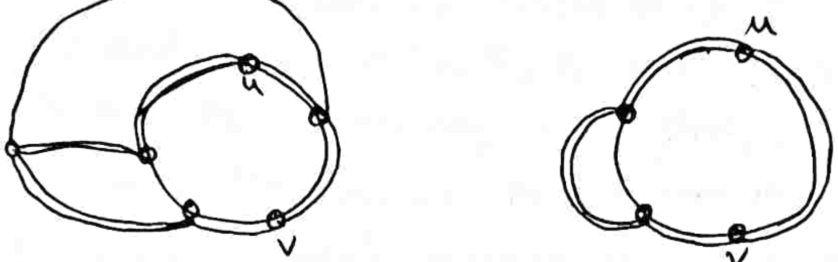
24) A graph is planar iff it contains no subdivision of K_5 or $K_{3,3}$.

Proof:-

If possible choose a non-planar graph G that contains no subdivision of K_5 or $K_{3,3}$ and has few edges as possible. Then G is simple and 3-connected. clearly G must also be a minimal nonplanar graph G.

Let uv be an edge of G, and let H be a planar embedding of the planar graph G-uv. Since G is 3-connected, H is 2-connected and u, v are contained together in a cycle of H. Choose a cycle C of H that contains u an v and is such that the number of edges in Int C is as large as possible.

Since H is simple and 2-connected, each bridge of C in H must have atleast two vertices of attachment. Now all outer bridges of C must be bridges that overlap uv because, if some outer bridge were a k-bridge for $k \geq 3$ or a 2-bridge that avoided uv, then there would be a cycle C' containing u and v with more edges in its interior than C, contradicting the choice of C. These two cases are given as



In fact all outerbridges of C in H must be single edges. For if a 2-bridge with vertices of attachement x and y had a third vertex, the set $\{x, y\}$ would be a 2-vertex cut of G, contradicting the fact that G is 3-connected.

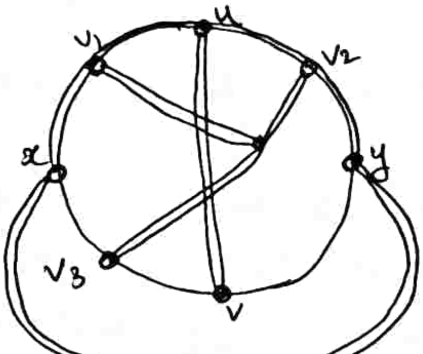
No two inner bridges overlap Therefore some inner bridge skew to uv must overlap some outer bridge. For otherwise all such bridges could be transfered and then the edge uv could be drawn in Int C to obtain a planar embedding of G; Since G is nonplanar, this is not possible. Therefore, there is an inner bridge B that is both skew to uv and skew to some outer bridge xy.

Two cases now arise, depending on whether B has a vertex of attachement different from u, v, x and y or not.

<u>Case 1</u>:- B has a vertex of attachement different from u, v, x and y. we can choose the notation so that B has a vertex of attachment v_1 in C. we consider two subcases, depending on whether B has a vertex of attachement in $C(y, v)$ or not.

Case 1(a):- B has a vertex of attachement v_2 in $C(y, v)$. In this case there is a (v_1, v_2)-path P in B that is internally-disjoint from C. But then $(C \cup P) - \{uv, xy\}$ is a subdivision of $K_{3,3}$ in G, a contradiction.

Case 1(b):- B has no vertex of attachement in $C(y, v)$. In this case since B is a skew to uv and to xy, B must have vertices of attachement v_2 in $C(u, y]$ and v_3 in $C[v, x)$. Thus B has three vertices of attachement v_1, v_2 and v_3. There exists a vertex v_0 in $V(B) \setminus V(C)$ and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3 respectively. Such that for $i \neq j$, P_i and P_j have only the vertex v_0 in common. But now $(C \cup P_1 \cup P_2 \cup P_3) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ a contradiction. This case is illustrated below. The



subdivision of $K_{3,3}$ is indicated by double lines.

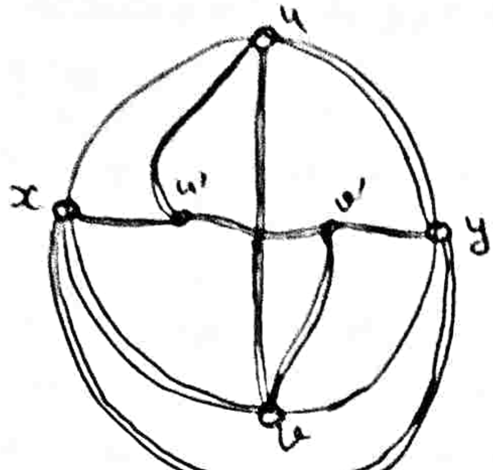
Case 2:-

B has no vertex of attachement other than u, v, x and y. Since B is skew to both uv and xy, it follows that u, v, x and y must all be vertices of attachement of B. Therefore there exists a (u, v)-path P and an (x, y)-path Q in B. Such that (i) P and Q are internally disjoint from C, and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two subcases, depending on whether P and Q have one or more vertices in common.

Case 2(a) $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of K_5 in G, again a contradiction

Case 2(b) $|V(P) \cap V(Q)| \geq 2$. Let u' and v' be the first and

and last vertices of P on Q and let P_1 and P_2 denote the (u, u')- and (v', v)-sections of P. Then $(C \cup P_1 \cup P_2 \cup Q) + \{uv, xy\}$ contains a subdivision of $K_{3,3}$ in G. Once more a contradiction.



Thus all possible cases lead to contradiction, and the proof is complete.

25) Every planar graph is 5-vertex-colourable.

26) The following three statements are equivalent:

i) every planar graph is 4-vertex-colourable;

ii) every plane graph is 4-face-colourable;

iii) every simple 2-edge-connected 3-regular graph is 3-edge colourable.

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UNIT-5:-

1) A directed graph D is an ordered triple $(V(D), A(D), \psi_D)$ consisting of non-empty set $V(D)$ of vertices, a set $A(D)$, disjoint from $V(D)$, of arcs and an incidence function ψ_D that associates with each arc of D an ordered pair of vertices of D.

2) A digraph D' is a subdigraph of D if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$ and $\psi_{D'}$ is the restriction of ψ_D to $A(D')$.

3) With each digraph D we can associate a graph G on the same vertex set; corresponding to each arc of D there is an edge of G with the same ends. This graph is the underlying graph of D.

4) Given any graph G, we can obtain a digraph from G by specifying, for each link, an order on its ends. Such a digraph is called an orientation of G.

5) A directed walk in D is a finite non-null sequence $W = (v_0, a_1, v_1, \ldots, a_k v_k)$, whose terms are alternately vertices and arcs, such that for $i = 1, 2, \ldots, k$ the arc a_i has head v_i and tail v_{i-1}.

6) A directed trail is a directed walk that is a trail. In a similar way directed path, directed cycles are defined.

7) If there is a directed (u, v)-path in D, vertex v is said to be reachable from u in D. Two vertices are diconnected in D if each is reachable from the other.

8) The subdigraphs $D[V_1], D[V_2], \ldots, D[V_m]$ induced by the resulting partitions $(V_1, V_2, \ldots, V_m)$ of $V(D)$ are called the dicomponents of D. A digraph D is diconnected if it has exactly one dicomponent.

9) The indegree $d_D^-(v)$ of a vertex v in D is the number of arcs with head v; the outdegree $d_D^+(v)$ of a vertex v is

the number of arcs with tail v. We denote the minimum and maximum indegrees and outdegrees in D by $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$ and $\Delta^+(D)$, respectively.

10) A digraph is strict if it has no loops and no two arcs with the same ends have the same direction.

11) Prove that A digraph D contains a directed path of length $\chi - 1$.

Proof:

Let A' be a minimal set of set of arcs D such that $D' = D - A'$ contains no directed cycle, and the length of a longest path in D' be k. Now assign colours $1, 2, \ldots, k+1$ to the vertices of D' by assigning colour i to vertex v if the length of the longest directed path in D' with origin v is $i-1$. Denote by V_i the set of vertices with colour i. We shall show that $(V_1, V_2, \ldots, V_{k+1})$ is a proper $(k+1)$-vertex colouring in D.

First observe that the origin and terminus of any directed path in D' have different colours. For let P be a directed (u,v)-path of positive length in D' and suppose $v \in V_i$. Then there is a directed path $Q = (v_1, v_2, \ldots v_i)$ in D', where $v_1 = v$. Since D' contains no directed cycle, PQ is a directed path with origin u and length atleast i. Thus $u \notin V_i$.

We can now show that the ends of any arc of D have different colours. Suppose $(u,v) \in A(D)$. If $(u,v) \in A(D')$ then (u,v) is a directed path in D' and so u and v have different colours. Otherwise $(u,v) \in A'$. By the minimality of A', $D' + (u,v)$ contains a directed cycle C. $C - (u,v)$ is a directed (v,u)-path in D' and hence in this case, too, u and v have different colours.

Thus $(V_1, V_2, \ldots, V_{k+1})$ is a proper vertex colouring of D. It follows that $\chi \leq k+1$, and so D has a directed path of length $k \geq \chi - 1$.

12) An orientation of a complete graph is called a tournament.

13) A directed Hamilton path of D is a directed path that includes every vertex of D.

14) Every tournament has a directed Hamilton path.

15) An in-neighbour of a vertex v in D is a vertex u such that $(u,v) \in A$; an out-neighbour of a vertex w such that $(v,w) \in A$. We denote the sets of in-neighbours and out-neighbours of v in D by $N_D^-(v)$ and $N_D^+(v)$, respectively

16) A loopless digraph D has an independent set S such that each vertex of D not in S is reachable from a vertex in S by a directed path of length atmost two.

Proof:

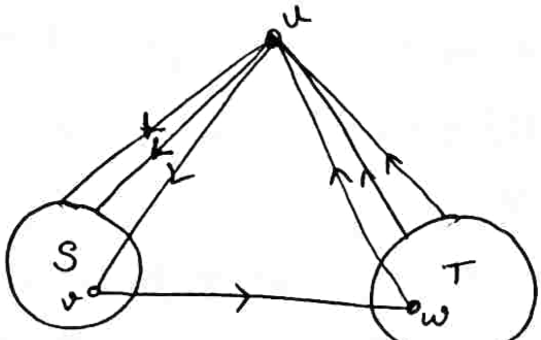
By induction on v. The theorem holds trivially for $v=1$. Assume that it is true for all digraphs with fewer than v vertices, and let v be an arbitrary vertex of D. By the induction of hypothesis there exists in $D' = D-(\{v\} \cup N^-(v))$ an independent set S' such that each vertex of D' not in S' is reachable from a vertex in S' by a directed path of length at most two. If v is an out-neighbour of some vertex u of S', then every vertex of $N^+(v)$ is reachable from u by a directed path of length two. Hence in this, case, $S=S'$ satisfies the required property. If on the other hand, v is not an out neighbour of any vertex of S', then v is joined to no vertex of S' and the independent set $S=S' \cup \{v\}$ has the required property.

17) A tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two.

18) Each vertex of a diconnected tournament D with $\nu \geq 3$ is contained in a directed k-cycle, $3 \leq k \leq \nu$.

Proof:-

Let D be a diconnected tournament with $\nu \geq 3$ and let u be any vertex of D. Set $S = N^+(u)$ and $T = N^-(u)$. We first show that u is in a directed 3-cycle. Since D is diconnected, neither S nor T can be empty; and for the same reason, (S,T) must be nonempty. There is thus some arc (v,w) in D with $v \in S$ and $w \in T$, and u is in the directed 3-cycle (u,v,w,u).



The theorem is now proved by induction on k. Suppose that u is in directed cycles of all lengths between 3 and n, where $n < \nu$. We shall show that u is in a directed $(n+1)$-cycle.

Let $C = (v_0, v_1, \ldots, v_n)$ be a directed n-cycle in which $v_0 = v_n = u$. If there is a vertex v in $V(D) \setminus V(C)$ which is both the head of an arc with tail in C and the tail of an arc with head in C, then there are adjacent vertices v_i and v_{i+1} on C such that both (v_i, v) and (v, v_{i+1}) are arcs of D. In this case u is in the directed ~~graph~~ $(n+1)$-cycle $(v_0, v_1, \ldots v_i, v, v_{i+1}, \ldots, v_n)$.

Otherwise, denote by S the set of vertices in $V(D) \setminus V(C)$ which are heads of arcs joined to C, and by T the set of vertices in $V(D) \setminus V(C)$ which are trails of arcs joined to C. As before since D is diconnected, S, T and (S,T) are all nonempty, and there is some arc (v,w) in D with $v \in S$ and $w \in T$. Hence u is in the directed $(n+1)$-cycle $(v_0, v, w, v_2, \ldots, v_n)$.

19) If D is strict and $\min\{\delta^-, \delta^+\} \geq \nu/2 > 1$, then D contains a directed Hamilton cycle.

Proof:

Suppose that D satisfies the hypothesis of the theorem, but does not contain a directed Hamilton cycle. Denote the length of a longest directed cycle in D by l, and let $C = (v_1, v_2, \ldots, v_l, v_1)$ be a directed cycle in D of length l. we note that $l > \nu/2$. Let P be a longest directed path in $D - V(C)$ and suppose that P has origin u, terminus v and length m. clearly $\nu \geq l + m + 1$ —— ①

and since $l > \nu/2$ $m < \nu/2$ —— ②

set $S = \{i : (v_{i+1}, u) \in A\}$ and $T = \{i : (v, v_i) \in A\}$

We first show that S and T are disjoint. Let $C_{j,k}$ denote the section of C with origin v_j and terminus v_k. If some integer i were in both S and T, D would contain the directed cycle $C_{i,j-1}(v_{i-1}, u)P(v, v_i)$ of length $l+m+1$, contradicting the choice of C thus

$$S \cap T = \phi \quad \text{—— ③}$$

Now because P is a maximal directed path in $D - V(C)$, $N^-(u) \subseteq V(P) \cup V(C)$. But the number of in-neighbours of u in C is precisely $|S|$ and so $d^-_D(u) = d^-_P(u) + |S|$. Since $d^-_D(u) \geq \delta^- \geq \nu/2$ and $d^-_P(u) \leq m$

$$|S| \geq \nu/2 - m \quad \text{—— ④}$$

A Similar argument yields

$$|T| \geq \nu/2 - m \quad \text{—— ⑤}$$

Note that by ②, both S and T are nonempty.

Adding ④ & ⑤, and using in ①, we obtain

$$|S| + |T| \geq l - m + 1.$$

and therefore by ③

$$|S \cup T| \geq l - m + 1 \quad \text{—— ⑥}$$

Since S and T are disjoint and non empty, there are positive integers i and k such that $i \in S$, $i+k \in T$ and

$$i+j \notin S \cup T \text{ for } 1 \leq j \leq k \quad \text{— (7)}$$

where addition is taken modulo j.

From (6) & (7) we see that $k \leq m$. Thus the directed cycle $C_{i+k, l-1}(v_{i-1}, v_1)\, P(v_1, v_{i+k})$, which has length $l+m+1-k$, is longer than C. This contradiction establishes the theorem.

—— x —— x ——.