M.Sc - STATISTICS

CORE COURSE - III

18KP1503

MEASURE AND PROBABILITY THEORY

UNIT- I

Events, algebra of sets, Fields - fields, Borel fields, Intersection and uniton of fieldsmonotone fields and necessary properties - minimal monotone class.

UNIT- II

Function, inverse function, measurable function. borel function, induced - field, indicator functions, elementary function, concept of random variable, Borel function of a vector random variable. Limits of random variables, continuity property of probability space. Caratheodory extension theorem (Statement only), induced probability space, probability as a measure

UNIT- III

Distribution function, Properties, Jordan decomposition theorem, distribution function of a random vector, Marginal and conditional distributions, correspondence theorem (statement only) empirical distribution. Expectation properties - Cramer Rao - inequality, Holder's inequality, Cauchy Schwartz's inequality. Minkowski inequality, Jenson's inequality, Basic inequality.

UNIT- IV

Convergence of random variables. Types of convergence: Monotone Convergence theorem, Dominated Convergence theorem, Characteristic function, properties, some inequalities on Characteristic functions, inversion theorem and simple problems.

UNIT-V

Limit theorems, Law of large members. Weak Law of large numbers. Bernoulli, Poisson and Khinchine's Law of large numbers; Strong law of large numbers, Levy - Cramer theorem, Central Limit Theorem, De-Moivre - Laplace, Liapounov's, Lindberg-Levy theorems. Statement of Lindberg - Feller theorem

Books for Study Reference :

1. B.R. Bhat - Modern probability theory : units 1,2,3,4,5,6 (up to 6.55 only), 7 (up to 7.4 only)
2. Mark Fisz - Probability theory and mathematical statistics : unit 6 (omitting 6.4, 6.5, 6.10, 6.13, 6.14, 6.15)

UNIT- I

EVENT :

Every non empty subset A of S, which is a discount union of single element subset of the sample space S of random experiment. E is called an EVENT.

For eg. i) In random toss of two coins the sample space

$S = \{HH, HT, TH, TT\}$, $S=4$

ii) Let us consider a random toss of a single dice Since, we can certain any one of the six face

$S = \{1, 2, 3, 4, 5, 6\} \Rightarrow S=6$

IMPOSSIBLE EVENT :

As the empty set of ϕ is a set of null set is $\{\phi\}$ also as event is known as impossible event

ELEMENTARY EVENT :

An event A in particular, can be a element, subset of S in which case, is known as elementary event.

SET :

Collection of objects under some investigation is called a set

SET OPERATOR :

$\cup \rightarrow$ Union $\quad$ $c \rightarrow$ Complement

$\cap \rightarrow$ Intersection $\quad$ $- \rightarrow$ Different

$\Delta \rightarrow$ Delta $\rightarrow$ Symmetric Difference

(i) $A \cup B = \{x / x \in A \text{ or } x \in B\}$

(ii) $A \cap B = \{x / x \in A \text{ and } x \in B\}$

(iii) $\cap$ = set of common elements

(iv) $A - B = \{x / x \in A, x \notin B\}$

(v) $B - A = \{x / x \in B, x \in A\}$

Eg:-

$\Omega = \{1,2,3,4,5,6,7,8\}$

$A = \{1,2,3,4,5,6\}$

$B = \{4,5,6,7,8\}$

$A - B = \{1,2,3\}$

$B - A = \{7,8\}$

$A \Delta B = (A-B) \cup (B-A)$

$= \{1,2,3,7,8\}$

$A^c = \Omega - A = \{7,8\}$

SEQUENCE:

An order set of events $\{E_n : n \geq 1\}$ is called Sequence. If $E_1 \subset E_2 \ldots \subset E_n$. Then, the sequence is called Increasing sequence or non-decreasing sequence and is denoted by $\{E_n : n \geq 1\} \uparrow$

Similarly,

$E_1 \supset E_2 \ldots \supset E_n$ then the sequence is called decreasing sequence or non-increasing sequence and it is denoted by $\{E_n : n \geq 1\} \downarrow$

LIMITS OF SEQUENCE:

If $\{E_n : n \geq 1\} \uparrow$ is an increasing sequence $\lim_{n \to \infty} E_n = \bigcup_{i=1}^{\infty} E_i$

Similarly,

If set of $\{E_n : n \geq 1\} \downarrow$ is an decreasing sequence $\lim_{n \to \infty} E_n = \bigcap_{i=1}^{\infty} E_i$

LIMIT SUPREMUM & LIMIT INFERMUM:

$\downarrow, \uparrow$. Let $\{E_n : n \geq 1\}$ is an orbitary sequence,

$$F_n = \bigcup_{i=1}^{\infty} E_i$$

Put $n = 1, 2, \ldots$

$$F_1 = \bigcup_{i=1}^{\infty} E_i$$

$$F_2 = \bigcup_{i=1}^{\infty} E_i \ldots$$

$$\Rightarrow F_1 > F_2 > \ldots$$

$\{F_n : n \geq 1\} \downarrow$ is an decreasing sequence

$$\lim_{n \to \infty} F_n = \bigcap_{i=1}^{\infty} F_n$$

$$= \bigcap_{i=1}^{\infty} \bigcup_{i=1}^{\infty} E_i$$

$$= \lim_{n \to \infty} \operatorname{Sup} E_n.$$

Let $\{E_n : n \geq 1\}$ be same of orbitary sequence define $G_n = \bigcap_{i=1}^{\infty} E_i$

Put $n = 1, 2, \ldots$

$$G_1 = \bigcap_{i=1}^{\infty} E_i$$

$$G_2 = \bigcap_{i=1}^{\infty} E_i \ldots$$

$\Rightarrow G_1 < G_2 < \ldots$

$\{G_n : n \geq 1\} \uparrow$ is an increasing sequence

$$\lim_{n \to \infty} G_n = \bigcup_{i=1}^{\infty} G_n$$

$$= \bigcup_{i=1}^{\infty} \bigcap_{i=1}^{\infty} E_i$$

$$= \lim_{n \to \infty} \operatorname{Inf} E_n$$

and it is denoted by $\lim_{n \to \infty} E_n$. Further $\overline{\lim} E_n \geq \underline{\lim} E_n$

DEMARGAN's RULE:

i) $\cap A_i^c = (\cup A_i)^c$

ii) $\cup A_i^c = (\cap A_i)^c$

FIELD:

A field is non-empty class of sets which are closed under complementation and finite intersection operation.

$$F = C\{c, \bigcap_{i=1}^{n}\}$$

PROPERTIES OF FIELD:

i) Let $A \subset F \Rightarrow A^c \in F$

$\Rightarrow A^c \in F$

$\Rightarrow A \cap A^c \in F$

i.e, $\phi \in F$

i.e, Every Field consists of a null set.

ii) Let $A_1, A_2, \ldots, A_n \in F$

$\Rightarrow A_1^c, A_2^c, \ldots, A_n^c \in F$

$\Rightarrow \bigcap_{i=1}^{n} A_i^c \in F$

$\Rightarrow (\bigcap_{i=1}^{n} A_i^c) \in F$

i.e, $\bigcup_{i=1}^{n} A_i \in F$

i.e, F is closed under Finite Union Operator.

iii) Let $A \in F$

$\Rightarrow A^c \in F$

$\Rightarrow A \cup A^c \in F$

i.e, $\Omega \in F$

i.e, Every Field consists of basic as universal set,

$F = c\{\phi, \Omega, c, \bigcup_{i=1}^{n}, \bigcap_{i=1}^{n}\}$

Example: 1

Consider $F_1 = \{A, A^c, \phi, \Omega\}$

i.e, Is F_1 is a field?

Proof:

i) ϕ and $\Omega \in F_i$

ii) $A \in F_1 \Rightarrow A^c \in F_1$

$A^c \in F_1 \Rightarrow A \in F_1$

$\phi^c \in F_1 \Rightarrow \Omega^c \in F_1$, F_1 is closed under finite completation operator

iii) $A \cap A^c = \phi \in F_1$

$A \cap \phi = \phi \in F_1$

$A \cap \Omega = A \in F_1$

$A^c \cap \Omega = A^c \in F_1$

i.e, F_1 is closed under finite intersection operator

iv) $A \cup A^c = \Omega \in F_1$

$A \cup \phi = A \in F_1$

$A \cup \Omega = \Omega \in F_1$

$A^c \cup \Omega = \Omega \in F_1$

i.e, F_1 is closed under finite union operator.

$\therefore F_1$ is a field.

Note:

(1) ly consider,

$F_2 = \{B, B^c, \phi, \Omega\}$

$\therefore F_2$ is a field.

ii) $F_1 \cap F_2$, Is it a field?

W.K.T:

$F_1 \cap F_2 = \{\phi, \Omega\}$

(i) $\phi, \Omega \in F_1, F_2$

(ii) $\phi \in F_1 \cap F_2$

$\therefore F_1 \cap F_2$ is a field.

Hence, the intersection of two fields is also a field.

Eg 2:

The union of any orbitary field need not be field.

Proof:

Consider.

$F_1 = \{A, A^c, \phi, \Omega\}$ $\Rightarrow F_1$ is a field

Similarly $F_2 = \{B, B^c, \phi, \Omega\}$ $\Rightarrow F_2$ is a field

$F_1 \cup F_2 = c\{A, A^c, B, B^c, \phi, \Omega\}$

i) ϕ and $\Omega \in F_1$

ii) $A \in F_1 \cap F_2 \Rightarrow A^c \in F_1 \cap F_2$

iii) $\phi \in F_1 \cap F_2 \Rightarrow \Omega \in F_1 \cap F_2$

$\therefore F_1 \cap F_2$ is closed under complementation operator

iv) $A \cap A^c = \phi \in F_1 \cap F_2$

$A \cap \phi = \phi \in F_1 \cap F_2$

$A \cap \Omega = A \in F_1 \cap F_2$

$B \cap B^c = \phi \in F_1 \cap F_2$

$B \cap \phi = \phi \in F_1 \cap F_2$

$B \cap \Omega = B \in F_1 \cap F_2$

$\therefore A \cap B$ is not given.

So, intersection is not satisfied. Then, the union of Any two orbitary field need not to be a field.

Eg 3:

Consider $F_1 = \{A, A^c, \phi, \Omega\}$ is a field.

Similarly $F_2 = \{B, B^c, \phi, \Omega\}$ F_2 is also a field

Consider $F_1 \cup F_2 = \{A, A^c, B, B^c, \phi, \Omega\}$

$F_1 \cup F_2$ is also a field.

Eg 4:

Statement:

The intersection of any orbitary finite number of field is also a field

Proof:

Let $C_i (i \in I)$ be a class of field $C_0 = \bigcap_{i \in I} C_i$. Here we have to show that C_0 is also a field (ie) we have to show that

$$C_0 = C\{\phi, \Omega, c, \bigcap_{i=1}^{n} \bigcup_{i=1}^{n}\}$$

i) Let $A \in C_0$

$\Rightarrow A \in C_i \ \forall i \in I$

$\Rightarrow A^c \in C_i \ \forall i = I$

$\Rightarrow A^c \in C_0$

$\therefore C_0$ is closed under complementation operator

ii) Let $A_k (k=1,2,\dots n) \in C_0$

$\Rightarrow A_k \ (k=1,2,\dots n) \in C_i \quad \forall i \in I$

$\Rightarrow \bigcap_{k=1}^{n} A_k \in C_0$

(i.e) C_0 is closed under finite intersection operation

iii) Let A_k $(k=1,2,\cdots n) \in C_0$

$\Rightarrow A_k \ (k=1,2,\cdots n) \in C_i \quad \forall i \in I$

$\Rightarrow \bigcap_{k=1}^{n} A_k^c \in C_i \quad \forall i \in I$

$\Rightarrow \left(\bigcap_{k=1}^{n} A_k^c\right) \in C_i \quad \forall i \in I$

i.e) $\bigcup_{i=1}^{n} A_k \in C_i \quad \forall i \in I$

$\Rightarrow \bigcup_{i=1}^{n} A_k \in C_0$

$\therefore C_0$ is closed under finite union operator

Since, C_i are field ϕ Ω $\in C_i \quad \forall i \in I$

$C_0 = \bigcap_{i \in I} C_i$

$\Rightarrow \phi$ Ω $\in C_0$

$C_0 \Rightarrow \{\phi, \Omega, C, \bigcap_{i=1}^{n} \bigcup_{i=1}^{n}\}$

$\therefore C_0$ is a field

(i.e) the intersection of any orbitary finite number field is also a field.

SIGMA FIELD:

(σ field)

A σ field is non-empty class of sets, which is closed under complementation and coutable intersection operator ie) $F = C\{G, \bigcap_{i=1}^{\infty}\}$

PROPERTIES OF σ- FIELD:

i) Let $A \in F$

$\Rightarrow A^c \in F$

$\Rightarrow A \cap A^c \in F$

$\phi \in F$

$\therefore$ Every σ field consists of a null set.

ii) Let $A_k\ [k=1,2,\cdots n] \in F$

$A_k^c\ (k=1,2,\ldots) \in k$

$$\Rightarrow \bigcap_{k=1}^{\infty} A_k^c \in F$$

$$\Rightarrow \left[\bigcap_{k=1}^{\infty} A_k^c\right]^c \in F$$

$$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in F$$

(i.e), F is closed under countable union operator.

iii) Let $A \in F$

$\Rightarrow A^c \in F$

$A \cup A^c \in F$

$\therefore \Omega \in F$

Every σ field consists a basic set

$$\therefore F \subset \{ \phi, \Omega, c, \bigcap_{i=1}^{\infty}, \bigcup_{i=1}^{\infty} \}$$

Book Work : 1

STATEMENT:

The intersection of any orbitary finite number of σ field is calso a field.

PROOF:

Let $C_i, C_i \in I$ be a class of σ field

$$C_0 = \bigcap_{i=I}^{Z} C_i ,$$

Here, We have to show that C_0 is also a σ field.

ie) We have to show that

$$C_0 = C \{ \phi, \Omega, c, \bigcap_{i=1}^{\infty}, \bigcup_{i=1}^{\infty} \}$$

i) Let $A \in C_0$

$A \in C_i \quad \forall i \in I$

$A^c \in C_i \quad \forall i \in I$

$\Rightarrow A^c \in C_0$

$\therefore C_0$ is closed under complementation operator.

ii) $A_k (k=1,2,\cdots \infty) \in C_0$

$\Rightarrow A_k (k=1,2,\cdots \infty) \in C_i \quad \forall i \in I$

$\bigcap_{k=1}^{\infty} A_k \in C_i \quad \forall i \in I$

$\Rightarrow \bigcap_{k=1}^{\infty} A_k \in C_0$

$\therefore C_0$ is closed under finite intersection operator.

iii) $A_k (k=1,2,\cdots \infty) \in C_0$

$\Rightarrow A_k (k=1,2,\cdots \infty) \in C_i \quad \forall i \in I$

$\Rightarrow A_k (k=1,2,\cdots \infty) \in C_i \ \forall i \in I$

$\bigcap_{k=1}^{\infty} A_k^c \in C_i \ \forall i \in I$

$\Rightarrow \left(\bigcap_{k=1}^{\infty} A_k^c\right)^c \in C_i \quad \forall i \in I$

$\bigcup_{k=1}^{\infty} A_k \in C_0$.

iv) Since C_i are field:

$\{\phi, \Omega, \in, a\} \ \forall i \in I$

Since $C_0 = \bigcap_{i \in I} C_i$

$C_0 = C\{\phi, \Omega, \bigcap_{i=1}^{\infty}, \bigcup_{i=1}^{\infty}\}$

$\therefore C_0$ is a σ field

(ie), the intersection of any orbitary finite no. of field is also field.

MONOTONE FIELD:-

Let F be a σ field

(ie) $F = c\{\phi, c, \Omega, \bigcap_{i=1}^{\infty}, \bigcup_{i=1}^{\infty}\}$

if more over $\forall A_k \in F$

$\overline{\lim} A_k$ and $\underline{\lim} A_k \in F$

Thus F is called a MONOTONE FIELD.

Book Mark 2:

EVERY σ IS A MONOTONE FIELD AND CONVERSELY:

Let F is a σ field

$$F = c\{\phi, \Omega, c, \bigcap_{i=1}^{\infty}, \bigcup_{i=1}^{\infty}\}$$

Let $\{A_k, k \geq 1\} \in F$ where $\{A_k\}$ is some arbitary sequence

W.K.T

If $\{A_k\}\uparrow$ then $\lim A_k = \bigcup_{i=1}^{\infty} A_k$ and if $\{A_k\}\downarrow$ then $\lim A_k = \bigcap_{i=1}^{\infty} A_k$

$\therefore$ if $\{A_k\}\downarrow \in F$

$\Rightarrow \bigcap_{i=1}^{\infty} A_k \in F$

$\lim A_k \in F$

Since F is closed under countable intersection operator

ii) If $\{A_k\}\uparrow \in F$

$\rightarrow \bigcup_{i=1}^{\infty} A_k \in F$

$\lim A_k \in F$

Since, f is closed countable union operator.

i.e) This implies f is monotone field

i.e) Every σ field is a monotone field.

CONVERSE FIELD PART :

Every monotane field is a σ field. Let f be a monotone field (ie) for any f this implies

$F \rightarrow \lim A_k \in F$

If $\{A_k\}\downarrow A_k \in F$. Then, $\lim A_k = \bigcap_{k=1}^{\infty} A_k \in F$

i.e) F is closed under countable Intersection operator

Similarly.

If $\{A_k\}\uparrow A_k \in F$ then

$\lim A_k = \bigcup_{i=1}^{\infty} A_k \in F$

i.e) F is closed under countable union operator.

Moreover $(\phi, \Omega) \in F$ and F is closed under complementation operator

$\therefore$ F is a σ field

MUTUALLY EXCLUSIVE EVENTS :

Two Events E_1 and E_2 are mutually exclusive If $E_1 \cap E_2 = \phi$ or E_1, E_2 are also called Disjoint Events.

NOTE:

1) The union of any arbitary, Events are denoted by $\cup$.
2) The union of mutually exclusive events are denoted by Σ

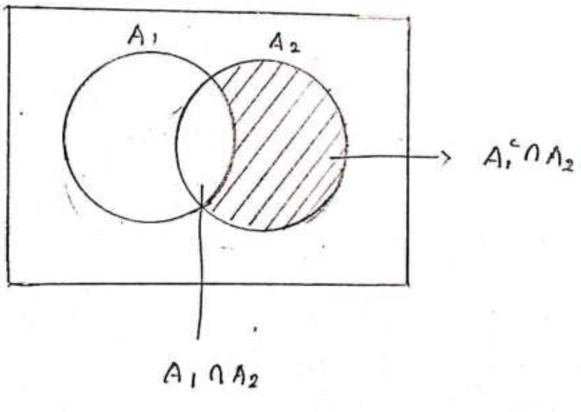
LEMMA :

If $\{A_n\}$ is an arbitary sequence of set, then there exists ($\exists$) an arbitary sequence. $\{B_n\}$ of disjoint sets,

So, that $\bigcup_{n \geq 1} A_n = \sum_{n \geq 1} B_n$

Proof:

First prove, Above Lemma for any arbitary two sets A_1 and A_2 for all $\forall$ $A_1 \cap A_2 = \phi$



$$A_1 \cap A_2 = A_1 \cup [A_1^c \cap A_2]$$

$$= B_1 + B_2 \qquad \text{Since } B_1 \cap B_2 = A_1 \cap (A_1^c \cap A_2) = \phi$$

$= B_1$ & B_2 are disjoint sets.

Similarly, for any $n = 3, 4, \ldots, m, m+1$

$$\bigcup_{i=1}^{m+1} A_i = \left(\bigcup_{i=1}^{m} A_i\right) \cup (A_{m+1})$$

$$= \left(\bigcup_{i=1}^{m} A_i\right) \cup \left[\left(\bigcup_{i=1}^{m} A_i\right)^c \cap A_{m+1}\right]$$

$$= \left(\bigcup_{i=1}^{m} A_i\right) \cup \left[\left(\bigcup_{i=1}^{m} A_i\right) \cap A_{m+1}\right]$$

$$= \sum_{n \geq 1}^{m} B_n \cup \left[(\Sigma B_n)^c \cap A_{m+1}\right]$$

$$\bigcup_{i=1}^{m+1} A_i = \sum_{n \geq 1}^{m} B_n + B_{m+1}$$

$$\bigcup_{i=1}^{m} A_i = \sum_{n \geq 1}^{m+1} B_n$$

There we have,

$$\bigcup_{n \geq 1}^{m+1} A_n = \sum_{n \geq 1}^{m+1} B_n$$

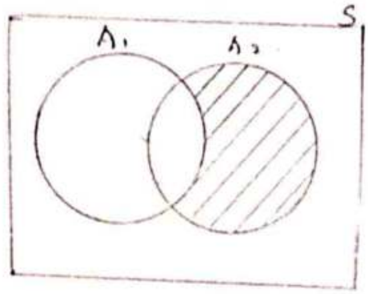
where $B_1 \cap B_2 = \phi \quad \forall\, i = j$

P.T $\bigcup_{i=1}^{n} A_i = A_1 + A_1^c \cap A_2 + A_1^c \cap A_2^c \cap A_3^c + \ldots$

where $\{A_i\}$ are some arbitary sequenc of sets.

Proof :

Let us prove for-any arbitary sets A_1 and A_2 if $A_1 \cap A_2 = \phi$



$A_1^c \cap A_2$

$A_1 \cap A_2$

Let $W \in A_1 \cap A_2 \Rightarrow W \in A_1$ (or) $W \in A_1^c \cap A_2$

Similarly, we have,

$A_1 \cup A_2 \cup A_3 = A_1 \cup (A_1^c \cap A_2^c \cap A_3^c)$

Let $W \in A_1 \cup A_2 \cup A_3 = W \in A$ (or) $W \in A_1^c \cap A_2$

$A_1 \cup A_2 \cup A_3 = A_1 + A_1 \cap A_2 + \ldots + A_1^c \cap A_2^c$ and intergal

$\bigcup_{i=1}^{n} A_i = A_1 + A_1^c \cap A_2 + \ldots$

INDEPENDENT EVENTS :

Two events A and B are said to be independent if the relations.

$P(A \cap B) = P(A) \cdot P(B)$

Similarly there events. A, B, C are said to be independent if the relation.

$P(A \cap B \cap C) = P(A).P(B).P(C)$

$P(A \cap B) = P(A).P(B)$

$P(A \cap C) = P(A).P(C)$

$P(B \cap C) = P(B).P(C)$

BOREL FIELD:

A σ field generated by class of intervals or the real line $R(-\infty, \infty)$ is called a Borel field and it is denoted by B.

MINIMAL σ FIELD:

Consider $C_0 = C\{\phi, \Omega, C, (AB)\}$ is a σ field $\in C_0$. Define $C_1 = C_0\{C_0, A^c \cap B, A \cap B^c, \ldots\} \in C_1$ is a field.

Similarly,

$C_2 = C\{C_1, C_2, \ldots\} \in C_2$ is also a field, and in general we can also define C_k. therefore C_k is also a field. Thus C_k is a field generated by the field such as $C_{k-1}, C_{k-2}, \ldots C_2, C_1, C_0$. Hence C_k is called σ field.

MONOTONE CLASS:

A class A of subset of Ω is said to be a monotone class, if (i) A is non-empty

(ii) For any $A_n \in A_{n \geq i}$ and $A_n \uparrow$ or $A_n \downarrow$. Then $\lim A_n = A$ exists and $A_n \in A$.

IMPORTANT QUESTIONS :

1. Define event with example.
2. Define Borel field.
3. Define Field.
4. Define σ field
5. Define Set operation.
6. Define Union and Intersection.
7. Define Monotone field.
8. Define the following set operation
   i) Complementation ii) Union and Intersection
9. Prove that sigma field is a monotone field and conversely.
10. Shows that the intersection of arbitary of fields is a field.
11. Prove that a field is closed under finite union.

UNIT - 2

MEASURABLE SPACE:

The space Ω of all outcomes of an experiment together will the specification of σ field A events is called Measurable space and it is denoted by (Ω, A)

Random Variable:

i) A Random variable is defined as a real valued function (i.e), $X : \Omega \to R$. It is called a real valued function.

ii) A function X is called a random variable if a) X must be a real valued function b) $\forall B \in \mathcal{B}, X^{-1} \subseteq A \subseteq \Omega$

ie) $(\Omega, A) \to (R, B)$

Here (Ω, A) is called measurable space and (R, B) is called induced measurable space (or) measurable space induced by X.

SIMPLE FUNCTION:-

Let $x (\Omega, A, P) \to (R, B, P_x)$ an extended real valued function. x is valued simple. if it is B measurable and assume only finite number of values by $x_1, x_2, \cdots x_n$

(i.e) $x = \sum_{i=1}^{n} x_i \; I_A(x_i)$

INVERSE FUNCTION:

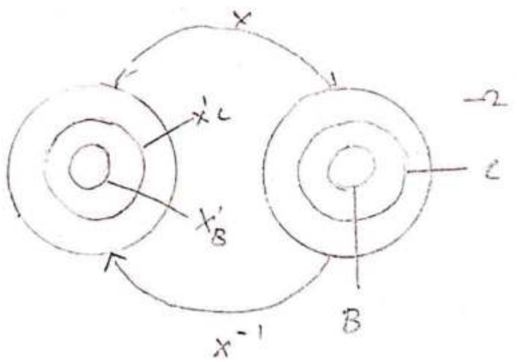
Let $x : \Omega \rightarrow \Omega'$

where $\Omega = \{W_1, W_2, \dots\}$ and $\Omega' = \{W_1', W_2', \dots\}$

(i.e) $\forall\, W \leq \Omega,\ x(W) = W' \leq \Omega'$

Then the function Ω' is to be Ω is called the inverse function of x (ie). A set of all points of $W \in \Omega$ whose image of W^{-1} and it is denoted by $x^{-1}(W)$, W $\forall\, W' \leq \Omega'$

Here x^{-1} is called the inverse function from Ω' to Ω.



SET FUNCTION:

A function defined as the relations between class of sets of elements, the relation $F : X \rightarrow Y : \sqrt{x} \leq X : (x)$

For Eg. $X = \{-3, -2, -1, 0, 1, 2, 3\}$

$Y = \{0, 1, 2, 3, 4, 5, 6, 9\}$

$F : X \rightarrow Y$ such that $\forall\, x \leq X : F(x) = x^2$

F is a function. Since Every $x \leq x$ has an image in y

Note: Define $F : X \rightarrow Y$ such that $F(x) = x + z$ here F is not a function.

PROPERTIES OF SET FUNCTION :

1) A set function is called a non-negative if $F \geq 0$

2) A set function F is finite and additive if $f\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} f(A_i)$

Similarly A set function F is σ additive $f\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} f(A_i)$

3) A set function f is called a normal function of $f(A_i) = 1$

MEASURE :

A normal probability non-negative a σ additive set function is called a measure

PROBABILITY AS A MEASURE :

Probability is a set function which normal non-negative and σ additive.

Eg :
A coin tossing expectations. The event space $\{N, T, H \cup T, H \cap T\}$

$$\begin{array}{ccc} A & & B \\ H & \longrightarrow & 1/2 \\ T & \longrightarrow & 1/2 \\ H \cup T & \longrightarrow & 1 \\ H \cap T & \longrightarrow & 0 \end{array}$$

PROBABILITY AS A MEASURE :

Definition :

$P : A \rightarrow B$

such that $P(H) = 1/2$; $P(T) = 1/2$

Hence, the set function is non-negative and

$$P(A \cup B) = P(A) + P(B)$$

$\Rightarrow$ (Finite additive property)

Further,

$P(H) + P(T) = 1$ (Normal property)

This set function

$P : A + B$ is a normal non-negative and sigma additive

i.e) Probability is a measure.

INDICATOR FUNCTION:

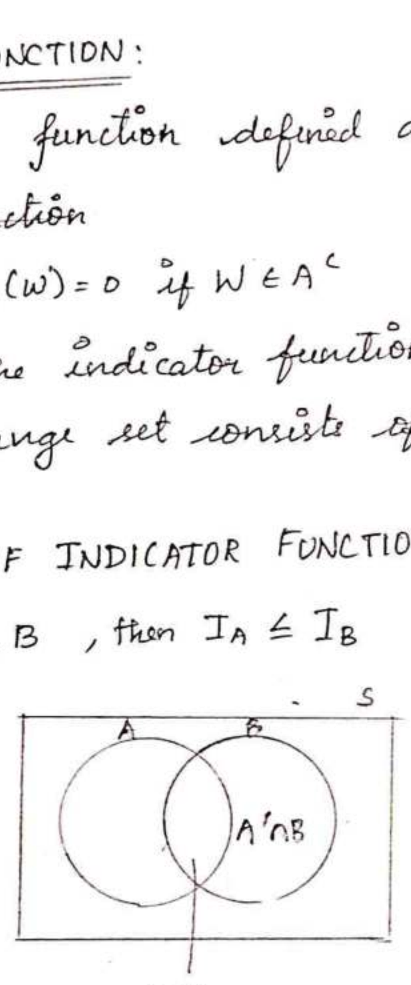
The function defined as $I_A(w) = 1$ if $w \in A$ is called the indicator function

$I_A(w) = 0$ if $w \in A^c$

is called the indicator function. This indicator function is one where the range set consists of only two values 0 and 1.

PROPERTIES OF INDICATOR FUNCTION:

i) If $A \leq B$, then $I_A \leq I_B$



Write $S = A \cup (A^c \cap B)$

$\Rightarrow I_B = I_A + I_{A^c \cap B}$

Since $A \cap (A^c \cap B) = \phi$

$\Rightarrow I_B \geq I_A$ (or) $I_A \leq I_B$

$I_A^c - 1 - I_A$

$P(I_{A^c}) = 1 - P(A)$

W.K.T

$A \cup A^c = \Omega$

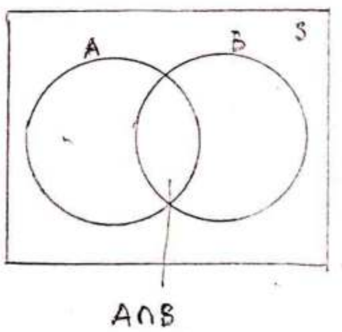
$I_{A \cup A^c} = I_\Omega = 1$

$I_A + I_{A^c} = 1$

$I_A \cancel{+}= 1 - I_A$

For any $A, B \rightarrow A \cap B = \phi$

then $I_{A \cup B} = I_A + I_B - I_{A \cap B}$



$A \cup B = A \cup \{A^c \cap B\}$

$I_{A \cup B} = I_{A \cup (A^c \cap B)}$

$= I_A + I_{A^c \cap B} \quad \rightarrow (1)$

Similarly that

$B = (A \cap B) \cup (A^c \cap B)$

$I_B = I_{A \cap B} + I_{A^c \cap B} \quad \rightarrow (2)$

From equ (1) and (2)

$$I_{A\cup B} = I_A + I_B - I_{A\cap B}$$

Similarly for events $A_1, A_2, \dots A_n$

$$I_{\bigcup_{i=1}^{n} A_i} = \Sigma I_{A_i} - \Sigma I_{A_i \cap A_j} \cdots (-1)^{n-1} I_{\bigcap_{i=1}^{n} A_i}$$

<u>Book Work</u> :

The inverse mapping preserves all set relatives

Proof :

Let $X : \Omega \to \Omega'$

i) Let $B \subset C \subset \Omega'$

$\Rightarrow X^{-1}(B) : \{W : X(w) \in B\} \subset \{W : X(W) \in C\}$

$\Rightarrow X^{-1}(C)$

(ie) For any $B \subset C \Rightarrow X^{-1}(B) \subset X^{-1}(C)$

ii) Let $B_k \subset \Omega'$ then,

$\Omega \in X^{-1}(\bigcap_{k\geq 1} B_k) \Leftrightarrow X(W) \in \bigcap_{k\geq 1} B_k$

$\Leftrightarrow X(W) \in B_1$ and $X(W) \in B_2$ and

$\Leftrightarrow W \in \bigcap_{k\geq 1} X^{-1}(B_k)$

$$W X^{-1}\left(\bigcap_{k\geq 1} B_k\right) \Leftrightarrow W \in \bigcap_{k\geq 1} X^{-1}(B_k)$$

Thus inverse mapping preserves interection operator

(iii) Let $B_k \in \Omega'$ then

$$W \in X^{-1}\left(\bigcup_{k\geq1} B_k\right) \Leftrightarrow X(W) \in \bigcup_{k\geq1} B_k$$

$$\Leftrightarrow X(W) \in B_1 \text{ (or) } X(W) \in B_2 \text{ (or) } \ldots$$

$$\Leftrightarrow W \in X^{-1}(B_1) \text{ (or) } W \in X^{-1}B_2 \text{ (or) } \ldots$$

$$\Leftrightarrow W \in \cup X^{-1}B_k$$

$$\therefore W \in X^{-1}\left(\bigcup_{k\geq1} B_k\right) \Leftrightarrow W \in \bigcup_{k\geq1} X^{-1}(B_k)$$

Thus inverse mapping preserves union operation

iv) Let $W \in X^{-1}(B)^c \Leftrightarrow X(W \notin B)$

$$\Leftrightarrow X(W) \notin B$$
$$\Leftrightarrow X \notin X^{-1}(B)$$
$$\Leftrightarrow W \in [X^{-1}(B)]^c$$

Thus inverse mapping preserves complementation operation.

$$X^{-1}(\Omega') = \{W\ X(W) \in \Omega'\} = \Omega$$

$$X^{-1}(\phi) = X^{-1}(\Omega')^c = [X^{-1}(\Omega')]^c = \Omega^c = \phi$$

Further if A and B are only two disjoint sets $\in \Omega'$, then $X^{-1}(A)$ but $X^{-1}(B)$ are also disjoint. If C' is a class of subset of Ω', then $X^{-1}C = \{C : X(W) \in \Omega'\}$ is also a class of subset of W. If C is closed under complementation and countable intersection operation then $X^{-1}(C')$ is also called under complementation and countable intersection operation.

(ie) If C' is a σ field $\in \Omega'$ then $X^{-1}(C')$ is also a σ field Ω.

Book Work 3:

If ϵ is a field of subset of Ω' and if $x : \Omega \to \Omega'$ then $x^{-1}(C')$ is also a field.

Proof:

Given that C is a field Ω' belonging to Ω' $c' = c\{\phi, \Omega, \bigcap_{i=1}^{n}, \bigcup_{i=1}^{n}, c\}$

We have prove that,

$$x^{-1}(c') = c\{\phi, \Omega, \bigcap_{j=1}^{n}, \bigcup_{j=1}^{n}, c\}$$

i) Let $C_k \in C'$ then,

$$W \in x^{-1}(\bigcap_{k} C_k) \Leftrightarrow x(W) \bigcap_{k=1} C_k$$

$$\Leftrightarrow x(W) \in C_k \quad \forall k$$

$$\Leftrightarrow W \in x^{-1}(C_k) \quad \forall k$$

$$\Leftrightarrow W \in \bigcap_{k} x^{-1}(C_k)$$

for $k = 1, 2, \dots, \Omega$

$C_k \in C' \subset \Omega$ given that,

$x^{-1}(C_k) \in C \subset \Omega$ and $\bigcap_{k \geq 1} x^{-1}(C_k) \in C\ \Omega'$

$\therefore x^{-1}(C') \in \Omega$

Satisfies finite intersection operator

ii) Let $C_k \in C$ then

$$W \in x^{-1}\left(\bigcup_{k \geq 1} C_k\right) \Leftrightarrow x(W) \in \bigcup_{k \geq 1} C_k$$

$$\Leftrightarrow x(W) \in C_1 \text{ (or) } x(W) \in C_2 \text{ (or)}$$

$$\Leftrightarrow W \in x^{-1}(C_1) \text{ (or) } (W)\ x^{-1}(C_2) \text{ (or)}$$

$\Leftrightarrow W \in \bigcup_{k \geq 1} x^{-1}(C_k)$ for $K = 1, 2, \ldots n$

$C_k \in C \subset \Omega'$ gives that $x^{-1}(C_k) \subset C \subset \Omega$ and

$\bigcup_{k \geq 1} x^{-1} c(C_k) \in C \subset \Omega$ $x^{-1}(C') \in \Omega$ satifies finite union operations

iii) Let $W \in x^{-1}(C_k)$

$\Leftrightarrow x(W) \in C_k^c$

$x(W) \notin C_k$

$\Leftrightarrow W \in x^{-1}(C_k)$

$\Leftrightarrow W \in [x^{-1}(C_k)]^c$

$x^{-1}(C) \in \Omega$ satisfy complement operation

Further,

$x^{-1}(\Omega') = \{W : x(W) \in \Omega'\} = \Omega$ and

$x^{-1}(\Omega')^c = [x^{-1}(\Omega')]^c = \Omega^c = \phi$

$x^{-1}(\Sigma') = c\{\phi, \Omega, \bigcap_{i=1}^{n}, \bigcup_{i=1}^{n}, c\}$ for all any field $C_k \in C'^c \Omega'$

$\Rightarrow x^{-1}(C')$ is also a field $\in \Omega$.

<u>INDICATOR FUNCTION</u>:

Consider the real valued function I_A or $I_{(A)}(W)$ defined as Ω follows

$$\left.\begin{array}{ll} I(A)W = I_A(W) = 1 & W \in A \\ I_A(W) = 0 & W \in A^c \end{array}\right\} \rightarrow (1)$$

then I_A (or) $I(A)$ is called indicator function of A, then straight range of $I_A(\Omega) = \{I_A(W) : W \in \Omega\} = \{0, 1\}$.

If $B \subset R$, the Range space then,

$$I_A^{-1}(B) = \{\phi, A, A^c, \Omega\} = \sigma\{A\} \rightarrow (3)$$

Evidently CI_A takes the values C and A and 0 an A^c and hence

$$C(I_A)^{-1}(B) = I^{-1}A(B) = \sigma(A)$$

Now $I\Omega$ takes the values 'i for all $w \in \Omega$. If $x(w) = C \ \forall \ w \in \Omega$ then $x = CI\Omega$ and this case,

(ii) $I^{-1}\Omega(B) = \{\phi, \Omega\} = (CI\Omega)^{-1}B \rightarrow (5)$

MEASURABLE FUNCTION, BOREL FUNCTION, INDUCEED SIGMA FIELD:

A important class of function is that of real valued function x on Ω to R. Let B be the Borel field of subsets of R and script be a σ field of subset of Ω.

DEFINITION:

If $x^{-1}B \leq A \ \forall$ boral sets $B \leq B$, x is said to be a A measurable function or a function measurable with respect to A.

If Ω is also the real line R, its subset and if x is measurable with respect to the borel field B on the domain. Then x is called Borel function. Thus, The mapping $x: R \rightarrow R$ is a borel function of $x^{-1}B \leq B \ \forall$ borel set B in the range space and borel field B is the domain is said to be a measurable function.

CANATHEODORY ERTENSION FUNCTION:

Probability defined on a field F can be extended uniquely to the minimal σ-field containg F.

An important measurable space is (R, B) where B is the Borel field of subset of the real line R.

B is the minimal σ-field containing the class of all intervals of the form (a, b) $a, b \leq k$ say. But this later class is closed under finite intersection operations. But it is not a field. So, it not clear, whether probability function defined on all such intervals (a, b) can be extended uniquely to all borel sets.

Limits of Random variable:

Let $\{X_n\}$ be a sequence of r.v's on $\{\Omega, A\}$. Let us define function Y_n & Z_n and σ-field Ω $Y_n(W) = \inf\limits_{k \geq n} X_n(W) =$ greatest lower bound of $X_k(W)$ for $k \geq n$.

$Z_n(W) = \sup\limits_{k \geq n} X_k(W) =$ Least upper bound of $X_k(W)$ for $k \geq n$

Then as $n \to \infty$

$$Y_n(W) \uparrow \sup Y_n(W) = \lim \inf X_n(W) = \underline{\lim}\, X_n(W)$$

$$Z_n(W) \downarrow \inf Z_n(W) = \lim \sup X_n(W) = \overline{\lim}\, X_n(W)$$

Thus function

$\underline{\lim}\, X_n(W)$ & $\overline{\lim}\, X_n(W)$ will exists $\forall\, W \in \Omega$.

More over $\lim X_n(W) \geq \lim X_n(W)$ $\forall\, W$

ELEMENTARY FUNCTION :

A countable linear combination of indicator function is called an elementary function. A few authors do not distensive simple and elementary functions and called them both as simple functions.

IMPORTANT QUESTION :

1. Define inverse function.
2. Define measurable function.
3. Write any two continuity property of probability space.
4. Define Indicator function.
5. Prove that inverse mapping preserves all set relations.
6. Describe Indicator function.
7. What are the Properties of Indicator function ?
8. Show that a field is finite intersection operations. a class closed under complementation and finite union is a field.
9. Shows that the σ-field induced by a simple function is the minimal σ-field.
10. Explain the limits of random variable.

UNIT - 3

DISTRIBUTION FUNCTION:

Def: Let x be a real r.v. In the probability (Ω, A, B) for $x \in R$ define (i) $P(X \leq x) = f_x(x)$ (ii) $P_x(a,b) = P(a \leq x \leq b) \Rightarrow P(X) = F_x(b) - f_x(a)$ $(b > a)$ then $f_x(x)$ is called the distribution function (or) function of x cumulative

PROPERTIES OF DISTRIBUTION FUNCTION:

In statistical practice, the generally take the relative frequency as an estimate A the probability or an event. It is n experiments x times the value x is deserved to less than are equal to x. k/n is an estimate of $f_x(x) = P(x \leq x)$. Hence, the probability distribution of x is estimated. If x is discreate and m is the number or items exist observed to have the value x. $P(x = x)$ is estimated by M/n. Thus distribution function from the foundation of the theory of probabilities and statistics.

JORDAN - HANN DE COMPOSITION THEOREM I:

Let ϕ be a σ additive set function on the measurable space (Ω, A). Such that, It can assume that most me of the value I_∞, without lass of generally we assume that it is bounded below (i.e), $-\infty \leq \phi(A) \leq \infty$. Define for $A_i, B_1, B_2, B \in A$

$$\phi^+ A = Sup\, \phi(B_1) \quad B_1 \in A \quad \phi^-(A) = Sup\left[-\phi(B_2)\right] B_2 \in A.$$

$$|\phi| A = \phi^+(A) + \phi^-(A) \Rightarrow \underset{B_1, B_2 \in A}{Sup} \left\{ \phi(B_1) - \phi(B_2) \right\}$$

then $\phi^+, \phi^-, |\phi|$ are called the positive variance, the negative variance and the total variance of ϕ respectively. It ϕ is σ additive we can prove that ϕ^+ and ϕ^-. $|\phi|$ are non-negative σ additive set function (i.e), measure the ϕ as a finite measure. More overs $\phi = \phi^+ - \phi^-$ this is called JORDAN-HANN DE Composition of the relation says that any σ additive set function can be considered as the difference or two measure of which atleast one is finite a set D is set to be negative with respect to ϕ if for every measurable Set A DA is measurable and $\phi(DA) \leq 0$. Similarly E is set to be positive with respect to ϕ if $\phi(EA) \geq 0$ $\forall A \in \mathcal{A}$ (script A) entention of positive and negative sets and their relation with ϕ is given by the following theorem is called JORDAN-HANN DE Composition of ϕ.

THEOREM: 2:

Let ϕ be a σ field an σ additive function definded on a σ field and sub set of Ω. Then there are two sets $E \in A$: (i) $D \cup E = \Omega$ (ii) $D \cap E = \phi$ (iii) $\phi^+(D) = \phi^-(E) = 0$. The last properties implies that the is negative and E is positive with respect to ϕ. Hence $\phi^-(A) = -\phi(AD)$, $\phi^+(A) = \phi(AE) = \phi(AD^c)$, $\phi(A) = \phi(AD) = -\phi(AD^c)$ for the proof of the theorem.

EXAMPLE 1:

If x is an integral function on $(\Omega, A,)$ an

$$\phi(A) = \int_A x\,du \,.\, D \Rightarrow [x < 0],\ D^c = [x \geq 0]$$

$$\phi^+(A) = \int_A x^+ = \int x_1 (AD^c) = \phi(AD^c)$$

$$\phi^-(A) = \int_A x^- = -\int_A x_1 (D) = \phi(AD)$$

$$|\phi|(A) = \int_A (x^+ + x^-) = \int_A |x| \, du.$$

CORRESPONDANCE THEOREM:

An arbitary distribution function is defineded as a non - decreasing finite function Continuous from the

$$F(x-0) = \lim_{x_n \uparrow x} F(x_n) = \sup_{x' < x} F(x')$$

$$F(x+0) = \lim_{x_n \downarrow x} F(x_n) = \inf_{x' > x} F(x') = F(x)$$

exist and bounded by $F(-\infty)$ and $F(\infty)$

DEFINITION:

A set D is set to Dense in R. If any point A, R is either in D (or) is a limit point of D.

for example: The set of all relations are dense in the real line countable dense subset is called a seperating subset relation also from a seperating subset of R.

EMPRICAL DISTRIBUTION FUNCTION:

We have stated that if k values of x are less than or equal to x out of n observed values, then k/n is an 'estimate' or the distribution function of x

Definition: Let $x_1, x_2, \cdots x_n$ be the n observed of x arranged in the increasing order or magnitude called ORDER STATISTICS and we define

$$F_n(x) = \begin{cases} 0 & , x < x_{(1)} \\ k/n & , x_{(k)} \le x \le x_{k+1} ; \quad k = 1, 2, \cdots n-1 \\ 1 & , x > x_{(n)} \end{cases}$$

Then $F_n(x)$ is called EMPIRICAL DISTRIBUTION FUNCTION of x.

It is a set function with jumps or magnitude $1/n$ at each of the observed values. If $F_n(x)$ is supposed to approximate $F(x)$ to which it convergence as $n \to \infty$. $\operatorname{Sup} |F_n(x) - F(x)|$ is measure or this approximation. This measure has the proportion or a 'distance' or a matric and is called KALMOGROV - SMIRNMOV DISTANCE BETWEEN F_n and F.

Here $f(x)$ is a theoritical distribution function x serving as a model which $F_n(x)$ is the observed distribution function KOLMORGOV - SMIRNMOR distance can be used to verify whether the thousand model is satisfaitory and agrees with the observed values.

GRAMER RAO INEQUALITY:

STATEMENT:

$$E|X+Y|^r \le C_r E|X|^r + C_r E|Y|^r$$

where, $C_r = 1$, if $r \le 1$

$$C_r = 2^{r-1}, \text{ if } r \ge 1$$

PROOF.

If $a>0$, $b>0$

$$\left[\frac{a}{a+b}\right]^r + \left[\frac{b}{a+b}\right]^r \ge 1, \text{ for } r \le 1$$

(i.e) $a^r + b^r \ge (a+b)^r$

Hence for all ω and $r \leq 1$ $|X(\omega)|^r + |Y(\omega)|^r \geq (|X(\omega)| + |Y(\omega)|)^r$
$\geq [|X(\omega) + Y(\omega)|]^r \quad \rightarrow (1)$

for $r \geq 1$. Consider the function,

$$\phi(P) = P^r + (1-P)^r, \quad (0 < P < 1)$$

which has a minimum when $p = 1/2$. Thus for $a > 0$, $b > 0$

$$\left(\frac{a}{a+b}\right)^r + \left(\frac{b}{a+b}\right)^r \geq 2^{-(r-1)}$$

$$2^{r-1}(a^r + b^r) \geq (a+b)^r$$

$$2^{r-1}\left[\,|X(\omega)|^r + |Y(\omega)|^r\right] \geq \left[|X(\omega)| + |Y(\omega)|\right]^r$$

$$\geq |X(\omega) + Y(\omega)|^r \quad \rightarrow (2)$$

Taking expectation of (1) and (2) we obtain. Hence, we have the Cr inequalities. This lemma implies that if the r^{th} absolute moments of x and y exist and are finite. So, also is the r^{th} absolute moment of $x + y$. There are restrictions on the distributions of x and y. This the class of random variables whose s^{th} (or) absolute moment is finite is close under addition.

HOLDER'S INEQUALITY:

STATEMENT:

$$E(XY) \leq E^{1/r}|X|^r \, E^{1/s}|Y|^s$$

where $r \geq 1$ and $r^{-1} + s^{-1} = 1$

Here and else where $E^{1/P}|X|^r$ denotes $\sqrt{(E|X|)^r}$

PROOF:

Consider $\phi(P) = \left(\frac{P^r}{r}\right) + \left(\frac{P^{-s}}{s}\right)$, $0 < P < \infty$

Now, $\phi(P)$ has a minimum when $P=1$, then $\phi(1) = 1$ putting $P = P_0 = \frac{a^{1/s}}{b^{1/r}}$ $(a > 0, b > 0)$

$$\phi(P_0) = (b^r)^{-1} a^{r/s} + (a^s)^{-1} b^{s/r} \geq 1 = \phi(1)$$

$$\Rightarrow r^{-1} a^{s + r/s} + s^{-1} b^{r + s/r} \geq ab$$

$$\Rightarrow r^{-1} a^r + s^{-1} b^s \geq ab$$

Since, $\frac{r+s}{rs} = 1$ putting $a = |X(\omega)| \, [E|X|^r]^{1/r}$; $b = |Y(\omega)| \, [E|X|^s]^{1/s}$ and taking expectation and using the assumption that $(1/r) + (1/s) = 1$ we have HOLDER'S INEQUALITY.

MINKOWSKI INEQUALITY:

THEOREM:

$$E^{1/r}|X+Y|^r \leq E^{1/r}|X|^r + E^{1/r}|Y|^r \quad \text{for } r \geq 1.$$

PROOF:

$$E|X+Y|^r = E\left[|X+Y| \cdot |X+Y|^{r-1}\right]$$

$$\leq E\left[|X| . |X+Y|^{r-1}\right] + E\left[|Y| \, |X+Y|^{r-1}\right]$$

$$\leq E^{1/r}|X|^r \, E^{1-1/r}|X+Y|^r + E^{1/r}|Y|^r \, E^{1-1/r}|X+Y|^r$$

using Holder's inequality with $S = r/r-1$ dividing by $E^{1-1/r}|X+Y|^r$ we have.

$$\leq \frac{E^{1/r} + |X|^r \, E^{1-1/r}|X+Y|^r + E^{1/r}|Y|^r \, E^{1-1/r}|X+Y|^r}{E^{1-1/r}|X+Y|^r} \leq \frac{\left[E^{1/r}|X|^r + E^{1/r}|Y|^r\right]}{E^{1-1/r}|X+Y|^r}$$

$$E^{1/r}|X+Y|^r \leq E^{1/r}|X|^r + E^{1/r}|Y|^r$$. Hence proved.

CAUCHY SCHWDRTZ'S INEQUALITY:

$$E|XY| \leq \sqrt{E|X^2| E|Y^2|} \longrightarrow (1)$$

which is called Schwortz's inequality. Since $E|XY| > |E(XY)|$ from equation (1) replacing x by $x - E(x)$ and y by $y - E(y)$ we have, the inequality $E(XY) - E(X)E(Y) \leq \sqrt{\text{Var}\,x, \text{Var}\,y} \rightarrow (2)$

(i.e) $E|x - E(x)| - E|y - E(y)|$

$E|(x - E(x))| - E|y - E(y)|$ called the Co-variance of x and y is less than the product of standard deviation of x and y in absolute value.

This implies that the Correlation Coefficient between x and y defined by,

$$r_{x,y} = \frac{Cov(x,y)}{(S.D\,x)(S.D\,y)}$$

has it absolute value always less than unity Janson's inequality.

JANSON'S INEQUALITY:

THEOREM:

If f is Convex and $E(x)$ is finite then $F(E(x) \leq E$ of $f(x) \rightarrow (1)$

PROOF:

Let x be a random variable whose values lie in I and F be on I Replacing x_0 and x by x we have, for all ω.

$E(x)\left[x(\omega)-E(Y)\right] \leq F\left[x(\omega)-F(E(x)\right]$ taking expectations since the L.H.S. vanishes we have $F[E(x)] \leq F f(x)$

If F is convex and monotone increasing and has an inverse F^{-1} then from equation (1)

$$Ex \leq F^{-1}\left[E f(x)\right]$$

for exampl is,

$E(x) = x^r, \quad r \geq 1, \; x \geq 0$ we have

$E x' \leq E^{1/r}_{|x|^r}$

In facts $Ex' \leq E^{1/r}_{|x|^r}$ is a particular case of the inequality.

$E^{1/r}_{|x|^r} \leq E^{1/p}_{|x|^p}$, $r < p$ called Liapounov's Inequality which can proved by using Schwartz inequality. This shows that $E^{1/r}$ is a non-decreasing function of r.

Hence proved

BASIC INEQUALITY :

Let x be an arbitary random variable g on R be a non-negative borel function. If g is 0 even and is non-decreasing on $[0, \infty]$ then for every $a > 0$.

$$\frac{E[g(x)] - g(a)}{a \cdot s\, Sup[g(x)]} \leq P[|x|] \geq a \leq \frac{E[g(x)]}{g(a)} \rightarrow (1)$$

If g is non-decreasing on R and R $P[|x|] \geq a$ is replaced by

$P[X \geq a]$ is true for every read a.

Here a.S $\operatorname{Sup} g(x)$ is the least among constants below which almost all values $X(\omega)$.

PROOF:

Since g is a borel function on R it follows that $g(x)$ is a measurable function on ω and is a r.v. Since g is non-negative its integer exists

Hence, $E g(x) = \int_A g(x) + \int_{A^c} g(x) \longrightarrow (2)$

where $A[|X| \geq a]$ on A. Since g is non-decreasing and even

$g(a) \leq g(x) \leq a.S.\operatorname{Sup} g(x)$

Hence,

$$g(a) \cdot PA \leq \int_A g(x) \leq a.s \operatorname{Sup} g(x)\ PA$$

Since,

$$\text{on } A;\ 0 \leq g(x)\ g(a)$$

$$0 \leq \int_{A^c} g(x) \leq g(a) pA^c \leq g(a)$$

Adding (4) & (5)

$$g(a) \cdot P^A \leq E[g(x)] \leq a.s.\operatorname{Sup} g(x) \cdot \phi^A + g(a)$$

Hence we have

$$\frac{E[g(x)] - g(a)}{a.s.\operatorname{Sup} g(x)} \leq PA \leq \frac{E(g(x))}{g(a)}$$

If g is non-negative and non-decreasing on R and if $A' = P[X \geq a]$ all the above statements are true for A' replacing A (*).

DISTRIBUTION FUNCTION:

Let x be a random variable the function f is defined for all real x by $f(x) = P(X \leq x) \Rightarrow P\{\omega : X(\omega) \leq x\}$.

PROPERTIES OF DISTRIBUTION FONCTION:

If F is the distribution function of the random variable and if $a < b$ then

$$P[a < x \leq b] = f(b) - f(a).$$

If f is a distribution function of the medimential random variable of x then

(i) $0 \leq f(x) \leq 1$ (ii) $F(x) \leq F(Y)$ if $x < y$.

DISTRIBUTION FUNCTION OF RANDOM VECTOR:

Let x and y be a random variable defined an a probability $(X+Y)(\omega) + Y(\omega)$ for $\omega \in \Omega$.

CONDITIONAL DISTRIBUTION:

The dimensional random variable (x, y) the joint distribution function $F_{XY}(x,y)$ for any real numbers x and y is given by $F_{XY}(x,y) = P[X \leq x, Y \leq y]$

MARGINAL DISTRIBUTION FUNCTION:

The joint distribution function $F_{XY}(x,y)$ it is possible to obtain the individual distribution function. $F_X(x)$ and $F_Y(y)$

which are termed as marginal distribution function $F_{XY}(x,y)$

$$F_X(x) = \sum_y P_{XY}(x,y)$$

$$F_Y(y) = \sum_x P_{XY}(x,y)$$

EXPECTATION:

If X be a discrete or continuous random variable which values $x_1, x_2, \ldots x_n$ with probabilities $P(x_1), P(x_2) \cdots P(x_n)$ (or) probability density function $f(x_1), f(x_2), \ldots f(x_n)$, then the mathematical value as follows

$$E(X) = \sum_{i=1}^{n} x\, f(x_i)\, dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x_i)\, dx .$$

CHARACTERISTIC OF FUNCTION:

(i) For all real we have (a) $\phi(0) = \int_{-\infty}^{\infty} f(x) = 1$ (b) $|\phi(t)| \leq 1 = \phi(0)$

(ii) $\phi(t)$ Continuous every where $\phi(t)$ continuous function of t in $(-\infty, \infty)$ rather $\phi(t)$ is uniformly continuous in t.

IMPORTANT QUESTIONS :-

1. Distribution function
2. Minkowski inequality
3. Marginal distribution function
4. Schwartz's inequality
5. State and prove Minkowski inequality
6. State and prove Schwartz's inequality
7. State and prove Jordan Decomposition theorem
8. Conditional Distribution
9. Marginal Distribution Function
10. State and prove Cauchy Schwartz's Inequality.
11. State and prove Janson's Inequality.
12. Emprical Distribution function.

UNIT - IV

TYPES OF CONVERGENCE:

There are four types of Convergence. They are given below (i) Convergence in probability (or) weak convergence (ii) Convergency with probability (or) Almost sure convergency (iii) Convergency in distribution (or) Convergency in law (iv) mean square convergence (or convergence in r^{th} mean.

(i) CONVERGENCE IN PROBABILITY (OR) WEAK CONVERGENCE :-

Def : A square of r.v $\{x_n\}$ is said to convergence of a random variable x in probability denoted by $x_n \xrightarrow{P} x$. If for any $\epsilon > 0$ and $n \to \infty$, the relation.

$$P[|x_n - x| \geq \epsilon] \to 0 \quad (\text{or}) \quad P[|x_n - x| < \epsilon] \to 1$$

(ii) CONVERGENCE WITH PROBABILITY (or) ALMOST SURE CONVERGENCE :-

Definition : 1 :

A sequence of random variable $\{x_n\}$ is said to convergence to a random variable x almost surely if $\forall\, \omega$ $x_n(\omega) \to x(\omega)$ expect for there points belongs to the null set.

Definition : 2 :

$x_n \xrightarrow{A.S} x$ if for any $\epsilon > \frac{1}{r}$ the relation,

$$P\left[\bigcup_{k=n}^{\infty} |x_k - x| \geq \frac{1}{r}\right] \to 0 \quad \forall\ r \text{ is satisfied}$$

(iii) CONVERGENCE IN DISTRIBUTION (OR) CONVERGENCE :-

Definition :

A sequence of random variable $\{X_n\}$ is said to converge to a random variable X in law denoted by $X_n \xrightarrow{L} X$

If $F_n \to f$ weakly (or) $F_n(x)$ $\forall$ $x \in (F)$ Cumulative function.

(iv) MEAN SQUARE CONVERGENCE (OR) CONVERGENCE IN r^{th} MEAN :-

Definition :

A sequence of random variable $\{X_n\}$ is said to convergence to a random variable X in r^{th} mean denoted by.

$X_n \xrightarrow{r} X$, if $E|X_n - X|^r \to 0$ as $n \to \infty$

If $r=1$ it's called Convergence in first mean, $r=2$ it's Convergence in second mean (or) mean Square Convergence

$X_k \leq \lim Y_n \leq X \to (3)$

$E X_k \leq E \lim Y_n \leq E X = E(\lim X_n)$

Since, $E \lim Y_n = \lim E Y_n$ and further as $k \to \infty$ $X \leq \lim Y_n \leq X$.

$\lim E X_k \leq E \lim Y_n \leq \lim E X_n$.

$E \lim Y_n = EX$.

(i.e) $E[\lim(\max X_{kn})] = EX$

(i.e) $E(X_k) = EX$

Thus for any increasing sequence of non-negative random variabl

Thus $0 \le X_n \uparrow X \Rightarrow EX_n \uparrow EX$

Corollary :- $0 \le X_n \downarrow X \Rightarrow EX_n \downarrow EX$

PROOF:

Given $X_n \downarrow X$

Consider $\{-X_n\}$ denote $Z_n = -X_n \Rightarrow Z_n \uparrow$

form part if $Z_n \uparrow \Rightarrow EZ_n \uparrow E_Z$

(i.e) $X_n \uparrow -X \Rightarrow E(-X_n) \uparrow E(-X)$

(i.e) $X_n \downarrow X \Rightarrow E(X_n) \downarrow E(X)$

Since, $E(-X_n) = -E(X_n)$ and $E(-X) = -E(X)$

MONOTONE CONVERGENCE THEOREM:

Statement:-

If $0 \le X_n \uparrow X$, then $E(X_n) \uparrow EX$. Here EX may be finite (or) infinite.

PROOF:

Let $\{X_k\}$ be a sequence of non-negative random variable (i.e) $X_k \ge 0 \; \forall k$. W.k.T any random variable can be written as a sequence of simple function.

(i.e) write

$X_k = \sum_m X_{km}$ Convergent to X_k as $m \to 0$, $X_{km} \to X_k$ (i.e).

$0 \le X_{11} \le X_{12} \le \ldots \quad \le X_{1n} \le \ldots \quad \to X_1$

$0 \le X_{21} \le X_{22} \ldots \ldots \le X_{2n} \le \ldots \quad \to X_2$

. . . .

. . . .

$0 \le X_{k1} \le X_{k2} \le \dots \le X_{kn} \le \dots \to X_k$

. . . .

$0 \le X_{n1} \le X_{n2} \le \dots \le X_{kn} \le \dots \to X_n$

Let $Y_n = \max\limits_{k \ge 1} X_{kn} \longrightarrow (1)$

Since X_{kn} is simple function $n \Rightarrow Y_n$ is also simple function

further from (1)

$$X_{kn} \le Y_n \le X_n \quad \longrightarrow (2)$$

$\Rightarrow E X_{kn} \le E Y_n \le E(X_n)$

Taking lim as $n \to \infty$ we have from (2) from (1) and (2)

if $\{X_n\}$ is any arbitary Sequence Convergence to X.

$E X_n \longrightarrow E X$.

THEOREM: DOMINATED CONVERGENCE

Statement:

If $|X_n| \le Y$ almost surely and Y is Integrable then

$X_n \xrightarrow{P} X \Rightarrow E X_n \longrightarrow E X$

PROOF:

Case (i) Take $X = 0$

(i.e) $X_n \xrightarrow{P} 0$

Consider the $E(x_n)$. $\lim E x_n$ exists

(i.e.) $[\overline{\lim}\, E x_n$ and $\underline{\lim}\, E x_n]$

Since $x_n \xrightarrow{P} 0$. There exists a sub. sequence $\{x_n'\}$ which also convergence to zero in probability

Then $E x_n' \to \lim E x_n$.

Since $x_n' \xrightarrow{P} 0$ there exists a another sub-sequence of $\{x_n''\}$ which also convergence almost surely to zero

i.e $x_n'' \xrightarrow{A.S} 0$

my^{ly}, choosing from a sub-sequence again from a sub-sequence we have.

$$E x_n'' \to 0$$

$$\lim E x_n = \overline{\lim}\, E x_n = \underline{\lim}\, E_n = 0$$

Thus, $E x_n = 0$

(ie) $x_n \xrightarrow{P} 0 \Rightarrow E x_n \to 0$

Case (ii)

If $x_n \xrightarrow{P} x$ where $x \neq 0$

$$x_n - x \xrightarrow{P} \phi r$$

$\therefore$ from previous result

$$x_n - x \xrightarrow{P} 0 \Rightarrow E(x_n - x) \to 0.$$

(i.e) $x_n \xrightarrow{P} 0 \Rightarrow E(x_n - Ex)$

Where $|x_n| \leq y$ almost surely and y is integrable.

Hence, the proof of dominated Convergence theorem.

CHARACTERISTIC FUNCTION DEFINITION:

In this selection, by a distribution function $F(x)$. We usually refer to a distribution function of a random variable with $F(-\infty) = 0$ and $F(\infty) = 0$. However, most of the results can be easily modified of the results can be easily modified of the results can be easily modified to general d.f's to a d.k $F(x)$ we correspond a complex valued function.

$$\phi F(x) = \int_R \exp(iux)\, dF(x), \quad u \in R, \quad i = \sqrt{-1} \quad \rightarrow (1)$$

Called the characteristic function on F. We, shall denote $\phi F(u)$ by $\phi(u)$ if there is no confusion. This is the fourier transform defined in section. The function $\phi(u)$ always exists and is well defined, for

$$\phi(u) = \int \exp(iux)\, dF(x)$$

$$= \int \cos ux\, dF(x) + i \int \sin ux\, dF(x) \quad \rightarrow (2)$$

If F is the d.F of a random variable of x. $\phi(u)$ may be written as $\phi_x(u)$ and is also called the characteristic further function of x.

PROPERTIES NORMAL DISTRIBUTION

Let x be a standard normal variate with pdf

$$F(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \quad . -\infty < x < \infty \longrightarrow (3)$$

The characteristic function of x is

$$\phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(iux) \exp(-x^2/2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos ux \ \exp(-x^2/2)\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin ux \ \exp(-x^2/2)\, dx$$

Since, $\exp(-x^2/2)$ is even and $\sin ux \exp(-x^2/2)$ is odd, the complex part of $\phi(u)$ vanishes. Evaluating the real part, $\phi(u) = \exp(-u^2/2)$ -(4)

In general, if x is symmetric about the origin (i.e), its p.d.f is even, with $f(x) = -f(x)$ then the corresponding characteristic function is real.

EXPONENTIAL DISTRIBUTION:

If x has exponential probability density function, with

$$f(x) = \begin{cases} \exp(-x) & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

The characteristic of x is

$$\phi(u) = \int_{0}^{\infty} \exp(iux) \exp(-x)\, dx = (1-iu)^{-1} \longrightarrow (5)$$

BINOMIAL DISTRIBUTION:

Let x be binomial random variable with

$$P(X=x) = \binom{n}{x} p^x q^{n-x} \qquad p, q > 0, \ p+q = 1 \quad (x = 0, 1, 2, \ldots n)$$

The characteristic function of x is

$$E[\exp(iux)] = \sum_{x=0}^{n} \exp(iux) \binom{n}{x} p^x q^{n-x} = [q + p\exp(iu)]^n \rightarrow (6)$$

POISSON DISTRIBUTION:

Let x be a discrete random variable with 1

$$P[x = x] = \exp(-\lambda) \frac{\lambda^x}{x!} [x = 0, 1, \dots \quad \lambda > 0]$$

The characteristic function of x is

$$E[\exp(iux)] = \sum_{x=0}^{\infty} \exp(iux) [\exp(-\lambda)\lambda^x] / x!$$

$$= \exp(-\lambda) \exp[\lambda \exp(iu)]$$

$$E[\exp(iux)] = \exp[-\lambda\{1 - \exp(iu)\}] \quad \rightarrow (7)$$

If x is degenerate at c, write $P[x = c] = 1$

$$E[\exp(iux)] = \exp(iuc) \quad \rightarrow (8)$$

Thus, $\exp(iuc)$ is the characteristic function of a d.f with jump of unit magnitude at $x = c$ unity is degenerate at the origin.

If $F(x) = \alpha F_1(x) + (1-\alpha) F_2(x)$, $0 < x < 1$, a m/x true of two d.f's with relative weights α and $(1-\alpha)$ $\phi F(u) = \alpha \phi F_1(u) + (1-\alpha)\phi F_2(u)$

In general, if

$$F(x) = \sum_{i=1}^{n} \alpha_i F_i(x), \quad [\alpha_i \geq 0, \ \Sigma \alpha_i = 1]$$

then $\phi F(u) = \sum_{i=1}^{n} \alpha_i \; \phi F_i(u) \longrightarrow (9)$

We may note that F_1 may be discrete and F_2 may be continuous. For example, if F is a mixture A a distribution degenerate at the origin and exponential distribution with weight P and $(1-P)$ respectively.

$$\phi(u) = P + (1-P)(1-iu)^{-1}.$$

As already stated in $E[\exp(\phi)]$ the m.g.f of x (where ϕ is real and belongs to a certain interval of R] may not exist. Therefore, we prefer to work with the characteristic function rather than the m.g.f. In the discrete case the p.g.f is very convenient and easy to handle. As already stated in Chapter 5, Characteristic is also called the fourier transforms, including the m.g.f and the P.g.f share most of the properties of the characteristic function we may also note have that the moments of Cauchy distribution do not exist. Hence, the m.g.f does not exist. But the Characteristic function exists.

SOME INEQUALITIES: LEMMA:1

For any characteristic function ϕ

$$\left.\begin{aligned}
&\text{(i) } Re\,[1-\phi(u)] \geq (1/4)\, Re\,[1-\phi(2u)] \\
&\text{(ii) } |\phi(u) - \phi(u+h)|^2 \leq 2\{1 - Re\,\phi(h)\} \\
&\text{(iii) } \int_{|x|} x^2 \, d f(x) \leq 3/u^2 \{1 - Re\,\phi(u)\}
\end{aligned}\right\} \longrightarrow (1)$$

Where $Re\,\phi$ is the real part of ϕ

PROOF :

(i) $1 - Re\,\phi(2u) = \int (1 - \cos 2ux)\, d.f(x)$

$$= 2\int (1 - \cos 2ux)(1 + \cos ux)\ d.F(x)$$

$$\leq 4 \int (1 - \cos ux)\ dF(x)$$

$$1 - Re\,\phi(2u) \leq 4\left[1 - Re\,\phi(u)\right]$$

(ii) $|\phi(u) - \phi(u+h)|^2 = \left[\int e^{iux}(1 - e^{ihx})\ dF(x)\right]^2 \leq \int |1 - e^{ihx}|^2\, dF(x)$

$$= 2\int (1 - \cos hx)\ dF(x) \Rightarrow 2\left[1 - Re\ \phi(h)\right]$$

Note that the right side of (ii) does not depend on u.

(iii) Since $1 - \cos x \geq (1/2)x^2\left[1 - (1/12)x^2\right]$

$$1 - Re\,\phi(u) = \int_{-\infty}^{\infty} [1 - \cos ux]\ d\ F(x) \geq \int_{-\infty} 1/2\,(ux)^2\left[1 - 1/_{12}(ux)^2\right] dF(x)$$

$$\geq \frac{11u^2}{24} \int_{|x| < u^{-1}} x^2\ dF(x)$$

$$\geq \frac{u^2}{a^3} \int_{|x| < u^{-1}} x^2\, d\ F(x)$$

CONVERGENCE :

The change of total probability value 0 (or) 1 is called Convergence.

LEMMA : 2

Let U_k, V_k $[K=1,2,\ldots n]$ be elements of a set of n arbitary numerical values, n also arbitary. Then for any function in [Real or Complex], defined on this set,

$$\sum_u \sum_v \phi(u-v)\, h(u)\, \bar{h}(v) \geq 0 \quad \rightarrow (1)$$

Where the summation is extended over all the n possible values of u and v.

PROOF:

$$\sum_u \sum_v \phi(u-v)\, h(u)\, \bar{h}(v) = \int_R \sum_u \sum_v \exp[i(u-v)x]\, h(u)\, \bar{h}(v)\; df(x)$$

$$= \int_R \left[\sum_u \exp(iux)\, h(u)\right]\left[\sum_v \exp(-ivx)\, \bar{h}(v)\right] df(x)$$

$$= \int_R \left|\sum_u \exp(iux)\, h(u)\right|^2 df(x) \geq 0$$

Definition:

A function g is saidtobe non-negative define, if for every finite set of n elements $\{u_1, u_2, \ldots u_n\} = \{v_1, v_2, \ldots v_n\}$ n arbitary and every real (or) complex valued function h.

$$\sum_u \sum_v g(u-v)\, h(u)\, \bar{h}(v) \geq 0.$$

Lemma (2) implies that a characteristic function is non-negative definite.

INVERSION THEOREM:

If a, b $(a<b)$ are points of continuity of F

$$F(b)-F(a) = F(a,b) = \lim_{u\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} \frac{\exp(-iua) - \exp(-iub)}{iu} \phi(u)\, du.$$

PROOF:

The integrand is defined at $u=0$ by continuity. The integrand is continuous every where and is bounded by $(b-a)\,\phi(0)$ its value at $u=0$ (prove) for every finite u, the integral.

$$I u = \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} \frac{e^{-iua} - e^{iub}}{iu} \phi(u)\, du, \quad a<b$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-u}^{u} \frac{e^{-iua} - e^{iub}}{iu} \int_{R} e^{iux}\, dF(x)\, du \longrightarrow (1)$$

is finite and using fubini's theorem we may interchange the order of integration (1). Thus,

$$I u = \frac{1}{\sqrt{2\pi}} \int_{R} \int_{-u}^{u} \frac{e^{-iua} - e^{iub}}{iu} e^{iux}\, du\, df(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{R} J(u)\, df(x) \rightarrow (2)$$

where $$J(u) = \int_{-u}^{u} \frac{e^{-iua} - e^{iub}}{iu} e^{iux}\, dx$$

$$= \left[\int_{-u}^{0} + \int_{0}^{u}\right] \frac{e^{iu(x-a)} - e^{iu(x-b)}}{iu}\, du$$

$$= 2\int_{0}^{u} \left[\frac{\sin u(x-a) - \sin u(x-b)}{u}\right] du$$

$$= 2\left[\text{sign}(x-a) \int_0^{u(x-a)} \frac{\sin v}{v}\, dv - \text{sgn}(x-b) \int_0^{u(x-b)} \frac{\sin v}{v}\, dv\right] \rightarrow (3)$$

where $\text{Sign}\ x = \begin{cases} 1 & , \text{if } x>0 \\ 0 & , \text{if } x=0 \\ -1 & , \text{if } x<0 \end{cases}$

Now, $a<b = (x-a) > (x-b)$

Case (i)

$$(x-b) > 0,\ (x-a) > 0$$

Then, $\frac{1}{2} J(u) = \int_{u(x-b)}^{u(x-a)} \sin v/v\, dv.$

we know, $\int_0^{\infty} \frac{\sin v}{v}\, dv = \pi/2 = \lim_{A\to\infty} \int_0^{A} \frac{\sin v}{v}\, dv$

Hence, there exists A_0 such that for $t_1, t_2 > A_0$.

$$\left| \int_{t_1}^{t_2} \frac{\sin v}{v}\, dv \right| < \epsilon$$

and for $t < A_0$. Splitting the range of integration into multiplies of π we get, an alternating the series with decreasing terms and hence.

$$\left| \int_0^{t} \frac{\sin v}{v}\, dv \right| \le H = \int_0^{\pi} \frac{\sin v}{v}\, dv < \infty$$

Hence for $t > A_0$ as well as for $t < A_0$

$$\left| \int_0^{t} \frac{\sin v}{v}\, dv \right| < H + \epsilon < \infty$$

Thus $\forall$ t_1, t_2 positive $\int_{t_1}^{t_2} \frac{\sin v}{v}\, dv$ in uniformly bounded and tends to zero as $t_1, t_2 \to 0$ [this may be compared with Cauchy criterion for convergence of an infinite series]. It follows that for $x > b$, $\lim_{u \to \infty} \frac{1}{2} J(u) = 0 \to (4)$

Case (ii) $(x-a) > 0$, $(x-b) > 0$, then as $u \to \infty$

$$\frac{1}{2} J(u) = \int_0^{u(x-a)} \frac{\sin v}{v}\, dv + \int_0^{-u(x-b)} \frac{\sin v}{v}\, dv$$

$$= \int_{u(x-b)}^{u(x-a)} \frac{\sin v}{v}\, dv \longrightarrow (5)$$

$$\int_0^{\infty} \frac{\sin v}{v}\, dv = \pi \longrightarrow (6)$$

Case (iii) $x < a$, then as in case (ii) as $u \to \infty$

$$\frac{1}{2} J(u) = 0 \longrightarrow (6)$$

Case (iv) : $x = a$ (or) $x = b$, Then as $u \to \infty$ $\frac{1}{2} J(u) \to \int_0^{\infty} \frac{\sin v}{v}\, dv = \pi/2 \to (7)$

Thus in general as $u \to \infty$ form (6) and (7)

$$\frac{1}{2} J(u) \Rightarrow \frac{1}{2} J(x) = \pi(a,b) + \pi/2 \left[J\{a\} + I\{b\} \right] \longrightarrow (8)$$

Since $J(u)$ is bounded by dominated convergence theorem from (2)

$$\lim_{u \to \infty} I_u = \frac{1}{2\pi} \int_R J(x)\, d f(x)$$

$$= \left[\{ F(b-0) - f(a+0) \} + \frac{1}{2} \{ f(a+0) - f(a-0) + f(b+0) - f(b-0) \} \right] \to (9)$$

This R.H.S of (9) equal $F(b) - F(a)$ if a and b are point continuity of F

CONVERGENCE OF DISTRIBUTION FUNCTION AND CHARACTERISTIC FUNCTION:

(a) Integral characteristic function:

The integral characteristic function $\hat{\phi}$ of F is defined

$$\hat{\phi} F(u) = \int_0^u \phi(u)\, du = \int_0^u \int_{-\infty}^{\infty} e^{iux} dF(x)\, du \longrightarrow (1).$$

Since $\phi(u)$ is bounded by $\phi(u)$ and $\hat{\phi} F(u)$ is bounded and hence we can interchange the order of integration in (1). Take care.

$$\hat{\phi} F(u) = \int_R \frac{e^{iux}-1}{ix}\, dF(x) \longrightarrow (2)$$

At $x=0$ the integrand of (2) is defined by conditions.

Since $\phi(u)$ is continuous there is (1-1) correspondence between $\phi(u)$ and $\hat{\phi} F(u)$ and by uniqueness property of characteristic function. There is one-to-one correspondence between $\hat{\phi} F(u)$ and F.

The following theorem gives a characteristic function for weak convergence.

THEOREM:

If $F_n \xrightarrow{w} F$ (u+ac) the $\hat{\phi}_n - \hat{\phi}_F$ conversely if $\hat{\phi}_n$ converges to some function $\hat{\psi}$, then exists a d.f F with $F_n \xrightarrow{w} F$ (u ± ac) and $\hat{\phi} F = \psi$ [$\hat{\phi}_n$ is the interval Ch. fn of F_n].

PROOF :

A function of x, $\left[\frac{e^{iux}-1}{ix}\right]$ is bounded by al $x=0$ and tends to zero $x \to \pm \infty$

$$\hat{\phi}_n(u) = \int_R \left(\frac{e^{iux}-1}{ix}\right) df_n(x)$$

$$\phi F(u) = \int_R \left(\frac{e^{iux}-1}{ix}\right) dF(x) \quad \to (1)$$

Conversely suppose $\hat{\phi}_n(u) \to \hat{\psi}(u)$

Let $\{f_n\}$ be the sequence d.f's corresponding to $\{\hat{\phi}_n(u)\}$.

Since $\{F_n\}$ contains a convergent sub-sequence by the $\{F_n\}$ converging to F_1 by (1) the corresponding integral [h.fn's]

$$\hat{\phi}_n(\omega) \longrightarrow \hat{\phi} R(u)$$

But $\hat{\phi}_n(\omega) \longrightarrow \hat{\psi}(u)$ implies

$$\phi F(u) = \hat{\psi}(u)$$

If there is another sub-sequence $F_n'' \xrightarrow{\omega} \bar{F}$, then as above $\phi F(\omega) = \hat{\psi}(u)$. But $\hat{\psi}(u)$ determine the distribution unequally (u+ac) Hence $F = \bar{F}(u+ac)$ Since, this is true for any convergent sub-sequence of $\{F_n\}$ $\quad F_n \xrightarrow{\omega} F$ with $\hat{\phi} F = \psi$

Corollary :

If $\phi_n \to \psi$ a.e (i.e) except possibly on a sequence null set then $F_n \xrightarrow{w} F(u+ac)$ with $\phi_F = \psi$ a.e (l the sequence measure)

PROOF :

Since ϕ_n and ψ are bounded and lebe sequence measurable by dominated convergence theorem in

$$\phi_n(u) = \int_0^u \phi_n(v)\, dv \longrightarrow \int_0^u \psi(v) dv = \hat{\psi}(u)$$

By the above theorem (1) $F_n \xrightarrow{\omega} F$ and $\phi F = \psi$ implying an differentation.

$$\phi F = \psi \text{ a.e}$$

IMPORTANT QUESTIONS :-

1. Types of Convergence
2. Convergence in Probability Define ?
3. Define convergence with Probability ?
4. State Monotone Convergence Theorem.
5. State Dominated Convergence.
6. Define : Characteristic function.
7. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Then prove that $X_n + Y_n \xrightarrow{P} X_n + Y$.
8. Obtain the characteristics function of the random variable X having the probability density function

$$f(x) = \begin{cases} e^{-x} & : x \geq 0 \\ 0 & : x < 0. \end{cases}$$

9. State and prove monotone convergence theorem.
10. Inverse theorem state and prove.
11. Dominated Convergence

LAW OF LARGE NUMBERS:

Def: Law of Large numbers explain the convergence of sequence of random variable where the limiting distribution is one point distribution

* Weak Law of Large number
* Strong Law of Large numbers

WEAK LAW OF LARGE NUMBERS:

A Law that is based on convergence probability is called weak law of large numbers.

STRONG LAW OF LARGE NUMBERS: [Or KOLMOGROV'S LAW OF LARGE NUMBER]

A Law that is based on Convergence with probability is called Strong Law of Large numbers convergence with probability is also called almost surely the convergence.

DIFFERENCE BETWEEN LAW OF LARGE NUMBERS AND CENTRAL LIMIT THEOREM:

LAW OF LARGE NUMBERS	CENTRAL LIMIT THEOREM
(i) Explain the convergence the sequence of random variable	(i) Explain the convergence of distribution function
(ii) Very easy the prove without any mild restriction.	(ii) In sequence terms mild restriction
(iii) Here the limiting distribution is one point distribution	(iii) Here the limiting distribution is standard normal distribution

WEAK LAW OF LARGE NUMBER (or) POISSON LAW OF LARGE NUMBER:

Let $\{X_k\}$ $k=1,2,\dots$ be a sequence of identifically independent distribution (i.i.d) random variable with $E(X_k) = m$ and $D^2(X_k) = \sigma^2$ $\forall k$ and if the Markov condition $\lim_{n\to\infty} \frac{\sigma^2}{n} = 0$ is satisfied then the sequence $\{Y_n - m\}$ is stocastically convergence to zero where

$$Y_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

PROOF:

Given the X_k are identically independent distribution and

$$Y_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$E(Y_n) = \cancel{Y_n} \frac{1}{n} \Sigma X_n$$

$$= \frac{1}{n} nm$$

$$E(Y_n) = m$$

$$D^2(Y_n) = \frac{1}{n^2} D^2(X_1 + X_2 + \dots + X_n)$$

$$= \frac{1}{n^2}(\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2)$$

$$D^2(Y_n) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

KCHINCHINE'S LAW:

Statement:

Let $\{X_k\}$ $k=1,2,\dots$ be a sequence of i.i.d random variable with respect value $E(X_k) = m$ $\forall k$.

Taking Log on both sides

$$\log \phi_{Y_k}(t) = n \log 1+z$$
$$= n[z - z^2/2 + \cdots] = n\left[\frac{itm}{n} + o(t/n)\right]$$
$$= itm + n.o(t/n)$$

$$\lim_{n\to\infty} \log \phi_{Y_k}(t) = itm \quad (or)$$

$$\lim_{n\to\infty} \log \phi_{Y_k}(t) = e^{itm}$$

which is the characteristic function of one point distribution.

Hence, Y_n is stocastically convergent of n.

KOLMOGROV'S LAW of LARGE NUMBER (OR) STRONG LAW OF LARGE NUNBERS:

Let $\{X_k\}$ $k=1,2,\ldots$ be a sequence of (i.i.d) random variable and let the variance $D^2(X_k)$ of X_k exists if $\sum_{i=1}^{\infty} \frac{D^2(X_k)}{k^2} < \infty$ then the sequence $\{X_k\}$ of strong Law of Large number with $C_n = \frac{1}{n}\sum_{k=1}^{n} X_k$

PROOF :

$$\text{Let } Z_n = \sum_{i=1}^{n} \frac{X_k - E(X_k)}{n}$$

Let m and m_0 be two Integers satisfying the inequality $2^{m_0} < \mu < 2^{m_0+1}$

Let $P_n = P[\text{Sub}|Z_n| > \epsilon]$

Hence for $\epsilon > 0$, $F_n > 0$ we have inequality $P[\text{Sup}|Z_n| \geq \epsilon] < m$ is satisfied. Hence, it is enough the prove that inequality $P[\text{Sup}|Z_n| \geq \epsilon] < m$ is satisfied in C_m consider KOLMOGROV'S Inequality

$$P[\max|Z_n| \geq \epsilon] \leq \frac{D^2(Y_n)}{E^2} \quad \rightarrow (1)$$

Replay $D^2(Y_n)$ by $D^2\left(\frac{Z_n}{Z^{m+1}}\right)$ in equation $\rightarrow (2)$

we have,

$$P[\max|Z_n| \geq \epsilon] \leq \frac{D^2(Z_n)}{Z^{2m+2}}.$$

(ie) $P[\sup|Z_n| \geq \epsilon] \leq \frac{D^2(Z_n)}{Z^{2m+2}E^2}$

Choosing m,n Large

$$P[\sup|Z_n| \geq \epsilon] < m$$

(i.e) $P_N < m$

The sequence $\{X_k\}$ obey's in Strong law of Large numbers.

Hence, the proved.

CENTRAL LIMIT THEOREM:

If the Limiting distribution is said to standard normal distribution the studies is called Central Limit Theorem.

Example: Binomial distribution, poisson distribution, normal distribution $\rightarrow$ as $n \rightarrow \infty$, χ^2 distribution and F distribution.

The following are variables central limit theorem:

i) De-moiver's Laplace central limit theorem

ii) Liapounov's central Limit theorem.

iii) Linberg-Levy central limit theorem.

iv) Linberg feller's central limit theorem

i) DE-MOIVRE's LAPLACE CENTRAL LIMIT THEOREM:-

Let $F_n(y)$ be the sequence of distribution function of a random variable Y_n where,

$$Y_n = \frac{X_n - np}{\sqrt{npq}} = \left[\frac{x - \bar{x}}{\sigma}\right]$$ is binomial distribution and

$P[X_n = r] = nC_r\, p^r (1-p)^{n-r}$; $0 < p < 1$, $r = 0, 1, 2, \ldots n$. for $0 < p < 1$ and every y the relation $\lim_{n\to\infty} F_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$ holds

PROOF :

Given that X_n follows binomial distribution with mean np and standard deviation $\sqrt{npq}$.

Let $\phi_x(t)$ be its characteristic function $\phi_x(t) = [q + pe^{it}]^n$ given

$$Y_n = \frac{X_n - np}{\sqrt{npq}}$$

Let $\phi_y(t)$ be its characteristic function

$$\phi_y(t) = E\left[e^{itY_n}\right] \Rightarrow E\left[e^{it\frac{X_n - np}{\sqrt{npq}}}\right]$$

$$= e^{it\left[\frac{np}{\sqrt{npq}}\right]} E\left[e^{it\frac{X_n}{\sqrt{npq}}}\right]$$

$$\phi_y(t) = e^{it\frac{np}{\sqrt{npq}}}\, \phi_x\left[\frac{t}{\sqrt{npq}}\right] \longrightarrow (1)$$

we know that

$$\phi_x(t) = (q + pe^{it})^n$$

Substitute the above values in equ (1) we get

$$\phi_Y(t) = e^{it\frac{np}{\sqrt{npq}}}\left[q + e^{it\frac{p}{\sqrt{npq}}}\right]^n$$

$$= \left[e^{-it\frac{p}{\sqrt{npq}}}\right]^n \left[q + p\,e^{\frac{it}{\sqrt{npq}}}\right]^n$$

$$\phi_Y(t) = \left[q\,e^{-it\frac{p}{\sqrt{npq}}} + p\,e^{it\frac{q}{\sqrt{npq}}}\right]^n \longrightarrow (2)$$

Let us considered $q\,e^{it\frac{p}{\sqrt{npq}}}$

$$q\,e^{\frac{itp}{\sqrt{npq}}} = q\left[1 - \frac{itp}{\sqrt{npq}} + \frac{i^2t^2q^2}{npq} - o\left(t^3/n\right)\right]$$

$$= q - \frac{itpq}{\sqrt{npq}} + \frac{t^2p^2q}{\sqrt{npq}} - q\left[o\left(t^3/n\right)\right] \longrightarrow (3)$$

$||^{ly}$

$$p\,e^{it\frac{q}{\sqrt{npq}}} = p\left[1 - \frac{itq}{\sqrt{npq}} + \frac{t^2q^2p}{\sqrt{npq}^2} + o\left(t^2/n\right)\right] \longrightarrow (4)$$

from equation (2), (3) & (4) we get

$$\phi_Y(t) = \left[1 + 0 - t^2/2n\,(p+q) + o\left(t^3/n\right)\right]^n$$

$$\phi_Y(t) = (1+z)^n$$

Where $z = \left[\frac{-t^2}{2n}(p+q)\right]$

taking Log on both sides

$$\phi_Y(t) = [1+z]^n$$

$$\log \phi_Y(t) = n \log[1+z]$$

$$\operatorname{Log} \phi_y(t) = n\left[z - \frac{z^2}{2} + \dots\right] = n\left[-\frac{t^2}{2n}\right]$$

$$= \left[-\frac{t^2}{2}\right]$$

Taking limit as $n \to \infty$ we have,

$$\lim_{n\to\infty} \operatorname{Log} \phi_y(t) = \frac{-t^2}{2} \quad (\text{or}) \quad \lim_{n\to\infty} \phi_y(t) = e^{-t^2/2}$$

which is a characteristic function of standard normal distribution

$$\lim_{n\to\infty} F_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \text{ holds.}$$

(ii) STATE AND PROVE THE LIAPOUNOV'S CENTRAL LIMIT THEOREM:

Let $\{X_k\}$ $k = 1, 2, \dots$ be a sequence of random variables whose moments of 3rd order exists and let m_k, $\sigma_k \neq 0$. a_k and b_k be mean, sandard deviation central moment of 3rd order and absolute central limit of 3rd order of X_n respectively for then. Let $B_n = S\sqrt{\sum_{k=1}^{m} b_k}$, $C_n = \sqrt{\sum_{k=1}^{n} \sigma_k^2}$

If the relation $\lim_{n\to\infty} \frac{B_n}{C_n} = 0$ is satisfied then the sequence $\{F_n(z)\}$ of distribution function of random variable Z_n where,

$$Z_n = \sum_{k=1}^{m} \left(\frac{X_k - m_k}{C_n}\right)$$

for every z relation,

$$\lim_{n\to\infty} F_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

PROOF:

Given that $Z_n = \sum_{k=1}^{n} \left(\frac{X_k - m_k}{C_n}\right)$ Let take $Y_k = \left(\frac{X_k - M_k}{C_n}\right)$. Consider the random variable, $\{X_k - M_k\}$

Let $\phi_{x_k}(t)$ be the characteristic function of $(X_k - M_k)$

$$\phi_{x_k}(t) = 1 - \frac{\sigma_t^2 t^2}{2} + \frac{1}{b} a_k (it)^3 + o(a_k)\, t^3 \longrightarrow (1)$$

The characteristic function of Y_k is

$$\phi_{Y_k}(t) = \phi_{x_k}(t) = 1 - \frac{\sigma_k^t t^2}{2C_n^2} + \frac{1}{b} a_k \frac{(it)^3}{C_n^3} + o\left[\frac{a_k t^3}{C_n^3}\right]$$

from this equation $1 + U_k \longrightarrow (2)$

where $U_k = \left\{ -\frac{\sigma_k^2 t^2}{2C_n^2} + \frac{1}{b} a_k \frac{(it)^3}{C_n^3} + o\left(\frac{a_k t^3}{C_n^3}\right)\right\}$

By Liapounov's inequality $\sigma_k < 3\sqrt{b_k}$

$$\therefore \lim_{n\to\infty} \left| \frac{-\sigma_k^2 t^2}{2C_n^2} \right| \le \lim_{n\to\infty} \left| \frac{3\sqrt{b_k}\, t^2}{2C_n^2} \right| \le \lim_{n\to\infty} \frac{1}{2} \frac{B_n^2}{C_n^2} t^2 = 0.$$

Similarly

$$\lim_{n\to\infty} \left| \frac{a_k (it)^2}{b} \right| \le \lim_{n\to\infty} \left(\frac{B_n^3}{C_n^3}\right) t^3 = 0$$

Similarly

$$\lim_{n\to\infty} \left| \frac{o(a_k t^3)}{C_n^3} \right| \le \lim_{n\to\infty} o\left(\frac{B_n^3}{C_n^3}\right) t^3 = 0$$

Hence, $\lim_{n\to\infty} U_k = 0$

$\therefore \{U_k\}$ is convergence series taking by an both sides of equation (2) we have,

$$\log \phi_{Y_k}(t) = \log(1 + U_k) = U_k - \frac{U_k^2}{2} + \frac{U_k^3}{3} + \frac{U_k^4}{4} + \cdots$$

$$\log \phi_{Y_k}(t) = Uk - \frac{Uk^2}{2} \cdot Uk$$

where $U_k = \left[1 - 2/3\, U_k + 2/4\, U_k^2 + \dots \right]$

$$|U_k| = 1 + |U_k| + |Uk^2| + \dots$$

$$\log Y_k(t) = U_k + U_k^2 \cdot U_k$$

Where ~~$Y_k(t$~~ $U_k = \frac{U_k}{2}$

$\log \phi_z(t)$ be the characteristic function of Z_n

$$\phi_z(t) = \prod_{k=1}^{n} \phi_{Y_k}(t)$$

$$\log \phi_z(t) = \sum_{k=1}^{n} \log \phi_{Y_k}(t)$$

$$= \sum_{k=1}^{n} \left(U_k + V \cdot U_k^2 \right)$$

where, $U_k = \left\{ \frac{\sigma_k^2}{2\delta^2 n^2} + \frac{a_k}{bC_n^3}(it)^3 + \sigma\left(\frac{a_k t^3}{C_n} \right) \right\}$

$$\Sigma U_k = \frac{-t^2}{2} + \frac{\Sigma a_k}{bC_n^3}(it)^3 + o\left(\frac{\Sigma a_k t^3}{C_n^3} \right)$$

$$\lim_{n \to \infty} \Sigma U_k = -t^2/2$$

$$\lim_{n \to \infty} \frac{\Sigma a_k}{bC_n^3}(it)^3 = 0$$

||ly $\lim_{n \to \infty} o\left[\frac{\Sigma a_k t^3}{C_n^3} \right] = 0$

Similarly $\lim_{n\to\infty} \Sigma U_i = 0$

$$\therefore \lim_{n\to\infty} \log \phi_z(t) = \frac{-t^2}{2} \quad (or)$$

$$\lim_{n\to\infty} \phi_z(t) = e^{-t^2/2}$$

which is the characteristic function standard normal distribution

$$\text{Hence, } \lim_{n\to\infty} F_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz.$$

Hence, the proved.

iii) LIBERG - LEVY CENTRAL LIMIT THEOREM:

If $x_1, x_2, x_3 \ldots x_n$ are independently randomvariable with the some distribution whose standard deviation $\sigma \neq 0$ exist the sequence of $\{F_n(z)\}$ of distribution function of random variable Z_n given by.

$$Z_n = \frac{y_n - nm}{\sigma\sqrt{n}} \quad ; \ nm = E(y_n)$$

$$E(x_n) = m \quad ; \ var(x) = 0.$$

iv) STATE LINBERG's FELLER CENTRAL LIMIT THEOREM

Let $\{X_k\}$ $k = 1, 2, \ldots$ be a sequence of independent random variable with whose variance exist and let $G_k(y)$. μ_k and $\sigma_k \neq 0$ respectively. The distribution function expanced value of the standard deviation of the random variable X_k and let $F_n(z)$ denoted the distribution between function of the standard random variable Z_n given by,

$$Z_n = \frac{X_k - \mu_k}{C_n}$$

where $C_n = \sqrt{\sum_{k=1}^{n} \sigma_k^2}$ then the relation

$$\lim_{n \to \infty} F_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \text{ holds}$$

BERNOULI LAW OF LARGE NUMBER :-

The sequence of random variable $\{X_k\}$ defined as $X_n = x_n$ and $P\left[X_n = \frac{r}{n}\right] = nC_r\, p^r q$ (or) $(1-p)^{n-1}$ $0 < p < r$; $r = 0, 1, \ldots n$.

Converges stochastically to zero.

(i.e.) for any $\epsilon > 0$.

$$\lim_{n \to \infty} P\left[|X_n| > \epsilon\right] = 0$$

PROOF :-

Given that,

$$X_n = Y_n - P$$

$$E(X_n) = E\left[(Y_n - P)\right] = E(Y_n) - P$$

$$= E\left(r/n\right) - P$$

$$= \frac{1}{n} \times nP - P$$

$$E(X_n) = P - P = 0$$

Similarly,

$$Var(X_n) = D^2(X_n)$$

$$= Var(Y_n - P)$$

$$= Var(Y_n)$$

$$= Var(r/n)$$

$$= 1/n^2 \; Var(r)$$

$$= 1/n^2 \times npq$$

$$= P(1-P)/n$$

$$Var(X_n) = P(1-P)/n \Rightarrow S.D = \sqrt{\frac{P(1-P)}{n}}.$$

we consider Chebyshew's inequality

$$P\left[|X_n| > k\sigma\right] \leq 1/k^2$$

$$\Rightarrow P\left[|X_n| > k\sqrt{\frac{P(1-P)}{n}}\right] \leq 1/k^2 \quad \rightarrow (1)$$

Choose $k = \frac{E}{\sqrt{P(1-P)/n}}$ using (1)

$$P\left[|X_n| > t\right] \leq \frac{P(1-P)}{nt^2} \leq \frac{1}{nt^2}$$

Taking lim as $n \to \infty$ in both sides

$$\lim_{n\to\infty} P\left[|X_n| > t\right] = 0$$

$\{X_n\}$ in Stochastically Converges Zero.

Hence, the proof.

IMPORTANT QUESTIONS :.

1. State strong law of large numbers.
2. State Liapounov's form of Central Limit theorem.
3. State Weak law of large numbers.
4. State Kchinchine's Law.
5. State Central Limit theorem.
6. State Liberg-Levy Central Limit theorem.
7. State and prove Bernoulli's Law of Large Numbers
8. State and prove Lindeberge Levy's form of central Theorem
9. Establish Kolmogorov strong law of Large numbers.
10 Establish Bernouli Law of Large Number.
11. State and prove the Liapounov's Central Limit theorem.
12. Differenie between Law of Large numbers and central limit