| Inst Hour | : 6 |
|---|---|
| Credit | : 4 |
| Code | : 18KP1MELM1 |

SEMESTER : I
ELECTIVE COURSE : I (1)

# DIFFERENTIAL GEOMETRY

**UNIT – I**
SPACE CURVES: Definition of a space curve – Arc length – tangent and osculating plane – normal and binormal – curvature and torsion – contact between curves and surfaces – tangent surface – involutes and evolutes – Intrinsic equations – Fundamental Existence Theorem for space curves – Helics.
**Chapter 1: Sections 1.1 to 1.7, 1.10, 1.13, 1.16, 1.17, 1.18**

**UNIT – II**
INTRINSIC PROPERTIES OF A SURFACE: Definition of a surface – curves on a surface – surface of revolution – Helicoids – Metric – Direction coefficients – families of curves – Isometric correspondence – Intrinsic properties.
**Chapter 2: Sections 2.1 to 2.11, 2.14, 2.15.**

**UNIT – III**
GEODESICS: Geodesics – Canonical geodesic and their differential equations – Canonical geodesic equations – geodesics on surface of revolution - Normal property of geodesics – Differential equations of geodesics using normal property -Existence Theorems
**Chapter 3: Sections 3.1 to 3.7**

**UNIT – IV**
GEODESICS: Geodesic parallels – Geodesics curvature – Gauss – Bonnet Theorem – Gaussian curvature – surface of constant curvature – Conformal mapping – Geodesic mapping.
**Chapter 3: Sections 3.8 to 3.15**

**UNIT – V**
THE SECOND FUNDAMENTAL FORM AND LOCAL NON INTRINSIC PROPERTIES OF A SURFACE: The second fundamental form – Classification of points on a surface - Principal curvature – Lines of curvature – The Dupin indicatrix - Developable surfaces – Developable associated with space curves and with curves on surface – Minimal surfaces – Ruled surfaces – Three fundamental forms.
**Chapter 4: Sections 4.1 to 4.12**

## TEXT BOOK
D. Somasundaram, Differential Geometry a First Course, Narosa Publishing House, New Delhi.

## REFERENCES

1. Struik, D.T. Lectures on Classical Differential Geometry, Addison – Wesley, Mass. 1950
2. Kobayashi S. and Nomizu. K. Foundations of Differential Geometry, Interscience Publishers, 1963.
3 Wihelm Klingenberg: A course in Differential Geometry, Graduate Texts in Mathematics, Springer Verlag, 1978.
4.J.A. Thorpe Elementary topics in Differential Geometry, Under – graduate Texts in Mathematics, Springer – Verlag 1979.
5. T.J. Willmore, An Introduction to Differential Geometry, Oxford University Press, (17th Impression) New Delhi 2002.(Indian Print).


## Question Pattern
**Section A :** 10 x 2 = 20 Marks, 2 Questions from each Unit.
**Section B :** 5 x 5 = 25 Marks, EITHER OR ( a or b) Pattern, One question from each Unit.
**Section C :** 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.



9/03/18
9/3/18
5/3/18
9.3.18

I M.Sc MATHEMATICS

DIFFERENTIAL GEOMETRY 18KP1MELM1

UNIT-I

THEORY OF SPACE CURVES

Definition-1:

Let I be a real interval and m be a positive integer. A real valued function f defined on I is said to be of class m, if f has continuous $m^{th}$ derivative at every point of I.

Definition-2:

A vector valued function $R = R(u)$ defined on I is said to be of class m, if it has continuous $m^{th}$ order derivative at every point of I.

Definition-3:

Two paths $R_1$ and $R_2$ of class $C^m$ defined on $I_1$ and $I_2$ are said to be equivalent if there exists a strictly increasing function $\phi$ of class m which maps $I_1$ onto $I_2$ such that $R_1 = R_2 \circ \phi$.

Definition-4:

Let $\gamma$ be a curve of class $m \geq 2$ and P and Q be two neighbouring points on $\gamma$. The limiting position of the plane that contains the tangential line at P passes through the point Q as $Q \to P$ is defined as the osculating plane at P.

Definition-5:

The point $P$ on the curve for which $r''=0$ is called a point of inflexion and the tangent line at $P$ is called inflexional.

Definition-6:

Let $P$ be a point on the curve $\gamma$. The plane through $P$ orthogonal to the tangent at $P$ is called the normal plane at $P$.

Definition-7:

The line of intersection of the normal plane and the osculating plane is called the principal normal at $P$. The unit vector along the principal normal is denoted by $n$.

Definition-8:

The normal at $P$ orthogonal to the osculating plane is called the binormal at $P$. The unit vector along the binormal is denoted by $b$.

Definition-9:

The arc rate at which the tangent changes direction as the point $P$ moves along the curve is called the curvature vector of the curve and it is denoted by $k$. $k=\frac{dt}{ds}$ and magnitude is $|k|$ and is denoted by $k$. $\rho=\frac{1}{k}$ is the radius of curvature.

Definition-10:

The torsion of a point at $P$ of a curve is defined as the arc rate at which the osculating plane turns about the tangent at $P$ as $P$ moves along the curve. It is denoted by $\tau$. Radius of Torsion is $\left|\frac{1}{\tau}\right|$.

Definition-11:

If $F'(u_0) \neq 0$, then $u_0$ is a simple zero of $F(u)=0$ Then the curve $\gamma$ and surface $S$ is said to have simple intersection at $r(u_0)$.

If $F''(u_0) \neq 0$ and $F'(u_0)=0$ then $u_0$ is a double zero of $F(u)$ and $\gamma$ and $S$ are said to have two point contact.

If $F'''(u_0) \neq 0$, $F'(u_0)=F''(u_0)=0$, $\gamma$ and $S$ are said to have three point contact at $u=u_0$ and $u_0$ is called a triple zero of $F(u)$.

If $F'(u_0)=F''(u_0)=\cdots\cdots=F^{n-1}(u_0)=0$ and $F^{n}(u_0)\neq 0$ $\gamma$ and $S$ are said to have $n$ point contact at $u=u_0$.

Definition-12:

A curve which lies on the tangent surface of $C$ and intersects the generators of the tangent surface orthogonally is called the involute of $C$ and is denoted by $\tilde{C}$.

If $\tilde{C}$ is an involute of a given curve $C$, then $C$ is defined to be the evolute of $\tilde{C}$.

Definition-13:

The equations expressing $k$ and $\tau$ as function of arcual length $s$ (ie) $k=f(s)$ and $\tau=g(s)$ are called intrinsic or natural equations of a curve.

Definition-14:

A space curve lying on a cylinder and cutting the generators of the cylinder at a constant angle is called a cylindrical Helix. The tangent to the curve makes a constant angle $\alpha$ with a fixed line known as the axis of the helix.

Definition-15:

If a curve on a sphere is a helix, then the curve is called a spherical helix.

Theorem:
The arc length of a curve is invariant under parametric transformation.

Proof:

Let $R_1(u)$ be the parametric representation of the given curve and $t = \phi(u)$ be the parametric transformation. Let the parametric representation corresponding to $t$ be $R_2(t)$. Since $R_1$ and $R_2$ are equivalent representations $R_1(u) = R_2(t) = R_2[\phi(u)]$.
As $u$ varies from $a$ to $b$, $t = \phi(u)$ varies from $\phi(a)$ to $\phi(b)$ and $\dot{R}_1(u) = \dot{R}_2(t) \cdot \frac{dt}{du} \longrightarrow ①$

$$\int_a^b |\dot{R}_1| du = \int_{\phi(a)}^{\phi(b)} |\dot{R}_2(t)| \left|\frac{dt}{du}\right| \cdot du \longrightarrow ②$$

Since $t = \phi(u)$ is a strictly increasing function $\dot{\phi}(u) \neq 0$, $\left|\frac{dt}{du}\right| = \frac{dt}{du} \longrightarrow ③$.

③ in ② $\int_a^b |\dot{R}_1| du = \int_{\phi(a)}^{\phi(b)} |\dot{R}_2| dt$ which proves that the arc length is invariant for a change of parameter from $u$ to $t$.

Example-1:

Find the arc length of one complete turn of the circular helix $r(u) = (a\cos u, a\sin u, bu)$, $-\infty < u < \infty$, where $a > 0$ and obtain the equation of the helix with $s$ as parameter.

Soln: $r(u) = (a\cos u, a\sin u, bu) \longrightarrow ①$.

$\dot{r}(u) = (-a\sin u, a\cos u, b)$

$$s = s(u) = \int_0^u |\dot{r}(u)| du = \int_0^u \sqrt{a^2+b^2}\, du = cu \longrightarrow ②$$

where $c = \sqrt{a^2+b^2}$. If a helix starts from $u_0$, it makes one complete turn when $u = u_0 + 2\pi$.
Hence the arc length corresponding to one complete turn is $s = \int_{u_0}^{u_0+2\pi} \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2} \cdot 2\pi = 2\pi c$

To obtain the equation of the helix with $s$ as parameter from ② $s = cu \Rightarrow u = \frac{c}{s} \longrightarrow$ ③.

using ③ in ①.

$r(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right)$ is the required equation.

**Theorem:**

If $u$ is the parameter of the curve $\gamma$, then the equation of the osculating plane at any point $P$ with position vector $r = r(u)$ is $[R - r, \dot{r}, \ddot{r}] = 0$.

**Proof:**

$$r' = \frac{dr}{ds} = \frac{dr}{du} \cdot \frac{du}{ds} = \frac{\dot{r}}{\dot{s}}$$

$$r'' = \frac{d}{du}\left(\frac{\dot{r}}{\dot{s}}\right)\frac{du}{ds} = \frac{\dot{s}\ddot{r} - \dot{r}\ddot{s}}{\dot{s}^2} \cdot \frac{1}{\dot{s}}$$

Using these values of $r'$ and $r''$ in $[R - r(0), r'(0), r''(0)] = 0$

we get $\left[R - r, \frac{\dot{r}}{\dot{s}}, \frac{\ddot{r}}{\dot{s}^2} - \frac{\dot{r}\ddot{s}}{\dot{s}^3}\right] = 0$.

Since $\dot{r} \times \dot{r} = 0$ and $\dot{s}$ is a scalar, and simplifying the above equation we obtain $[R - r, \dot{r}, \ddot{r}] = 0$ as the equation of the osculating plane.

**Example:**

Find the equation of the osculating plane at a point $u$ of the circular helix $r = (a\cos u, a\sin u, bu)$.

**Soln:** $r = (a\cos u, a\sin u, bu) \longrightarrow$ ①.

$\dot{r} = (-a\sin u, a\cos u, b) \longrightarrow$ ②.

$\ddot{r} = (-a\cos u, -a\sin u, 0) \longrightarrow$ ③.

From ①, ② and ③ the equation of the osculating plane is

$$\begin{vmatrix} X - a\cos u & Y - a\sin u & Z - bu \\ -a\sin u & a\cos u & b \\ -a\cos u & -a\sin u & 0 \end{vmatrix} = 0$$

Expanding the above determinant along the last column and simplifying, the equation of the osculating plane is $b(X\sin u - Y\sin u - au) + aZ = 0$.

Theorem:

A necessary and sufficient condition for a curve to be a straight line is that $k=0$ at all points of the curve.

Proof:

The condition is necessary. Let us take the curve to be a straight line and let its vector equation be $r=as+b$, $a$ and $b$ are constant vectors.

$\frac{d}{ds}(r)=\frac{d}{ds}(as+b) \Rightarrow r'=a \Rightarrow r'=t=a \Rightarrow r''=0.$

Since the curvature vector $r''$ vanishes at all points of the curve, its magnitude $k=0$ at all points of the curve.

To prove the sufficiency, let us assume $k=0$ at all points of the curve. This implies that $t'=r''=0$. Integrating $t'=r''$ twice we obtain $r=as+b$, where $a$ and $b$ are constants. This proves that the curve is a straight line.

Theorem:

A necessary and point condition that a given curve be a plane curve is that $\tau=0$ at all points of the curve.

Proof:

The condition is necessary. Let us take the curve to be a plane curve and show that $\tau=0$. Since the curve lies in a plane, the osculating plane at every point of the curve is the plane containing the curve itself so that the binormal $b$ is a constant. Since $b$ is constant $\frac{db}{ds}=0$ which implies $\left|\frac{db}{ds}\right|=0$. Hence $\tau=\left|\frac{db}{ds}\right|=0$ at all points of the curve.

Conversely let $\tau=0$ at all points of the curve. To prove that the curve is a plane curve. Since $\tau=0$ $\frac{db}{ds}=0$ at all points of the curve so that $b$ is a constant vector. For any vector $r$,

$$\frac{d}{ds}(r\cdot b)=\frac{dr}{ds}\cdot b+r\cdot\frac{db}{ds}=t\cdot b+r\cdot b'.$$

Since $t\cdot b=0$ and $b'=0$, $\frac{d}{ds}(r\cdot b)=0$ for any point $r$ on the curve. Hence $r\cdot b=\text{constant}=c$. If $r=(x(s),y(s),z(s))$ and $b=(b_1,b_2,b_3)$ then

$r \cdot b = c$ gives $xb_1 + yb_2 + zb_3 = c$ which shows that $r(s) = (x(s), y(s), z(s))$ lies on the plane $b_1X + b_2Y + b_3Z = C$. This proves that the curve is plane curve and the condition is sufficient.

Theorem:

If $r = r(s)$ is the position vector of a point $p$ with arc-length as parameter on a curve $c$ then

(i) $k^2 = r'' \cdot r''$ (ii) $\tau = \dfrac{[r', r'', r''']}{r'' \cdot r''}$ or $k^2\tau = [r', r'', r''']$

Proof:

We know that $r' = t$ and $r'' = kn \longrightarrow (1)$

Hence $r'' \cdot r'' = (kn) \cdot (kn) = k^2$, since $n \cdot n = 1 \longrightarrow (2)$

(ii) Now $r' \times r'' = t \times kn = kb \longrightarrow (3)$

Differentiate bothsides of (3) with respect to $s$,

$$r' \times r''' + r'' \times r'' = k'b + kb' \longrightarrow (4)$$

Since $r'' \times r'' = 0$ and $b' = -\tau n$ (4) becomes,

$$r' \times r''' = (k'b - k\tau n) \longrightarrow (5)$$

Taking dot product with $r''$ on bothsides of (5),

$$(r' \times r''') \cdot r'' = -k^2\tau \text{ as } r'' = kn \text{ and } n \cdot n = 1 \longrightarrow (6)$$

But $(r' \times r''') \cdot r'' = -r' \cdot (r'' \times r''') = -[r', r'', r'''] \longrightarrow (7)$

using (7) in (6), $k^2\tau = [r', r'', r'''] \longrightarrow (8)$

substituting for $k^2$ from (i)

$$\tau = \frac{[r', r'', r''']}{r'' \cdot r''}$$

Example:

Find the curvature and Torsion of the circular Helix $r = (a\cos u, a\sin u, bu)$.

Soln:

From the equation of the helix, $r = (a\cos u, a\sin u, bu)$

$\dot{r} = (-a\sin u, a\cos u, b)$

$\ddot{r} = (-a\cos u, -a\sin u, 0)$ and $\dddot{r} = (a\sin u, -a\cos u, 0)$

$\dot{r} \times \ddot{r} = (ab\sin u, -ab\cos u, a^2)$.

$|\dot{r}|^2 = a^2 + b^2$, $|\dot{r} \times \ddot{r}|^2 = a^2(a^2 + b^2)$.

Now, $k = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \dfrac{a}{a^2+b^2}$

$(\dot{r} \times \ddot{r}) \cdot \dddot{r} = a^2 b$ so that $[\dot{r}, \ddot{r}, \dddot{r}] = a^2 b$

Hence $\tau = \dfrac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \dfrac{b}{a^2+b^2}$

Theorem:

The osculating plane at any point $P$ has three point contact with the curve at $P$.

Proof:

Let $P$ be any point on the curve and let the arc length be measured from $P$ so that $s=0$ at $P$ and let the equation of the curve be $r = r(s)$. The osculating plane at $P$ is $[r(s) - r(0), r'(0), r''(0)] = 0$ and let

$$F(s) = [r(s) - r(0), r'(0), r''(0)] \longrightarrow ①$$

We shall show that $F'(s) = F''(s) = 0$ and $F'''(s) \neq 0$ at $P$ where $s=0$ and this proves that the osculating plane has three point contact with the curve.

Expanding $r(s)$ by Taylor's theorem in the neighbourhood of $P$,

$$r(s) = r(0) + \frac{r'(0)}{1!} s + \frac{r''(0)}{2!} s^2 + \frac{r'''(0)}{3!} s^3 + O(s^4) \longrightarrow ②.$$

Neglecting powers of $s$ greater than 3 in ② and substituting it in ①.

$$F(s) = \left[\frac{s\,r'(0)}{1!} + \frac{r''(0)}{2!} s^2 + \frac{r'''(0)}{3!} s^3, r'(0), r''(0)\right] \longrightarrow ③.$$

$$= \left[\frac{s\,r'(0)}{1!}, r'(0), r''(0)\right] s + [r''(0), r'(0), r''(0)] \frac{s^2}{2!} + [r'''(0), r'(0), r''(0)] \frac{s^3}{3!}$$

The first two terms of ③ vanish. Using $k^2\tau = [r', r'', r''']$ in ③, $F(s) = -\dfrac{k^2\tau}{6} s^3$.

Hence $F'(0) = 0$, $F''(0) = 0$, $F'''(0) = -k^2\tau \neq 0$ provided $k$ and $\tau$ do not vanish at $P$. This proves that the curve and the osculating plane have three point contact at $P$.

Corollary:

The distance between corresponding points of two involutes is constant.

Proof:

Let $P$ be a fixed point on $C$. Let $r_1$ and $r_2$ be the position vectors of the corresponding points on two involutes $c = c_1$ and $c = c_2$.

Then $r_1 = r + (c_1 - s)t$, $r_2 = r + (c_2 - s)t$

$$r_1 - r_2 = r + (c_1 - s)t - r - (c_2 - s)t$$

$$r_1 - r_2 = (c_1 - c_2)t \Rightarrow |r_1 - r_2| = |c_1 - c_2| \text{ a constant.}$$

Thus the length between two such corresponding points is constant.

**Theorem:**

The locus of the centres of spherical curvature is an evolute if and only if the curve is a plane curve.

**Proof:**

The equation of the evolute is

$$r_1 = r + \rho n + \rho \cot(\psi + c) b, \ \psi = \int \tau ds \longrightarrow (1).$$

Comparing (1) with the equation of the locus of the centre of curvature $r_1 = r + \rho n$ the two curves coincide if and only if $\rho \cot(\psi + c) = 0 \longrightarrow (2)$

Since $\rho \neq 0$, (2) $\Rightarrow \int \tau ds + c = n\frac{\pi}{2}$ or $\int \tau ds = \frac{n\pi}{2} - c \rightarrow (3)$

Differentiating bothsides of (3) with respect to $s$, we get $\tau = 0$ which is the necessary and sufficient condition for a curve to be a plane curve.

**Example:**

Show that when the curve $r = r(s)$ has constant torsion $\tau$ the curve $r_1 = -\frac{1}{\tau} n + \int b ds$ has constant curvature $\pm \tau$.

**Soln:**

Given $r = r(s) \Rightarrow r_1 = r_1(s_1) \longrightarrow (1)$

Differentiating bothsides of (1) with respect to $s$

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = \frac{-n'}{\tau} + b = -\frac{\tau b - kt}{\tau} + b = \frac{k}{\tau} t \rightarrow (2)$$

$$t_1 \frac{ds_1}{ds} = \frac{k}{\tau} t \Rightarrow k_1 n_1 \cdot \left(\frac{k}{\tau}\right) = \pm kn.$$

Thus $k_1 n_1 = \pm \tau n \Rightarrow n_1$ is parallel to $n$ and $k_1 = \pm \tau$.

**Theorem:**

A spherical helix projects on a plane perpendicular to its axis in an arc of an epicycloid.

**Proof:**

To find the intrinsic equation of the projection curve using the relation between the curvature and arc-length of the helix and the projected curve. Form of the intrinsic equation of the projection on the plane perpendicular to the axis gives the solution.

If the helix lies on a sphere of radius r, the sphere is the osculating sphere at every point of the helix so that radius of the sphere is the same as the radius of the osculating sphere.

Hence $R^2 = \rho^2 + (\rho'\sigma)^2 \longrightarrow ①$ For a helix $\sigma = \rho \tan\alpha \longrightarrow ②$

② in ① $\Rightarrow \dfrac{\rho\rho'}{\sqrt{R^2-\rho^2}} = \pm\cot\alpha \Rightarrow \dfrac{\rho\, d\rho}{\sqrt{R^2-\rho^2}} = \pm\cot\alpha\, ds \longrightarrow ③$

Integrating ③ with respect to s and choosing the integrating constant as zero, $R^2 = \rho^2 + s^2\cot^2\alpha \longrightarrow ④$

Consider the projection of a helix on a plane perpendicular to the axis, $k = k_1 \sin^2\alpha$ and $s_1 = s\sin\alpha \longrightarrow ⑤$.

⑤ in ④ $\Rightarrow r^2\sin^4\alpha = \rho_1^2 + s_1^2\cos^2\alpha \longrightarrow ⑥$

Since $\cos\alpha < 1$, ⑥ represents the intrinsic equation of an epicycloid. Thus a spherical helix projects on a plane perpendicular to its axis in an arc of an epicycloid.

Theorem:

A Necessary and Sufficient Condition for a curve to be helix is that the ratio of the curvature to torsion is constant at all points.

Proof:

To prove the necessity of the condition, let a be the unit vector in the direction of the axis. Since the helix cuts the generators at a constant angle, let the angle between the generator and the tangent at any point P on the helix be $\alpha$. We have $t\cdot a = \cos\alpha \longrightarrow ①$

Differentiating ①, $t'\cdot a + t\cdot a' = 0 \longrightarrow ②$.

$t' = kn$ and ② $\rightarrow kn\cdot a = 0 \longrightarrow ③$.

If $k=0$, the curve is a straight line and the conclusion is obvious. If $k=0$ ③ $\Rightarrow n\cdot a = 0 \Rightarrow n \perp a$

Since a passes through P making a constant angle $\alpha$ with the tangent t at P and perpendicular to the normal at P, it lies in the rectifying plane at P. Hence $(\cos\alpha, \sin\alpha)$ are the components of $\alpha$ in the rectifying plane, $a = t\cos\alpha + b\sin\alpha \longrightarrow ④$.

Differentiating ④ with respect to $s$ and using $a' = 0$

$(t'\cos\alpha + b'\sin\alpha) = (k\cos\alpha - \tau\sin\alpha)n = 0.$

$n \neq 0 \Rightarrow k\cos\alpha - \tau\sin\alpha = 0 \Rightarrow \frac{k}{\tau} = \tan\alpha$ which is constant proving the necessity of the condition.

To prove the converse assume $\frac{k}{\tau} = \lambda$ and prove that the curve is a helix. $\therefore \tan\alpha = \lambda. \Rightarrow \frac{k}{\tau} = \tan\alpha$

giving $(k\cos\alpha - \tau\sin\alpha) = 0 \longrightarrow ⑤.$

$n \neq 0$ ⑤ $\Rightarrow n(k\cos\alpha - \tau\sin\alpha) = 0 \longrightarrow ⑥.$

⑥ $\Rightarrow \frac{d}{ds}[n(k\cos\alpha - \tau\sin\alpha)] = 0 \Rightarrow \frac{d}{ds}(nk\cos\alpha - n\tau\sin\alpha) = 0$

$\therefore \frac{d}{ds}(t\cos\alpha + b\sin\alpha) = 0$

This proves that, $t\cos\alpha + b\sin\alpha$ is a constant vector $a$.

Then $a \cdot t = (t\cos\alpha + b\sin\alpha) \cdot t \Rightarrow a \cdot t = \cos\alpha \; (t \cdot b) = 0.$

which proves that the curve is a helix.

**Example:**

Find the equation of the curve whose curvature and Torsion are constant.

Soln:

From the Fundamental theorem of space curves,

$$\frac{d\alpha}{ds} = k\beta,\ \frac{d\beta}{ds} = \tau\gamma - k\alpha,\ \frac{d\gamma}{ds} = -\tau\beta \longrightarrow ①$$

where $\alpha, \beta, \gamma$ are functions of $s$ and $k, \tau$ are constants,

From ①, $\frac{d^2\alpha}{ds^2} = k\frac{d\beta}{ds} = k(\tau\gamma - k\alpha) =$

$$\frac{d^3\alpha}{ds^3} = k\tau\frac{d\gamma}{ds} - k^2\frac{d\alpha}{ds} = -(k^2+\tau^2)k\beta = -(k^2+\tau^2)\frac{d\alpha}{ds}$$

$$\frac{d^3\alpha}{ds^3} = -c^2\frac{d\alpha}{ds},\ c^2 = k^2+\tau^2,\ \text{then } \frac{d^3\alpha}{ds^3} + c^2\frac{d\alpha}{ds} = 0 \longrightarrow ②.$$

Take $X = \frac{d\alpha}{ds}$, ② $\Rightarrow \frac{d^2X}{ds^2} + cX = 0$ and the solution of this equation is $X = A\sin cs + B\cos cs$.

$X = \frac{d\alpha}{ds}$, $\alpha = \int(A\sin cs + B\cos cs)\,ds$

$$\alpha = \frac{-A\cos cs}{c} + \frac{B\sin cs}{c} + d \Rightarrow \alpha c = -A\cos cs + B\sin cs + cd$$

$C = cd \Rightarrow \alpha c = -A\cos cs + B\sin cs + C \longrightarrow ③$

From ③, $\frac{d\alpha}{ds} = A\sin cs + B\cos cs$

using ① in the above equation,

$$\beta k = A\sin cs + B\cos cs \longrightarrow ④.$$

Hence $\frac{d\gamma}{ds} = -\frac{\tau}{k}(A\sin cs + B\cos cs)$

$$\int d\gamma = \frac{-\tau}{k}\int (A\sin cs + B\cos cs)\,ds$$

$$\gamma = \frac{-\tau}{k}\left(\frac{-A\cos cs}{c} + \frac{B\sin cs}{c}\right) + D$$

$$\gamma = \frac{\tau}{k}\left(\frac{A\cos cs}{c} + \frac{B\sin cs}{c}\right) + D \longrightarrow ⑤.$$

By choosing the constants $A, B, C$ & $D$ we find $(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, 3$.

(i) To find $(\alpha_1, \beta_1, \gamma_1)$ let $A_1 = C_1 = D_1 = 0$ and $B_1 = -k$

Then $\alpha_1 = \frac{-k}{c}\sin cs$, $\beta_1 = -\cos cs$, $\gamma_1 = \frac{\tau}{c}\sin cs$.

(ii) To find $(\alpha_2, \beta_2, \gamma_2)$ let $B_2, C_2, D_2 = 0$ and $A_2 = -k$.

Then $\alpha_2 = \frac{k}{c}\cos cs$, $\beta_2 = -\sin cs$, $\gamma_2 = \frac{-\tau}{c}\cos cs$.

(iii) To find $(\alpha_3, \beta_3, \gamma_3)$ let $A_3, B_3 = 0$, $C = \tau$, $D = \frac{k}{c}$

Then $\alpha_3 = \frac{\tau}{c}$, $\beta_3 = 0$, $\gamma_3 = \frac{k}{c}$.

$t = (\alpha_1, \alpha_2, \alpha_3) = \left(-\frac{k}{c}\sin cs, \frac{k}{c}\cos cs, \frac{\tau}{c}\right)$

$n = (\beta_1, \beta_2, \beta_3) = (-\cos cs, -\sin cs, 0)$.

$b = (\gamma_1, \gamma_2, \gamma_3) = \left(\frac{\tau}{c}\sin cs, -\frac{\tau}{c}\cos cs, \frac{k}{c}\right)$

To find the position vector, using $t$,

$$r = \int_0^s t\,ds = \int_0^s \left(-\frac{k}{c}\sin cs, \frac{k}{c}\cos cs, \frac{\tau}{c}\right)ds$$

$$= \left(\frac{k}{c^2}\cos cs, \frac{k}{c^2}\sin cs, \frac{\tau}{c}s\right)$$ which can be easily identified with a circular helix with,

$k = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$ namely,

$$r = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right), \quad c = \sqrt{a^2+b^2}.$$

Example

Find the involutes and evolutes of the circular helix $r = (a\cos\theta, a\sin\theta, b\theta)$.

Soln:

$$r = (a\cos\theta, a\sin\theta, b\theta) \longrightarrow (1).$$

The equation of the involute is $r_1 = r + (\lambda - s)t \longrightarrow (2)$.

Differentiating (1), $\frac{dr}{ds}\cdot\frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, b)$

$$t\cdot\frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, b).$$

Hence $\left(\frac{ds}{d\theta}\right)^2 = a^2 + b^2 = c^2$ (say), $\frac{ds}{d\theta} = c \Rightarrow ds = c\cdot d\theta \Rightarrow s = c\theta$.

Then $t = \frac{1}{c}(-a\sin\theta, a\cos\theta, b) \longrightarrow (3)$.

Using (3) in (2), $r_1 = (a\cos\theta, a\sin\theta, b\theta) + \frac{1}{c}(\lambda - c\theta)(-a\sin\theta, a\cos\theta, b)$

The equation of the evolute is,

$r_1 = r + \rho n + \rho\cot[\psi + \lambda]b$, where $\psi = \int \tau\, ds \longrightarrow (4)$.

To find $r_1$ we have to find $n, b, \rho, \tau$ and $\psi$.

For the circular helix, we know that $k = \frac{a}{c^2}, \tau = \frac{b}{c^2}$.

Hence $\psi = \int \frac{b}{c^2} ds = \frac{bs}{c^2} \longrightarrow (5)$.

Using (3) $\frac{dt}{ds} = \frac{1}{c}(-a\cos\theta, -a\sin\theta, 0)\frac{d\theta}{ds}$.

Hence $n = \frac{1}{k}\cdot\frac{1}{c^2}(-a\cos\theta, -a\sin\theta, 0) = (-\cos\theta, -\sin\theta, 0) \longrightarrow (6)$

Further $b = t\times n = \frac{1}{c}(b\sin\theta, -b\cos\theta, a) \longrightarrow (7)$.

Using (5), (6), (7) in (4)

$$r_1 = (a\cos\theta, a\sin\theta, b\theta) + \frac{c^2}{a}(-\cos\theta, -\sin\theta, 0)$$
$$+ \frac{c^2}{a}\cot\left[\frac{b}{c}\theta + \lambda\right]\frac{1}{c}(b\sin\theta - b\cos\theta, a)$$

This equation can be written by Componentwise as,

$$x = \left[a\cos\theta - \frac{c^2}{a}\cos\theta + \frac{c}{a}b\sin\theta\cdot\cot\left(\frac{b\theta}{c} + \lambda\right)\right]$$

$$y = \left[a\sin\theta - \frac{c^2}{a}\sin\theta - \frac{c}{a}b\cos\theta\cdot\cot\left(\frac{b\theta}{c} + \lambda\right)\right]$$

$$z = \left[b\theta + c\cot\left(\frac{b\theta}{c} + \lambda\right)\right].$$

Theorem: Serret-Frenet Formulae:

If $(t, n, b)$ is the moving orthogonal triad of unit vectors at a point $P$ on a space curve $\gamma$, then

(i) $\frac{dt}{ds} = kn$ (ii) $\frac{dn}{ds} = \tau b - kt$ (iii) $\frac{db}{ds} = -\tau n$.

Proof:

To prove (i), differentiating $t\cdot t = 1$ with respect

to $s$ at a point $P$ on the curve, $t \cdot t' = 0 \Rightarrow t \perp r t'$.
Since $t = \frac{dr}{ds}$, $t' = r''$. As $r''$ lies in the osculating plane, $t'$ also lies in the osculating plane. Therefore $t'$ is a vector perpendicular to $t$ and lies in the osculating plane. Hence $t'$ is perpendicular to the principal normal. $|t'| = k$ being the curvature at $P$ on the curve, $t' = \pm kn$ and $t' = kn$.

(ii) As in the above case we find the vector $b'$. Differentiating $b \cdot b = 1 \Rightarrow b \cdot b' = 0 \Rightarrow b' \perp b$ and $b'$ lies in the osculating plane. Since $b \cdot t = 0$, differentiating this and using (i) $b' \cdot t + b \cdot t' = 0$
$b' \cdot t + b \cdot (kn) = 0$. As $b \cdot n = 0 \Rightarrow b' \cdot t = 0$ and $b' \perp t$ and $b'$ lies in the osculating plane.

Hence $b'$ is parallel to the principal normal at $P$. By definition $|b'| = \tau$, being the torsion at $P$. Since the magnitude $\tau$ and the direction $n$ of $b'$, we can write $b' = -\tau n$ where the negative sign is introduced because as a convention torsion is regarded as positive when the rotation of the osculating plane as $s$ increases in the direction of a right handed screw moving in the direction of $t$.

To prove (iii), let us consider $n = b \times t$.
Differentiating both sides of the above vector with respect to $s$, we obtain $\frac{dn}{ds} = \frac{db}{ds} \times t + b \times \frac{dt}{ds}$
using (i) and (ii) in the above equations,

$$\frac{dn}{ds} = (-\tau n) \times t + b \times (kn)$$

Since $n \times t = -b$ and $b \times n = -t$ we have $\frac{dn}{ds} = \tau b - kt$

I M.Sc MATHEMATICS

DIFFERENTIAL GEOMETRY 18KP1MELM1

UNIT – II

# THE FIRST FUNDAMENTAL FORM AND LOCAL INTRINSIC PROPERTIES OF A SURFACE

Definition:

A surface is the locus of a point $P(x,y,z)$ in $E_3$ which is satisfying some restrictions on $x,y,z$ expressed by a relation of the type $F(x,y,z)=0$. The equation $F(x,y,z)=0$ is called the implicit or constraint equation of the surface.

Definition:

A parametric transformation is said to be proper if (i) $\phi$ and $\psi$ are single valued functions, and

(ii) The Jacobian $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in some domain $D$.

Definition:

Let $r=r(u,v)$ be the given surface of class r. Let $v=c$ where $c$ is an arbitrary constant. Then the position vector $r=r(u,c)$ is a function of a single parameter t and hence $r=r(u,c)$ represents a curve lying on the surface $r=r(u,v)$. This curve is called the parametric curve $v=$ Constant.

Definition:

Let $u=c_1$ and $v=c_2$, when the constants $c_1$ and $c_2$ vary, the whole surface is covered with a net of parametric curves, two of which pass through every point $(u,v)$ are called the curvillinear coordinates of P. The parametric curves are called coordinate curves.

Definition.

The parametric curves through a point P are said to be orthogonal if $r_1 \cdot r_2 = 0$ at P.

Definition:

Two surfaces $S$ and $S'$ are said to be isometric or applicable if there exists a correspondence $u' = \phi(u,v)$, $v' = \psi(u,v)$ between their parameters where $\phi$ and $\psi$ are single valued and $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ such that the metric of $S$ is transformed into a metric of $S'$. The correspondence itself is called an isometry.

Theorem:

The notion of r-equivalence of representations of a surface is an equivalence relation.

proof:

Let $R$ be a representation of $S$ and let $S$ be composed to overlapping parts $\{S_j\}$. Since the change of parameters from $S_i$ to $S_j$ is given by a proper parametric transfor-mation of class $r$, the relation of r-equivalance of representation $R$ is reflexive.

Since $R$ and $R'$ are equivalent, there exists a proper parametric transformation $\phi$ at $P$ from $S_j$ to $S'_j$. Since the proper parametric transformation is locally one to one and possess inverse transformation $\phi^{-1}$ exists at the point $P$ of overlap of $S_j$ to $S'_j$. In otherwords there exists $\phi^{-1}$ from $S'_j$ to $S_j$. Thus $R'$ is equivalent to $R$ so that the relation of r-equivalence of class $r$ is symmetric.

Let $R, R', R''$ be any three representations of class $r$ of a surface $S$ and let them be r-equivalent such that $R \sim R'$, $R' \sim R''$. Since $R$ & $R'$

are equivalent, there exists a proper parametric transformation $\phi$ at the common point $P_1$ in the overlap of the family $\{S_j', S_j\}$. Since $R' \sim R''$, the change of parameter of a point $P_1$ in the overlap family $S_j'$ & $S_j''$ is given by a proper parametric transformation $\psi$ from $S_j'$ to $S_j''$. Since $\phi$ and $\psi$ are locally one-to-one, $\psi \circ \phi$ is locally one-to-one giving the change of parameter from $S_j$ to $S_j''$. Hence the representation $R$ and $R''$ are equivalent so that the relation of equivalence of class $r$ of surfaces is transitive.

Since the notion of the relation of equivalence of class $r$ is reflexive, symmetric and transitive, it is an equivalence relation.

Theorem:

The equation of a tangent plane at $P$ on a surface with position vector $r = r(u, v)$ is either $R = r + a r_1 + b r_2$ or $(R - r)\cdot(r_1 \times r_2) = 0$ where $a$ and $b$ are parameters.

Proof:

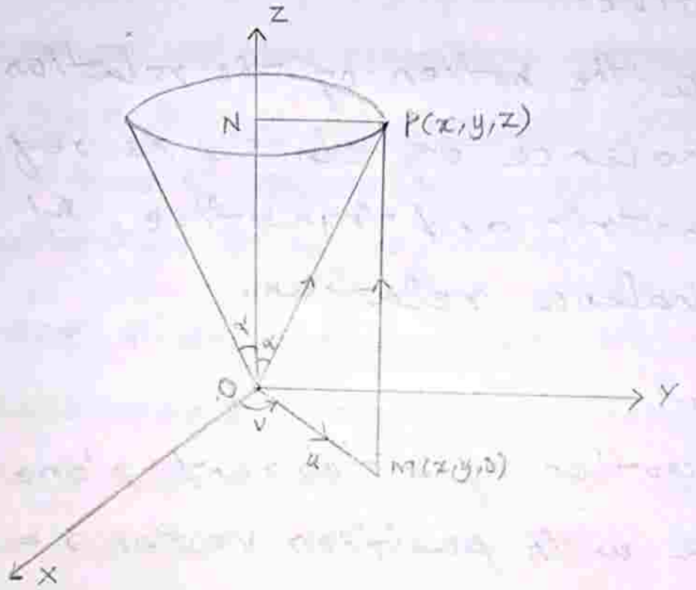
Let $r = r(u, v)$ be the position vector of a point $P$ on the surface. The tangent plane $P$ passes through $r$ and contains the vectors $r_1$ and $r_2$. So if $R$ is the position vector of any point on the tangent plane at

P, then $R-r$, $r_1$, $r_2$ are coplanar. Hence we have $R-r = ar_1 + br_2$ where $a$ and $b$ are arbitrary constants. $r_1 \times r_2$ is perpendicular to the tangent plane at P. Hence $r_1 \times r_2$ is perpendicular to $R-r$ lying in the tangent plane so that $(R-r)\cdot(r_1 \times r_2) = 0$ is another form of the equation of the tangent plane at P.

**Example:**

Obtain the surface equation of a cone with semi-vertical angle $\alpha$ and find the singularities, parametric curves, tangent plane at a point and the surface normal.

**Proof:**




Taking the axis of the cone as the z-axis, let $P(x,y,z)$ be any point on the cone. Draw PN and PM perpendicular to the axis of the cone and XOY plane. Let $NP = u$ and $v$ be the angle of rotation of the plane so that $XOM = v$.

$x = u\cos v$, $u\sin v = y$ and $z = u\cot\alpha$.

$r = (u\cos v, u\sin v, u\cot\alpha)$.

$r_1 = (\cos v, \sin v, \cot\alpha)$ and $r_2 = (-u\sin v, u\cos v, 0)$

Now $r_1 \times r_2 = 0$ at $u=0$. Hence the vertex of the cone is the only singularity, it is the essential singularity.

When $u$ = constant, the distance of $P$ from the Z-axis is constant so $P$ describe a circle. Hence the system of parametric curves when $u$ = const. is a system of parallel circles with centres on the Z-axis. When $v$ = constant, the plane of rotation through Z-axis makes a constant angle with the x-axis so that the parametric curves are the intersection of this plane with the cone along a generator. Hence $v$ = constant, the system of parametric curves are the generators of the cone through the origin.

$r_1 \cdot r_2 = 0 \Rightarrow$ the parametric curves are orthogonal.

$r_1 \times r_2 = i(-u\cos v\cot\alpha) + j(-u\sin v\cot\alpha) + ku$.

Hence the equation to the tangent plane at any point is,

$(X-x)(-u\cos v\cot\alpha) + (Y-y)(-u\sin v\cot\alpha) + (Z-z)u = 0$

Further $|r_1 \times r_2| = u\operatorname{cosec}\alpha$.

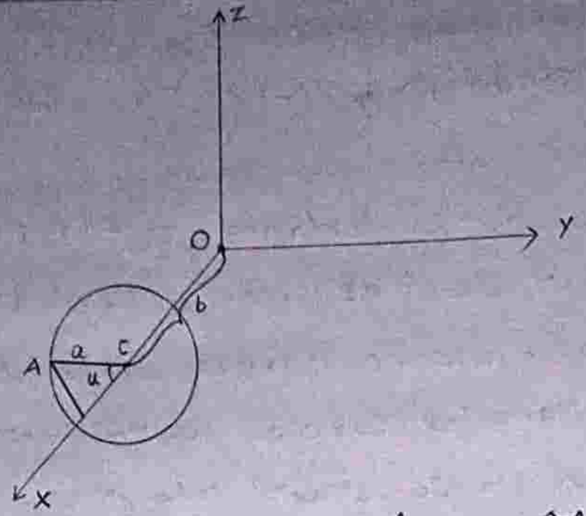
Thus $N = \dfrac{r_1 \times r_2}{|r_1 \times r_2|} = (-\cos v\cos\alpha, -\sin v\cos\alpha, \sin\alpha)$

Theorem:

The position vector of a point on the anchor ring is $r = [(b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u)]$ where $(b,0,0)$ is the centre of the circle and Z-axis is the axis of rotation.

Proof:

Taking the axis of revolution as the Z-axis and the generating circle in the XOZ plane with centre $C(b,0,0)$ on the x-axis, we shall find the coordinates of the point $A$ on the circle.




Let CA make an angle $u$ with the $x$-axis. The $x$-coordinates of A is $b+a\cos u$ and the $z$-coordinate of A is $a\sin u$. Hence as A has coordinates $(b+a\cos u, 0, a\sin u)$.

Let $P(x,y,z)$ be the position of the point A after the generating circle has revolved through an angle $v$. Since the point A describes a a circle about the $z$-axis, the distance of P from the $z$-axis is the radius of this circle given by $(b+a\cos u)$. Since it has been revolved through an angle $v$, its $x$ and $y$ coordinates are

$(b+a\cos u)\cos v, (b+a\cos u)\sin v$.

For the point A as well as for the point P, its $z$-coordinate $a\sin u$ is always a constant. Hence the position vector of the point P on the anchor ring is given by

$[(b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u]$

where $0<u<2\pi$ and $0<v<2\pi$.

When $u$ = constant, CA is fixed and revolves about the $z$-axis. Hence it is a circle on the anchor ring and these curves are parallels. When $v$ = constant, the rotating plane is fixed. Hence the parametric curve for $0<u\leq 2\pi$ is the intersection or the cross section of this plane and the anchor ring so that it is a generating circle. Thus the meridians are circles.

Further $r_1 = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$r_2 = (-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)$

Since $r_1 \cdot r_2 = 0$, the parametric curves are orthogonal.

$r_1 \times r_2 = -(b + a\cos u)[a\cos u \cos v, a\cos u \sin v, a\sin u]$

Since $b > a$, the above vector is negative for the range of values of $u$ and $v$ so that the normal is directed inside the anchor ring, since $|r_1 \times r_2|$ is always positive.

Theorem:

The metric is invariant under a parametric transformation.

Proof:

Let $u' = \phi(u, v)$ and $v' = \varphi(u, v)$ be the parametric transformation and $r = r(u', v')$ be the equation of the surface.

$$r_1' = \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u} \cdot \frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v} \cdot \frac{\partial v}{\partial v'} = r_1 \frac{\partial u}{\partial u'} + r_2 \frac{\partial v}{\partial u'} \longrightarrow (1)$$

Similarly $$r_2' = r_1 \frac{\partial u}{\partial v'} + r_2 \frac{\partial u}{\partial v'} \longrightarrow (2)$$

$$du = \frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv' \longrightarrow (3)$$

$$dv = \frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv' \longrightarrow (4)$$

If $E'$, $F'$ and $G'$ are the fundamental coefficients in the new parametric system, then we have

$$E' du'^2 + 2F' du' dv' + G' dv'^2$$

$$= r_1'^2 du'^2 + 2 r_1' \cdot r_2' du' dv' + r_2'^2 dv'^2$$

$$= [r_1' du' + r_2' dv']^2.$$

Using (1) and (2) in the above step, we obtain

$$= \left[\left(r_1 \frac{\partial u}{\partial u'} + r_2 \frac{\partial v}{\partial u'}\right) du' + \left(r_1 \frac{\partial u}{\partial v'} + r_2 \frac{\partial v}{\partial v'}\right) dv'\right]^2$$

$$= \left[r_1 \left(\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'\right) + r_2 \left(\frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv'\right)\right]^2$$

Using ⑤ and ④ in the above step,

$E'du'^2 + 2F'du'dv' + G'dv'^2$

$= (r_1 du + r_2 dv)^2$

$= r_1^2 du^2 + 2r_1 \cdot r_2 du dv + r_2^2 dv^2$

$= E du^2 + 2F du dv + G dv^2.$

Example:

Find $E, F, G$ and $H$ for the paraboloid $x = u,\ y = v,\ z = u^2 - v^2$.

Soln:

Any point on the paraboloid has position vector $r = (u, v, u^2 - v^2)$

Hence $r_1 = (1, 0, 2u)$ and $r_2 = (0, 1, -2v)$.

$E = r_1 \cdot r_1 = 1 + 4u^2;\ F = r_1 \cdot r_2 = -4uv.$

$G = r_2 \cdot r_2 = 1 + 4v^2.$

Further $r_1 \times r_2 = (-2u, 2v, 1)$.

Hence, $H = |r_1 \times r_2| = \sqrt{4u^2 + 4v^2 + 1} = \sqrt{EG - F^2}$.

Theorem:

If $(l, m)$ and $(l', m')$ are the direction coefficients of the two directions at a point $P$ on the surface and $\theta$ is the angle between the two direction at $P$, then

(i) $\cos\theta = E l l' + F(lm' + l'm) + G mm'$

(ii) $\sin\theta = H(lm' - l'm)$.

Proof:

If $(l, m)$ and $(l', m')$ are the direction coefficients of the two directions at the same point $P$ on the surface $r = r(u, v)$, then the corresponding unit vectors along these directions at $P$ are.

$$a = l r_1 + m r_2,\ a' = l' r_1 + m' r_2 \longrightarrow ①.$$

Let $\theta$ be the angle between the two directions.

$a \cdot a' = \cos\theta$, $a \times a' = \sin\theta N$ $\longrightarrow$ ②

$a \cdot a' = (l r_1 + m r_2) \cdot (l' r_1 + m' r_2)$

$= l l' r_1^2 + (l m' + l' m) r_1 \cdot r_2 + m m' r_2^2$

$= E l l' + F(l m' + l' m) + G m m'$ $\longrightarrow$ ③

② in ③

$\cos\theta = E l l' + F(l m' + l' m) + G m m'$

$a \times a' = (l r_1 + m r_2) \times (l' r_1 + m' r_2)$

$r_1 \times r_1 = r_2 \times r_2 = 0$ and $a \times a' = (l m' - m l')(r_1 \times r_2)$.

$r_1 \times r_2 = NH \Rightarrow a \times a' = (l m' - m l') NH$ $\longrightarrow$ ④

② in ④

$\sin\theta = (l m' - m l') H$, $H = \sqrt{EG - F^2}$

When the two directions are orthogonal, then $\cos\theta = 0$ which gives in terms of direction co-efficients, $E l l' + F(l m' + l' m) + G m m' = 0$.

Example:

Find the parametric directions and the angle between the parametric curves.

Soln:

For the parametric curve $v$ = constant, the parametric direction ratio $(du, 0)$ by $(v)$.

Direction coefficients $(l, m) = \dfrac{(du, 0)}{\sqrt{E du^2}} = \dfrac{(1, 0)}{\sqrt{E}}$

For direction ratio $(0, dv)$.

Direction coefficients $(l', m') = \dfrac{(0, dv)}{\sqrt{G dv^2}} = \dfrac{(0, 1)}{\sqrt{G}}$

Let $\theta$ be the angle between the parametric curves.

$\cos\theta = \dfrac{F}{\sqrt{EG}}$, $\sin\theta = \dfrac{H}{\sqrt{EG}}$.

When $\theta = \pi/2$, $\cos\theta = 0$ so the condition of orthogonality of parametric curves is $F = 0$.

**Definition:**

If $(l, m)$ are the direction coefficients of a direction at a point $P$ on the surface, the scalars $(\lambda, \mu)$ which are perpendicular to $(l, m)$ are called the direction ratios of the direction.

**Definition:**

If $P, Q, R$ are continuous functions of $u$ and $v$ which do not vanish together and if $Q^2 - PR > 0$ then the quadratic differential equation $P du^2 + 2Q du dv + R dv^2 = 0$ represents two families of curves on the given surface.

**Theorem:**

The two directions given by

$$P du^2 + 2Q du dv + R dv^2 = 0 \longrightarrow (1)$$

are orthogonal on a surface iff $ER - 2QF + GP = 0$

**Proof:**

If $(l, m)$ and $(l', m')$ are the direction coefficients of the two families of curves of (1) at $P$ then $\frac{l}{m}$ and $\frac{l'}{m'}$ are the roots of the quadratic $P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \longrightarrow (2)$.

$$\frac{l}{m} + \frac{l'}{m'} = -\frac{2Q}{P}, \quad \frac{ll'}{mm'} = \frac{R}{P} \longrightarrow (3)$$

$(l, m)$ and $(l', m')$ are orthogonal iff

$$E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G = 0 \longrightarrow (4).$$

(3) in (4) $\rightarrow$ $ER - 2FQ + GP = 0$.

**Example:**

Show that the curves $du^2-(u^2+c^2)dv^2=0 \longrightarrow (1)$ form an orthogonal system or the right helicoid.

**Soln:**

$$r=(u\cos v, u\sin v, cv) \longrightarrow (2).$$

(1) is a double family of curves on (2) with $P=1, Q=0$ and $R=-(u^2+c^2)$.

For the right helicoid (2),

$$r_1=(\cos v, \sin v, 0),\ r_2=(-u\sin v, u\cos v, c).$$

$$E=r_1\cdot r_1=1;\ F=r_1\cdot r_2=0;\ G=r_2\cdot r_2=(u^2+c^2).$$

$ER-2FQ+GP=1[-(u^2+c^2)+(u^2+c^2)]=0$ So the condition of orthogonality is satisfied. Hence (1) form an orthogonal system of (2).

**Definition:**

Two surfaces $S$ and $S'$ are said to be isometric or applicable if there exists a correspondence $u'=\phi(u,v)$, $v'=\psi(u,v)$ between their parameters where $\phi$ and $\psi$ are single valued and $\frac{\partial(\phi,\psi)}{\partial(u,v)}\neq 0$ such that the metric of $S$ is transformed into a metric of $S'$. The correspondence itself is called an isometry.

# I. M.Sc., MATHEMATICS

TITLE OF THE PAPER : DIFFERENTIAL GEOMETRY

PAPER CODE : 18KP1MELM1

## UNIT III

## GEODESICS ON A SURFACE

**Definition:**

Let A and B be two given points on a surface S and let these points be joined by curves lying on S. Then any curve possessing stationary length for small variation over S is called a Geodesic

**Theorem:**

If $\dot{u} \neq 0$ in the neighbourhood of a point on a geodesic, then taking $u(t) = t$ the curve $v = v(u)$ is a geodesic iff v satisfies the second order differential equation $\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0$ where P, Q, R and S are functions of u and v determined by E, F and G.

**Proof:**

Using the condition $U\dfrac{\partial T}{\partial \dot{v}} - V\dfrac{\partial T}{\partial \dot{u}} = 0$ we shall derive the differential equation of the geodesic. Now $T = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2]$

$$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v} = E + \dot{v}F, \ \dot{u} = 1 \longrightarrow (1).$$

$$\frac{\partial}{\partial T}\left(\frac{\partial T}{\partial \dot{u}}\right) = \frac{dE}{dt} + \frac{dF}{dt}\dot{v} + F\ddot{v}$$

$$= \frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\frac{dv}{dt} + \left(\frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt}\right)\dot{v} + F\ddot{v}$$

$\dot{u} = 1$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = E_1 + (E_2 + F_1)\dot{v} + F_2\dot{v}^2 + F\ddot{v}$

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left(E_1 + 2F_1\dot{v} + G_1\dot{v}^2\right), \ \dot{u} = 1$$

Hence $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = F\ddot{v} + \left(F_2 - \frac{1}{2}G_1\right)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1 \longrightarrow (2)$

Let us find V. $\frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v} = F + G\dot{v} \longrightarrow (3)$

Hence $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = \left(\frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial v}\dot{v}\right) + \left(\frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial v}\dot{v}\right) + G\ddot{v}$

As $\dot{u} = 1$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v}$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{dT}{dv}$$

$$= F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v} - \frac{1}{2}\left(E_2 + 2F_2\dot{v} + G_2\dot{v}^2\right)$$

$$= G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + \frac{1}{2}E_2 \longrightarrow (4)$$

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0 \longrightarrow (5)$$

(1), (2), (3), (4) in (5)

$$(E + F\dot{v})\left(G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2\right)$$

$$-(F + G\dot{v})\left(F\ddot{v} + \left(F_2 - \frac{1}{2}G_1\right)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1\right) = 0$$

Writing the above as 2nd order differential equation

$$(EG - F^2)\ddot{v} + \frac{1}{2}(GG_1 + FG_2 - 2GF_2)\dot{v}^3$$

$$+ \frac{1}{2}(G_2E + 3G_1F - 2FF_2 - 2E_2G)\dot{v}^2$$

$$+ \frac{1}{2}(2EG_1 + 2F_1F - 3E_2F - E_1G)\dot{v}$$

$$+ \frac{1}{2}(2EF_1 - E_2E - E_1F) = 0.$$

which can be rewritten as

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = H^2(\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S) = 0$$

$$P = \frac{1}{H^2}\cdot\frac{1}{2}(GG_1 + FG_2 - 2GF_2).$$

$$Q = \frac{1}{H^2}(G_2E + 3G_1F - 2FF_2 - 2E_2G)$$

$$R = \frac{1}{H^2}\cdot\frac{1}{2}(2G_1E + 2FF_1 - 3E_2F - E_1G).$$

$$S = \frac{1}{H^2}\cdot\frac{1}{2}(2EF_1 - E_2E - E_1F)$$

Hence the equation of the geodesic is given by

$$\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + S = 0.$$

Example:

prove that the curves of the family $\frac{v^3}{u^2} =$ constant are geodesics on a surface with the metric; $v^2du^2 - 2uvdudv + 2u^2dv^2$, $u>0$ & $v>0$.

Soln:

$\frac{v^3}{u^2} = c$ is a geodesic at any point on the surface iff $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0.$

Choosing $t$ as a parameter, the parametric representation of the curve can be taken as

$u = ct^3,\ v = ct^2. \longrightarrow (1)$

Hence $\dot{u} = 3t^2(c) = 3ct^2$ & $\dot{v} = 2ct \longrightarrow (2).$

$T = \frac{1}{2}(v^2\dot{u}^2 - 2uv\dot{u}\dot{v} + 2u^2\dot{v}^2) \longrightarrow (3)$

From (1) & (2)

$$\frac{\partial T}{\partial u} = -v\dot{u}\dot{v} + 2u\dot{v}^2 = 2c^3t^5$$

$$\frac{\partial T}{\partial v} = v\dot{u}^2 - u\dot{u}\dot{v} = 3c^3t^6$$

$$\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6$$

$$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2u^2\dot{v} = c^3t^7$$

Hence $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}$

$$= \frac{d}{dt}(c^3t^6) - 2c^3t^5$$

$$U = 4c^3t^5$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(c^3t^7) - 3c^3t^6$$

$$V = 4c^3t^6.$$

Hence $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 4c^3t^5 \cdot c^3t^7 - 4c^3t^6 \cdot c^3t^6 = 0$

$\therefore$ The curve $\frac{v^2}{u^3} = c$ is a geodesic on the surface for all values of $c$.

Theorem:

The geodesics on a right circular cylinder are helices.

Proof:

To show that the geodesics on a right circular cylinder is a helix, we prove that the geodesic cut the generators at a constant angle.

If the surface of revolution is a right circular cylinder, the meridian are generators and the distance between the meridian and the axis of the cylinder is a constant ($\alpha$) say. Hence if $\varphi$ is the angle between the geodesic and the generator, then by Clairaut's theorem $\alpha \sin\psi = h$ where $h$ is a constant. Thus $\sin\psi = \frac{h}{\alpha} \Rightarrow \psi$ is constant. Hence a geodesic on a right circular cylinder cuts the generators at a constant angle and therefore it is a helix.

Theorem:

Any curve $u = u(t)$, $v = v(t)$ on a surface $r = r(u,v)$ is a geodesic iff the principal normal at every point on the curve is normal to the surface.

Proof:

Let $r = r(u,v)$ to be a geodesic at a point $P$ on the surface $r'' \cdot r_1 = 0$ and $r'' \cdot r_2 = 0$ showing the principal normal $r'' = kn$ of the curve is orthogonal to the tangential direction $r_1$ & $r_2$ at $P$ and the principal normal of the curve coincides with the surface normal.

$$\frac{\partial T}{\partial \dot{u}} = \dot{r} \cdot r_1, \quad \frac{\partial T}{\partial \dot{v}} = \dot{r} \cdot r_2 \quad U(t) = \ddot{r} \cdot r_1, \; V(t) = \ddot{r} \cdot r_2 \longrightarrow (1).$$

Consider,

$$\dot{r} = \frac{dr}{dt} = r_1 \dot{u} + r_2 \dot{v} \longrightarrow (2).$$

$$\dot{r} \cdot \dot{r} = (r_1 \dot{u} + r_2 \dot{v})^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

Take $T = \frac{1}{2} \dot{r}^2 \longrightarrow (3).$

Differentiating (3) partially,

$$\frac{\partial T}{\partial u} = \dot{r}\cdot\frac{\partial \dot{r}}{\partial u}, \quad \frac{\partial T}{\partial v} = \dot{r}\cdot\frac{\partial \dot{r}}{\partial v}$$

$$\frac{\partial T}{\partial \dot{u}} = \dot{r}\cdot\frac{\partial \dot{r}}{\partial \dot{u}}, \quad \frac{\partial T}{\partial \dot{v}} = \dot{r}\cdot\frac{\partial \dot{r}}{\partial \dot{v}} \longrightarrow (4)$$

Differentiating (2) partially,

$$\frac{\partial \dot{r}}{\partial \dot{u}} = r_1 \quad \& \quad \frac{\partial \dot{r}}{\partial \dot{v}} = r_2$$

$$\frac{\partial \dot{r}}{\partial u} = r_{11}\dot{u} + r_{21}\dot{v}, \quad \frac{\partial \dot{r}}{\partial v} = r_{12}\dot{u} + r_{22}\dot{v} \longrightarrow (5).$$

where $r_{11} = \frac{\partial^2 r}{\partial u^2}$, $r_{12} = \frac{\partial^2 r}{\partial u \partial v}$, $r_{21} = \frac{\partial^2 r}{\partial v \partial u}$, $r_{22} = \frac{\partial^2 r}{\partial v^2}$.

Using (5) in (4)

$$\frac{\partial T}{\partial u} = \dot{r}\cdot(r_{11}\dot{u} + r_{21}\dot{v}), \quad \frac{\partial T}{\partial v} = \dot{r}\cdot(r_{12}\dot{u} + r_{22}\dot{v})$$

$$\frac{\partial T}{\partial \dot{u}} = \dot{r}\cdot r_1, \quad \frac{\partial T}{\partial \dot{v}} = \dot{r}\cdot r_2 \longrightarrow (6).$$

Let us find $U(t)$ and $V(t)$ as follows,

$$U(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}$$

Using (6)

$$U(t) = \frac{d}{dt}(\dot{r}\cdot r_1) - \dot{r}\cdot(r_{11}\dot{u} + r_{21}\dot{v})$$

$$= \ddot{r}\cdot r_1 + \frac{dr_1}{dt}\dot{r} - \dot{r}\cdot(r_{11}\dot{u} + r_{21}\dot{v})$$

$$= \ddot{r}\cdot r_1 + \dot{r}\cdot(r_{11}\dot{u} + r_{12}\dot{v}) - \dot{r}\cdot(r_{11}\dot{u} + r_{21}\dot{v})$$

$r_{12} = r_{21} \Rightarrow U(t) = \ddot{r}\cdot r_1 \longrightarrow (7).$

$$V(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}$$

$$V(t) = \frac{d}{dt}(\dot{r}\cdot r_2) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v})$$
$$= \ddot{r}\cdot r_2 + \dot{r}\frac{d}{dt}r_2 - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v})$$
$$= \ddot{r}\cdot r_2 + \dot{r}\cdot(r_{21}\dot{u} + r_{22}\dot{v}) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v})$$

Hence $V(t) = \ddot{r}\cdot r_2 \longrightarrow (8)$.

$U(s) = r''\cdot r_1$ & $V(s) = r''\cdot r_2$

The Canonical geodesic equation

$U(s) = 0$, $V(s) = 0$ implies,

$r''\cdot r_1 = 0$ and $r''\cdot r_2 = 0 \longrightarrow (9)$.

(9) $\Rightarrow r''$ ⊥r $r_1$ & $r'' \perp r_2$ lying in the tangent at $P$. Hence $r''$ is the surface normal at $P$.

$$r'' = \frac{d^2 r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{dt}{ds} = kn,$$ Shows that $r''$ is along the principal normal of the geodesic at $P$. Hence the principal normal at every point of the geodesic is normal to the surface at $P$.

Theorem:

The geodesic equations are

$$Eu'' + Fv'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \longrightarrow (1)$$
$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1 u'v' + \frac{1}{2}G_2 v'^2 = 0 \longrightarrow (2).$$

Proof:

Let the equation of the surface be $r = r(u,v)$ where $u = u(s)$ and $v = v(s)$.

$$r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}\cdot\frac{du}{ds} + \frac{\partial r}{\partial v}\cdot\frac{dv}{ds}$$

$$r' = r_1 u' + r_2 v' \longrightarrow (1)$$

Differentiating (1) with respect to s,

$$r'' = (r_{11}u' + r_{12}v')u' + r_1 u'' + (r_{21}u' + r_{22}v')v' + r_2 v''$$

$$r'' = r_1 u'' + r_2 v'' + r_{11}u'^2 + 2r_{12}u'v' + r_{22}v'^2 \longrightarrow (2).$$

$$r''\cdot r_1 = 0 \text{ and } r''\cdot r_2 = 0 \longrightarrow (3).$$

Taking scalar product of (2) with $r_1$ and $r_2$ and using (3)

$$r_1\cdot r_1 u'' + r_2\cdot r_1 v'' + r_{11}\cdot r_1 u'^2 + 2r_{12}\cdot r_1 u'v' + r_{22}\cdot r_1 v'^2 = 0 \longrightarrow (4).$$

$$r_1\cdot r_2 u'' + r_2\cdot r_2 v'' + r_{11}\cdot r_2 u'^2 + 2r_{12}\cdot r_2 u'v' + r_{22}\cdot r_2 v'^2 = 0 \longrightarrow (5)$$

$$r_1\cdot r_1 = E,\ F = r_1\cdot r_2,\ r_2\cdot r_2 = G$$

$$r_1\cdot r_{11} = \frac{1}{2}\frac{\partial}{\partial u}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial u} = \frac{1}{2}E_1 \longrightarrow (6).$$

$$r_1\cdot r_{12} = \frac{1}{2}\frac{\partial}{\partial v}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial v} = \frac{1}{2}E_2 \longrightarrow (7).$$

$$r_2\cdot r_{22} = \frac{1}{2}\frac{\partial}{\partial v}(r_2^2) = \frac{1}{2}\frac{\partial G}{\partial v} = \frac{1}{2}G_2 \longrightarrow (8)$$

$$r_2\cdot r_{21} = \frac{1}{2}\frac{\partial}{\partial u}(r_2^2) = \frac{1}{2}\frac{\partial G}{\partial u} = \frac{1}{2}G_1 \longrightarrow (9).$$

$$\frac{\partial}{\partial u}(r_1\cdot r_2) = r_{11}\cdot r_2 + r_1\cdot r_{21}$$

$$r_2\cdot r_{11} = F_1 - \frac{1}{2}E_2 \longrightarrow (10).$$

$$\frac{\partial}{\partial v}(r_1\cdot r_2) = r_{12}\cdot r_2 + r_1\cdot r_{22}$$

$$r_1\cdot r_{22} = F_2 - \frac{1}{2}G_1 \longrightarrow (11).$$

(6), (7), (11) in (4)

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \longrightarrow (I)$$

8, 9, 10 in 5 $\Rightarrow$

$$Fu'' + Gv'' + (F_1 - \tfrac{1}{2}E_2)u'^2 + G_1u'v' + \tfrac{1}{2}G_2v'^2 = 0 \longrightarrow \text{II}.$$

Theorem:

A geodesic can be found to pass through any given point and have any given direction on a Surface. The geodesic is uniquely determined by the initial conditions.

proof:

Using I and II of previous theorem and deduce the existence of a geodesic at a point from the uniqueness of solution of the initial value problem of such a differential equation $\frac{dv}{du} = \frac{dv}{ds}\cdot\frac{ds}{du}$

and
$$\frac{d^2v}{du^2} = \frac{d}{du}\left(\frac{dv}{ds}\cdot\frac{ds}{du}\right) = \frac{d^2v}{ds^2}\cdot\left(\frac{ds}{du}\right)^2 + \frac{dv}{ds}\cdot\frac{d}{du}\left(\frac{ds}{du}\right)$$

$$= \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 + \frac{dv}{ds}\cdot\frac{d}{du}\left(\frac{1}{u'}\right)$$

$$= \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 - \frac{1}{u'^2}u''\cdot\frac{ds}{du}\cdot\frac{dv}{ds}$$

Hence
$$\frac{d^2v}{du^2} = \frac{d^2v}{ds^2}\left(\frac{ds}{du}\right)^2 - \left(\frac{ds}{du}\right)^2\cdot u''\cdot\frac{dv}{du} \longrightarrow (1)$$

$$u'' = -(lu'^2 + 2mu'v' + nv'^2) \longrightarrow (2)$$

$$v'' = -(\lambda u'^2 + 2\mu u'v' + \nu v'^2) \longrightarrow (3)$$

$(3)\left(\frac{ds}{du}\right)^2 - (2)\cdot\frac{dv}{du}\left(\frac{ds}{du}\right)^2 \Rightarrow$

$$v''\left(\frac{ds}{du}\right)^2 - u''\frac{dv}{du}\left(\frac{ds}{du}\right)^2 = -\left[\lambda + 2\mu\frac{dv}{du} + \nu\left(\frac{dv}{du}\right)^2\right]$$

$$+\left[l\frac{dv}{du}+2m\left(\frac{dv}{du}\right)^2+n\left(\frac{dv}{du}\right)^3\right] \longrightarrow (4)$$

using (3) in LHS of (4) and simplifying,

$$\frac{d^2v}{du^2}=n\left(\frac{dv}{du}\right)^3+(2m-v)\left(\frac{dv}{du}\right)^2+(l-2\mu)\frac{dv}{du}-\lambda \longrightarrow (5)$$

From the existence and uniqueness of solution of the initial value problem of an ordinary differential equation of second order, there exists a unique solution for $v$ of (5) with the initial conditions $v=v_0$ and $\frac{dv}{du}=v_1$ at $u=u_0$. Thus any solution $u, v$ of (5) gives the direction coefficient of tangent at $P$. Hence a geodesic is uniquely determined by the initial point $P$ and the tangent at $P$ under the given conditions.

Definition:

A region $R$ on a surface is said to be convex if any two points of the surface can be joined by a geodesic arc lying wholly in $R$.

Definition:

The region $R$ is said to be simple if there is only one geodesic arc joining any two points and lying entirely in it.

I M.Sc MATHEMATICS

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UNIT-IV

GEODESICS ON A SURFACE.

3.8 GEODESIC PARLLELS:

Definition:

The orthogonal trajectories of the given family of geodesics $v$ = constant on a surface are called geodesic parallels, $u$ & $v$ are called geodesic parameters.

Theorem:

If a surface admits two orthogonal families of geodesics, then it is isometric with the plane.

Proof:

$v$ = Constant be a family of geodesics. Then the family of trajectories is $u$ = Constant. Take $u$ = Constant also as a family of geodesics Then $v$ = Constant is a family of orthogonal trajectories So that the surface admits the two orthogonal family of geodesics. Measuring the distance along a geodesic $v$ = constant, let the distance from some fixed parallels to the neighbouring parallel be du. Hence ds = du and dv = 0 so that

$ds = E du \Rightarrow du = E du \Rightarrow E(u) = 1$.

In a similar manner measuring the distance along the geodesic u = Constant, let the distance from some fixed parallel to the neighbouring parallel be dv. Hence ds = dv and du = 0 so that

$ds = G(u,v) dv \Rightarrow dv = G(u,v) dv \Rightarrow G(u,v) = 1$.

Thus the metric becomes $ds^2 = du^2 + dv^2$ which is the metric of the plane. This proves that the surface admitting two families of orthogonal geodesics is isometric with the plane.

Theorem:

With s as parameter, the components of the geodesic curvature vector are given by

$$\lambda = \frac{1}{H^2} \frac{U}{V'} \frac{\partial T}{\partial v'} = \frac{-1}{H^2} \frac{V}{u'} \frac{\partial T}{\partial v'}$$

$$\mu = \frac{1}{H^2} \frac{V}{u'} \frac{\partial T}{\partial u'} = -\frac{1}{H^2} \frac{U}{v'} \frac{\partial T}{\partial u'}$$

proof:

$$\lambda = \frac{1}{H^2}(UG - VF), \quad \mu = \frac{1}{H^2}(EV - FU) \longrightarrow (1)$$

$$Uu' + Vv' = 0 \Rightarrow V = \frac{-Uu'}{v'} \text{ and } \frac{U}{v'} = \frac{V}{u'} \longrightarrow (2)$$

$$(1) \Rightarrow \lambda = \frac{1}{H^2}\left(GU + \frac{FUu'}{v'}\right) = \frac{U}{H^2 v'}(v'G + u'F) \longrightarrow (3)$$

To complete the proof it is enough to show that

$$(u'F + v'G) = \frac{\partial T}{\partial v'}$$

Now $T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$

$$\frac{\partial T}{\partial v'} = Fu' + Gv';\ \frac{\partial T}{\partial u'} = Eu' + Fv' \longrightarrow (4)$$

(4) in (3) $\Rightarrow \lambda = \frac{1}{H^2} \cdot \frac{U}{v'}\frac{\partial T}{\partial v'} = \frac{-1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}$

(2) in (1) $\Rightarrow \mu = \frac{-1}{H^2}\left(\frac{u'U}{v'}E + FU\right) = \frac{-1}{H^2}\frac{U}{v'}(u'E + v'F) \longrightarrow (5)$

using (4) in (5)

$$\mu = -\frac{1}{H^2}\frac{U}{v'}\cdot\frac{\partial T}{\partial u'} = \frac{V}{H^2 u'}\frac{\partial T}{\partial u'}.$$

Hence the proof.

Example

Obtain the geodesic curvature vector of a curve on a right helicoid $r = (u\cos v, u\sin v, \alpha v)$ using different formulae for $\overline{r}$.

Soln:

$$r_1 = (\cos v, \sin v, 0);\ r_2 = (-u\sin v, u\cos v, \alpha)$$

$$E = r_1 \cdot r_1 = 1,\ F = 0,\ G = \alpha^2 + u^2;\ H = \sqrt{\alpha^2 + u^2}$$

(i) $\lambda = \frac{1}{H^2}(UG - VF),\ \mu = \frac{1}{H^2}(EV - FU)$

$$T = \frac{1}{2}(u'^2 + (u^2 + \alpha^2)v'^2)$$

$$\frac{\partial T}{\partial u'} = u';\ \frac{\partial T}{\partial v'} = (u^2 + \alpha^2)v'.$$

$$\frac{\partial T}{\partial u} = uv'^2,\ \frac{\partial T}{\partial v} = 0 \longrightarrow (1)$$

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = u'' - uv'^2 \longrightarrow (2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = (u^2 + \alpha^2)v'' + 2uu'v' \longrightarrow (3)$$

Hence $\lambda = \frac{1}{\alpha^2 + u^2}\left((u^2 + \alpha^2)\cdot(u'' - uv'^2)\right) = u'' - uv'^2$

$$\mu = \frac{1}{\alpha^2 + u^2}\left((u^2 + \alpha^2)v'' + 2uu'v'\right)$$

(ii) $\lambda = u'' + \Gamma_{11}^{1} u_1'^2 + 2\Gamma_{12}^{1} u'v' + \Gamma_{22}^{1} v'^2$

$\mu = v'' + \Gamma_{11}^{2} u'^2 + 2\Gamma_{12}^{2} u'v' + \Gamma_{22}^{2} v'^2$.

$\Gamma_{111} = \frac{1}{2}E_1 = 0;\ \Gamma_{112} = \frac{1}{2}E_2 = 0,\ \Gamma_{122} = F_2 - \frac{1}{2}G_1 = -u$.

$\Gamma_{211} = F_1 - \frac{1}{2}E_2 = 0,\ \Gamma_{212} = \frac{1}{2}G_1 = u;\ \Gamma_{222} = \frac{1}{2}G_2 = 0$.

Hence $\Gamma_{11}^{1} = \frac{1}{H^2}(G\Gamma_{111} - F\Gamma_{211}) = 0,\ \Gamma_{12}^{1} = \frac{1}{H^2}(G\Gamma_{112} - F\Gamma_{212}) = 0$

$$\Gamma_{22}^{1} = \frac{1}{H^2}(G\Gamma_{122} - F\Gamma_{222}) = \frac{1}{\alpha^2+u^2}(\alpha^2+u^2)(-u) = -u.$$

$\therefore \lambda u'' - uv'^2$.

$\Gamma_{11}^{2} = \frac{1}{H^2}[E\Gamma_{211} - F\Gamma_{111}] = 0,\ \Gamma_{12}^{2} = \frac{1}{H^2}(E\Gamma_{212} - \Gamma_{112}F) = \frac{u}{\alpha^2+u^2}$.

$\Gamma_{22}^{2} = \frac{1}{H^2}(E\Gamma_{222} - F\Gamma_{122}) = 0$.

$$\therefore \mu = v'' + \frac{2u'v'u}{\alpha^2+u^2} = \frac{1}{\alpha^2+u^2}\left(v''(\alpha^2+u^2) + 2uu'v'\right)$$

(iii) $\lambda = \frac{1}{H^2}\frac{v}{v'}\frac{\partial T}{\partial v'} = -\frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}$

$\mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = -\frac{1}{H^2}\frac{U}{V'}\frac{\partial T}{\partial u'}$

Using (1), (2) & (3),

$$\lambda = \frac{1}{\alpha^2+u^2}\cdot\frac{u'' - uv'^2}{v'}v'(\alpha^2+u^2) = u'' - uv'^2.$$

$$\lambda = -\frac{1}{\alpha^2+u^2}\left((u^2+\alpha^2)v'' + 2uu'v'\right)\frac{v'}{u'}(\alpha^2+u^2)$$

$$= \frac{-v'}{u'}\left((u^2+\alpha^2)v'' + 2uu'v'\right)$$

using (1), (2) & (3)

$$\mu = \frac{1}{\alpha^2+u^2}\left((u^2+\alpha^2)v'' + 2uu'v'\right) \text{ and}$$

$$\mu = -\frac{1}{\alpha^2+u^2}\left(\frac{u'' - uv'^2}{v'}\right)u'.$$

Theorem (Liouville's Formula):

If $\theta$ is the angle which the curve C makes with the parametric curve v = Constant, then $k_g = \theta' + Pu' + Qv'$ where $P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$.

$Q = \frac{1}{2HE}(EG_1 - FE_2)$.

Proof:

The direction co-efficients of the curve at (u, v) and v = Constant are respectively $(u', v')$ and $\left(\frac{1}{\sqrt{E}}, 0\right)$. If $\theta$ is the angle between the two directions $\left(\frac{1}{\sqrt{E}}, 0\right)$ and $(u', v')$, from the formulae

$$\cos\theta = Ell' + F(lm' + ml') + Gmm', \quad \sin\theta = H(lm' - l'm)$$

$$\cos\theta = \frac{1}{\sqrt{E}}[Eu' + Fv'], \quad \sin\theta = \frac{Hv'}{\sqrt{E}} \longrightarrow (1)$$

$$T = \frac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)$$

$$\frac{\partial T}{\partial u'} = Eu' + Fv' \text{ and } \frac{\partial T}{\partial u} = \frac{1}{2}[E_1 u'^2 + 2F_1 u'v' + G_1 v'^2] \longrightarrow (2)$$

(2) in (1) $\Rightarrow$ $\cos\theta = \frac{1}{\sqrt{E}} \frac{\partial T}{\partial u'} \longrightarrow (3)$

Differentiating (3) with respect to s,

$$-\sin\theta\,\theta' = \frac{1}{\sqrt{E}} \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{1}{2E^{3/2}} \frac{\partial T}{\partial u'} \frac{dE}{ds} \longrightarrow (4)$$

$$\frac{dE}{ds} = \frac{\partial E}{\partial u}\frac{du}{ds} + \frac{\partial E}{\partial v}\frac{dv}{ds} = E_1 u' + E_2 v' \longrightarrow (5)$$

(5) in (4) $\Longrightarrow$

$$-\sqrt{E}\sin\theta\,\theta' = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{1}{2E}(E_1 u' + E_2 v')\frac{\partial T}{\partial u'} \longrightarrow (6)$$

Substitute $\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right)$ from $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u}$

$$-\sqrt{E}\sin\theta\,\theta' = U + \frac{\partial T}{\partial u} - \frac{1}{2E}(E_1 u' + E_2 v')\frac{\partial T}{\partial u'}$$

using the value of $\sin\theta$ in (1),

$$-Hv'\theta' = U + \frac{\partial T}{\partial u} - \frac{1}{2E}(E_1 u' + E_2 v')\frac{\partial T}{\partial u'} \longrightarrow (7)$$

using ② in ⑦

$$-HV'\theta' = U + \frac{1}{2}(E_1u'^2 + 2F_1u'v' + G_1v'^2) - \frac{1}{2E}(E_1u' + E_2v')(Eu' + Fv')$$

$$= U + \frac{1}{2E}\left(u'v'(2EF_1 - E_1F - EE_2) + v'^2(EG_1 - FE_2)\right) \longrightarrow ⑧$$

Taking $P = \dfrac{2EF_1 - E_1F - EE_2}{2HE}$, $Q = \dfrac{1}{2HE}(EG_1 - FE_2)$

From ⑧, $-\theta' = \dfrac{U}{V'H} + u'P + v'Q \longrightarrow ⑨$

Since $k_g = -\dfrac{U}{V'H'}$ ⑨ becomes $-\theta' = -k_g + u'P + v'Q$.

$\therefore k_g = \theta' + Pu' + Qv'$

Theorem:

Let P and Q be two neighbouring points of a curve C on a surface with the arcual length $\delta s$. Let $\bar{C}$ be the geodesic arc PQ of length $\delta \bar{s}$. If $\delta\theta$ is the angle between C and $\bar{C}$ at P and $\delta\phi$ is the between $\bar{C}$ and C at Q, then the geodesic curvature at P is $k_g = \lim\limits_{\delta s \to 0} \dfrac{\delta\theta + \delta\phi}{\delta s}$.

proof:

Let the unit tangent vectors to the curve C at Q and P be $(u', v')$ and $(u_0', v_0')$. For the geodesic arc $\bar{C}$ let the unit tangent vectors at Q and P be $(\bar{u}', \bar{v}')$ and $(\bar{u}_0', \bar{v}_0')$. Using the formula $\sin\theta = H(lm' - l'm)$ for the angle between the two curves we have

$$\sin\delta\theta = H(u_0, v_0)(u_0'\bar{v}_0' - v_0'\bar{u}_0') \longrightarrow ①$$

$$\sin\delta\phi = H(u, v)(v'\bar{u}' - u'\bar{v}') \longrightarrow ②$$

To find Taylor's approximation of various terms in ③ & ④ and by neglecting the terms of $\delta s^2$ and higher order,

$$u' = u_0' + u_0''\delta s, \quad v' = v_0' + v_0''\delta s \longrightarrow ③$$

$$\overline{u}' = \overline{u}_0' + \overline{u}_0''\delta\overline{s}, \quad \overline{v}' = \overline{v}_0' + \overline{v}_0''\delta\overline{s} \longrightarrow ④$$

Since $(\overline{u}, \overline{v})$ belong to the geodesic $\overline{C}$, they satisfy the geodesic differential equation of second order.

Hence $\overline{u}_0'' = f(u_0, v_0, \overline{u}_0', \overline{v}_0'), \; \overline{v}_0'' = g(u_0, v_0, \overline{u}_0', \overline{v}_0') \longrightarrow ⑤$.

using ⑤ in ④

$$\overline{u}' = \overline{u}_0' + f(u_0, v_0, \overline{u}_0', \overline{v}_0')\delta\overline{s}$$

$$\overline{v}' = \overline{v}_0' + g(u_0, v_0, \overline{u}_0', \overline{v}_0')\delta\overline{s}$$

If $H(u_0, v_0) = H_0$, then $H(u, v) = H_0 + \delta H$ where $\delta H = O(\delta s)$ as $\delta s \longrightarrow 0$

Further $\delta\overline{s} = \delta s + O(\delta s^2)$ as $\delta s \longrightarrow 0$ and $\overline{u}_0' = u_0'$ and $\overline{v}_0' = v_0'$.

Let us find the approximate value of $\sin\delta\phi$ in terms of $(\overline{u}', \overline{v}', \overline{u}_0', \overline{v}_0')$ as follows,

$$(H_0 + \delta H)\{(\overline{u}_0' + f(u_0, v_0, \overline{u}_0', \overline{v}_0')\delta\overline{s}\}(v_0' + v_0''\delta s)$$
$$- \{\overline{v}_0' + g(u_0, v_0, \overline{u}_0', \overline{v}_0')\delta\overline{s}\}(u_0' + u_0''\delta s)\}.$$
$$= H_0\{(\overline{u}_0' v_0' - u_0'\overline{v}_0') + (\overline{u}_0' v_0'' - u_0''\overline{v}_0')\delta s +$$
$$v_0' f(u_0, v_0, \overline{u}_0' \overline{v}_0') - u_0' g(u_0, v_0, \overline{u}_0', \overline{v}_0')\}\delta s \longrightarrow ⑥$$

using ① in ⑥,

$$\sin\delta\theta + \sin\delta\phi = H_0\delta s\{\overline{u}_0'(v_0'' - g(u_0, v_0, \overline{u}_0', \overline{v}_0'))$$
$$- \overline{v}_0'(u_0'' - f(u_0, v_0, \overline{u}_0, \overline{v}_0'))\} \longrightarrow ⑦$$

as $\delta H \longrightarrow 0$

$\delta\bar{s} = \delta s$ as it is of order $O(\delta s^2)$.

We have approximately, $\sin\theta = \delta\theta$, $\sin\delta\phi = \delta\phi$ $\longrightarrow$ (8)

From the Theorem of the curvature vector,

$$\lambda = u_0'' - f(u_0, v_0, u_0', v_0'),\ \mu = v_0'' - g(u_0, v_0, u_0', v_0') \longrightarrow (9).$$

using (8) and (9) in (7),

$$\frac{\delta\theta + \delta\phi}{\delta s} = H_0(u_0'\mu - v_0'\lambda) + O(\delta s) \longrightarrow (10)$$

(10) is true at any point on the surface. Hence taking the limit as $\delta s \to 0$, we get

$$\lim_{\delta s \to 0} \frac{\delta\theta + \delta\phi}{\delta s} = H(u'\mu - v'\lambda) = k_g.$$

Theorem

If $E$, $F$ and $G$ are the fundamental co-efficients of a surface, then

$$K = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2 - EG_1}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right)$$

Proof:

From the definition of $K$,

$$K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$$ where $P$ and $Q$ are given by the Liouville's formula $k_g = \theta' + Pu' + Qv'$

Substituting for $P$ and $Q$ in $K$,

$$K = \frac{-1}{H}\frac{\partial}{\partial u}\left(\frac{EG_1 - FE_2}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right) \text{ or}$$

$$K = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2 - EG_1}{2HE}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1 - FE_1 - EE_2}{2HE}\right)$$

When the parametric curves are orthogonal, $F = 0$

Hence $K$ assumes the form,

$$K = -\frac{1}{2H}\left\{\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right\}.$$

If a mapping of a surface $S$ onto a surface $S^*$ is both geodesic and conformal, then it is an isometry or a similarly mapping.

Proof:

If we consider a geodesic on a surface, it may be either a parametric curve $v$ = constant or a curve cutting the parametric curve $v$ = constant at an angle $\theta$.

Let us consider a system of geodesic coordinate on $S$ so that the metric of $S$ becomes.

$$ds^2 = du^2 + G(u,v)dv^2 \longrightarrow (1).$$

Since the mapping $S \rightarrow S^*$ is conformal the metric $ds^{*2}$ on $S^*$ is proportional to (1). Taking the constant of proportionality to be $\lambda(u,v)$, $ds^{*2} = \lambda(u,v)(du^2 + G dv^2) \longrightarrow (2)$.

The parametric curves $v$ = constant are geodesic with respect to the metric (1) on $S$. Since the mapping is geodesic, the image of geodesic $v$ = constant must be a geodesic on $S^*$ where the same parameter is used for both the surfaces.

The conditions for the images curve $v$ = constant to be a geodesic is $E^*E_2^* + F^*E_1^* - 2E^*F_1^* = 0 \longrightarrow (3)$.

Since $F^* = 0$ and $E^* = \lambda(u,v)$ we get from (3),

$\lambda(u,v)\dfrac{\partial\lambda}{\partial v} = 0$. Since $\lambda \neq 0$, $\dfrac{\partial\lambda}{\partial v} = 0$ so that on $\lambda$

is a function of $u$ only. Hence $\lambda$ is independent of $v$.

Now consider any geodesic on $S$ making an angle $\theta$ with the curve $v$ = constant.
By Liouville's formula.

$$k_g = \frac{d\theta}{ds} + Pu' + Qv' \text{ where } P \text{ and } Q \text{ are}$$

$$P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2),\ Q = \frac{1}{2HE}(EG_1 - FE_2).$$

Using the metric ① and noting $k_g = 0$ for a geodesic, the Liouville's formula becomes.

$$d\theta + \frac{G_1}{2H} dv = 0 \longrightarrow ④.$$

Since the mapping is conformal and geodesic the equation corresponding to ④ on $S^*$ is,

$$d\theta + \frac{G_1^*}{2H^*} dv = 0 \longrightarrow ⑤.$$

From ④ and ⑤ $\dfrac{G_1}{H} = \dfrac{G_1^*}{H^*} \longrightarrow ⑥.$

For the metrices ① + ② we have $H = \sqrt{G}$ and

$H^* = \lambda(u,v)\sqrt{G} \longleftrightarrow ⑦$

⑦ in ⑥. $\dfrac{G_1}{\sqrt{G}} = \dfrac{(\lambda G)_1}{\lambda\sqrt{G}} = \dfrac{\lambda G_1 + G\lambda_1}{\lambda\sqrt{G}} = \dfrac{G_1}{\sqrt{G}} + \sqrt{G}\cdot\dfrac{\lambda_1}{\lambda}$

which gives $\sqrt{G}\dfrac{\lambda_1}{\lambda} = 0.$

Since $G \neq 0$ we have $\lambda_1 = 0$ which shows that $\lambda$ is independent of $u$. Thus $\lambda$ is independent of $u$ and $v$ so that it is a constant. Let it be $c$. Thus $ds^2 = c\, ds^{*2}$ which shows that the mapping is a similarly mapping. If $c = 1$, the mapping becomes an isometry.

I M.Sc MATHEMATICS

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UNIT V

4.2 Definition:

The quadratic form $Ldu^2 + 2Mdudv + Ndv^2$ is called second fundamental form of the surface and L, M, N which are functions of u, v are called second fundamental coefficients.

Example 1:

Find L, M, N for the sphere

$r = (a\cos u\cos v, a\cos u\sin v, a\sin u)$ where u is the latitude and v is the longitude.

Soln:

$$r = (a\cos u\cos v, a\cos u\sin v, a\sin u).$$

$$r_1 = (-a\sin u\cos v, -a\sin u\sin v, a\cos u), E = a^2$$

$$r_2 = (-a\cos u\sin v, a\cos u\cos v, 0), G = a^2\cos^2 u.$$

$F = r_1 \cdot r_2 = 0$, $H = \sqrt{EG - F^2} = a^2\cos u$.

$N = \frac{r_1 \times r_2}{H} = (-\cos u\cos v, -\cos u\sin v, -\sin u)$, shows that normal is directed inside the sphere.

$HN = r_1 \times r_2 = (-a^2\cos^2 u\cos v, -a^2\cos^2 u\sin v, -a^2\sin u\cos u)$

$N_1 = (\sin u\cos v, \sin u\sin v, -\cos u)$

$N_2 = (\cos u\sin v, -\cos u\cos v, 0)$.

$L = -N_1 \cdot r_1 = a$, $M = 0$ and $N = -N_2 \cdot r_2 = a\cos^2 u$.

Theorem:

The conditions for an elliptic, parabolic or hyperbolic point are independent of the particular parametric representation.

Proof:

We shall find $LN-M^2$ in the new coordinate system for the parametric transformation $u=\phi(u',v')$, $v=\psi(u',v')$. To find $LN-M^2$

$$L=-r_1\cdot N_1,\ M=-r_1\cdot N_2=-r_2\cdot N_1,\ N=-r_2\cdot N_2$$

$$\text{Now } r_1'=\frac{\partial r}{\partial u'}=\frac{\partial r}{\partial u}\frac{\partial u}{\partial u'}+\frac{\partial r}{\partial v}\frac{\partial v}{\partial u'}=\frac{\partial r}{\partial u}\phi_1+\frac{\partial r}{\partial v}\psi_1$$

$$r_1'=r_1\phi_1+r_2\psi_1$$

In a similar manner, $r_2'=r_1\phi_2+r_2\psi_2$ and

$$N_1'=N_1\phi_1+N_2\psi_1,\ N_2'=N_1\phi_2+N_2\psi_2$$

As we have already noted $LN-M^2=(N_1\times N_2)\cdot(r_1\times r_2)$

To find $L'N'-M'^2$ after parametric transformation

$$N_1'\times N_2'=(N_1\phi_1+N_2\psi_1)\times(N_1\phi_2+N_2\psi_2)$$

$$N_1\times N_1=N_2\times N_2=0.$$

$$N_1'\times N_2'=(\psi_2\phi_1-\phi_2\psi_1)(N_1\times N_2)$$

$$r_1'\times r_2'=(r_1\phi_1+r_2\psi_1)\times(r_1\phi_2+r_2\psi_2)$$

$$r_1\times r_1=r_2\times r_2=0$$

$$r_1'\times r_2'=(\phi_1\psi_2-\psi_1\phi_2)(r_1\times r_2)$$

$$\text{Hence } L'N'-M'^2=(N_1'\times N_2')\cdot(r_1'\times r_2')$$

$$=(\phi_1\psi_2-\psi_1\phi_2)^2(N_1\times N_2)\cdot(r_1\times r_2)$$

$$=\begin{vmatrix}\phi_1 & \phi_2\\ \psi_1 & \psi_2\end{vmatrix}^2(LN-M^2)=J^2(LN-M^2).$$

Since $J^2$ is always positive after parametric transformation, $L'N'-M'^2$ is positive, zero or negative according as $LN-M^2$ is positive, zero or negative at P. Thus the conditions for the point to be elliptic, parabolic or hyperbolic are independent of a particular coordinate system.

Definition:

The directions at P in which normal curvature has maximum or minimum values are called principal directions at P. The principal curvatures at P are the normal curvatures at P along the principal directions.

Theorem:

The principal curvatures are given by the roots of the equation $k^2(EG-F^2)-k(EN+GL-2FM)+LN-M^2=0$.

Proof:

Let the direction coefficients in direction at $P(u,v)$ on a surface be $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$. Then we have

$$k = \frac{L\left(\frac{du}{ds}\right)^2+2M\frac{du}{ds}\frac{dv}{ds}+N\left(\frac{dv}{ds}\right)^2}{E\left(\frac{du}{ds}\right)^2+2F\frac{du}{ds}\frac{dv}{ds}+G\left(\frac{dv}{ds}\right)^2}$$

using the notation $l = \frac{du}{ds}$, $m = \frac{dv}{ds}$.

$$k = Ll^2 + 2Mlm + Nm^2 \longrightarrow (1)$$

$$El^2 + 2Flm + Gm^2 = 1 \longrightarrow (2).$$

Since the fundamental Coefficients are fixed at $P$, $k$ varies as $l, m$ vary subject to the constraining condition (2). Hence to determine the maximum and minimum values of normal curvature, the method of Lagrange's multipliers is to be used.

Let $k = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2 - 1)$

when $k$ is stationary, we should have,

$$\frac{\partial k}{\partial l} = 0 \text{ and } \frac{\partial k}{\partial m} = 0.$$

So $\frac{1}{2}\frac{\partial k}{\partial l} = Ll + Mm - \lambda El - \lambda Fm = 0 \longrightarrow (3)$

$$\frac{1}{2}\frac{\partial k}{\partial m} = Ml + Nm - \lambda Fl - \lambda Gm = 0 \longrightarrow (4).$$

To find the extreme values of $k$, let us find the values of $\lambda$ from the above equations.

$(3)l + (4)m \Rightarrow (Ll^2 + 2Mlm + Nm^2) - \lambda(El^2 + 2Flm + Gm^2) = 0$

using (1) & (2) in the above equation, $k - \lambda = 0$ or $\lambda = k$.

using $\lambda = k$ in (4).

$$(L - kE)l + (M - kF)m = 0 \longrightarrow (5)$$

$$(M - kF)l + (N - kG)m = 0 \longrightarrow (6)$$

Eliminating $l$ and $m$ between the above equations, using determinants

$$(L - kE)(N - kG) - (M - kF)^2 = 0.$$

Rewriting the above equation as quadratic in $k$,

$$k^2(EG-F^2)-k(EN+GL-2FM)+(LN-M^2)=0$$

The roots of the above equation gives the principal curvatures at $P$, and let them be $k_a$ and $k_b$. One value must be a maximum and the other value must be a minimum.

**Definition:**

If $k_a$ and $k_b$ are the principal curvatures at a point $P$ on the surface, then the mean curvature denoted by $\mu$ is defined as

$$\mu=\frac{1}{2}(k_a+k_b)=\frac{EN+GL-2FM}{2(EG-F^2)}.$$

**Definition:**

If $k_a$ and $k_b$ are principal curvatures, the Gaussian curvature denoted by $k$ is defined as $k=k_ak_b=\dfrac{LN-M^2}{EG-F^2}$.

**Definition:**

A point on a surface is called an umbillic ~~if~~ at that point $\dfrac{L}{E}=\dfrac{M}{F}=\dfrac{N}{G}$ is true.

**Example:**

Show that all points on a sphere are umbilics.

**Proof:**

The representation of a point on a sphere with colatitude $u$ and longitude $v$ as parameters is $r=(a\sin u\cos v, a\sin u\sin v, a\cos u)$

For this parametric representation of a point on a sphere we shall find first and second fundamental forms at a point and show that $\frac{L}{E}=\frac{M}{F}=\frac{N}{G}$ so that every point on a sphere is an umbilic.

$r=(a\sin u\cos v, a\sin u\sin v, a\cos u)$

$r_1=(a\cos u\cos v, a\cos u\sin v, -a\sin u)$

$r_2=(-a\sin u\sin v, -a\sin u\cos v, 0)$

$E=r_1\cdot r_1=a^2,\ F=r_1\cdot r_2=0,\ G=r_2\cdot r_2=a^2\sin^2 u.$

$r_1\times r_2=i(a^2\sin^2 u\cos v)+j(a^2\sin^2 u\sin v)+k(\sin u\cos u)$

$H^2=a^4\sin^2 u$. we shall use the scalar triple product formula to find $L, M, N$

$$LH=[r_{11}, r_1, r_2]=H=\begin{vmatrix} -a\sin u\cos v & -a\sin u\sin v & -a\cos u \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix}$$

Hence $L=\dfrac{-a^3\sin u}{a^2\sin u}=-a.$

In a similar manner, $M=\dfrac{[r_{12}, r_1, r_2]}{H}=0.$

$$N=\frac{[r_{22}, r_1, r_2]}{H}=\frac{-a^3\sin^3 u}{a^2\sin u}=-a\sin^2 u.$$

Now $\frac{L}{E}=\frac{M}{F}=\frac{N}{G}$ gives $\frac{-a}{a^2}=\frac{0}{0}=\frac{-a\sin^2 u}{a^2\sin^2 u}=-\frac{1}{a}.$

Thus all points on a sphere are umbilics and the principal directions are indeterminate at any point on the sphere.

Example:

Show that the meridians and parallels of a surface of revolution are its lines of curvature.

proof:

The position vector of any point on the surface of revolution is $r = (u\cos v, u\sin v, f(u)) \rightarrow (1)$.
We know that the meridians $v$ = constant and the parallel $u$ = constant are parametric curves.
The parametric curves are lines of curvature if and only if $F = 0$ and $M = 0$. So it is enough to prove that $F = 0, M = 0$. From (1) we have

$$r_1 = (\cos v, \sin v, f_1) \text{ and } r_2 = (-u\sin v, \cos v, 0).$$

Hence $r_1 \cdot r_2 = F = 0$. Further $r_1 \times r_2 = (-f_1 u\cos v$

$$r_1 \times r_2 = (-f_1 u\cos v, -f_1 u\sin v, u)$$

$r_{12} = (-\sin v, \cos v, 0)$ since $f$ is a function of $u$ only. Now $M = \frac{r_{12} \cdot (r_1 \times r_2)}{H}$

$$= \frac{1}{H}\left[u f_1 \sin v\cos v - u f_1 \sin v\cos v\right]$$

Since $H \neq 0$, $M = 0$.

Theorem:

The edge of regression of the polar developable of a space curve is the locus of centres of spherical curvature.

Proof:

Let $r = r(s)$ be the given space curve and

R be any point on the normal plane. Then the equation of normal plane is, $(R-r)\cdot t = 0 \longrightarrow (1)$.

Since $r$ and $t$ are functions of a single parameters, (1) is a single parameter family of planes whose envelope is the polar developable. To find the edge of regression, we shall find the characteristic point from the following equations.

Differentiating (1) with respect to $s$,

$(R-r)\cdot t' - r'\cdot t = 0$

Since $t' = kn$, $r' = t$ and $t\cdot t = 1$ we have

$(R-r)\cdot kn - 1 = 0 \Rightarrow (R-r)\cdot n = \frac{1}{k} = \rho \longrightarrow (2)$

Differentiating (2) with respect to $s$,

$(R-r)\cdot n' - r'\cdot n = \rho'$

$n' = \tau b - kt$ and $r' = t$ we have

$(R-r)\cdot(\tau b - kt) - t\cdot n = \rho'$

using (1) and $t\cdot n = 0$ we obtain $(R-r)\cdot b = \frac{\rho'}{\tau} = \rho'\sigma \longrightarrow (3)$

The point of intersection of (1), (2) and (3) is the characteristic point and its locus is the edge of regression of the polar developable.

Since $(R-r)$ is orthogonal to $t$, $(R-r)$ lies in the plane of $n$ and $b$ so that we can take,

$(R-r) = \lambda n + \mu b$ or $R = r + \lambda n + \mu b \longrightarrow (4)$

where we determine the scalars $\lambda$ and $\mu$.

Taking dot product with $n$ on both sides of ④ we get $(R-r)\cdot n = \lambda \longrightarrow$ ⑤.

Comparing ② + ⑤ we obtain $\lambda = \rho$

Taking dot product with $b$ on both sides of ④ $(R-r)\cdot b = \lambda n\cdot b + \mu b\cdot b$

Since $n\cdot b = 0$, $b\cdot b = 1$ using ③ $\mu = \sigma\rho'$.

Substituting the values of $\lambda$ and $\mu$ in ④, we get, $R = r + \rho n + \sigma\rho' b \longrightarrow$ ⑥.

which gives the position vector of the characteristic point but we know that $R$ gives the position of the centre of spherical curvature. Thus the characteristic point of the polar developable coincides with the centre of spherical curvature. Hence the edge of regression of the polar developable is the locus of the centres of spherical curvature.

Example:

Find the third fundamental form of the right helicoid, $r = (u\cos v, u\sin v, \alpha v)$ and verify the relation $K\,I - 2\mu\,II + III = 0$.

Soln: $r = (u\cos v, u\sin v, \alpha v)$

$r_1 = (\cos v, \sin v, 0)$; $r_2 = (-u\sin v, u\cos v, \alpha)$.

$E = r_1\cdot r_1 = 1$, $F = r_1\cdot r_2 = 0$, $G = r_2\cdot r_2 = u^2+\alpha^2$.

Hence $I = du^2 + (u^2+\alpha^2)dv^2 \longrightarrow$ ①.

$r_1 \times r_2 = (a\sin v, -a\cos v, u)$, $H = \sqrt{u^2+\alpha^2}$.

$$N = \left( \frac{\alpha}{\sqrt{u^2+\alpha^2}} \sin v, \frac{-\alpha \cos v}{\sqrt{u^2+\alpha^2}}, \frac{u}{\sqrt{u^2+\alpha^2}} \right)$$

$$N_1 = \left( \frac{-\alpha u \sin v}{(u^2+\alpha^2)^{3/2}}, \frac{\alpha u \cos v}{(u^2+\alpha^2)^{3/2}}, \frac{\alpha^2}{(u^2+\alpha^2)^{3/2}} \right)$$

$$N_2 = \left( \frac{\alpha \cos v}{\sqrt{u^2+\alpha^2}}, \frac{\alpha \sin v}{\sqrt{u^2+\alpha^2}}, 0 \right).$$

Hence $L = -N_1 \cdot r_1 = 0$; $M = -N_2 \cdot r_1 = \dfrac{-\alpha}{\sqrt{u^2+\alpha^2}}$

$N = -N_2 \cdot r_2 = 0 \longrightarrow (4)$

using (4), $\mathrm{II} = \dfrac{-2\alpha \, du \, dv}{\sqrt{\alpha^2+u^2}}$

Gaussian Curvature $k = \dfrac{LN - M^2}{EG - F^2} = \dfrac{-\alpha^2}{(u^2+\alpha^2)^2} \longrightarrow (5)$

Mean Curvature $\mu = \dfrac{EN + GL - 2FM}{2(EG - F^2)} = 0 \longrightarrow (6)$

using $N$ we find $dN \cdot dN$

$$dN = \frac{\partial N}{\partial u} du + \frac{\partial N}{\partial v} \cdot dv$$

$$\mathrm{III} = dN \cdot dN = \left(\frac{\partial N}{\partial u}\right)^2 du^2 + 2 \frac{\partial N}{\partial u} \cdot \frac{\partial N}{\partial v} + \left(\frac{\partial N}{\partial v}\right)^2 dv^2 \longrightarrow (7).$$

Let us find $\left(\frac{\partial N}{\partial u}\right)^2$, $\frac{\partial N}{\partial u} \cdot \frac{\partial N}{\partial v}$ and $\left(\frac{\partial N}{\partial v}\right)^2$ as follows.

Taking dot product of (2) with itself,

$$\left(\frac{\partial N}{\partial u}\right)^2 = \frac{\alpha^2 u^2}{(u^2+\alpha^2)} + \frac{\alpha^4}{(u^2+\alpha^2)^2} = \frac{\alpha^2}{(u^2+\alpha^2)^2} \longrightarrow (8).$$

In a similar manner, $\left(\frac{\partial N}{\partial v}\right)^2 = \frac{\alpha^2}{(u^2+\alpha^2)}$ and $\frac{\partial N}{\partial u} \cdot \frac{\partial N}{\partial v} = 0$.

(8) & (9) in (7) $\Rightarrow \mathrm{III} = \dfrac{\alpha^2}{(u^2+\alpha^2)^2} \left[ du^2 + (\alpha^2 + u^2) dv^2 \right] \longrightarrow (10)$

using (1), (4), (5), (6) in (10)

$$k\mathrm{I} - 2\mu \mathrm{II} + \mathrm{III} = -\frac{\alpha^2}{(u^2+\alpha^2)} \left( du^2 + (u^2+\alpha^2) dv^2 \right)$$

$$+ \frac{\alpha^2}{(u^2+\alpha^2)^2} \left( du^2 + (u^2+\alpha^2) dv^2 \right) = 0.$$

$\therefore$ The three relations among the fundamental forms are verified.