# Kunthavai Naachiyaar Govt. Arts College (W) Autonomous, Thanjavur

## M.Stat

18KP1S02

CORE COURSE - II

REAL ANALYSIS AND MATRIX THEORY

Hours:6
Credit:5

Unit – I

Functions – Real valued function, Equivalence, countability, least upper bounds Sequence of real numbers – Definition, limit of a sequence, convergent sequence, divergent sequence, bounded sequence, monotone sequence, limit superior and limit inferior, Series of real numbers – convergence and divergence, series with non-negative terms, alternating series, conditional convergence and absolute convergence.

Unit – II

Calculus – Sets of Measure zero, Definition of the Riemann integral, existence of the Riemann integral, Derivatives, Rolle's Theorem, the law of the mean, Fundamental theorems of calculus, improper integrals.

Unit – III

Rank of a matrix – elementary transformation of a matrix, Equivalent Matrices, Elementary matrices, Echelon matrix – Hermite Canonical form, sylvester's law, Frobenius inequality, certain results on the rank of an Idempotent matrix.

Unit – IV

Eigen values and Eigen Vectors – properties, Cayley – Hamilton theorem, application of Cayley-Hamilton theorem – simple problems.

Unit – V

Generalized inverse of a matrix: Definition, different classes of generalized inverse, Properties of G- inverse – classes properties – properties of Moore and Penrose – application of generalized inverse in the solution of system of linear equations. Quadratic forms – Definition, classification of the quadratic form, positive semi-definite quadratic form and Canonical reduction.

Text Books:


1. Goldberg. R. (1963). Method of Real Analysis. Oxford & IBH publishers. New Delhi (Unit – I: Chapter – 1: Page no.3 – 17, 21-75, Unit – II: Chapter-7 Page no.(156 – 194).Biswas.S(1996) , A Text book of Matrix Algebra, New Age International Publishers, New Delhi. (Unit – III: Chapter – 5, Unit – IV: Chapter – 7: Page no.185 – 198, 208 – 209, 213 – 227. Unit – V: Chapter – 8; page no.228 – 245 and Chapter – 9: page no. 267 – 268, 317 – 323).

# REAL ANALYSIS AND MATRIX THEORY

UNIT- 1

## Functions :

" If to each $x$ (in a set $S$) there corresponds one and only one value of $y$, then we say that $y$ is a function of $x$".

**Definition :**

If $A, B$ are sets, then the Cartesian Product of A and B (denoted $A \times B$) is the set of all ordered pairs $\langle a, b \rangle$ where $a \in A$ and $b \in B$.

**Definition :** [THE COMPOSITION OF FUNCTIONS]

If $f : A \to B$ and $g : B \to C$, then we define the function $g \circ f$ by,

$$g \circ f(x) = g[f(x)] \qquad (x \in A)$$

That is the image of $x$ under $g \circ f$ is defined to the image of $f(x)$ under $g$. Then function $g \circ f$ is called the composition of function of $f$ with $g$.

## Real valued Functions :

The range of a given function $f$ is contained in the set of all real numbers. (We hence forth denote the set of all real numbers by $R$). If $f : A \to R$, we call of a real valued function. If $x \in A$, then $f(x)$ (heretofore called the image of $x$ under $f$) is also called the value of $f$ at $x$.

Definition :

If $f : A \to R$ and $g : A \to R$, We define $f+g$ as the function whose value at $x \in A$ is equal to $f(x)+g(x)$. That is

$$(f+g)(x) = f(x) + g(x) \qquad (x \in A)$$

EQUIVALENCE AND COUNTABILITY :

Definition :

If $f : A \Rightarrow B$ and $f$ is 1-1 then $f$ is called a 1-1 correspondence (between A and B). If there exists a 1-1 correspondence between the sets A and B, then A and B are called equivalence.

1. Every Set A is equivalent to itself.
2. If A and B are equivalent, then B and A are equivalent.
3. If A and B are equivalent B and C are equivalent then A and C are equivalent.

Definition :

The Set A is said to be countable (or denumerable) if A is equivalent to the Set I of Positive integers. An uncountable Set is an infinite Set Which is not countable.

The element of A are then the integers of images $f(1), f(2), \ldots,$ of the Positive integers,

$$A = \{ f(1), f(2), \ldots \}$$

## Least Upper Bounds:

**Definition:**

If $A \subset R$ is bounded above, then N is called an upper bound for A if $x \leq N$ for all $x \in A$. If $A \subset R$ is bounded below, the M is called a lower bound for A if $M \leq x$ for every $x \in A$.

**Definition:**

Let the Subset A or R be bounded above. The number L is called the upper bound for A if (1) L is an upper bound for A, and (2) no number smaller than L is an upper bound for A.


## Sequence of Real Numbers:

A Sequence of real Numbers is a function from the Set N of natural numbers to the Set R of the real numbers. If $f: N \to R$ is a Sequence, and if $a_n = f(n)$ for $n \in N$, then we write the Sequence f as $(a_n)$ (or) $(a_1, a_2 \cdots)$. A sequence of real numbers is also called a real sequences.

Convergence Sequence and Divergent :

A Sequence $(a_n)$ in $\mathbb{R}$ is Said to Converge to a real Number $a$ if for every $\epsilon > 0$, there exists Positive integer $N$ (ingeneral depending on $\epsilon$) Such that

$$|a_n - a| < \epsilon \quad \forall\, n \geq N,$$

and in that case, the number $a$ is called a limit of a Sequence $(a_n)$ and $(a_n)$ is called a Convergent sequences.

A Sequence which does not converge is called a divergent Sequence.

Bounded Sequence :

A Sequence $(a_n)$ is Said to be bounded above if there exists a real number $M$ Such that $a_n \leq M$ for all $n \in N$; and the Sequence $(a_n)$ is Said to be bounded below if there exists a real number $M'$ Such that $a_n \geq M'$ for all the $\in N$.
A Sequence Which is bound above and bounded below is Said to be a bounded Sequence.

Monotonic Sequences :

We can infer the convergence or divergence of a Sequence in Certain cases by observing the way the terms of the sequence Varies.

Definition :

Consider a Sequence $(a_n)$

(i) $(a_n)$ is Said to be monotonically increasing, if $a_n \leq a_{n+1}$ for all $n \in N$.

(ii) $(a_n)$ is Said to be monotonically decreasing, if $a_n \geq a_{n+1}$ for all $n \in N$.

If Strict inequality occur in (i) and (ii) then we say that the Sequences is strictly increasing and decreasing.

Definition :

Let $S$ be a subset of $R$, Then

(i) $S$ is Said to be bounded above if there exists $b \in R$ such that $x \leq b$ for all $x \in S$, and in that case $b$ is called a upper bound of $S$.

(ii) $S$ is Said to be bounded below if there exists $a \in R$ Such that $a \leq x$ for all $x \in S$, and in that case of $a$ is called a lower bound of $S$.

Limit Superior :

If $\{S_n\}_{n=1}^{\infty}$ is a Sequence of real numbers that is not bounded above, We Write the,

$$\lim \sup_{n \to \infty} S_n = \infty$$

$$\lim \sup_{n \to \infty} n = \infty.$$

Limit inferior :

If $\{S_n\}_{n=1}^{\infty}$ is a Sequence of real numbers that is not bounded below, we write $\lim\inf_{n\to\infty} S_n = -\infty$

Thus $\lim\inf_{n\to\infty} (-1)^n = -1$, $\lim\inf_{n\to\infty} n = \infty$, $\lim\inf_{n\to\infty} (-n) = -\infty$. The sequence $1, -1, 1, -2, 1, -3, 1, -4, \ldots$, has $\lim\inf = -\infty$.

Real Number:

A Series of Number is an expression of the form $a_1 + a_2 + a_3 + \cdots$, or more compactly as $\sum_{n=1}^{\infty} a_n$; where $(a_n)$ is a Sequence of real numbers. The number is called the n-th term of the Series and the Sequence $S_n = \sum_{i=1}^{n} a_i$ is called the $n^{th}$ Partial Sum of the Series $\sum_{n=1}^{\infty} a_n$.

Convergence and divergence of Series :

A Series $\sum_{n=1}^{\infty} a_n$ is Said to Converge (to $S \in \mathbb{R}$) if the Sequence $\{S_n\}$ of Partial Sum of the Series converge (to $S \in \mathbb{R}$).

If $\sum_{n=1}^{\infty} a_n$ converges to $S$, then we write $\sum_{n=1}^{\infty} a_n = S$.

A Series which does not converge is called a divergent Series

Alternating Series :

A Series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ where $(u_n)$ is a Sequence of Positive terms is called an alternating Series.

Absolute convergence and conditional convergence :

A Series $\sum_{n=1}^{\infty} a_n$ is Said to be a converge abosultely, if $\sum_{n=1}^{\infty} |a_n|$ converges.

Every absolutely convergent Series convergences, Let $S_n$ and $\sigma_n$ be the n-th Partial Sums of the Series $\sum_{n=1}^{\infty} |a_n|$ respectively, Then for $n > m$, We have

$$|S_n - S_m| = \left| \sum_{j=m+1}^{n} a_n \right| \leq \sum_{j=m+1}^{n} |a_n| = |\sigma_n - \sigma_m| .$$

Since $\{\sigma_n\}$ Converges, it is a sequences. Hence from that $\{S_n\}$ is also a Cauchy Sequences.

Definition :

A Series $\sum_{n=1}^{\infty} a_n$ is Said to be Converge conditionally, if $\sum_{n=1}^{\infty} a_n$ converges but not absolutely to the Sequences.

Example :

(i) The Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is Conditionally convergent sequences.

(ii) The Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent Sequences.

(iii) The Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent.

For any $\alpha \in R$ the Series $\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2}$ is absoluttely convergent :

$$\left| \frac{\sin(n\alpha)}{n^2} \right| \leq \frac{1}{n^2} \quad \forall n \in N.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges,

By Comaparison test, Hence the value is

$\sum_{n=1}^{\infty} \left| \frac{\sin(n\alpha)}{n^2} \right|$ also Convergent to the

Sequences.

Series with non-negative terms:

The easiest series to deal with are those with non-negative terms. For these series all the theory on the convergence and divergence is embodied in the following theorem:

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative numbers with $S_n = a_1 + \cdots + a_n$ $(n \in I)$, then,

(a) $\sum_{n=1}^{\infty} a_n$ converges if the sequences $\{S_n\}_{n=1}^{\infty}$ is bounded;

(b) $\sum_{n=1}^{\infty} a_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ is not a bounded.

Definition:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

(a) If $\sum_{n=1}^{\infty} |a_n|$ converges we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Unit-2

## Sets of Measure Zero:

Definition :

The Subset E of $\mathbb{R}$ is Said to be of Measure Zero If for each $\varepsilon > 0$, there exists a finite or a Countable number of a open intervals $I_1, I_2$ Such that

$$E \subset \cup_n I_n \quad \text{and} \quad \Sigma |I_n| < \varepsilon .$$

Definition :

A Statement is Said to hold at almost every Point of $[a,b]$ (or almost every where in $[a,b]$) if the set. of Points of which the Statement does not hold is of Measure Zero.

## Definition of the Riemann Integral:

Definition:

Let $I$ be a bounded interval of real numbers and let $f$ be a bounded function defined on $I$. We define $M[f;I]$ and $m|f:I|$ by

$$M|f:I| = \sup_{x \in I} f(x)$$

$$M[f:I| = \inf_{x \in I} f(x)$$

Definition : 2

A Subdivision of the closed bounded interval $[a,b]$ we mean a finite subset $\{x_0, x_1, \dots x_n\}$ of $[a,b]$ Such that,

$$a = x_0 < x_1 < x_2 < \dots x_n = b$$

If $\sigma$ and $T$ are two Subdivisions of $[a,b]$ we say that $\gamma$ is a refinement of $\sigma$ if $\sigma \subset T$.

If $\sigma = \{x_0, x_1, \dots x_n\}$ is a Subdivision of $[a,b]$ then the closed intervals.

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2] \dots I_n = [x_{n-1}, x_n]$$

are called the components of $\sigma$.

Definition : 3

Let $f$ be a bounded function on $[a,b]$ and let $\sigma$ be a Subdivision of $[a,b]$ with components $I_1, I_2, \dots I_n$ We define $U[f;\sigma]$, called the Upper Sum for $f$ corresponding to $\sigma$, by

$$U|f;\sigma| = \sum_{k=1}^{n} M[f:I_k]\,|I_k|.$$

Here $|I_k|$ is the length of $I_k$ which is given by $I_k = x_k - x_{k-1}$.

$$L[f:\sigma] = \sum_{k=1}^{n} m[f:I_k]\,|I_k|.$$

It is denoted to the lower sum of the corresponding to $\sigma$.

Definition : 4

Let $f$ be a binomial function on $[a,b]$ we define the upper integral of $f$ over $[a,b]$ by,

$$\overline{\int_a^b} f(x)dx = g.l.b\ U[f;T]$$

We define the lower integral of $f$ over $[a,b]$ by

$$\underline{\int_a^b} f(x)(dx) = l.u.b\ L[f;T]$$

It is common to denote upper integrals and lower integrals of $f$;

$$\overline{\int_a^b} f \text{ and } \underline{\int_a^b} f.$$

Definition : 5

If $f$ is bounded on $[a,b]$ we say that $f$ is Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

In this case, we define the Riemann integrals,

$$\int_a^b f(x)dx = \int_a^b f = \underline{\int_a^b} f \leq \overline{\int_a^b} f$$

The class of the Riemman integral is denoted by $R[a,b]$.

## Existence of the Reimann Integral :

Let $f$ be a bounded function on $[a,b]$ then $f \in R$ if and only if $f$ is continuous at almost every Point Of $[a,b]$.

## Properties of Riemann Integral :

(i) If $f \in R\,[a,b]$ and $a < c < b$, then $f \in R\,[a,c]$, $f \in R[c,b]$ and $$\int_a^b f = \int_a^c f + \int_c^b f.$$

(ii) If $f \in R\,[a,b]$ and $\lambda$ is any real number than $\lambda f \in R\,[a,b]$ and $$\int_a^b \lambda f = \lambda \int_a^b f.$$

(iii) If $f, g \in R\,[a,b]$ then $f+g \in R\,[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

If $f \in R\,[a,b]$ and if $f(x) \geq 0$. almost where in $[a,b]$ then

$$\int_a^b f \geq 0 .$$

## Derivative :

Definition :

Let $f$ be defined on an interval $J$. If $c \in J$, We Say that $f$ has a derivative at $c$,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x-c}$$

(i) If $f$ has a derivative at $c$, then $f$ is continuous at $c$.

(ii) If $f$ and $g$ both have derivatives at $c$, then do $f+g$, $f-g$, $fg$, and

$$(f+g)'(c) = f'(c) + g'(c)$$

$$(f-g)'(c) = f'(c) - g'(c)$$

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

If $g(c) \neq 0$, then $f/g$ has derivative at $c$,

$$(f/g)'(c) = \frac{f'(c)\,g(c) - f(c)g'(c)}{g(c)^2}$$

(iii) If $g$ has derivative at $c$ and $f$ has derivative at $g(c)$ then $f \circ g$ has derivative at $c$ and,

$$(f \circ g)'c = f'(g(c)\, g'(c))$$

(iv) Let $f$ be one-to-one function and let $\phi$ be the inverse of $f$. If $f$ is continuous at $c$ and if $\phi$ has derivative at $d = f(c)$ with $\phi(d) \neq 0$, then $f'(c)$ exists and,

$$f'(c) = \frac{1}{\phi(d)}.$$

Rolle's Theorem :

(i) Let f be a continuous function on $[a,b]$. If the maximum value of f is attained at $c \in (a,b)$ and if $f'(c)$ exists then $f'(c) = 0$.

(ii) Let f be a continuous function on $[a,b]$. If the minimum value of f is attained at $c \in (a,b)$ and if $f'(c)$ exists, then $f'(c) = 0$.

(iii) Let f be continuous on $[a,b]$ with $f(a) = f(b) = 0$.

If $f'(x)$ exists for every $x \in (a,b)$ then there exists some point $c \in (a,b)$ such that $f'(c) = 0$.

(iv) If f has derivatives at every point of $[a,b]$ then $f'$ takes value between $f'(a)$ and $f'(b)$.

Mean Value Theorem :

Theorem 1 : If f is continuous on $[a,b]$ and differentiable on $(a,b)$ then there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Theorem 2 : If f is continuous on $[a,b]$ and differentiable on $(a,b)$ and $f'(x) > 0$ for all $x \in (a,b)$ then f strictly increasing on $[a,b]$

Theorem 3 : If g are continuous on $[a,b]$ $g(a) \neq g(b)$, if f, g are both differentiable on $(a,b)$ and if $f'(x)$ and $g'(x)$ are both zero for any $x \in (a,b)$ then three exists a points $c \in (a,b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g b - g(a)}.$$

The Mean Value Theorem:

Theorem 1: If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ then there exists a points $c \in (a,b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Theorem 2: If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$ and if $f'(x) > 0$ for all $x \in (a,b)$ then $f$ is strictly increasing on $[a,b]$.

Theorem 3: If $f, g$ are continuous on $[a,b]$, $g(a) \neq g(b)$ if $f, g$ are both differentiable on $(a,b)$ and if $f'(x)$ and $g'(x)$ are not both zero for any $x \in (a,b)$ then there exists a point $c \in (a,b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The Fundamental Theorem of Calculus:

First fundamental Theorem:

Theorem :

If $f$ is continuous on $[a,b]$ and if,

$$F(x) = \int_a^x f(t)\,dx.$$

for each $x \in (a,b)$ then $F'(x) = f(x)$ for all $x \in [a,b]$.

Theorem : 2

If $f'(x) = 0$ for every $x \in (a,b)$ then $f(x) = f(a)$ $\forall$ $x \in (a,b)$

Theorem : 3

If $f'(x) = g'(x)$ for all $x \in (a,b)$ then there exists a Constant $C$ Such that,

$$f(x) = g(x) + C \qquad \forall\ x \in (a,b).$$

The Second fundamental Theorem of Calculus :

If $f$ is a Continuous function on $[a,b]$ and if,

$\phi'(x) = f(x)$, $\forall$ $x \in (a,b)$

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Theorem : 2

Let $\phi$ be a function on $[a,b]$ Such that $\phi$ is continuous on $[a,b]$ let $A = \phi(a)$ and $B = \phi(b)$, Then if $f$ is continuous on $\phi([a,b])$ we have,

$$\int_A^B f(x)\,dx = \int_a^b f[\phi(u)]\,\phi'(u)\,du.$$

## Improper Integrals :

**Definition :**

Suppose $f \in R[a,s]$ for every $s > a$, we define

$$F(s) = \int_a^s f(x)\,dx$$

$\int_a^\infty f(x)\,dx$ is convergent to A,

if $\lim_{s \to \infty} F(s) = A$

If $\int_a^\infty f(x)\,dx$ does not converge, we say it diverges.

**Definition :**

If $f \in R[a,s]$ for every $s > a$ and if $\int_a^\infty |f(x)|\,dx$ converges, we say that, $\int_a^\infty f$ converges absolutely

If $\int_a^\infty f$ converges but $\int_a^\infty |f|$ divergres we say that converges conditionally.

**Definition :**

We define $\int_{-\infty}^a f(x)\,dx = \lim_{s \to \infty} \int_{-s}^a f(x)\,dx$ Provided the limits exists. If the limit exists we say $\int_{-\infty}^a f$ converges and if the limits does not exist we say the $\int_{-\infty}^a f$ diverges.

## Improper Integral (continued) :

Definition :

Suppose $f \in R[a+\epsilon, b]$ for every $\epsilon$ such that $0 < \epsilon < b-a$ but $f \notin R[a, b]$,

Define $f(\epsilon) = \int_{a+\epsilon}^{b} f(x)\,dx$.

We say the Improper integral,

$\int_{a}^{b} f(x)\,dx$ Converges to A, if

$\lim_{\epsilon \to 0} f(\epsilon) = A$.

If the limit does not exist we say $\int_{a}^{b} f$ diverges.

An improper integral of the form $\int_{a}^{b} f$ is called an improper integral of second kind.

Unit - 2

1) Rank of Matrix :

Let, $A = (a_{ij})_{m\times n}$ $(i = 1, 2, \ldots m, j = 1, 2, \ldots n)$
be a matrix of order $m \times n$.
Where $a_{ij} \in R$, $R$ is a real field (or complex field) ie,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The whole array can be imagined as a set of m row vectors or as a set of n column vectors, the number of linearly independent rows of columns is called the rank of the matrix.

2) Elementary Transformation of a Matrix :

$E_1 \rightarrow$ Replacing a row/column say $r^{th}$ row of A by a scalar multiple of the same, In notation. $R_1 \rightarrow CR_1$ or $C_1 \rightarrow CC_1$ where $R_r$ and $C_r$ stands for $r^{th}$ row and $r^{th}$ column respectively.

$E_2 \rightarrow$ Replacing the $r^{th}$ row/column plus $\lambda$ times the $s^{th}$ row/column, i.e., $R_r \rightarrow R_r + \lambda R_s$ (or) $C_r \rightarrow C_r + \lambda C_s$.

$E_3 \rightarrow$ Interchanging $r^{th}$ row/column with $s^{th}$ row/column i.e., $R_r \leftrightarrow R_s$ (or) $C_r \leftrightarrow C_s$.

3) EQUIVALENT MATRICES :

Any Matrix obtained from a given Matrix by Performing a Series of elementary transformations is called as equivalent to the given matrix.

4) Elementary Matrices :

A Matrix obtained from a Performing a Single elementary transformation on a unit Matrix is called an elementary matrix.

Problem :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix}$$

$$e_{23} = \lambda$$

$r = 2 \ , S = 3$

$$E_{rs}(\lambda)\, A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4+5\lambda & 5+7\lambda & 6+9\lambda \\ 5 & 7 & 9 \end{pmatrix}$$

which Consists Of replacing the Second row to second row+$\lambda$ times of the row,

$E_{rs}(\lambda)$ is non-Singular

Which Consists the Second Column + $\mu$ times of the third of third column.

ECHELON MATRIX :

A Matrix A is Said tobe echlon if elements in a row are either Zero or a unity Preceded by Zeros (but not necessarily Succeed by Zeros) and if in a Column there is a unity which is the first unity of a row then all the other elements of the Columns are Zeros,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & \delta_2 \\ 0 & 0 & 1 & \delta_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

HERMITE CANONICAL FORM :

A Matrix H of order mxm is Said to be in Hermite Canonical form if its Principal diagonal Consists of only Zeros or unities and all elements below the diagonal are Zero, Such that when a diagonal element is zero, the entire row or Zero and when a diagonal element is unity, the rest of the elements in the column are all Zeros.

$$H = [\delta_{ij}]_{m \times m}$$

$$\delta_{ij} = \left.\begin{matrix} 0 \\ 1 \end{matrix}\right\} \text{ and } \delta_{ki} = 0 \;\forall\; k < i \;(i, k = 1, 2 \ldots m)$$

Subject to,

(i) $\delta_{ii} = 0 \Rightarrow \delta_{ij} = 0 \;\forall\; j$

(ii) $\delta_{ii} = 1 \Rightarrow \delta_{ij} = 0 \;\forall\; i$

PROBLEM:

$$H = \begin{pmatrix} 1 & \delta_{12} \\ 0 & 0 \end{pmatrix} \text{ is a Hermite canonical form}$$

Which is an idempotent matrix.

$$H_2 = \begin{pmatrix} 1 & \delta_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \delta_{12} \\ 0 & 0 \end{pmatrix} = H$$

FROBENIUS INEQUALITY:

Theorem:

If A, B, C are any three matrices such that Products considered are defined, then.

$$P(AB) + P(BC) \leq P(ABC) + P(B)$$

Proof: $P(BC) \geq P(ABC) \rightarrow 1$

$P(B) \geq P(AB) \rightarrow (2)$

$P(B) \geq P(BC), \; P(AB) \geq P(ABC).$

Sub (1) from (2) we get,

$$P(B) - P(BC) \geq P(AB) - P(ABC)$$

$$\Rightarrow P(B) + P(BC) \geq P(AB) + P(BC)$$

$$\Rightarrow P(AB) + P(BC) \leq P(ABC) + P(B).$$

Hence Proved.

CERTAIN RESULTS ON THE RANK OF AN IDEMPOTENT MATRIX :

A Matrix $(A)_{n\times n}$ is Said to be idempotent if $A^2 = A$.

I. The rank of an idempotent matrix A is equal to its trace (i.e sum of the diagonal elements).

$P(A)$ be the rank of A

$P(A) = r$

$\exists$ two non-singular matrices P and Q $\ni$

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = P^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Let us denote $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = [I_r, 0]_{n\times n}$.

$$A^2 = A = P^{-1} I_{r,0} Q^{-1}, P^{-1} I_{r,0} Q^{-1} = P^{-1} I_{r,0} Q^{-1}$$

$$= PP^{-1} I_{r,0} Q^{-1} P^{-1} I_{r,0} Q^{-1} Q = PP^{-1} I_{r0}, Q^{-1}Q = I_{r,0}.$$

$$= I_n I_{r,0} Q^{-1} P^{-1} I_{r,0} I_n = I_n I_{r,0} I_n$$

$$= I_{r,0}(Q^{-1}P^{-1} - I_n) I_{r,0} = 0$$

$$= Q^{-1}P^{-1} = I_n$$

$$= Q^{-1} = P$$

$$PAP^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$PAP^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$tr(PAP^{-1}) = tr(P^{-1}PA) = tr(A)$$

$$= tr\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = r = \rho(A)$$

---

Problems :

(i) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$E_1$ consists of multiplying the 2nd row by 2.

Here, $r=2$; $C=2$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{=\Delta} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix}$$

$\Delta = [\delta_{ij}]$ is non-singular.

(ii) Suppose we want to replace the second column by 3 times the second column.

$$C = 3 \; ; \; r = 2$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 3 & 12 \end{pmatrix}$$

$E_2$ : Replacing any row (column) say $r^{th}$ row (column) by $r^{th}$ row/column any other row/column say $s^{th}$ row/column.

In this case we require to pre or post multiply A by a non-replacement.

$E_{rs}(\lambda)$ where $E_{rs}(\lambda)$ is defined by,

$$E_{rs}(\lambda) = [e_{ij}]$$

$$e_{i1} = 1 \; \forall i$$

$$e_{ij} = 0 \qquad i \neq j$$

$$e_{rs} = \lambda$$

Which the consists of replacing of the rows and columns by the singular matrix.

Example :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad r=1,\ s=2.$$

$$E_{sr}(-1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$E_{rs}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$E_{sr}(-1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_{sr}(-1)\ E_{sr}(1)\ E_{sr}(-1)\ \Delta$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} +0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$E_{sr}(-1)\ E_{rs}(1)\ E_{sr}(-1)\ \Delta$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

# Unit - 4

## Eigen Values and Eigen Vectors :

Let,

$$(A)_{n\,n}(X)_{n-1} = \lambda (X)_{n-1}$$

Then $\lambda$ is called the characteristic root (or eigen value or the Latent root of A and X is called the characteristic vector (or eigen vector) of A or the right characteristic Vector of A.

If the A is non-Symmetric (or non-Hermitian) then $A^- \neq A$ and if it statisfies from some Scalar $\lambda$.

$$A'X = \lambda X$$

Then X will be called the left characteristic vector corresponding to $\lambda$. If $\lambda$ is a symmetric Matrix then left and right characteristic Vectors are the Same.

$$AX = \lambda X$$

$$(A - \lambda I) X = 0.$$

is a system of n linear and homogenous equation.

$|A - \lambda I|$ is called a characteristic polynomial of A. If $A = (a_{ij})_{n \times n}$ then,

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$= (-\lambda)^n + (-\lambda)^{n-1}\sigma_1 + (-\lambda)^{n-2}\sigma_2 + \cdots + a_n,$$

Where, $\sigma_0 = 1$, $\sigma_n = |A|$

$\sigma_i$ = Sum of the principal minor of older, $i$ in A.

$i = 1, 2, \ldots n-1$.

Problem :

Show that a skew symmetric matrix has no real characteristic roots other than zero.

A is a skew symmetric matrix

$AX = \lambda X$ holds

$iAX = (i\lambda)X$.

If A is a skew symmetric then $iA$ is Hermitian whose characteristic roots are real $i\lambda$ is real $\lambda$ is imaginary unless $\lambda = 0$.

the characteristic root of A is same as the characteristic root of A.

$$|A-\lambda I| = 0 = |A' - \lambda I| = 0$$

$$= -A - \lambda I = 0 \text{ if } A \text{ is Skew Symmetric}$$

$$= |A + \lambda I| = 0$$

$$= \lambda^n + \lambda^{n-1}\sigma_1 + \lambda^{n-2}\sigma_2 + \cdots + \underbrace{|A|}_{=\sigma_n} = 0$$

$$(-\lambda)^n + (-\lambda)^{n-1}\sigma_1 + (-\lambda)^{n-2}\sigma_2 + \cdots + (-)^{n-1}|A| = 0$$

This shows that if $\lambda = \lambda_0$ is a root then $\lambda = -\lambda_0$ is also a root of the characteristics equation which implies that either $\lambda_0$ is imaginary or zero.

CAYLEY HAMILTON THEOREM:

Every Matrix Satisfies its characteristic equation.

Proof: We have,

$$[A-\lambda I] = \frac{\text{adj}[A-\lambda I]}{[A-\lambda I]} = I_n$$

$$= [A-\lambda I]\,\text{adj}\,[A-\lambda I] = [A-\lambda I]$$

$$\text{adj}\,[A-\lambda I] = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

$C_1$'s are matrices,

$$[A-\lambda I] = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

$a_1$'s are Scalars,

$$[A-\lambda I]\left(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}\right)$$

$$= a_0 + a_1\lambda + \cdots + a_n\lambda^n$$

Equating co-efficients of $\lambda, \lambda^2, \ldots$ etc on both sides we have,

$$AC_0 = a_0$$

$$AC_1 - C_0 I = a_1 = A^2C_1 - AC_0 I = Aa_1$$

$$= A^{n-1}C_{n-2} - A^{n-2}C_{n-3} I = A^{n-2}a_{n-2}$$

$$= AC_{n-1} - C_{n-2}I = a_{n-1}$$

$$= A^nC_{n-1} - A^{n-1}C_{n-2} I = A^{n-1}a_{n-1}$$

$$= -C_{n-1} I = a_n.$$

$$= -A^nC_{n-1} = A^n a_n$$

Adding up, we have,

$$a_0 + a_1 A + a_2 A^2 + \cdots + a_{n-1}A^{n-1} + a_n A^n = 0.$$

Application of Cayley Hamilton Theorem :

We have if,

$$|A-\lambda I| = a_0 = a_1\lambda + a_2\lambda^2 + \cdots + a_n A^n = 0$$

$$a_0 + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0 .$$

For a non-singular matrix, Premultiplying both sides by $A^{-1}$,

$$a_0 A^{-1} + a_1 I + a_2 A + \cdots + a_n A^{n-1} = 0 .$$

The equation can be employed to obtain $A^{-1}$ when the Powers of A are known.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$ ; ch equation

$$\lambda^2 - (1+3)\lambda + 1 = 0$$

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\Rightarrow A^2 - 4A + I = 0$$

$$= A - 4I + A^{-1} = 0$$

$A - 4I + A^{-1} = 0$

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = -A^{-1}$$

$$= \begin{bmatrix} 1-4 & 2-0 \\ 1-0 & 3-0 \end{bmatrix} = -A^{-1}$$

$$= \begin{bmatrix} 3 & -2 \\ -1 & -3 \end{bmatrix} = A^{-1}.$$

Example Problems:

[S is a real - skew symmetric matrix $\Rightarrow (I-S)$ is non-singular and $A = (I+S)(I-S)^{-1}$ is orthogonal].

Proof: $|I-S| \neq 0$

$$A' = [(I+S)(I-S)^{-1}]'$$

$$= [(I-S)^{-1}]'(I+S)'$$

$$= (I+S)^{-1}(I-S)$$

$$Q^{-1}BP^{-1} = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

Partitioned in the same way as $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$$PAQQ^{-1}BP^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ 0 & 0 \end{bmatrix}$$

It follows that the characteristic roots of $AB$ are either those of $C_{11}$ or zeros.

$Q^{-1}BAQ$ where $Q$ is non-singular

$$= Q^{-1}BP^{-1} \, PAQ$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & 0 \\ C_{21} & 0 \end{bmatrix}$$

Again the characteristics roots of $BA$ are either those of $C_{11}$ or zeros. Hence the characteristic roots of $AB$ and $BA$ are identical.

$$A'A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$$

$$= (I+S)^{-1}(I^2-S^2)(I-S)^{-1}$$

$$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$$

$$= I \quad (\because A \text{ is orthogonal}).$$

Problem :

Show that AB and BA have same characteristic roots :

Proof : A and its transform $P^{-1}AP$ (P is a non-singular)

$$AX = \lambda X$$

$$= P^{-1}AX = \lambda P^{-1}X$$

$$= P^{-1}APP^{-1}X = \lambda P^{-1}X$$

$$= \lambda(P^{-1}X).$$

Which shows that $P^{-1}AP$ and A have the same characteristic roots.

$$\rho(A) = r, \quad PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$P(AB)P^{-1}$ which has the characteristic roots.

$$P(AB)P^{-1} = PAQQ^{-1}BP^{-1}.$$

Q is non-singular matrix of the same order such that, Unit - IV

## Generalised Inverse of Matrix :

Let, $(A)_{m \times n} (X)_{n \times 1} = (Y)_{m \times 1}$

be a consistent system of equation. A generalised inverse (or a g- inverse) of A is a $n \times m$ matrix denoted by $A^-$ (or $A^g$) Such that,

$X = A^- Y$ (or $X = A^g Y$) is a solution of the equation.

$Y \in \mu(A)$.

Example : 1

$(A)_{m \times n}$ and $\rho(A) = r$.

$\exists$ a non-singular sub matrix $(B)_{r \times r} \ni \rho(B) = r$

By a suitable interchange of rows and columns of A,

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

$$A^- = \begin{pmatrix} B^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$ then one may show

$$\left.\begin{array}{l} A A^- A = A \\ A^- A A^- = A^- \end{array}\right\} \rho(A^-) = \rho(A)$$

$A^-$ is a reflexive generalised inverse. It is also easy to see $E = DB^{-1}C$.

Example : 2

Obtain a g-inverse of the matrix given by.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Sol : It may be seen that $\rho(A) = 2$

$|A| = 0$

$\rho(A) = 2$ since $\exists$ one

minor of A viz $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \neq 0$

A generalised inverse of is obtained by arbitary values of U, V and W.

$$\begin{bmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^{-1} & (U)_{2\times 1} \\ 0 & \\ V' & W' \end{bmatrix}$$

$U = 0$, $V' = 0$, $W = 0$

We have, $$A^{g} = \begin{bmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## DIFFERENT CLASSES OF GENERALISED INVERSE :

We define matrices over a complex number field clearly analogus results are obtainable when the matrices are defined over a real field.

$$AXA = A$$

$$XAX = X$$

$$(XA)^* = XA$$

$$(AX)^* = AX$$

$*$ = Conjugate transpose.

Example :

Show that,

$A+ = (A^*A^{-1})\, A^*$ if $(A^*A)^{-1}$ exists.

$B+ = B^*(BB^*)^{-1}$ if $(BB^*)^{-1}$ exists.

Sol: Put $B = I$ in Solution,

$$(AI)^+ = I^*(II^*)^{-1}(A^*A)^{-1}A^*$$

$$\Rightarrow A^* = ((A^*A)^{-1})\, A^*$$

$A = I$.

$$(IB^+) = B^*(BB^*)^{-1}(I^*I)^{-1}I^*$$

$$\Rightarrow B^+ = B^*(BB^*)^{-1}.$$

PROPERTIES OF G-INVERSE ($A^g$)

(a) $\rho(A^g) \geq \rho(A)$

(b) One choice of $(A^*)^g$ is $(A^g)^*$

(c) $A(A^*A)^g A^* A = A$

(d) $A^*A(A^*A)^g A^* = A^*$

(e) If A is Idempotent Matrix

$A^g = A$ is a choice of $A^g$

(f) $A(A^*A)^g A^*$ is a Hermitian idempotent Matrix Projecting vectors onto the Subspace Spanned by the columns of A.

(g) $A^g A$ and $AA^g$ are Hermitian.

PROPERTIES OF G-INVERSE ($A^r$)

(a) All the Properties from (b) to (g) for generalised-inverse

(b) $A^g$ is reflective g-inverse if and only if

$\rho(A^g) = \rho(A)$.

PROPERTIES OF LEFT WEAK G-INVERSE ($A^{W}$):

(a) Properties (c)(d)(f) and of replacing $A^g$ with $A$

(b) One choice for $A^{W}$ is $A^*(AA^*)^g$ and for $(A^*)^{W}$ is $(A^n)^*$ $a \neq 0$ $(A^*A)^{W}$ is $A^{W}(A^*)^{W}$

(c) One choice for $(UAV)^{W}$ is $V^*A^{W}U^*$

Where U and V are unitary matrices.

(d) $A^+A = A^{W}A$

for any choice of $A^{W}$.

(e) If $B_1$ and $B_2$ are any two choices of $A^{W}$ then $(B_1 - B_2)A = 0$

(f) If $P(A) = n = \min(m, n)$

Then, $\forall$ $A^*$ is $A^{W}$.

g) If A is Hermitian and indempotent of the matrix.

PROPERTIES of RIGHT WEAK GI:

The following properties are true for of $A(A^{n})$.

(a) Properties (c),(d),(f)(g) of ($A^g$) in Replacing of $A^g$

(b) One choice for $A^n$ is $(A^*A)^g A^*$

(c) One choice for $(AA^*)^n$ is $(A^*)^n A^n$.

(d) One choice for $(UAV)^n$ is $V^* A^n V^*$ Where U and V are unitary matrices.

(e) $AA^+ = AA^n$ for any choice of $A^n$.

(f) If $B_1$ and $B_2$ are any two choices of $A^n$ then $A(B_1 - B_2) = 0$.

(g) If $P(A) = m = \min(m,n)$ then every $A^g$ is $A^n$.

(h) If A is hermitian and idempotent one choice for $A^n$ is A.

APPLICATIONS OF GENERALISED INVERSES IN THE SOLUTION OF SYSTEMS OF LINEAR EQUATIONS :

Theorem :

A Necessary and Sufficient condition for the equation

$(A)XB = C$

to have a solution is

$A(A^{+}CB^{+})B = C$ in which case the general solution is

$X = A^{+}CB^{+} + Y - A^{+}AYBB^{+}$

Y is arbitary.

Proof : $AXB = C$

$C = AXB = AA^{+}AX(BB^{+}B$

$\Rightarrow C = AA^{+}CB^{+}B \qquad (\because AXB = C)$

$AXB = 0$

$AXB = A(Y - A^{+}AYBB^{+})B$

$= AYB - AA^{+}AYBB^{+}B$

$= AYB - AYB = 0.$

The general solution of the non-homogenous equation.

$AXB = C$

$\Rightarrow$ general Solution of $AXB = 0 +$ a particular solution of $AXB = C$

Hence the general of $AXB = C$ is given by

$$X = Y - A^{+}AYBB^{+} + A^{+}CB^{+} (Q.E.D).$$

Theorem 6

If $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ $P^{-1} = [X, Y]$

$Q = [Q_1, Q_2]$ $Q^{-1} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$

(a) $P_1 X = I$, $P_2 X = 0$.

(b) $XQ_1 = I$, $XQ_2 = 0$ have a unique Common Solution.

$$X = (Q_1 - Q_2 Q_2 + Q_1) + (I - Q Q_2^{+}).$$

Proof :

$$P^{-1} = [Y, Y]$$

$$PP^{-1} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} [X, Y] = \begin{bmatrix} P_1 X & P_1 Y \\ P_2 X & P_2 Y \end{bmatrix}$$

$P_1 X = I$ and $P_2 X = 0$

$= I$ (identically).

$$P_2 X = (P_2 - P_2 P_2^{+} P_2)(P_1 - P_2^{+} P_2) +$$

$$= 0.$$

Similarly it can easily be verified that the common solution of $XQ_1 = I$ and $XQ_2 = 0$.
is given by

$$X = (Q_1 - Q_2 Q_1 + Q_1)^{+} (I - Q Q_2^{+}).$$

## Quadratic forms :

If $X$ is a column vector of order $n \times 1$ and there is a symmetric matrix $A = [a_{ij}]_{n \times n}$ i.e, $a_{ij} = a_{ji}$

Then $Q = (X')_{1-n} (A)_{n-n} (X)_{n-1}$ is called a quadratic form.

## Classification of the Quadratic form :

If we apply linear transformation say $X = PY$ then

$$Q = Y' (P'AP) Y$$

$$P'AP = I_n$$

where $P$ is a non-singular matrix

$$\rho(A) = n, \text{ then } Q = \sum_{i=1}^{n} y_i^2$$

$Q > 0$, unless all $y$'s are zero and $y$'s cannot be zeros unless all $x$'s are zeros.

Positive Semi - Definite Quadratic form :

If $\rho(A) = r < n$ and if we apply the Same linear transformation $X = PY$, P being a non-Singular Matrix Then

$$Q = X'AX$$
$$= Y'P'APY$$
$$= Y'I_rY \quad (\because \rho(A) = r)$$
$$= \sum_{i=1}^{r} y_i^2 \geq 0$$

is called a Positive Semi-definite Quadratic form.

For eg: Correlation matrix A is Positive definite or Semi definite matrix in case of Multinomial distribution is positively Semi-definite.

When Q transformed in the form,

$$Q = -\sum_{i=1}^{r} y_i^2 < 0.$$

then Q is called Negative Semi-definite Quadratic form.

The Number of Positive terms S is called the index of the Qudratic forms.

Compute a $g$-inverse :

$$\begin{pmatrix} 18 & 2 & 46 \\ 2 & 1 & 2 \\ 46 & 2 & 130 \end{pmatrix}$$

Solution : Note $\rho(A) = 2$

$$\begin{bmatrix} 18 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{14} \begin{bmatrix} 1 & -2 \\ -12 & 18 \end{bmatrix}$$

$$\therefore A^g = \frac{1}{14} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 18 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2) Compute the right weak $g$-inverse($A^n$) of $A$ given by,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}_{4\times 2}$$

Solution : $\rho(A) = 2$, $\min(m,n) = n$

A right weak generalised inverse $= (A^*A)^g A^*$

$$= \left( \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}_{4\times 2} \right)$$

$$= \begin{pmatrix} 7 & 4 \\ 4 & 7 \end{pmatrix}^{g} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{33} \begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{33} \begin{bmatrix} -1 & 10 & 3 & 11 \\ 10 & -1 & 3 & -11 \end{bmatrix}$$

$$\therefore A^{n} = \frac{1}{33} \begin{bmatrix} -1 & 10 & 3 & 11 \\ 10 & -1 & 3 & -11 \end{bmatrix}.$$