SEMESTER : II
CORE COURSE : V

Inst Hour	: 6
Credit	: 5
Code	: 18KP2M05

COMPLEX ANALYSIS

UNIT-I

Arcs & closed curves – Analytic functions in regions – Conformal mapping – Length and area - Line integrals – Rectifiable arcs – Line integrals as functions of arcs – Cauchy's Theorem for a Rectangle – Cauchy's Theorem in a disk.

Chapter – III : Sec 2.1 to 2.4 Chapter - IV : Sec 1.1 to 1.5

UNIT-II

Cauchy's Integral Formula :- The Index of a point with respect to a Closed Curve - The integral formula – Higher Derivatives – Morera's theorem – Liouville's theorem - Cauchy's estimates – Fundamental theorem of algebra.

Chapter-IV : Sections 2.1 to 2.3

UNIT-III

Local properties of Analytical functions: Removable singularities – (Taylor's theorem) - Zeros and Poles – Meromorphic functions – Essential singularities – The Local Mapping (Theorem – The Maximum Principle).

Chapter –IV : Sec 3.1 – 3.4

UNIT-IV

The General form of Cauchy's theorem: Chains and Cycles – Simply connected sets – Homology – The general statement of Cauchy's theorem and it's proof – Locally exact differentials – Multiply connected Regions. The Calculus of Residues: The Residue Theorem - The Argument Principle - Evaluation of Definite Integrals.

Chapter 4 : Sec 4.1 to 4.7 & 5.1 to 5.3

UNIT-V

Harmonic functions :- Definition and Basic properties – The mean value property – Poisson's formula – Schwartz's Theorem – Reflection Principle – Weirstrass's Theorem – The Taylor series – The Laurent series.

Chapter 4 : Sec 6.1 to 6.5 & Chapter 5 : Sec 1.1 to 1.3

TEXT BOOK

L.V.Ahlfors – Complex Analysis – Third Edition Mc Graw Hill Education (India) Edition 2013.

REFERENCES

1. J.N. Sharma, Functions of Complex Variables.
2. SergeLang, Complex Analysis, Addison Wesley, 1977.
3. S.Ponnusamy, Foundations of Complex Analysis, Narosa Publishing House, 1977.
4. Dr.V Karunakaran, Complex Analysis, Narosa Publishing House

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

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Complex Analysis

Unit - I

Line Integral.

If $f(t) = u(t) + iv(t)$ is a continuous function defined in an interval (a, b) we set by definition $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

Result :-

If $a < b$ then prove that $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

If c is any complex number constant then we know that $c\int_a^b f(t)dt = \int_a^b cf(t)\,dt$.

Take $c = e^{-i\theta}$, (θ real)

$$e^{-i\theta}\int_a^b f(t)dt = \int_a^b e^{-i\theta} f(t)dt.$$

$$\text{Re } e^{-i\theta}\int_a^b f(t)\,dt = \text{Re}\int_a^b e^{-i\theta} f(t)\,dt$$

$$\text{Re } e^{-i\theta}\int_a^b f(t)\,dt = \int_a^b \text{Re } e^{-i\theta} f(t)\,dt$$

Since $\text{Re } z \leq |z|$

$$\leq \int_a^b |e^{-i\theta} f(t)|\,dt.$$

$$\text{Re } e^{-i\theta}\int_a^b f(t)\,dt \leq \int_a^b |f(t)|dt \longrightarrow (1)$$

choosing $\theta = \arg\int_a^b f(t)\,dt$

$$\int_a^b f(t)\,dt = \left|\int_a^b f(t)\,dt\right| e^{i\theta}$$

(1) becomes

$$Re \left\{ e^{-i\theta} \left| \int_a^b f(t)\,dt \right| e^{i\theta} \right\} \leq \int_a^b |f(t)|\,dt$$

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$$

Definition: Complex integral over the arc.

(1/x) Let ~~go~~ γ be a piece wise differentiable arc defined by the equation $z = z(t)$, $a \leq t \leq b$. Let $f(t)$ be defined and continuous of γ. Then the integral of $f(z)$ over γ is defined as

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$z = z(t)$
$\frac{dz}{dt} = z'(t)$
$dz = z'(t)\,dt$

Definition:

Negative of an arc.

If γ is an arc defined by $z(t)$, $a \leq t \leq b$. Then $-\gamma$ is defined by $z(-t)$ $-b \leq t \leq -a$

2m 1) Prove that $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$.

~~We k~~

If $z = z(t)$ then.

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

If ~~z(~~ $z = z(-t)$, $-b \leq t \leq -a$ then.

$$\int_{-\gamma} f(z)\, dz = \int_{-b}^{-a} f(z(-t)) \times -z'(-t)\, dt.$$

Put $t = -u$, $-t = u$

$-dt = du.$

When $t = -b$ $u = b$

$t = -a$ $u = a$

$$\int_{-\gamma} f(z)\, dz = \int_{b}^{a} f(z(u))\, z'(u)\, du.$$

$$= -\int_{a}^{b} f(z(u))\, z'(u)\, du.$$

$$= -\int_{a}^{b} f(z(t))\, z'(t)\, dt.$$

$$\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz.$$

Result.

we can subdefine an arc γ into a finite number of subarcs. A subdivision can be indicate by symbolic equation.

$$\gamma = \gamma_1 + \gamma_2 + \dots\dots + \gamma_n$$ and the corresponding integral satisfies the relation.

$$\int_{\gamma_1+\gamma_2+\dots\gamma_n} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \dots + \int_{\gamma_n} f(z)\, dz.$$

Result.

We know $\int_\gamma f\, d\bar{z} = \overline{\int_\gamma \bar{f}\, dz}$

using this notation, line integrals with respect to x or y can be introduced by.

$$\int_\gamma f\, dx = \frac{1}{2}\left[\int_\gamma f\, dz + \int_\gamma f\, d\bar{z}\right]$$

$$\int_\gamma f\, dy = \frac{1}{2i}\left[\int_\gamma f\, dz - \int_\gamma f\, d\bar{z}\right]$$

$$\int_\gamma f\, dz = \frac{1}{2}\left[\int_\gamma f\, dz + \int_\gamma f\, d\bar{z}\right] + \frac{1}{2i}\left[\int_\gamma f\, dz - \int_\gamma f\, d\bar{z}\right]$$

If $f = u + iv$ then.

$z = x+iy$

$dz = dx + idy$

$$\int_\gamma f\, dz = \int_\gamma \cancel{(du + idv)}\,(u+iv)$$

$$= \int_\gamma (u+iv)(dx + idy)$$

$$= \int_\gamma u\,dx + iu\,dy + iv\,dx - v\,dy$$

$$= \int_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

$$\int_\gamma f\, dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx .$$

Definition:- Integral with respect to arc length

If s is length of the arc γ then the integral of $f(z)$ with respect to arc length s is defined as $\int_\gamma f\,ds = \int f|dz|$ $\quad f(z)|dz|$

$$= \int f(z(t))\,|dz(t)| \qquad z = z(t), \quad dz = z'(t)dt.$$

$$\int_\gamma f\,ds = \int f(z(t))\,|z'(t)|dt$$

Note:-

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12.10.09
13.10.09
20.10.09

1) $\int_{-\gamma} f\,|dz| = \int_\gamma f|dz|$

2) $\left|\int_{-\gamma} f|dz|\right| \leq \int_\gamma |f|\,|dz|.$

3) $\int_\gamma |dz|$ denotes the length of γ

Example:-

compute the length of a circle take the circle $c : |z-a| = \rho$

$$z - a = \rho e^{i\theta}$$

$$z = a + \rho e^{i\theta}$$

$$dz = \rho e^{i\theta} i\,d\theta$$

$$|dz| = |\rho e^{i\theta} i\,d\theta| \qquad |e^{i\theta}| = 1$$

$$|dz| = \rho d\theta$$

$$\int_c |dz| = \int_0^{2\pi} \rho\,d\theta$$

$$\int_C |dz| = \rho\,[\theta]_0^{2\pi}$$

$$= \rho(2\pi)$$

$$= 2\pi\rho$$

Rectifiable Arc

2m Define Rectifiable Arc

The length of an arc can also be defined as the least upper bound of all sums.

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \dots + |z(t_n) - z(t_{n-1})|$$

Where $a = t_0 < t_1 < t_2 < \dots \dots < t_n = b$ if this least upper bound is finite, we say that the arc is rectifiable

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Result:-

An arc γ defined by $z = z(t)$ is rectifiable iff its real and imaginary part $x(t)$ and $y(t)$ are functions are bounded variation.

Proof:

$z = x + iy$.

$z(t) = x(t) + iy(t)$

$Re(z) \leq |z|$

$\Rightarrow |x(t_i) - x(t_{i-1}) \leq |z(t_i) - z(t_{i-1})|$

$$\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \leq \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| \rightarrow ①$$

Similarly.

$Im(z) \leq |z|$

$\Rightarrow |y(t_i) - y(t_{i-1})| \leq |z(t_i) - z(t_{i-1})|$

$$\Rightarrow \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})| \leq \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| \rightarrow ②$$

$$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| \leq \sum_{i=1}^{n} |x(t_i) + iy(t_i) - x(t_{i-1}) - iy(t_{i-1})]$$

$$= \sum_{i=1}^{n} \left| x(t_i) - x(t_{i-1}) + i[y(t_i) - y(t_{i-1})] \right|$$

$$\leq \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| + \sum_{i=1}^{n} |y(t_i) - y(t_{i-1})] \rightarrow ③$$

From ①, ② and ③ it is cleared that this sums.

$\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})$, $\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})$, $\sum_{i=1}^{n} |y(t_i) - y(t_{i-1})|$ are bounded at the same time. that is when $x(t)$'s and $y(t)$ are function of bounded variation.

The sum $\sum_{i=1}^{n} |z(t_i) - z(t_{i-1})|$ is bounded and conversely.

Hence the result follows.

Note:-

If γ is rectifiable and if $f(t)$ is continuous on γ. then.

$$\int_\gamma f(z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} \{ f(z(t_i)) \, |z(t_i) - z(t_{i-1})| \}$$

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Theorem :-

The line integral $\int_\gamma P\,dx + q\,dy$, defined in Ω depends only on the end points of γ iff $\exists$ a function $U(x,y)$ in Ω with the partial derivatives. $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = q$

Proof.

The condition are necessary.

Let the value of $\int_\gamma P\,dx + q\,dy$ depend only on the end points of γ

Choose a fixed point (x_0, y_0) in Ω and join it to a point (x,y) in Ω by a polygon γ in Ω. whose sides are parallel to the co-ordinate axis

Define a function $u(x,y)$ by.

$$U(x,y) = \int_{\gamma} Pdx + qdy.$$

This function is well defined because the integral depends only on the end points. If we choose the last segment of the horizontal we can keep y constant (Hence $dy=0$) and x varies without changing the other segment. choosing x as parameter on this segment, we get.

$$U(x,y) = \int^{x} Pdx + 0 + \text{a constant}.$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = P$$

Similarly by choosing the final segment as. vertical (keeping x constant and y as parameter) we can show that.

$$\frac{\partial u}{\partial y} = q.$$

Hence the condition are necessary.

The condition are sufficient

Let the function $U(x,y)$ exist in Ω such that $\frac{\partial u}{\partial x} = P$ and $\frac{\partial u}{\partial y} = q$. Let a and b be the end points of γ.

then. $$\int_{\gamma} Pdx + qdy = \int_{\gamma} \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

$$= \int_\gamma \frac{\partial u}{\partial x} \frac{dx}{dt} dt + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \cdot dt.$$

$$= \int_\gamma \frac{\partial u}{\partial x} x'(t)\,dt + \frac{\partial u}{\partial y} y'(t)\,dt.$$

$$= \int_a^b \frac{d}{dt} \{u(x(t), y(t))\}\,dt$$

$$= [u(x(t), y(t))]_a^b$$

$$= u(x(b), y(b)) - u(x(a) - y(a)).$$

That is the value of the integral depends only on the end points of the γ.

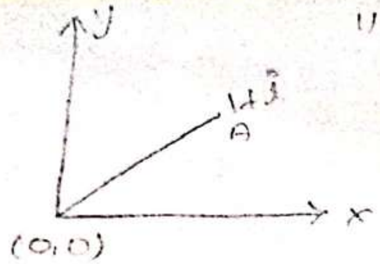
<u>Note:-</u>

1) The line integral $\int_\gamma f(z)\,dz$ (with $f(z)$ is continuous in Ω) depends only on the end points of γ iff $f(z)$ is the derivative of analytic function in Ω.

2) If $f(z)\,dz$ is an exact differential and if γ is closed curve in Ω then $\int_\gamma f(z)\,dz = 0$.

U.Q eg:-

1. Compute $\int_\gamma x\,dz$ where γ is the directed line segment from 0 to $1+i$



γ is the line joining (0,0) to (1,1) in x,y plane. on γ. (i.e. on OA)

x varies from 0 to 1
y varies from 0 to 1.

$x = y$ on OA.

$dy = dx$.

$$\int_\gamma x\,dz = \int_\gamma^1 x\,(dx + i\,dy)$$

$$= \int_0^1 x\,(dx + i\,dx)$$

$$= (1+i)\int_0^1 x\,dx.$$

$$= (1+i)\left(\frac{x^2}{2}\right)_0^1$$

$$= (1+i)(1/2)$$

$$\int_\gamma x\,dz = \frac{1+i}{2}$$

12-07.

2. Compute $\int_{|z|=r} x\,dz$ for the +ve sense of the circle, in two ways: First, by use of a parameter and second by observing that $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

Solution

(i) $|z| = r \Rightarrow z = re^{i\theta}$

$dz = re^{i\theta}\, i\, d\theta.$

$$I = \int_{|z|=r} x\,dz$$

$$= \int_0^{2\pi} r\cos\theta \times re^{i\theta}\, i\,d\theta.$$

$$= r^2 i \int_0^{2\pi} \cos\theta\,(\cos\theta + i\sin\theta)\,d\theta.$$

$$= r^2 i \int_0^{2\pi} (\cos^2\theta + i\sin\theta\cos\theta)\,d\theta.$$

$$= r^2 i \int_0^{2\pi} \left(\cos^2\theta + \frac{i\sin 2\theta}{2}\right) d\theta.$$

$$= 2r^2 i \int_0^{\pi} \cos^2\theta\,d\theta + 0. \quad = 2r^2 i \int_0^{\pi} \cos^2\theta\,d\theta + 0$$

$$= 4r^2 i \int_0^{\pi/2} \cos^2\theta\,d\theta.$$

$$= 4r^2 i \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I = \pi r^2 i$$

$e^{i\theta} = \cos\theta$

$\sin 2\theta = 2\sin$

$\sin\theta\cos\theta = \sin$

(ii) $z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$.

$\therefore x = \frac{1}{2}(z + \bar{z})$

$\therefore |z|^2 = r^2$

$z\bar{z} = r^2$

$z = \frac{r^2}{z}$

$\therefore x = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$

$\therefore |z| = r \Rightarrow z = re^{i\theta}$

$$dz = re^{i\theta}\, i\, d\theta.$$

$$x = \frac{1}{2}\left(re^{i\theta} + \frac{r^2}{re^{i\theta}}\right)$$

$$= \frac{1}{2e^{i\theta}}\left(re^{i2\theta} + r\right)$$

$$= \frac{r}{2e^{i\theta}}\left(e^{i2\theta} + 1\right)$$

$$= \int_0^{2\pi} \frac{r}{2.e^{i\theta}} \cdot \left(e^{2i\theta} + 1\right) \times re^{i\theta}\, i\, d\theta.$$

$$= \frac{r^2 i}{2}\int_0^{2\pi} \left(e^{2i\theta} + 1\right) d\theta.$$

$$= \frac{r^2 i}{2}\left[\frac{e^{2i\theta}}{2i} + \theta\right]_0^{2\pi}$$

$$= \frac{r^2 i}{2}\left[\left(\frac{e^{4i\pi}}{2i} + 2\pi\right) - \frac{1}{2i}\right]$$

$$= \frac{r^2 i}{2}\left(\frac{\cos 4\pi i + i\sin 4\pi}{2i} + 2\pi - \frac{1}{2i}\right)$$

$$= \frac{r^2 i}{\cancel{2}}\left[\cancel{\frac{1}{2i}} + \cancel{2}\pi - \cancel{\frac{1}{2i}}\right]$$

$$= r^2 \pi i.$$

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2. Compute $\int_{|z|=1} |z-1| . |dz| \quad |z|$

$|z| = 1$

$z = e^{i\theta}$

$dz = e^{i\theta}\, i\, d\theta$

$|dz| = d\theta$

$|z-1|^2 = |e^{i\theta} - 1|^2$

$= |\cos\theta + i\sin\theta - 1|^2$

$= |(\cos\theta - 1) + i\sin\theta|^2$

$= (\cos\theta - 1)^2 + \sin^2\theta$.

$= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$

$= 1 - 2\cos\theta + 1$

$= 2(1 - \cos\theta)$

$= 2(2\sin^2 \theta/2)$

$|z-1|^2 = (2\sin\theta/2)^2$

Taking square root

$|z-1| = 2\sin\theta/2$

$\int_{|z|=1} |z-1|\,|dz| = \int_0^{2\pi} 2\sin\theta/2 \, d\theta$.

$= 2\times 2 \int_0^{\pi} \sin\theta/2 \, d\theta$

$= 4\times 2 \int_0^{\pi/2} \sin\theta/2 \, d\theta$

$= 8 \times \dfrac{-\cos\theta/2}{}$

$= 2\left[-\dfrac{\cos\theta/2}{1/2} \right]_0^{2\pi}$

$= 4[-\cos\pi + \cos 0]$

$= 8$

UQ 10m

Cauchy's theorem for a rectangle :-

State and prove Cauchy's theorem for a rectangle.

If the function $f(z)$ is analytic in R. Then.

$$\int_{\partial R} f(z)\,dz = 0.$$

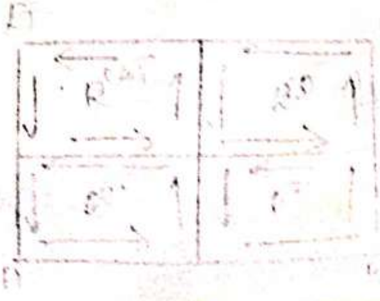
Where R is a rectangle and ∂R denotes the boundary of R.

Proof:

consider a rectangle R defined by $a \le x \le b$, $c \le y \le d$. So that R lies to the left of the four directed segments of sides AB, BC, CD, DA.

Let $\eta(R) = \int_{\partial R} f(z)\,dz$.

Subdivide R into four congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$ as shown in the figure.



R

Let $\partial R^{(i)}$ denoted the boundary $R^{(i)}$, $i = 1,2,3,4$. Then $\partial R \leq \partial R + \partial R$

$$\partial R = \partial R^{(1)} + \partial R^{(2)} + \partial R^{(3)} + \partial R^{(4)}$$

Hence $\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}) \rightarrow (1)$.

Now for atleast (1) ~~area~~ $R^{(i)}$ $i = 1,2,3,4$.

We have,

$$|\eta(R^{i})| \geq \frac{1}{4} |\eta(R)| \rightarrow (2).$$

If not, let $|\eta(R^{i})| < \frac{1}{4} |\eta(R)|$ $i = 1,2,3,4$ $\rightarrow (3)$.

Then

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

$$|\eta(R)| = \left| \sum_{i=1}^{4} \eta(R^{(i)}) \right|$$

$$\leq \sum_{i=1}^{4} |\eta(R^{(i)})|$$

$$\leq \sum_{i=1}^{4} \frac{1}{4} |\eta(R)|$$

$$\leq \frac{1}{4} \cdot 4 |\eta(R)|$$

$$|\eta(R)| < |\eta(R)|$$

This is a contradiction.

Hence (2) is true.

Let R_1 ~~R^{i}~~ be the rectangle for which.

$$|\eta(R_1)| \geq \frac{1}{4}|\eta(R)|$$

Repeating the above procedure indefinitely we get a sequence of nested rectangles. such that

$$R \supset R_1 \supset R_2 \supset \ldots\ldots \supset R_{n-1} \supset R_n \supset \ldots\ldots$$

with the property $|\eta(R_n)| \geq \frac{1}{4}|\eta(R_{n-1})|$.

$$\left[\because \text{ since } |\eta(R_{n-1})| \geq \frac{1}{4}|\eta(R_{n-2})|\right]$$

$$|\eta(R_n)| \geq \frac{1}{4}\cdot\frac{1}{4}|\eta(R_{n-2})|$$

$$\geq \frac{1}{4^2}|\eta(R_{n-2})|$$

$$\geq \frac{1}{4^3}|\eta(R_{n-3})|$$

$$\ldots\ldots\ldots$$

$$\ldots\ldots\ldots$$

$$|\eta(R_n)| \geq \frac{1}{4^n}|\eta(R_0)|. \longrightarrow (4)$$

by contour's theorem $\exists$ a point z_0 common ~~to the~~ all the rectangle's of the above sequence. Next $f(z)$ is analytic on R.

$\Rightarrow f(z)$ is continuous at z_0

Therefore Give $\epsilon > 0$, $\exists$ a $\delta > 0$ such that.

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon \text{ Where } |z-z_0| < \delta.$$

$$\Rightarrow |f(z)-f(z_0)-(z-z_0)f'(z)| < \epsilon|z-z_0| \text{ When } |z-z_0| < \delta \longrightarrow (5)$$

If $N(z_0, \delta)$ be a neighbourhood of z_0

Such that, $|z - z_0| < \epsilon$, then for large N,

$R_n \subset N(z_0, \delta)$.

Then ⑤ is true for all $z \in R_n$

Further $\int_{\partial R_n} dz = 0 \longrightarrow$ ⑥

$$\int_{\partial R_n} z\,dz = 0 \longrightarrow ⑦.$$

$$\int_{\partial R_n} f(z)\,dz = \int_{\partial R_n} f(z)\,dz - f(z_0)\int_{\partial R_n} dz$$

$f(z) - f(z_0^*) - (z - z_0^*) f'(z_0^*)$

$$- f'(z_0)\int_{\partial R_n} (z - z_0)\,dz$$

$\int_{\partial R_n} f(z)\,dz - \int_{\partial R_n} f(z_0^*)\,dz - \int_{\partial R_n} (z - z_0^*) f'(z_0^*)\,dz$

$$\int_{\partial R_n} f(z)\,dz = \int [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,dz$$

$\int_{\partial R_n} f(z)\,dz - f(z_0)\int_{\partial R_n} dz - f'(z_0)\int_{\partial R_n} (z - z_0)\,dz$

$$\left|\int_{\partial R_n} f(z)\,dz\right| \leq \int_{\partial R_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)|\,|dz|$$

$$\leq \int_{\partial R_n} \epsilon\, |z - z_0|\,|dz|$$

$\leq \epsilon d_n \int_{\partial R_n} |dz|$ Where $|z - z_0| < d_n$ and

d_n is the diagonal of R_n

$\leq \epsilon d_n L_n \longrightarrow$ ⑧ Where L_n is the length of R_n

If d is the ~~dia~~ length of the diagonal R.

and L is the length of R (Perimeter).

Then $d_n = \frac{d}{2^n}$, $L_n = \frac{L}{2^n}$

(8) becomes.

$$\left| \int_{\partial R_n} f(z)\,dz \right| \le \epsilon \frac{d}{2^n} \cdot \frac{L}{2^n}$$

$$\therefore |\eta(R_n)| \le \frac{\epsilon d L}{4^n}$$

(4) becomes.

$$|\eta(R_n)| \le 4^n |\eta(R_n)|$$

$$\le \epsilon d L$$

Since as ϵ is arbitary, it follows that

$$\eta(R) = 0.$$

ie, $\eta(R) = \int_{\partial R} f(z)\,dz = \underline{0}$.

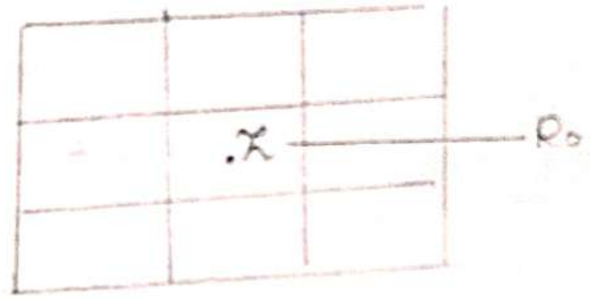
1.01 **Theorem:-**

Let $f(z)$ be analytic. on the set R obtain from the rectangle R. by omitting. a finite number of interior points z_j. If it is true that $\lim_{z \to z_j} (z - z_j) f(z) = 0$ for all j then $\int_{\partial R} f(z)\,dz = 0$.

Proof.

Sino R can be Subdivided into Smaller rectangular which contain atmost one z_j it is sufficient if we prove the theorem for one exceptional ~~exceptial~~ point z_j

Divide R into nine rectangles and assume that z_0 is the centre of a rectangle R_0



$$\int_{\partial R} f(z)dz = \sum \int_{\partial R_j} f(z)dz$$

$\therefore$ The other rectangle cancel each other

$$= \int_{\partial R_0} f(z)dz. \longrightarrow (1)$$

By the given hypothesis

$\therefore$

$$\lim_{z\to z_j} (z-z_j)f(z) = 0.$$

Given $\epsilon > 0$ we can choose the rectangle R_0 So small such that

$$|(z-z_j)f(z) - 0| < \epsilon$$

$\Rightarrow |f(z)| < \dfrac{\epsilon}{|z-z_j|} \rightarrow (2)$.

If $2a$ is the side of R_0. Then for all z on ∂R_0.

$$\therefore |z-z_j| > a.$$

$$\frac{1}{|z-z_j|} < \frac{1}{a} \rightarrow (3).$$

(2) becomes.

$$|f(z)| < \frac{\epsilon}{a}.$$

$$(1) \Rightarrow \quad \left| \int_{\partial R} f(z)dz \right| = \left| \int_{\partial R_0} f(z)dz \right|$$

$$\leq \int_{\partial R_0} |f(z)|dz.$$

$$\leq \int_{\partial R_0} \frac{\epsilon}{a}|dz|$$

$$\leq \int_{\partial R_0} \frac{\epsilon}{a}|dz|$$

$$\leq \frac{\epsilon}{a}(\text{length of } \partial R_0)$$

$$\leq \frac{\epsilon}{a} \times 8a.$$

$$\therefore \left| \int_{\partial R} f(z)dz \right| \leq 8\epsilon$$

Since ϵ is arbitary, it follows that $\int_{\partial R} f(z)dz = 0.$

Hence the theorem.

Definition -

Open disc or circular disc.

An open disc (or a circular disc) is a region given by $|z - z_0| < \rho$ Such region are denoted by Δ. Thus.

$$\Delta : |z - z_0| < \rho$$

---

(5) **Cauchy's theorem in a disc.**

If $f(z)$ is analytic in an open disc Δ then $\int_\gamma f(z)dz = 0$. for every closed curve γ in Δ.

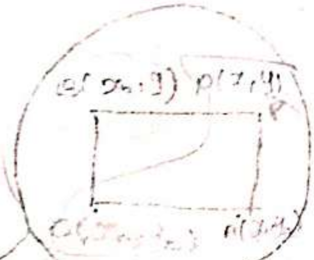
Proof

Define $F(z) = \int_\sigma f(z)dz$.

Where σ is a polygonal arc consisting of.

1) A horizontal segment OA from the centre (x_0, y_0) to (x, y_0)

2) A Vertical segment ~~OB~~ AP from (x, y_0) to (x, y)



$$\therefore F(z) = \int_{OAP} f(z)dz.$$

$$F(z) = \int_{OA} f(z)dz + \int_{AP} f(z)dz \longrightarrow (1)$$

On OB

$z = t + iy_0$

$dz = dt$.

t varies from x_0 to x.

On AP

$z = x+it$

$dz = i\,dt$

t varies from y_0 to y

$$F(z) = \int_{x_0}^{x} f(t+iy_0)\,dt + \int_{y_0}^{y} f(x+it)\,i\,dt.$$

$$\frac{\partial F}{\partial y} = i f(x+iy) = \underline{i f(z)}. \rightarrow (2)$$

By Cauchy's theorem.

$$\int_{OAPB} f(z)dz = 0.$$

$$\int_{OAPB} f(z)dz + \int_{PBO} f(z)dz = 0.$$

$$\int_{OAP} f(z)dz = -\int_{PBO} f(z)dz.$$

$$\int_{OAP} f(z)dz = -\int_{OBP} f(z)dz \rightarrow (3)$$

$$F(z) = -\int_{OB} f(z)\,dz + \int_{BP} f(z)\,dz \rightarrow (3)$$

On OB.

$z = x_0 + it$

$dz = i\,dt$, t Varies from y_0 to y

On BP

$z = t + iy$

$dz = dt$, t Varies from x_0 to x.

$\therefore$ (3) becomes.

$$F(z) = \int_{y_0}^{y} f(x_0 + it)\, i\,dt + \int_{x_0}^{x} f(x+iy)\,dt.$$

$$\frac{\partial F}{\partial x} = F(x+iy)$$

$$\frac{\partial F}{\partial x} = f(z) \rightarrow (4).$$

from (2) and (4) we get.

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

If $F(z) = u + iv$ then.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Equating the real and imaginary part.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \&$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are CR equation of $f(z)$.

Further $f(z)$ is continuous and $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$

$\Rightarrow U_x, U_y, V_x, V_y$ are continuous.

Hence $F(z)$ is also analytic and

$F'(z) = f(z)$

Thus in a circular disc we have proved that $f(z)$ is the derivative of an analytic function $F(z)$.

Hence for any closed curved curve γ in Δ.

$\int_\gamma f(z)\,dz = 0$.

Hence the theorem.

Theorem.

Let $f(z)$ be analytic in the Δ' obtained by omitting a finite number of points z_j from an open dis Δ if $f(x)$ satisfies the condition

$\lim_{z \to z_j} (z - z_j) f(z) = 0$ for all j then $\int_\gamma f(z)\,dz = 0$

for any closed curve γ in Δ'

Proof.

It is enough if we prove the theorem for one exceptional point z_j. let Δ be the open disc defined by.

Define $F(z) = \int_\sigma f(z)\,dz$.

(i) σ doesnot pass through z_j

(ii) σ consisting of 3 segment OA, AB, BP, as shown in fig assume that z_0 is the exceptional pt.

By the previous theorem.

F(z) is independent of the middle segment and final segment may be vertical (or horizontal)

Thus F(z) is a indefinite integral of f(z) and hence it can be shown that

$F'(z) = f(z)$ in Δ'

Hence

$F(z) = \int_{\gamma} f(z)dz = 0$ for any closed curve.

γ in Δ'

Thus the theorem is proved.

UNIT II

Thus the theorem

UNIT - II

The Index Of The Point With Respect To A Closed curve:-

LEMMA-1

If the piece-wise differentiable closed curved γ doesnot pass through a point 'a'. Then the value of integral

$$\int_{\gamma} \frac{dz}{z-a}$$ is a multiple of $2\pi i$

Proof.

Let γ be defined by $z = z(t)$, $\alpha \leq t \leq \beta$

consider, $h(t) = \int_{\alpha}^{t} \frac{z'(t)dt}{z(t)-a} \rightarrow ①$.

This is defined and continuous on $[\alpha, \beta]$

$h'(t)$ is $= \frac{z'(t)}{z(t)-a} \rightarrow ②$.

$\Rightarrow z'(t) = h'(t)\,[z(t)-a] \rightarrow (2i)$

wherever $z'(t)$ is continuous.

consider,

$$\frac{d}{dt}\left[e^{-h(t)}(z(t)-a)\right] = e^{-h(t)}z'(t) - e^{-h(t)} \times h'(t)(z(t)-a)$$

$$= e^{-h(t)}z'(t) - e^{-h(t)}z'(t).$$

$$\frac{d}{dt}\left[e^{-h(t)}(z(t)-a)\right] = 0 \qquad [\text{using } ②]$$

Since $e^{-h(t)} \times [z(t)-a]$ is continuous and its derivative is equal to zero implies that.

$$e^{-h(t)}[z(t)-a] = \text{a constant}$$

$$e^{-h(t)}[z(t)-a] = C \text{ (Say)} \rightarrow ③.$$

In particular,

$$e^{-h(\alpha)}[z(\alpha)-a] = C$$

Since $h(\alpha) = 0$, $z(\alpha) - a = C$.

③ becomes,

$$e^{-h(t)}[z(t)-a] = z(\alpha) - a.$$

$$e^{-h(t)} = \frac{z(\alpha)-a}{z(t)-a}.$$

when $t=\beta$

$$e^{-h(\beta)} = \frac{z(\alpha)-a}{z(\beta)-a} \rightarrow (4).$$

Since γ is a closed curve, $\alpha=\beta$ or.

$$\therefore z(\alpha) = z(\beta)$$

(4) becomes.

$$e^{-h(\beta)} = 1.$$

$$e^{h(\beta)} = 1$$

$$e^{h(\beta)} = e^{2n\pi i}$$

$$h(\beta) = 2n\pi i$$

$$\therefore (1) \Rightarrow \int_\alpha^\beta \frac{z'(\beta)\,d\beta}{z(\beta)-a} = 2n\pi i$$

$$\int_\gamma \frac{dz}{z-a} = 2n\pi i.$$

$\Rightarrow \int_\gamma \frac{dz}{z-a}$ is a multiple of $2n\pi i$

Definition :-

[Define the index 'a' w.r.to a curve γ or define the winding number of γ w.r.to 'a']

The index of the point 'a' w.r.to curve γ by the equation.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}.$$

The index is also called winding number of γ, w.r.to 'a'

Result

Prove that $n(-\gamma, a) = -n(\gamma, a)$

By definition.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

$$n(-\gamma, a) = \frac{1}{2\pi i} \int_{-\gamma} \frac{dz}{z-a}$$

$$= -\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} \qquad \text{Since } \int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$$

$$n(-\gamma, a) = -n(\gamma, a)$$

Result : 2.

If γ lies inside the disc and a lies outside the disc. then Prove that $n(\gamma, a) = 0$.

By definition.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}.$$

Since a lies outside the disc, $\frac{1}{z-a}$ is analytic inside the disc. and γ is any closed curve in Δ. Hence by Cauchy's theorem.

$$\int_{\gamma} \frac{dz}{z-a} = 0.$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}=0.$$

$n(\gamma,a)=0.$

RESULT: 3

As a function of 'a', the index $n(\gamma,a)$ is constant in each of the region determined by γ. and zero in the unbounded region.

Proof.

If a and b belong to the same region determined by γ, then we have to prove that.

$$n(\gamma,a)=n(\gamma,b)$$

Join a and b by a line segment, not intersecting γ. Outside this line segment $\log\left(\frac{z-a}{z-b}\right)$ is analytic and its derivative its $\frac{1}{z-a}-\frac{1}{z-b}$.

Hence $\int_{\gamma}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)dz=0.$

$$\int_{\gamma}\frac{1}{z-a}dz=\int_{\gamma}\frac{1}{z-b}dz$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}=\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b},$$

$\therefore\ n(\gamma, a) = n(\gamma, b)$.

Next if a lies outside the region (unbounded) determined γ, then $\frac{1}{z-a}$ is analytic inside γ

Hence $\int_{\gamma} \frac{dz}{z-a} = 0$.

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = 0.$$

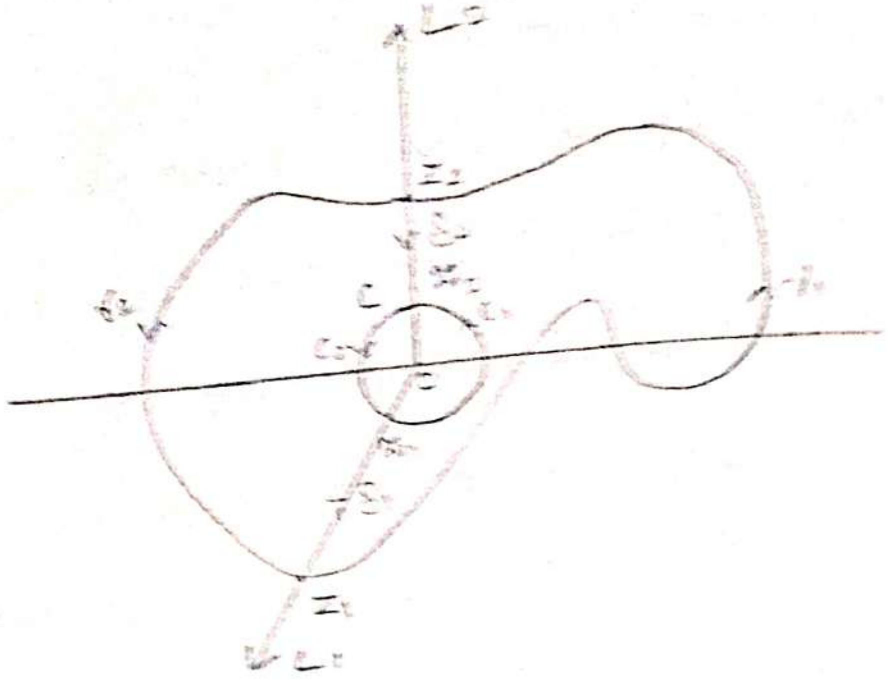
$$n(\gamma, a) = 0.$$

LEMMA:

Let z_1, z_2 be two points and a closed curve γ which does not pass through the origin denote the sub arc and from z_1 and z_2 in the direction of the curve by γ_1, and the subarc from z_2 and z_1 by γ_2 Suppose that z_1 lies in the lower half plane and z_2 lies in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis. Then P.T $n(\gamma, 0) = 1$

Proof.

Draw the half lines L_1 and L_2 from the origin through z_1 and z_2. Let C be a circle.



with origin as centre. Intersecting L_1 and L_2 in z_{01} and z_{02}. Let C_1 be the sub arc of the circle C from z_{01} and z_{02}, which does not meet the negative real axis.

Let C_2 be the sub arc of the circle from z_{01} and z_{02} which does not meet the positive real axis.

Denote the directed line segment from z_1 to z_{01} and from z_2 to z_{02} by δ_1, δ_2.

Introducing the closed curves

$$\sigma_1 = \gamma_1 + \delta_2 - C_1 - \delta_1$$

$$\sigma_2 = \gamma_2 + \delta_1 - C_2 - \delta_2$$

$$\sigma_1 + \sigma_2 = \gamma_1 + \gamma_2 - (C_1 + C_2)$$

$$\sigma_1 + \sigma_2 = \gamma - C$$

$\gamma = \sigma_1 + \sigma_2 + C.$

$n(\gamma, 0) = n(\sigma_1, 0) + n(\sigma_2, 0) + n(C, 0).$

By our discription σ_1 does not meet the -ve real axis. Hence zero lies in the unbounded region determined by σ_1. Hence $n(\sigma_1, 0) = 0$.

$|||^{ly}$.
$n(\sigma_2, 0) = 0.$

Since zero lies inside C

$\therefore n(C, 0) = 1$

Hence $n(\gamma, 0) = 1$.

Cauchy's integral formula.

(5) State and prove cauchy's integral formula.

Suppose that $f(z)$ is analytic in an open disc Δ and let γ be a closed curve in Δ for any point 'a' not on γ.

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\, dz.$$

Where $n(\gamma, a)$ is a index of 'a' respect to γ.

Proof.

Let $f(z)$ be an analytic on open disc Δ. and γ be a closed curve in Δ. Let a belongs to Δ which does not lies on γ.

consider,

$$F(z) = \frac{f(z) - f(a)}{z - a}.$$

This function is analytic at $z \neq a$. When $z = a$. The function is not defined

by but,

$$\lim_{z\to a} (z-a) F(z) = \lim_{z\to a} \cancel{(z-a)} \left[\frac{f(z) - f(a)}{\cancel{z-a}}\right]$$

$$= \lim_{z\to a} (f(z) - f(a))$$

$$= f(a) - f(a)$$

$$\lim_{z\to a} (z-a) F(z) = 0.$$ $\therefore$ by a $f(z)$ is continuous at $z=a$.

$$\therefore \int_\gamma F(z)\, dz = 0.$$

$$\int_\gamma \left(\frac{f(z) - f(a)}{z-a}\right) dz = 0.$$

$$\int_\gamma \frac{f(z)}{z-a} dz - f(a) \int_\gamma \frac{dz}{z-a} = 0.$$

$$\int_\gamma \frac{f(z)}{z-a} dz = f(a) \int_\gamma \frac{dz}{z-a}.$$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = f(a) \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Note: -

If $n(\gamma, a) = 1$ then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Compute: $\int_{|z|=1} \frac{e^z}{z}\, dz$.

Here $c: |z| = 1$ is a closed curve in an open disc. Infact this is the circle.

$n(\gamma, a) = n(c, 0) = 1$.

By cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

$$\int_{|z|=1} \frac{e^z}{z}\, dz = n(c,0)\, f(0) \cdot 2\pi i \quad \{\because f(z) = e^z,\ f(0) = 1.$$

$$= 2\pi i$$

Compute $\int_{|z|=2} \frac{dz}{z^2+1}$

$c: |z| = 2$ is a circle.

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\int_\gamma \frac{dz}{z^2+1} = \frac{1}{2i}\left[\int_c \frac{dz}{z-i} - \int_c \frac{dz}{z+i}\right]$$

$$= \frac{1}{2i}\left[2\pi i \times 1 - 2\pi i \times 1\right]$$ $n(c, i) = 1$, $n(c, -i) = 1$

$$= 0$$

8.1.2008.

Evaluate : $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$ on the circle

1) $|z| = 4$, (2) $|z| = 1$.

Solution

(i) $C : |z| = 4$.

$z - 3 = 0$.

$z = 3$

$z = 3$ lies inside C.

$f(z) = z^2+5$

$= 9+5$

$= 14$.

By cauchy's integral formula.

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = f(3)$$

$$= 14.$$

(ii) $C : |z| = 1$.

$z = 3$ lies outside the circle.

By cauchy's theorem.

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz = 0.$$

2.

Evaluate $\int_C \frac{3z-1}{z^3-z} dz$ using cis

(i) $|z| = \frac{1}{2}$

(ii) $|z| = 2$.

Solution.

(i) $C : |z| = \frac{1}{2}$.

$$I = \int_C \frac{3z-1}{z^3-z} dz.$$

$$= \frac{3z-1}{z(z+1)(z-1)} dz$$

$\therefore z(z+1)(z-1) = 0$.

$z = 0, 1, -1$

$z = 0$ lies inside c

$$f(z) = \frac{3z-1}{(z+1)(z-1)}$$

$$f(0) = \frac{-1}{-1} = 1$$

By cauchy's integral formula.

$$I = 2\pi i f(0)$$

$$= 2\pi i.$$

(ii) $c : |z| = 2$.

$z = 0, 1, -1$ all the points lies inside c.

$$\frac{3z-1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}.$$

$$3z-1 = A(z-1)(z+1) + B(z+1)z + Cz(z-1)$$

But $z=0$

$$-1 = -A.$$

$$\boxed{A=1}$$

But $z=1$

$$3-1 = 2B.$$

$$2 = 2B.$$

$$\boxed{B=1}$$

$z=-1$

$$-3-1 = 2C$$

$$-4 = 2C$$

$$\boxed{C=-2}$$

$$\frac{3z-1}{z(z-1)(z+2)} = \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z+1}$$

$f(z)=1$
$f(0)=1$
$f(-1)=1$
$f(1)=1$

$$I = \int_C \frac{dz}{z} + \int_C \frac{dz}{z-1} - 2\int_C \frac{dz}{z+1}$$

$$= 2\pi i\, f(0) + 2\pi i\, f(1) - 2\times 2\pi i\, f(-1$$

$$= 2\pi i + 2\pi i - 4\pi i$$

$$= 0.$$

Evaluate $\int \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle

$|z| = 3/2$.

39, 27, 48

Solution

$c : |z| = 3/2$.

$(z-1)(z-2) = 0$

$z = 1, 2$.

$z=1$ lies inside C.

$$f(z) = \frac{\cos \pi z^2}{z-2}.$$

$$f(1) = \frac{-1}{-1} = 1$$

By cauchy's integral formula.

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i \, f(1)$$

$$= 2\pi i.$$

S

Lemma:-

Suppose that $\phi(\zeta)$ is continuous on the arc γ. Then the function $F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} d\zeta$

is analytic in each of the regions determined by γ. and its derivatives is $F_n'(z) = n F_{n+1}(z)$

Proof. , Step: 1

First we have to prove that $F_1(z)$ is continuous at z_0.

Let z_0 be a point not at γ, and choose to the neighbourhood $|z-z_0|<\delta$. Show that it does not meet γ, by restricting z to the smaller neighbourhood. $|z-z_0|<\delta/2$.

We attain that $|\zeta - z| > \delta/2$ for all $\zeta \in \gamma$.

$$F_1(z) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z}\, d\zeta.$$

$$F_1(z_0) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z_0}\, d\zeta$$

$$F_1(z) - F_1(z_0) = \int_\gamma \phi(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right] d\zeta$$

$$= (z-z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)}\, d\zeta$$

$$|F_1(z) - F_1(z_0)| \leq |z-z_0| \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)}\, |d(\zeta)|$$

$$\leq |z-z_0| \frac{4M}{\delta^2} \int_\gamma |d\zeta|$$

$$|F_1(z) - F_1(z_0)| \leq |z-z_0| \frac{4M}{\delta^2} \cdot L$$

$|F_1(z) - F_1(z_0)| \leq \epsilon$ an $z - z_0$.

$\Rightarrow F_1(z)$ is continuous at z_0.

Let us prove the result by the principle of induction

Step : 2.

Next we have to prove that $F_1'(z) = 1 \cdot F_2(z)$

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} d\zeta.$$

$$F_1'(z_0) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)(\zeta - z_0)} d\zeta$$

$$= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^2} d\zeta$$

$$F_1'(z_0) = F_2(z_0)$$

$$F_1'(z_0) = 1 \cdot F_2(z_0)$$

This is true for all $z_0 \in \gamma$.

$\therefore F_1'(z) = 1 \cdot F_2(z)$.

Step : 3.

Assume that $F_{n-1}'(z) = (n-1) F_{n}(z)$

finally we have to prove that $F_n'(z) = n F_{n+1}(z)$

$$F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} d\zeta \qquad F_n(z_0) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta$$

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta$$

$$= \int_\gamma \cancel{\phi(\zeta - z_0)}$$

$$= \int_\gamma \frac{\phi(\zeta)(\zeta - z_0)}{(\zeta - z)^{n-1}(\zeta - z)(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta.$$

$(\zeta - z)^{n-1}$

$$= \int_\gamma \frac{\phi(\zeta)(\zeta - z + z - z_0)}{(\zeta - z)^{n-1}(\zeta - z)(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta$$

$$= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} + (z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta$$

$$- \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta$$

Now $F_{n-1}'(z)$ exist $\Rightarrow F_{n-1}(z)$ is continuous at:

i.e., $\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)}$ is continuous at z_0

Similarly.

$$\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}} d\zeta \text{ is continuous at } z_0$$

Hence as $z \to z_0$.

$$\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z_0)^n} d\zeta \to 0.$$

Further.

$$\left| (z - z_0) \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n (\zeta - z_0)} d\zeta \right| \le |z - z_0| \cdot M \frac{2^{n+1}}{\delta^{n+1}} L$$

$$(z - z_0)\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n(\zeta - z_0)} d\zeta \to 0 \text{ as } z - z_0.$$

$\therefore F_n(z) - F_n(z_0) \to 0$ as $z \to z_0$

$$\lim_{z \to z_0} F_n(z) = F_n(z_0)$$

$\Rightarrow F_n(z)$ is continuous at z_0.

Further

$$\int_\gamma \frac{\phi(\xi)}{(\xi-z)^n(\xi-z_0)}\,d\xi \text{ is continuous at } z_0.$$

$$\lim_{z\to z_0}\int_\gamma \frac{\phi(\xi)}{(\xi-z)^n(\xi-z_0)}\,d\xi = \int_\gamma \frac{\phi(\xi)}{(\xi-z_0)^{n+1}}\,d\xi. = \psi_n(z_0)$$

$$= F_{n+1}(z_0).$$

Let $\psi_{n-1}(z) = \int_\gamma \frac{\phi(\xi)}{(\xi-z)^{n-1}(\xi-z_0)}\,d\xi$ $(\xi-z_0)^{n-1}(\xi-z_0)^1$

(I)

$$\psi_{n-1}(z_0) = \int_\gamma \frac{\phi(\xi)}{(\xi-z_0)^n}\,d\xi.$$

$\psi_{n-1}(z)$ is continuous on z_0

$$\lim_{z\to z_0}\frac{\psi_{n-1}(z)-\psi_{n-1}(z_0)}{z-z_0} = \psi_{n-1}'(z_0)$$

$$= (n-1)\,\psi_n(z_0)$$

$$= (n-1)\int_\gamma \frac{\phi(\xi)}{(\xi-z_0)^{n+1}}\,d\xi.$$

$$\lim_{z\to z_0}\frac{\psi_{n-1}(z)-\psi_{n-1}(z_0)}{z-z_0} = (n-1)F_{n+1}(z_0)$$

$$\lim_{z\to z_0}\frac{1}{z-z_0}\left[\int_\gamma \frac{\phi(\xi)\,d\xi}{(\xi-z)^{n-1}(\xi-z_0)} - \int_\gamma \frac{\phi(\xi)}{(\xi-z_0)^n}\,d\xi\right] = (n-1)F_{n+1}(z_0).$$

$$\lim_{z\to z_0}\frac{F_n(z)-F_n(z_0)}{z-z_0} = (n-1)F_{n+1}(z_0) + F_{n+1}(z_0)$$

$$F_n'(z_0) = n\,F_{n+1}(z_0)$$

This is true for any arbitrary z_0

Hence $F_n'(z) = n F_{n+1}(z)$.

4-01-08. Theorem:-

An analytic function defined in a region Ω as derivative of all order and there derivative are analytic in Ω.

Proof.

Let $a \in \Omega$ and $f(z)$ be analytic in Ω consider a δ-neighbourhood Δ above a in Ω

Take a circle c in $\delta\Delta$

For all z inside c, $n(c, z) = n(c, a) = 1$.

By cauchy's integral formula.

$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

By the lemma.

$$F_n'(z) = n F_{n+1}(z)$$

Where

$$F_n(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n} \, d\zeta$$

we get.

$$f'(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

again applying the above lemma

we get,

$$f''(z) = \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

i.e., $$f''(z) = \frac{\lfloor 2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

Ill^{ly} $$f'''(z) = \frac{\lfloor 3}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^4} dz.$$

. . . .

. . . .

$$f^n(z) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(z)}{(\zeta - z)^{n+1}} d\zeta$$

Morera's Theorem:-

If $f(z)$ is defined and continuous in a region Ω and if $\int_\gamma f(z)\,dz = 0$ for all closed curve γ in Ω then $f(z)$ is analytic in Ω.

Proof.

$\int_\gamma f(z)dz = 0$, for all closed curve γ in Ω

$\Rightarrow$ $f(z)dz$ is an exact differential in Ω

$\Rightarrow$ $\exists$ an analytic function $F(z)$ in Ω

Such that $F'(z) = f(z)$

Since the derivative an analytic function is analytic it follows that $f(z)$ is analytic in Ω.

5) Cauchy's Inequality

If $f(z)$ is analytic in Ω and C is any circle with centre 'a' and radius r in Ω then $|f^n(a)| \le \frac{M n!}{r^n}$, where $|f(z)| \le M$ on C

Proof:

For Cauchy's integral formula for n^{th} derivative,

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

C is $|z-a| = r$.

Since $f(z)$ is analytic in Ω, $f(z)$ is continuous in Ω and hence on C.

Since C is compact, $|f(z)| \le M$ on C.

$$|f^n(a)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \right|$$

$$\le \frac{n!}{2\pi} \int_C \frac{|f(z)|}{|z-a|^{n+1}} |dz|$$

$$\le \frac{n!}{2\pi} \frac{M}{r^{n+1}} \int_C |dz|$$

$$\le \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} 2\pi r$$

$$|f^n(a)| \le \frac{M n!}{r^n}$$

Entire function.

A function $f(z)$ which is analytic in the whole plane is called an entire function. (integral function).

Example:-

Any polynomial in $z, e^z, \sin z, \cos z$ are some examples to entire function.

$\log z$ is not an entire function.

State and prove Liouville's theorem.

A function which is analytic and bounded in the whole plane must reduced the constant

(or)

Any bounded entire function is constant

Proof -

Let a be the any point in the plane.

take circle 'c' with centre 'a' and radius 'r'

$c : |z-a| = r$.

By cauchy's integral formula for the first derivative.

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$$

$$|f'(a)| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz\right|$$

$$\leq \frac{1}{2\pi}\int_C \frac{|f(z)|}{|z-a|^2}|dz|$$

Since $f(z)$ is bounded on the whole plane.

$|f(z)| \leq M, \forall z.$

$|f(z)| \leq M$ on C.

$$|f'(a)| \leq \frac{M}{2\pi r^2}\int_C |dz|$$

$$\leq \frac{M}{\cancel{2\pi} r^{\cancel{2}}} \times \cancel{2\pi} \cancel{r}.$$

$$\leq M/r.$$

This is true of any circle and with any radius r. when $r \to \infty$ (we get the whole plane) $|f'(a)| \leq 0$

$\Rightarrow f'(a) = 0$

$f(a) =$ constant

This is true for any point a in the plane.

$\therefore$ $f(z)$ is constant in whole plane.

---

5 State and prove fundamental theorem of Algebra.

Statement

If $P(z)$ is a polynomial of degree $n \geq 1$ with real and complex co-efficient then

The equation $P(z)=0$ has atleast one root.

Proof:

Let $P(z) = a_0 + a_1(z) + a_2 z^2 + \dots + a_n z^n$

Suppose that $P(z)=0$ has no root.

Then $P(z) \neq 0$. for any z

Now,

$$P(z) = z^n\left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)$$

for sufficiently large values of $|z|$ we have.

$$\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots \frac{a_{n-1}}{z}\right| \leq \frac{|a_n|}{2}.$$

$$\therefore \left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} + a_n\right| \geq |a_n| - \left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z}\right|$$

$$\left|\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} + a_n\right| \geq |a_n| - \frac{|a_n|}{2}$$

$$\geq \frac{|a_n|}{2}.$$

$$\left|z^n\left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_{n-1}}{z^{n-1}} + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)\right| \geq \frac{|a_n||z|^n}{2}$$

$$|P(z)| \geq \frac{|a_n||z|^n}{2}$$

$|P(z)| \to \infty$ as $z \to \infty$ $\Rightarrow |P(z)| \to \infty$ as $z \to \infty$

$|P(z)| \geq M \;\forall\; |z| > R$.

$|Q(z)| \leq \frac{1}{M} \;\forall\; |z| > R$, Where $Q(z) = \frac{1}{P(z)}$.

$\Rightarrow Q(z)$ is bounded When $|z| > R$

Further $Q(z) = \frac{1}{P(z)}$ and $P(z) \neq 0$ for any z

$\Rightarrow$ that $Q(z)$ is analytic at whole plane

If $|z| \leq R$ then $Q(z)$ is analytic in

$|z| \leq R$.

$\Rightarrow$ $Q(z)$ is bounded when $|z| \leq R$. $\rightarrow$ ②

From ① and ② we get.

$Q(z)$ is bounded for all z

By Liouville's theorem. $Q(z)$ is a constant

$\Rightarrow$ $P(z)$ is also a constant.

This is a contradiction is our hypothesis that $P(z)$ is non constant polynomial.

Hence $P(z)$ is zero as atleast one root.

---

① compute $\int_{|z|=1} e^{z} z^{-n} \, dz$

$$I = \int_{|z|=1} e^{z} . z^{-n} dz.$$

$C : |z| = 1$.

$$I = \int_C \frac{e^z}{z^n} dz$$

$z = 0$ lies inside C

$$\begin{cases} f(z) = e^z \\ f^{n-1}(z) = e^z \\ f^{n-1}(z) = e^0 = 1 . \end{cases}$$

$$I = \frac{2\pi i}{\lfloor n-1} . f^{n-1}(0)$$

$$I = \frac{2\pi i}{\lfloor n-1}$$

2) Evaluate: $\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz$ where c is the circle $|z|=1$

$$I = \int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz.$$

$f(z) = \sin^2 z$

$f'(z) = 2\sin z\cos z$

$= \sin 2z$

$f''(z) = 2\cos 2z$

$\therefore C : |z| = 1.$

$z = \pi/6$ lies inside

$f''(\pi/6) = 2\cos 2\pi/6.$

$= 2\cos \pi/3 = 2 \times \frac{1}{2}$

$= 1$

$$I = \frac{2\pi i}{2!} f''(\pi/6)$$

$= \pi i.$

3 Evaluate $\int_C \frac{e^{2z}}{(z-1)^4} dz$ where c is $|z| = 3/2.$

$$I = \int_C \frac{e^{2z}}{(z-1)^4} dz$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f(1) = 8e^2.$

$C : |z| = 3/2.$

$z=1$ lies inside C

$$I = \frac{2\pi i}{3!} f'''(1)$$

$= \frac{2\pi i}{6_3} \cdot 8e^2$

$= \frac{8\pi i}{3} e^2.$

$\int \frac{dz}{z^2(z-3)}$ where c is $|z|=2$.

$$I = \int \frac{dz}{z^2(z-3)}$$

$f(z) = \frac{1}{z-3}$

$c : |z| = 2.$

$f'(z) = -\frac{1}{(z-3)^2}$

$z=0 \quad z=3.$

$z=0$ lies inside c.

$f'(0) = -1/9.$

$$I = \frac{2\pi i}{1!} \cdot f'(0)$$

$$= \frac{2\pi i}{1!} \cdot -1/9.$$

$$= -\frac{2\pi i}{9}$$

08

*. Evaluate $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ where $|a| \neq \rho$

Solution

$$I = \int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

Since $|z| = \rho$

$$z = \rho e^{i\theta}$$

$$dz = \rho e^{i\theta}\, i\, d\theta$$

$$|dz| = |\rho e^{i\theta}\, i\, d\theta|$$

$$|dz| = \rho\, d\theta$$

$$|z-a|^2 = |\rho e^{i\theta} - a|^2$$

$$= |\rho(\cos\theta + i\sin\theta) - a|^2$$

$$= |(\rho\cos\theta - a) + \rho i \sin\theta|^2$$

$$= (\rho\cos\theta - a)^2 + \rho^2\sin^2\theta.$$

$$|z-a|^2 = \rho^2 + a^2 - 2a\rho\cos\theta.$$

Since $\cos(2\pi-\theta) = \cos\theta$

$$I = \int_0^{2\pi} \frac{\rho\, d\theta}{\rho^2 + a^2 - 2a\rho\cos\theta}.$$

Put $t = \tan\theta/2$

$d\theta = \frac{2dt}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$$= 2\rho \int_0^{\pi} \frac{d\theta}{\rho^2 + a^2 - 2a\rho\cos\theta}$$

When $\theta = 0, t = 0$

$\theta = \pi, t = \infty$

$$= 2\rho \int_0^{\pi} \frac{\frac{2dt}{1+t^2}}{\rho^2 + a^2 - 2a\rho\,\frac{(1-t^2)}{(1+t^2)}}$$

$$= 2\rho \int_0^{\infty} \frac{dt}{(\rho^2+a^2)(1+t^2) - 2a\rho(1-t^2)}$$

$$= 4\rho \int_0^{\infty} \frac{dt}{\rho^2 + a^2 - 2a\rho + (\rho^2 + a^2 + 2a\rho)t^2}$$

$$= 4\rho \int_0^{\infty} \frac{dt}{(\rho - a)^2 + (\rho + a)^2 t^2}$$

$$= \frac{4\rho}{(\rho+a)^2} \int_0^{\infty} \frac{dt}{\left(\frac{\rho-a}{\rho+a}\right)^2 + t^2}$$

$$= \frac{4\rho}{(\rho+a)^2}\, \frac{1}{\frac{\rho-a}{\rho+a}} \tan^{-1}\left(\frac{t}{\frac{\rho-a}{\rho+a}}\right)\Bigg]_0^{\infty}$$

$$= \frac{4\rho}{\rho^2 - a^2} \cdot \left(\frac{\pi}{2} - 0\right)$$

$$I = \frac{2\pi\rho}{\rho^2 - a^2}$$

2. Evaluate $\int_C z^2 dz$ where C is the arc of the circle $|z| = r$ from $\theta = \alpha$ to $\theta = \beta$

Solution.

2.

$$I = \int_C z^2 dz.$$

$$|z| = r$$

$$z = re^{i\theta}.$$

$$dz = re^{i\theta}\, i\, d\theta.$$

$$|dz| = |re^{i\theta}\, i\, d\theta|$$

$$|dz| = r\, d\theta.$$

θ varies from α to β.

$$I = \int_\alpha^\beta r^2 e^{i2\theta}\, re^{i\theta}\, i\, d\theta.$$

$$= r^3 i \int_\alpha^\beta e^{3i\theta}\, d\theta.$$

$$= r^3 i \left.\frac{e^{3i\theta}}{3i}\right)_\alpha^\beta$$

$$= \frac{r^3}{3}\left(e^{3i\beta} - e^{3i\alpha}\right).$$

Unit-2