SEMESTER : II
CORE COURSE : VI

Inst Hour	: 6
Credit	: 5
Code	: 18KP2M06

LINEAR ALGEBRA

UNIT-I

Introduction – Vector Spaces – Subspaces – Linear Combinations and Systems of Linear Equations – Linear Dependence – Linear Independence – Basis and Dimension – Maximal Linearly Independent Subspaces.

Chapter 1: Sections 1.1 to 1.7

UNIT – II

Linear Transformations – Null Spaces and Ranges – The Matrix Representation of a Linear Transformation – Composition of Linear Transformation and Matrix Multiplication – Invertibility and Isomorphism – The change of Coordinate Matrix – Dual Spaces – Homogeneous Linear Differential Equations with Constant Coefficients.

Chapter 2: Sections 2.1 to 2.7

UNIT- III

Elementary Matrix Operations and Elementary Matrices – The Rank of Matrix and Matrix Inverses – System of Linear Equations – Theoretical and Computational Aspects. Determinants of Order 2 and Order n – Properties of Determinants.

Chapter 3: Sections 3.1 to 3.4 and Chapter 4: Sections 4.1 to 4.3

UNIT – IV

Eigen Values and Eigen Vectors – Diagonalizability – Matrix Limits and Markov Chains – Invariant Subspaces and The Cayley Hamilton Theorem.

Chapter 5: Sections 5.1 to 5.4

UNIT – V

Inner Product and Norms – The Gram Schmidt Orthogonalization Process and Orthogonal Complements – The Adjoint of a Linear Operator – Normal and Self – Adjoint Operators – Unitary and Orthogonal Operators and their Matrices

Chapter 6: Sections 6.1 to 6.5

TEXT BOOK

[1] Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Edition, Pearson New International Limited, 2014.

REFFERENCES

1. I.N. Herstein, Topics in Algebra, Wiley Eastern Limited, New Delhi1975.
2. I.S.Luther and I.B.S.Passi , Algebra, Vol I-Groups , Vol.II- Rings, Narosa Publishing House (Vol.I-1996,Vol.II-1999)
3. N.Jacobson, Basic Algebra, Vol.I& II.Freeman, 1980 Hindustan Publishing Company.
4. Kenneth Hoffman and Ray Kunze, Linear Algebra, Second Edition, Prentice– Hall of India Private Limited, New Delhi, 2014.

Question Pattern

Section A : 10 x 2 = 20 Marks, 2 Questions from each Unit.
Section B : 5 x 5 = 25 Marks, EITHER OR (a or b) Pattern, One question from each Unit.
Section C : 3 x 10 = 30 Marks, 3 out of 5, One Question from each Unit.

9.3.18

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1. Vector Space :

A vector space V over a field F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V, and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold.

1. For all x, y in V, $x+y = y+x$ (commutatively of addition).
2. For all x, y, z in V, $(x+y)+z = x+(y+z)$ (associativity of addition).
3. There exists an element in V denoted by 0 such that $x+0 = x$ for each x in V.
4. For each element x in V there exists an element y in V such that $x+y = 0$
5. For each element x in V, $1x = x$.
6. For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$
7. For each element a in F and each pair of elements x, y in V, $a(x+y) = ax + ay$.
8. For each pair of elements a, b in F and each element x in V, $(a+b)x = ax + bx$.

The elements $x+y$ and ax are called the sum of x and y and the product of a and x, respectively.

2. Subspace :

A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

In any vector space V, V and $\{0\}$ are subspaces. The latter is called the zero subspace of V.

Thus a subset W of a vector space V is a subspace of V if and only if the following four properties hold.

1. $x+y \in W$ whenever $x \in W$ and $y \in W$. (W is closed under addition).
2. $cx \in W$ whenever $c \in F$ and $x \in W$. (W is closed under scalar multiplication)
3. W has a zero vector.
4. Each vector in W has an additive inverse in W.

3. Theorem:

Let V be a vector space and W be a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

(a) $0 \in W$

(b) $x+y \in W$ whenever $x \in W$ and $y \in W$.

(c) $cx \in W$ whenever $c \in F$ and $x \in W$.

Proof:

If W is a subspace of V, then W is a vector space with the operations of addition and scalar multiplication defined on V. Hence conditions (b) and (c) hold, and there exists a vector $0' \in W$ such that $x+0'=x$ for each $x \in W$. But also $x+0=x$, and thus $0'=0$ by cancellation law of vector addition. So condition (a) holds.

Conversely, if conditions (a), (b), and (c) hold, the discussion preceding this theorem shows that W is a subspace of V if the additive inverse of each vector in W lies in W.

But if $x \in W$, then $(-1)x \in W$ by condition (c), and $-x=(-1)x$.

Hence W is a subspace of V.

4. **Theorem :**

Any intersection of subspaces of a vector space V is a subspace of V.

Proof :

Let C be a collection of subspaces of V, and let W denote the intersection of the subspaces in C. Since every subspace contains the zero vector, $0 \in W$.

Let $a \in F$ and $x, y \in W$. Then x and y are contained in each subspace in C. Because each subspace in C is closed under addition and scalar multiplication, it follows that $x+y$ and ax are contained in each subspace in C. Hence $x+y$ and ax are also contained in W, so that W is a subspace of V.

5. **Linear Combination :**

Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if there exists a finite number of vectors $u_1, u_2, \ldots, u_n$ is in S and scalars $a_1, a_2, \ldots, a_n$ in F, a linear combination of $u_1, u_2, \ldots u_n$ such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2, \ldots u_n$ and call $a_1, a_2, \ldots, a_n$ the coefficiets of linear combination.

Observe that in any vector space V, $0v = 0$ for each $v \in V$. Thus the zero vector is a linear combination of any nonempty subset of V.

6. **Span :**

Let S be a nonempty subset of a vector space V. The span of S, denoted span (S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span $(\phi) = \{0\}$.

7. **Theorem :**

The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.

Proof :

This result is immediate if $S = \phi$ because span $(\phi) = \{0\}$, which is a subspace that is contained in any subspace of V.

If $S \neq \phi$, then S contains a vector z. So $0z = 0$ is in span (S). Let $x, y \in$ span (S). Then there exists vectors $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$, such that

$$x = a_1u_1 + a_2u_2 + \cdots + a_mu_m \text{ and } y = b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

Then,

$$x+y = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

and, for any scalar c,

$$cx = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_m)u_m$$

are clearly linear combinations of the vectors in S; so $x+y$ and cx are in span (S). Thus span (S) is a subspace of V.

Now, let W denote any subspace of V that contains S. If $w \in$ span (S), then w has the form $w = c_1w_1 + c_2w_2 + \cdots + c_kw_k$ for some vectors $w_1, w_2, \ldots, w_k$ in S and some scalars $c_1, c_2, \ldots, c_k$.

Since $S \subseteq W$, we have $w_1, w_2, \ldots, w_k \in W$. Therefore $w = c_1w_1 + c_2w_2 + \cdots + c_kw_k$ is in W. Because w, an arbitrary vector in span(S), belongs to W, it follows that span(S) $\subseteq W$.

8. **Linearly dependent Set :**

A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$$

In this case we also say that the vectors of S are linearly dependent.

9. **Linearly independent Set :**

Let V be a vector space over a field F. A finite set of vectors $v_1, v_2, \ldots, v_n$ in V is said to be linearly independent if

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0.$$

10. **Theorem :**

Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in$ span(S).

Proof :

If $S \cup \{v\}$ is linearly dependent, then there are vectors $u_1, u_2, \ldots, u_n$ in $S \cup \{v\}$ such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$ for some nonzero scalars $a_1, a_2, \ldots, a_n$. Because S is linearly independent, one of the u_i's, say u_1, equals v.

Thus $a_1v + a_2u_2 + \cdots + a_nu_n = 0$, and so,

$$v = a_1^{-1}(-a_2u_2 - \cdots - a_nu_n) = -(a_1^{-1}a_2)u_2 - \cdots - (a_1^{-1}a_n)u_n.$$

Since v is a linear combination of $u_2, \ldots, u_n$, which are in S, we have $v \in \text{span}(S)$.

Conversely, let $v \in \text{span}(S)$. Then there exists vectors $v_1, v_2, \ldots, v_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $v = b_1v_1 + b_2v_2 + \cdots + b_mv_m$.

Hence

$$0 = b_1v_1 + b_2v_2 + \cdots + b_mv_m + (-1)v.$$

Since $v \neq v_i$ for $i = 1, 2, \ldots, m$, the coefficient of v in this linear combination is non-zero, and so that the set $\{v_1, v_2, \ldots, v_m, v\}$ is linearly dependent.

Therefore $S \cup \{v\}$ is linearly dependent.

11. **Basis:**

A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

12. **Theorem:**

Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof:

Let β be a basis for V. If $v \in V$, then $v \in \text{span}(\beta)$, because $\text{span}(\beta) = V$.

Thus v is a linear combination of the vectors of β.

Suppose that,

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \quad \text{and} \quad v = b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

are two such that representations of v. Subtracting the second equation from the first gives

$$0 = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_n - b_n)u_n.$$

Since β is linearly independent, it follows that $a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$. Hence $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$, and so v is uniquely expressible as a linear combination of the vectors of β.

Conversely,

$$v \in V, \quad \text{Span}(S) = V$$

uniquely expressed as a linear combination of vectors of β.

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

$$a_1u_1 = -a_2u_2 = -a_3u_3 = \cdots = 0,$$

linearly independent subset of V.

Hence β is a basis for V.

13. **Theorem:**

If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

Proof:

If $S = \phi$ or $S = \{0\}$, then $V = \{0\}$ and ϕ is a subset of S that is a basis for V. Otherwise S contains a nonzero vector u_1. $\{u_1\}$ is a linearly independent set.

Continue, if possible, choosing vectors $u_2, \ldots, u_k$ in S

Such that $\{u_1, u_2, \ldots u_k\}$ is linearly independent. Since S is a finite set, we must eventually reach a stage at which $\beta = \{u_1, u_2, \ldots, u_k\}$ is a linearly independent subset of S, but adjoining to β any vector in S not in β Produces a linearly dependent set.

We claim that β is a basis for V. Because β is linearly independent by construction, it suffices to show that β spans V. We need to show that $S \subseteq \text{span}(\beta)$.

Let $v \in S$: If $v \in \beta$, then clearly $v \in \text{span}(\beta)$. Otherwise, if $v \notin \beta$, then the preceeding constructions show that $\beta \cup \{v\}$ is linearly dependent. So $v \in \text{span}(\beta)$.

Thus $S \subseteq \text{span}(\beta)$.

14. Replacement Theorem:

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V.

Proof:

The Proof is by mathematical induction on m. The induction begins with $m=0$; for in this case $L = \phi$, and so taking $H = G$ gives the desired result.

Now suppose that the theorem is true for some integer $m \geq 0$.

We prove that the theorem is true for $m+1$. Let $L = \{v_1, v_2, \dots, v_{m+1}\}$ be a linearly independent subset of V consisting of $m+1$ vectors. $\{v_1, v_2, v_3, \dots v_m\}$ is linearly independent, and so we may apply the induction hypothesis to conclude that $m \leq n$ and that there is a subset $\{u_1, u_2, \dots, u_{n-m}\}$ of G such that $\{v_1, v_2, \dots v_m\} \cup \{u_1, u_2, \dots, u_{n-m}\}$ generates V. Thus there exist scalars $a_1, a_2, \dots a_m, b_1, b_2, \dots, b_{n-m}$ such that

$$a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + b_2 u_2 + \dots + b_{n-m} u_{n-m} = v_{m+1} \qquad (1)$$

Note that $n-m > 0$, lest v_{m+1} be a linear combination of $v_1, v_2, \dots v_m$ contradicts the assumption that L is linearly independent.

Hence $n > m$; that is, $n \geq m+1$. Moreover, some b_i, say b_1, is nonzero, for otherwise we obtain the same contradiction.

Solving (1) for u_1 gives

$$u_1 = (-b_1^{-1}a_1)v_1 + (-b_1^{-1}a_2)v_2 + \dots + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1}$$
$$+ (-b_1^{-1}b_2)u_2 + \dots + (-b_1^{-1}b_{n-m})u_{n-m}.$$

Let $H = \{u_2, \dots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$, and because $v_1, v_2, \dots v_m, u_2, \dots, u_{n-m}$ are clearly in $\text{span}(L \cup H)$, it follows that

$$\{v_1, v_2, \dots v_m, u_1, u_2, \dots u_{n-m}\} \subseteq \text{span}(L \cup H).$$

Because $\{v_1, v_2, \dots v_m, u_1, u_2, \dots, u_{n-m}\}$ generates V. $\text{Span}(L \cup H) = V$. Since H is a subset of G that contains $(n-m)-1 = n-(m+1)$ vectors, the theorem is true for $m+1$. This completes the induction.

15. Finite dimensional :

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by dim(V).

16. Maximal Linearly independent subset :

Let S be a subset of a vector space V. A maximal linearly independent subset of S is a subset B of S satisfying both of the following conditions.

(a) B is linearly independent

(b) The only linearly independent subset of S that contains B is B itself.

① UNIT-II

Linear Transformation (3.1)

Defn:-

Let V and W be vector spaces over the field F. A linear transformation from V into W is a fn T from V into W such that
" $T(c\alpha+\beta) = c(T\alpha) + T\beta$ " for all α and β in V and all scalars c in F

Ex:-1
If V is any vector space, the identity transf. I, define by $I\alpha = \alpha$, is a lin. transf. V into V. The zero transf. 0, defined by $0\alpha = 0$ is a lin. transf. from V into V

Ex:2 Let F be a field and let V be the space of polynomial fn f from F into F, given by
$$f(x) = c_0 + c_1x + \cdots + c_k x^k$$
Let $(Df)(x) = c_1 + 2c_2x + \cdots + kc_k x^{k-1}$
Then D is a lin. transf. from V into V - the differentiation transf.

Ex:-3
Let A be a fixed m×n matrix with entries in the field F. The fn T defined by $T(X) = AX$ is a lin. transf. from $F^{n\times 1}$ into $F^{m\times 1}$. The fn U defined by $U(\alpha) = \alpha A$ is a lin. from F^m into F^n

② Ex:4
Let P be a fixed m×m matrix with entries in the field F and let Q be a fixed n×n matrix over F. Define a fn T from the space $F^{m\times n}$ into itself by $T(A) = PAQ$. Then T is a lin. transf. from $F^{m\times n}$ into $F^{m\times n}$, because
$$\begin{aligned} T(cA+B) &= P(cA+B)Q \\ &= (cPA+PB)Q \\ &= cPAQ + PBQ \\ &= cT(A) + T(B) \end{aligned}$$

Ex:-5
Let R be a field of real number and let V be the space of all fn from R into R which are continuous. Defined T by
$$(Tf)(x) = \int_0^x f(t)\,dt$$
Then T is a lin. transf. from V into V. The fn Tf is not only continuous but has a continuous 1st derivative. The linearity of integration is one of its fundamental properties.

NOTE:
(i) T is a lin. transf. from V into W, then T(0)=0
$$T(0) = T(0+0) = T(0) + T(0)$$
(ii) In addition property T(0)=0. The general lin. transf. T, such a transf. 'preserves' lin. combt.; that is if $\alpha_1 \cdots \alpha_n$ are vectors V and $c_1 \cdots c_n$ are scalars, then $T(c_1\alpha_1 + \cdots c_n\alpha_n) = c_1(T\alpha_1) + \cdots + c_n(T\alpha_n)$

For exp
$$\begin{aligned} T(c_1\alpha_1 + c_2\alpha_2) &= T(c_1\alpha_1) + T(c_2\alpha_2) \\ &= c_1(T\alpha_1) + c_2(T\alpha_2) \end{aligned}$$

(3) Theorem:-1

Let V be a finite-dim vectorspace over the field F and let $\{\alpha_1 \dots \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1 \dots \beta_n$ be any vectors in W. Then there is precisely one lin. transf. T from V into W such that
$T\alpha_j = \beta_j \quad j = 1, 2 \dots n$

Proof:-

To prove there is some lin. transf. T with $T\alpha_j = \beta_j$ we proceed as follows. Given α in V, there is a unique n-tuples $(x_1, x_2 \dots x_n)$ such that
$\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$
For this vector α we define

$$T\alpha = x_1\beta_1 + \dots + x_n\beta_n$$

Then T is a well-defined rule for associating with each vector α in V a vector $T\alpha$ in W. From the defn. it is clear that $T\alpha_j = \beta_j$ for each j.
To see that T is linear, let

$$\beta = y_1\alpha_1 + \dots + y_n\alpha_n \quad \text{be in } V.$$

and let c be any scalar. Now

$$c\alpha + \beta = c(x_1\alpha_1 + \dots + x_n\alpha_n) + y_1\alpha_1 + \dots + y_n\alpha_n$$

$$= c(x_1 + y_1)\alpha_1 + \dots + c(x_n + y_n)\alpha_n$$

and so by defn.

$$T(c\alpha + \beta) = (cx_1 + y_1)\beta_1 + \dots + (cx_n + y_n)\beta_n$$

(4) On the other hand,

$$c(T\alpha) + T\beta = c\sum_{i=1}^{n} x_i\beta_i + \sum_{i=1}^{n} y_i\beta_i$$

$$= \sum_{i=1}^{n} (cx_i + y_i)\beta_i$$

$$= (cx_1 + y_1)\beta_1 + \dots + (cx_n + y_n)\beta_n$$

$$= T(c\alpha + \beta)$$

Thus $\quad T(c\alpha + \beta) = c(T\alpha) + T\beta$

If U is a lin. transf. from V into W with $U\alpha_j = \beta_j$, $j = 1 \dots n$ then for the vector $\alpha = \sum_{i=1}^{n} x_i\alpha_i$, we have

$$U\alpha = U\left(\sum_{i=1}^{n} x_i\alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i U\alpha_i$$

$$= \sum_{i=1}^{n} x_i \beta_i$$

$$= T\alpha$$

So that U is exactly the rule T which we defined above. This shows that the lin. transf. T with $T\alpha_j = \beta_j$ is unique.

Ex:-6 The vector $\alpha_1 = (1, 2)$ $\alpha_2 = (3, 4)$ are lin. indept. and therefore form a basis for R^2. According to thm 1, there is a unique lin. transf. from R^2 into R^3 such that $T\alpha_1 = (3, 2, 1)$, $T\alpha_2 = (6, 5, 4)$

If so, we must be able to find $T(\epsilon_1)$. We find scalars c_1, c_2 such that $\epsilon_1 = c_1\alpha_1 + c_2\alpha_2$ and then we k.T
$T\epsilon_1 = c_1 T\alpha_1 + c_2 T\alpha_2$.

If $(1,0) = c_1(1,2) + c_2(3,4)$ then $c_1 = -2$, $c_2 = 1$

Thus $T(0,1) = -2(7,2,1) + 1(6,5,4)$

$= (0,1,2)$

18/2/18 Ex: 7

Let T be a lin. transf. from the m tuples in to R^n n triple space then T is uniquely determine by the sequence of vector $\beta_1, \beta_2 \dots \beta_m$ where $\beta_i = T\epsilon_i$, $i = 1, 2 \dots m$

T is uniquely determine by the images of the standard basic vectors. The determation is $\alpha = (x_1, x_2 \dots x_m)$

$$T\alpha = (x_1\beta_1 + x_2\beta_2 + \dots + x_m\beta_m)$$

If B is the $m \times n$ matrix which has row vectors $\beta_1, \beta_2, \dots$ this says that $T\alpha = \alpha B$. In other words $\beta_i = B_{i1}, B_{i2} \dots B_{in}$ then $T(x_1, x_2 \dots x_n) = [x_1, x_2 \dots] \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & & & \\ B_{m1} & B_{m2} & \dots & B_{mn} \end{pmatrix}$

NOTE

If T is a lin. transf V into W then the range of T is not only a subset of W it is a subspace of W. Let R_T be the range of T is the set of all vectors β in W s.t $\beta = T\alpha$ for some α in V.

Pardors

Let β_1 & β_2 be the in R_T and let c be a scalar. There are vectors α_1 & α_2 in V such that $T\alpha_1 = \beta_1$ and $T\alpha_2 = \beta_2$.

Since T is lin. $T(c\alpha_1 + \alpha_2) = c(T\alpha_1) + T\alpha_2$

$= c\beta_1 + \beta_2$

which is shows that $c\beta_1 + \beta_2$ is also R_T

Note :- 2

The lin. transf. T is the set N consisting of the vectors α in V such that $T\alpha = 0$

It is a subspace of V, because

a) $T(0) = 0$ S.T N is non-empty

b) If $T(\alpha_1) = T(\alpha_2) = 0$, then $T(c\alpha_1 + \alpha_2) = c(T\alpha_1) + T\alpha_2 = 0$

Show that $c\alpha_1$ and α_2 in N.

Defn:- Null space

Let V & W be a vectors spaces over a field F and let T be a lin. transformation V into W.

The null space of T is the set of all vectors α in V such that $T\alpha = 0$

If V is finite dim., the rank of T is the dimension of the range of T and the nullity of T is the dimension of null space of T

7 Theorem:- 2

Let V and W vectorspace over the field F and T be a lint transf- from V into W. Suppose that V is finite dimt. then rank T + nullity T = dim V.

Proof:-

Let $(\alpha_1, \alpha_2 \dots \alpha_k)$ be a basis for N, the null space of T. There are n vectors $(\alpha_{k+1}, \alpha_{k+2} \dots \alpha_n)$ in V such that $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ is a basis for V

Now we P.T $\{T\alpha_{k+1}, T\alpha_{k+2}, \dots T\alpha_n\}$ is a basis for the range of T.

The vectors $(T\alpha_1 + T\alpha_2 + \dots T\alpha_n)$ suddenly span the range T

Since $T\alpha_j = 0$ for $j \le k$

$\therefore T\alpha_{k+1}, T\alpha_{k+2} \dots T\alpha_n$ span the range.

Therefore these vectors are indept suppose the scalar C_i such that $\sum_{i=k+1}^{n} C_i(T\alpha_i) = 0$

This says that $T(\sum_{i=k+1}^{n} C_i\alpha_i) = 0$ and accordingly the vector $\alpha = \sum_{i=k+1}^{n} C_i\alpha_i$ is the null space of T

8 Since $\alpha_1, \alpha_2 \dots \alpha_k$ forms a basis for N. There must be scalars $b_1, b_2 \dots b_k$ such that $\alpha = \sum_{i=1}^{k} b_i\alpha_i$

Thus $\sum_{i=1}^{k} b_i\alpha_i - \sum_{j=k+1}^{n} c_j\alpha_j = 0$ since $\alpha_1, \alpha_2 \dots \alpha_n$ be lint indept. we must have $b_1 = b_2 = \dots = b_k = c_{k+1} = c_{k+2} = \dots = c_n = 0$

If r is the rank of T in fact that $T\alpha_{k+1}, T\alpha_{k+2} \dots T\alpha_n$ form a basis for the range of T

Therefore $r = n - k$

Since k is the nullity of T and n is dimt of V Therefore

rank T + nullity T = dim V.

Thm:-3

If A is an $m \times n$ matrix with entries in the field F, then row rank of A = column rank of A

Proof:-

Let T be lint transf from $F^{n\times1}$ into $F^{m\times1}$ defined by $T(X) = AX$. The null space of T is the solution space for the syst. $AX = 0$ i.e) the set of all column matrices X such that $AX = 0$

9 The range of T is the set of all $m \times 1$ Column matrices Y such that $AX = Y$ has a solution for X.

If $A_1, A_2 \cdots A_n$ are column of A, Then $AX = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$ Show that the range of T is the subspace spanned by the columns of A.

In other words the range of T is the column space of A.

Therefore ~~T~~ rank T = Column rank A.

If ~~S~~ S is the solution space for the system $AX = 0$.

Then dim ~~S~~ = Column rank $A - r$.

If ~~r~~ r is the dim of the row space of A.

then the solution space S has a basis consisting of $(n-r)$ vectors

Therefore dim S = n - row rank of A

∴ row rank of A = Column rank of A

10 3.2 The Algebra of Lin. Transf

19-8-

Thm:- 4

~~Let~~ V and W be a vector space over the field F. Let T and U be lin. transf from V into W. The fn. $(T+U)$ defined by $(T+U)\alpha = T\alpha + U\alpha$ is a lin. transf. from V into W. If c is any element of F, the cT defined by $(cT)\alpha = c(T\alpha)$ is a lin. transf. from V into W. The set of all lin. transf from V into W, together with the addl. and scalar multi. defined above, is a vector space over the field F.

Proof:- Suppose T and U lin. transf. from V into W and that is defined $(T+U)$ as above.

Then $(T+U)(c\alpha+\beta) = T(c\alpha+\beta) + U(c\alpha+\beta)$

$= c(T\alpha) + T\beta + c(U\alpha) + U\beta$

$= c(T\alpha + U\alpha) + T\beta + U\beta$

$= c(T+U)\alpha + (T+U)\beta$

Which is shows that $(T+U)$ is a lin. transf. $|||^{ly}$,

$cT(d\alpha+\beta) = c[T(d\alpha+\beta)]$

$= c(T(d\alpha) + T\beta)$

$= c(d(T\alpha) + T\beta)$ $\quad \because (cT)\alpha = c(T\alpha)$

$= d\,c(T\alpha) + cT\beta$

$= d\,(cT)\alpha + (cT)\beta$

$cT(d\alpha+\beta) = d(cT)\alpha + (cT)\beta$

III^{ly}

$(T+U)(c\alpha+\beta) = c(T+U)\alpha + (T+U)\beta$

Note

We shall denote the space of lin. transf from V into W by $L(V,W)$

Thm:-5

Let V be n dim. vector space over the field F. and let W be m dim. vector space over F. Then this space $L(V,W)$ is fint. dim. and has dim mn

Proof:

$\mathcal{B} = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$

$\mathcal{B}' = \{\beta_1, \beta_2 \ldots \beta_m\}$ be order basis ~~for~~ V and W. respectively

For each pair of integr (p,q) with $1 \le p \le m$ and $1 \le q \le n$. we define a lin. transf E^{pq} from V into W by

$E^{(p,q)}(\alpha_i) = \begin{cases} 0 & \text{if } i \ne q \\ \beta_p & \text{if } i = q \end{cases}$

$= \delta_{iq}\beta_p$

There is an unique lin. transf from V into W satisfying these conditions.

We claims that mn transformations $E^{(pq)}$ form a basis for $L(V,W)$

Let T be a lin. transf from V into W. For each j, $1 \le j \le n$

Let $A_{1j} \ldots A_{mj}$ be the co-ordinates of the vector $T\alpha_j$ in the order basis ~~[illegible]~~ $\mathcal{B}'$.

ie) $T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$ ——— (1)

Let as S.T,

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq} E^{pq} \text{ ——— (2)}$$

Let U be the lin. transf in the right hant not eqn (2)

Then for each j $\quad U\alpha_j = \sum_p \sum_q A_{pq} E^{pq}(\alpha_j)$

$= \sum_p \sum_q A_{pq}\,\delta_{jq}\,\beta_p$

$= \sum_{p=1}^{m} A_{pj}\beta_p$

$= T\alpha_j$

And consequently U

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Now eqn ⑥ shows that $E^{p,q}$ span $L(V,W)$ we must prove that they are indept. Therefore the transf $U = \sum_p \sum_q A_{pq} E^{p,q}$ is the 0 transformation, $U\alpha_j = 0$ for each j

$\therefore \sum_p A_{pj} \beta_p = 0$

and the independent of β_p implies that

$A_{pj} = 0$ for every p, j

Thm:-6

If V, W, Z be the vector spaces of the field F. T be a lint transf V into W and U be a lint transf W into Z then the composed fn UT defined by $(UT)\alpha = U(T\alpha)$ is a lint transf V into Z.

23.08.16

$$UT(c\alpha+\beta) = U[T(c\alpha+\beta)]$$
$$= U(cT(\alpha) + T\beta)$$
$$= U(cT(\alpha)) + UT\beta$$
$$= c(UT)(\alpha) + (UT)\beta$$

$\therefore UT$ is lint trans

Ex:

If V is a vectorspace over the field F, a lint operator on V is a lint transf from V into V.

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NOTE.

Incase the thm 6 $V = W = Z$, s.t U, T are lint operator on the space V. The composition UT is again lint operator on V.

Therefore this space $L(V,V)$ has a mult defined on it by composition.

In this case the operator TU is also defined and in general $UT \neq TU$

ie) $UT - TU \neq 0$

Lemma:-

Let V be a vectorspace over F. Let U, T_1 and T_2 be a lint operators on V and let c be an elements F.

a) $IU = UI = U$

b) $U(T_1+T_2) = U(T_1) + U(T_2)$

c) $(T_1+T_2)U = T_1U + T_2U$

d) $c(UT_1) = (cU)T_1 = U(cT_1)$

Proof:-

a) $(IU)\alpha = I(U\alpha) = 1(U\alpha) = U\alpha$

$(UI)\alpha = (U\alpha)I = (U\alpha)1 = U\alpha$

Hence $(IU) = UI = U$

b) $[U(T_1+T_2)]\alpha = U[(T_1+T_2)\alpha]$

$= U[T_1\alpha + T_2\alpha]$

$= U(T_1\alpha) + U(T_2\alpha)$

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$= (UT_1)\alpha + (UT_2)\alpha$

Hence $U(T_1+T_2) = UT_1 + UT_2$

III'ly $(T_1+T_2)U = T_1U + T_2U$

c) $C(UT_1)\alpha = C(U(T_1\alpha))$

$= C(UT_1\alpha)$

$= C(UT_1)\alpha$.

Thm:-7 Ex:-8

If A is an $m \times n$ matrix with entries in F, we have the lin. transf. T defined by $T(X) = AX$ from $F^{n\times 1}$ into $F^{m\times 1}$ if B is a $p\times m$ matrix we have the lin. transf. U from $F^{m\times 1}$ into $F^{p\times 1}$ defined by $U(Y) = BY$. Find the composition of UT.

$(UT)X = U(T(X))$

$= U(A(X))$

$= B(AX)$

$= (BA)X$.

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Ex: 9

$f(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$

Let F be a field V be the vector space of all polynomial fn. from F into F.

Let D be the differentiation operator defined by $(Df)x = c_1 + 2c_2x + \dots + nc_nx^{n-1}$ and

Let T be the lin. operator multi. by x

Therefore $(Tf)x = xf(x)$ then $DT \neq TD$

Thm:-7.

Let V and W be vector space over the field F and let T be a lin. transf. from V into W. If T is invertible, then the inverse fn. T^{-1} is a lin. transf. from W onto V

Proof:

Let T is one-one and on-to there is a uniquely determinant inverse fn. T^{-1} which map W onto V such that $T^{-1}T$ is the identity fn. on V. and TT^{-1} is the identity fn. on W.

Therefore if a lin. fn. T is invertible, then the inverse T^{-1} is also lin.

Let β_1 & β_2 be vector in W and c be a scalar we show that $T^{-1}(c\beta_1+\beta_2)$ $= cT^{-1}(\beta_1) + T^{-1}(\beta_2)$

Let $\alpha_i = T^{-1}\beta_i$, $i = 1,2$.

ie) Let α_i be the unique vector in V.

Such that $T\alpha_i = \beta_i$

Since T is lin/-

$\therefore T(c\alpha_1+\alpha_2) = c(T\alpha_1) + T\alpha_2$

$= c\beta_1 + \beta_2$

Thus $c\alpha_1+\alpha_2$ is a unique vector in V which is sent by T into $c\beta_1+\beta_2$.

$T^{-1}(c\beta_1+\beta_2) = c\alpha_1+\alpha_2$

(e) $= c(T^{-1}\beta_1) + T^{-1}(\beta_2)$

$\therefore T^{-1}$ also lin/-.

24.8.

Note:-

* Suppose that we have an invertable lin/- transf T from V onto W and an invertable lin/- transf U from W onto Z then UT is invertable $(UT)^{-1} = T^{-1}U^{-1}$. That conclusion does not require the linearity. Therefore UT is 1-1 and onto.

* $T^{-1}U^{-1}$ is both left & Right inverse for UT

* If T is lin/, then $T(\alpha-\beta) = T\alpha - T\beta$ hence $T\alpha = T\beta$ iff $T(\alpha-\beta)=0$. Therefore T is one to one.

Let us call T non-singular if $T\gamma = 0 \rightarrow \gamma = 0$

ie) If null spaces of T is $\{0\}$

Evidently T is one-one iff T is non-singular.

* The non-singular lin/- transf are lin/- independent.

Thm:-

If T be a lin/ transf from V into W then T is non-singular iff T carries each lin/- indep/- subset of V onto lin/- indep/- subset of W

Proof:-

Suppose that T is non-singular.

Let S be lin/ indep/- subset of V

If $\alpha_1, \alpha_2 \ldots \alpha_k$ are vectors in S.

Then the vector $T\alpha_1, T\alpha_2 \ldots T\alpha_k$ are lin/- independent for iff

$c_1(T\alpha_1) + c_2(T\alpha_2) + \cdots + c_k(T\alpha_k) = 0$

$T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0$.

Since T is non-singular.

$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$.

From which it follows that each $c_i = 0$ because S is an independent set.

This arguments implies that the image of S under T is indep/-.

Suppose that T carries Indepe/- subsets onto Independ- subset.

Let α be the non-zero vector in V. Then the set S consisting of the one vector α is independent. The image of S is the set consisting of one vector $T\alpha$ and this set is independent.

Therefore $T\alpha \neq 0$ Because the set consisting of the zero vector along is dependent.

This shows that the null space of T is the 0 subspace. Therefore T is non-singular.

Ex:-1

Let F be a subfield of complex no- at B be the space of polynomial fn over F. Consider the differential operator D and mult- by X the operator T defined as follows $(Tf)x = Tf(x)$ $\therefore DT \neq TD$

Since T sends all constants to 0. T is singular however T is not finite dim- the range of T is all of V and that is possible to define a right inverse for T.

For ex- If E is the infinite Igral operator.

$$E(c_0 + c_1x + c_2x^2 + \cdots + c_nx^n) = c_0x + \frac{c_1x^2}{2} + \frac{c_2x^3}{3} + \cdots + \frac{c_nx^{n+1}}{n+1}$$

Then E is lin- operator on E and $DE = I$.

On the other hand $ED \neq I$ Because E, D sends the constant into 0. The operator T is called the reverse situation. If $f(f(\alpha)) = 0 \ \forall x$, then $f = 0$. Thus T is non-singular and it is possible to find the left inverse of T

Ex:-12

Let F be a field and T be the lin- operator F^2 defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$ Then T is non-singular if $T(x_1, x_2) = 0$ therefore $x_1 + x_2 = 0$ and $x_1 = 0$

S.T $x_1 = x_2 = 0$ and T is onto.

Let (x_1, x_2) be any vector in F^2. to S.T z_1, z_2 is in the range of T we must find scalars x_1 & x_2 $\ni$ $x_1 + x_2 = z_1$ and $x_1 = x_2$. Therefore the solns are $x_1 = z_2$, $x_2 = z_1 - z_2$ This last computation gives as formular for T^{-1} namely $T^{-1}(z_1, z_2) = z_2, z_1 - z_2$