UNIT : I

12. Topological Spaces

Topology:

A topology on a set X is a collection $\mathcal{J}$ of subsets of X having the following properties

(i) ϕ and X are in $\mathcal{J}$

(ii) The union of the elements of any subcollection of $\mathcal{J}$ is in $\mathcal{J}$.

(iii) The ~~union~~ intersection of elements of any finite subcollection of $\mathcal{J}$ is in $\mathcal{J}$.

Topological Space : $(X, \mathcal{J})$

A set X for which a topology $\mathcal{J}$ has been specified is called a topological space

Example:

$X = \{a, b, c\}$

$\mathcal{J} = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$

$\mathcal{J} = \{\phi; X, \{b\}, \{c\}, \{a, b\}\}$

Let X be a set and $\mathcal{J}$ be the collection of all subsets of X (i.e $\mathcal{J}$ is a power set of X). Then $\mathcal{J}$ is called the discrete topology.

Power set $= 2^n$

Example: 1 $X = \{a, b\}$

$2^2 = 4$ element

$\mathcal{J} = \{X, \phi, \{a\}, \{b\}\}$

Example: 2 $X = \{a, b, c\}$

$2^3 = 8$ elements

$\mathcal{J} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

ifine B. Basis for a topology:

Let X be a set a basis $\mathcal{B}$ for a topology on X is a collection of subset of (X) called basis element such that

2M (4.)

(i) For each $x \in X$, there is atleast one basis element $B \in \mathcal{B}$ such that $x \in B$

(ii) If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $x \in B_3 \subseteq B_1 \cap B_2$.

Example: Arithmetic Progression Basis:

Let X be the set of positive integers and consider the collection $\mathcal{B}$ of all arithmetic progression of positive integers then B is a Basis.

If $m \in X$ then $\mathcal{B} = \{m+(n-1)p\}$ contains m.

Next consider two arithmetic progression $B_1 = \{a_1 + (n-1)p_1\}$

$B_2 = \{a_2 + (n-1)p_2\}$

containing an integer m. Then $B_3 = \{m + (n-1)p\}$.

Thus the job $p = L.C.M \{p_1, p_2\}$

m = X member
n = total member
p = difference

$B_1 = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

$B_2 = \{4, 8, 12, 16, 20\}$

$B_1 \cap B_2 = \{4, 8, 12, 16, 20\}$

$B_3 = \{4, 8, 12, 16, 20\}$

That is $8 \in B_3 \subseteq B_1 \cap B_2$

Example:

The topology $\{\phi, \{a\}, \{b\}, \{a,b\}\}$ on the set $\{a,b\}$ has the following basis

1. $\{\phi, \{a\}, \{b\}, \{a,b\}\}$
2. $\{\{a\}, \{b\}, \{a,b\}\}$
3. $\{\phi, \{a\}, \{b\}\}$
4. $\{\{a\}, \{b\}\}$

Example:

Let $X = \{a, b, c, d, e\}$ and let $\mathcal{T} = \{\phi, \{a,b\}, \{c,d\}, \{a,b,c,d\}, X\}$ be a topology defined on X.

$\mathcal{B} = \{\{a,b\}, \{c,d\}, X\}$ is a subcollection of $\mathcal{T}$ which meets the requirement for a basis because each member of $\mathcal{T}$ is a union of members of B.

Topology $\mathcal{T}$ generated by $\mathcal{B}$.

If $\mathcal{B}$ is a basis then the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows

A subset U of X is said to be open in X [i.e to be an element of $\mathcal{T}$].

If for each $x \in U$, there is a Basis element $B \in \mathcal{B}$ such that $x \in B$ & $B \subset U$

Example:

consider the set $X = \{a, b, c, d\}$ and the set $\mathcal{B} = \{\{a\}, \{c, d\}, \{a, b, c\}\}$ determine whether there exist a topology $\mathcal{J}$ on such that $\mathcal{B}$ is a base for $\mathcal{J}$.

All possible unions of elements from $\mathcal{B}$ are given below

$$\left\{ \bigcup_{B \in \mathcal{B}} : \mathcal{B}' \subset \mathcal{B} \right\} = \{\phi, \{a\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\}$$

If $\mathcal{J}$ is a topology generated $\mathcal{B}$, then

$\mathcal{J} = \{\phi, \{a\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\}$

Note:

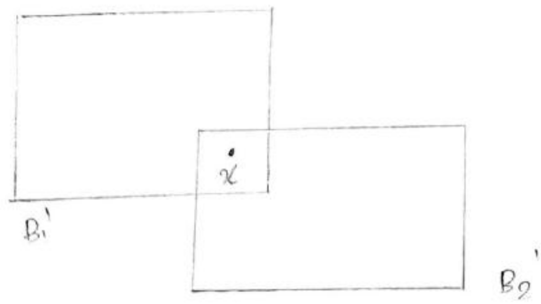
Each basis element is itself an element of $\mathcal{J}$.

Example:

Let $\mathcal{B}$ be a collection of all circular regions. [interiors of circles] in the plane. Then $\mathcal{B}$ satisfies both conditions for a basis. The second condition is illustrated in figure. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every x in U lies in some circular region contained in U.

Example:

Let $\mathcal{B}'$ be the collection of all rectangular regions [interiors of rectangles] in the plane, where the rectangles have parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both conditions for a bases in this case the conditions is trivial, because the intersections of any two basis element is itself a basis element (or empty). The basis $\mathcal{B}'$ generates the same topology on the plane as the basis $\mathcal{B}$



1 Lemma 13.1

Let X be a set; let $\mathcal{B}$ be basis for a topology $\mathcal{J}$ on X. Then $\mathcal{J}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof:

Given a collection of elements of $\mathcal{B}$. they are also elements of $\mathcal{J}$. Because $\mathcal{J}$ is a topology, their union is in $\mathcal{J}$. conversely, given $U \in \mathcal{J}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$. so, U equals a union of elements of $\mathcal{B}$.

Example:

If X is any set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

To check the collection $\mathcal{J}$ generated by the basis $\mathcal{B}$ is a topology on X.

(i) If U is the empty set, it satisfies the defining condition of openness vacuously.

(ii) Likewise X is in $\mathcal{J}$, since for each $x \in X$ there is some basis elements B, such that $x \in B$ and $B \subset X$.

(iii) Now, take an indexes family $\{U_\alpha\}_{\alpha \in \mathcal{J}}$ of elements of $\mathcal{J}$ & show that $U = \bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{J}$.

Given $x \in U$, there is an index α such that $x \in U_\alpha$, since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$.

Then $x \in B$ & $B \subset U$, so that U is open by defin

Now take two elements U_1 & U_2 such that $U_1 \cap U_2 \in \mathcal{J}$. is in $\mathcal{J}$.

Given $x \in U_1 \cap U_2$

choose a basis element B_1 such that $x \in B_1$, & $B_1 \subset U_1$ similarly $x \in B_2$ and $B_2 \subset U_2$

The second condition for a basis, choose a basis element B_3 such that $x \in B_3$ & $B_3 \in B_1 \cap B_2$ (see figure). Then $x \in B_3$ & $B_3 \in U_1 \cap U_2$ so $U_1 \cap U_2$ belongs to $\mathcal{J}$ by definition.

Finally show that by induction any finite intersection $U_1 \cap U_2 \cap \dots U_n$ of elements of $\mathcal{T}$ is in $\mathcal{T}$.

This is trivially for $n=1$,
Suppose that is true for $n-1$ and prove it for n.

New $(U_1 \cap U_2 \cap \dots U_n) = (U_1 \cap U_2 \cap \dots U_{n-1}) \cap U_n$

By hypothesis $U_1 \cap U_2 \cap \dots U_{n-1}$ belongs to $\mathcal{T}$ by the result just proved, the intersection of $U_1 \cap \dots \cap U_{n-1}$ and U_n also belongs to $\mathcal{T}$. Thus the collection of open sets generated by a basis $\mathcal{B}$ is a topology.

<u>Note</u>:

7. Easy to remember

<u>$B' \subset B$ out $\mathcal{T}' \supset \tau$</u>

Topological space is a truck load full gravel pebbles as the basis elements of the topology; after the pebble's are smashed to dust. The dust particles are the basis elements of the new topology The new topology is finer than the odd one and each dust particle was contained inside a pebble.

Lemma : 13.2

Let $\mathcal{B}$ & $\mathcal{B}'$ be bases for the topology $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following are ~~requirement~~ equivalent

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(ii) For each $x \in X$ & each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}$ such that $x \in B' \subset B$.

Proof:

$2 \Rightarrow 1$

Given an element U of $\mathcal{T}$.

To show that $U \in \mathcal{T}'$

Let $x \in U$ since $\mathcal{B}$ generates $\mathcal{T}$. There is an element $B \in \mathcal{B}$ such that $x \in B \subset U$ condition 2 tells there exists an element $B' \in \mathcal{B}'$ such that $x \in B' \in B$. Then $x \in B' \subset U$. So $U \in \mathcal{T}'$ by definition. Hence $\mathcal{T}'$ is finer that $\mathcal{T}$.

$1 \Rightarrow 2$

Given $x \in X$ & $B \in \mathcal{B}$ with $x \in B$

Now $B \in \mathcal{T}$ by definition & $\mathcal{T} \in \mathcal{T}'$ by condition (i)

$\therefore$ $B \in \mathcal{T}'$

Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$ [By the property $x \in B_1 \cap B_2$ & $x \in B_3$, $B_3 \in B_1 \cap B_2$].

Lemma 13.2

Let X be a topological space suppose that $\mathcal{C}$ [Here $\mathcal{C}$ is not a basis] is a collection of open sets of X such that for each open set U of X & each x in U, there is an element c of $\mathcal{C}$ such that $x \in c \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof:

To show that $\mathcal{C}$ is a basis first condition. Given $x \in X$, since X is itself an open set. there is by hypothesis an element c of $\mathcal{C}$ such that $x \in \mathcal{C} \subset X$

Second condition:

Let $x \in C_1 \cap C_2$ where C_1 & C_2 are elements of $\mathcal{C}$ since C_1 & C_2 are open so $C_1 \cap C_2$ is also open. Therefore there exists by hypothesis an element C_3 of $\mathcal{C}$. Such that $x \in C_3 \subset C_1 \cap C_2$

$\therefore$ $\mathcal{C}$ is the basis of the topology of X

$\mathcal{J} = \mathcal{J}'$

Let $\mathcal{J}$ be the collection of open sets of X. To show that the topology $\mathcal{J}'$ generated $\mathcal{C}$ equals the topology $\mathcal{J}$. First if U belongs to $\mathcal{J}$ and if $x' \in U$, then there is by hypothesis an element c of $\mathcal{C}$ such that $x \in \mathcal{C} \subset U$.

It follows that U belongs to the topology $\mathcal{J}'$ by definition [$\because x' \in U$]

Conversely, if W belongs to the topology $\mathcal{J}'$. Then W equals a union of elements of $\mathcal{C}$

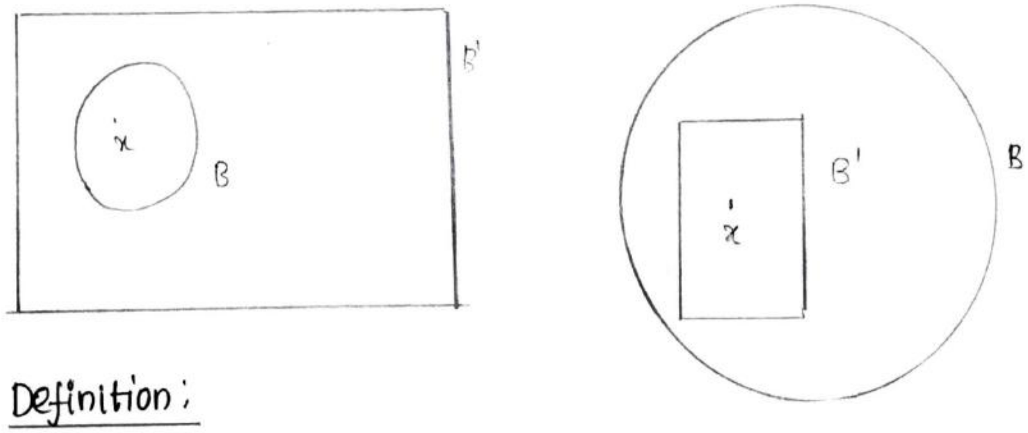
the lemma let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all union of elements $\mathcal{B}$

Since each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and $\mathcal{T}$ is a topology. W also belongs to $\mathcal{T}$.

$\therefore$ The topology $\mathcal{T}'$ generated by $\mathcal{C}$ equal the topology $\mathcal{T}$!

Example:

One can now see that the collection $\mathcal{B}$ of all circular regions in the plane generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions. Figure illustrates the proof.



Definition:

If $\mathcal{B}$ is the collection of all open intervals in the real line, $(a,b) = \{x / a < x < b\}$

2m the topology generated by $\mathcal{B}$ is called the Standard topology $\mathbb{R}$ on the real line.

If $\mathcal{B}'$ is the collection of all half-open intervals of the form $[a,b) = \{x / a \leq x < b\}$. Where $a < b$, the topology generated by $\mathcal{B}'$ is called the lower limit topology ~~on~~ $\mathbb{R}$.

2m

Finally let K denotes the set of all members of the form $\frac{1}{n}$, for $n \in Z_+$ [Z_+ +ve integers]. and let $\mathcal{B}''$ be the collection of all open intervals (a, b). along with all sets of the form $\boxed{(a,b)-K}$.

M The topology generated by $\mathcal{B}''$ will be called the K- topology $\mathbb{R}_K$ on $\mathbb{R}$.

All three of these collections are basis.

Lemma: 13.4

The topologies of $\mathbb{R}_l$ & $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ but are not comparable with one another.

Proof:

Let $\mathcal{J}$, $\mathcal{J}'$ & $\mathcal{J}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ & $\mathbb{R}_K$ respectively.

Given a basis element (a,b) for $\mathcal{J}$ and a point $x \in (a,b)$.

the basis element $[x,b)$ for $\mathcal{J}'$, $x \in [x,b) \in (a$

On the other hand, given the basis element $[x,d)$ for $\mathcal{J}'$, there is open set interval (a,b) and $x \notin (a,b)$ but $x \in [x,d)$ & $(a,b) \subset [x,d]$

Thus $\mathcal{J}'$ is strictly finer than $\mathcal{J}$ [by Lemma

let $\mathcal{B}$ & $\mathcal{B}'$ be bases for $\mathcal{J}$ & $\mathcal{J}'$ respectively on x

Then the following equivalent (i) $\mathcal{J}'$ is finer than $\mathcal{J}$. (ii) $x \in B' \subset B$

Example of standard topology:

If $X = R$, then the standard topology is the topology whose open set are the unions of open intervals.

Definition:

A ~~Sub basis~~ $\mathcal{S}$ for a topology on X is a collection of subsets (elements) of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

4. Totally ordered set:

A set in which a relation as "less than or equal to" holds for all pairs of elements of the set. Also called chain, linearly ordered set, simply ordered set.

Intervals:

Suppose that X is set having a simple order relation $<$. Given elements a & b of X such that $a < b$, there are four subsets of X that are called the intervals determined by a & b.

(i) (a,b) $\{x \,/\, a < x < b\}$ open intervals in X

(ii) $(a,b]$ $\{x \,/\, a < x \leq b\}$ half open intervals

(iii) $[a,b)$ $\{x \,/\, a \leq x < b\}$

(iv) $[a,b]$ $\{x \,/\, a \leq x \leq b\}$ closed interval in X

These are intervals in an arbitrary ordered set.

Note:

The term "open" in this connection suggests that open intervals in X should turn out to be open sets when we put a topology on X.

Definition:

Order topology:

Let X be a set with a simple order relation. Let $\mathcal{B}$ be the collection of all sets of the following types.

1) All open intervals (a, b) in X.

2) All intervals of the form $[a_0, b)$ where a_0 is the smallest element [if any] of X.

3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Example: 3

The positive integers Z_+ form an ordered set with a smallest element. The order topology on Z_+ is the discrete topology, for every one point set is open.

If $n > 1$, then the one point set $\{n\} = (n-1, n+1)$ is a basis element and if $n = 1$, the one point set $\{1\} = [1, 2)$ is a basis element.

Note:

Let X be set and $a<b$ total ordering on X. That is $<$ is a relation with the Properties

1] If $a<b$ & $b<c$ then $a<c$

2] For no a does $a<a$

3] If $a<b$ then, it is not the case that $b<c$

An example if an ordered set $(N,<)$ with the standard ordering or $(R,<)$ with the some ordering.

Definition: Rays

If X is an ordered set and a is an element of X, there are four subsets of X that are called the rays determined by a

(i) $(a, +\infty) = \{x / x > a\}$

(ii) $(-\infty, a) = \{x / x < a\}$

(iii) $[a, \infty) = \{x / x \geq a\}$

(iv) $(-\infty, a] = \{x / x \leq a\}$

sets of the first two types are called open rays, and sets of ~~at~~ the last two types are called closed rays.

open} → open
closed} → closed

15. The product topology on X and Y

Definition: Product topology 2M X

Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as the Basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

To check that $\mathcal{B}$ is a basis

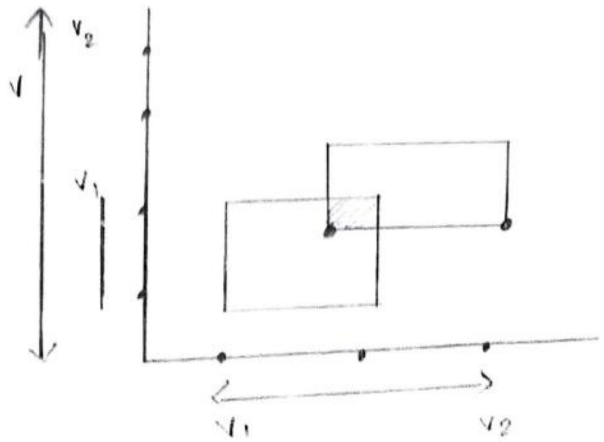
The first condition is trivial since $X \times Y$ is itself a basis element.

<u>second condition:</u>

Since the intersection of any two basis elements $U_1 \times V_1$ and $U_2 \times V_2$ is another basis elements.

For $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ and the latter set is a basis element. Because $U_1 \cap U_2$ & $V_1 \cap V_2$ are open in X and Y respectively.

Example:



The collection $\mathcal{B}$ is not a topology on $X \times Y$. The union of the two rectangles pictured in fig. It is not a product of two sets is $(U_1 \times V_1) \cup (U_2 \times V_2) \neq (V_1 \cup U_2) \times (V_1 \cup V_2)$. So it cannot belongs to $\mathcal{B}$ but it is open in $X \times Y$.

Theorem : 15.1

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $D = \{B \times C \,/\, B \in \mathcal{B} \;\&\; C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Proof :

Given an open set W of $X \times Y$ and a point $x \times y \in W$.

By definition of the product topology there is basis element $U \times V$ such that $x \times y \in U \times V \subset W$

Because $\mathcal{B}$ & $\mathcal{C}$ are of bases for X and Y respectively.

We can choose an element B of $\mathcal{B}$ such that $y \in \mathcal{C} \subset V$.

Then $x \times y \in B \times \mathcal{C} \subset W$ where $B \times C \in D$

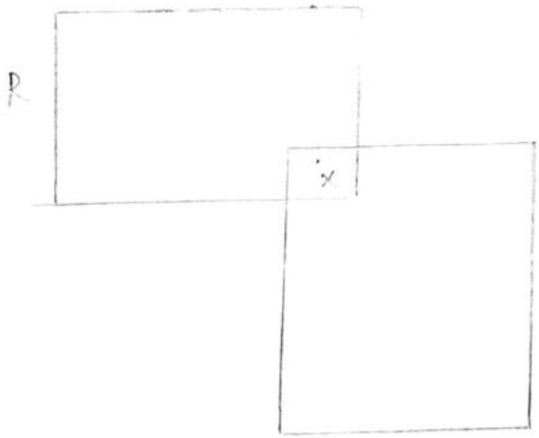
Thus the collection D the criterian of lemma [let X be a topological space. Suppose that for each x in X and open set V of X, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X].

So, D is a basis for $X \times Y$.

Example :

(1) The product of standard topology on $\mathbb{R}$ with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of opensets of $\mathbb{R}$.

But the theorem "If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y then the collection $D = \{B \times C \mid B \in \mathcal{B}$ and $C \in$ is a basis for the topology of $X \times Y$". tells us that the much smaller collection of all products $(a \times b) \times (c \times d)$ of open intervals in R will also serve as a basis for the topology of R^2. Each such set can be pitured as the interior of a rectangle in R^2. Thus the Standard topology on R



R

Definition:

Projection of $X \times Y$.

Let $\pi_1 : X \times Y \to X$ defined by

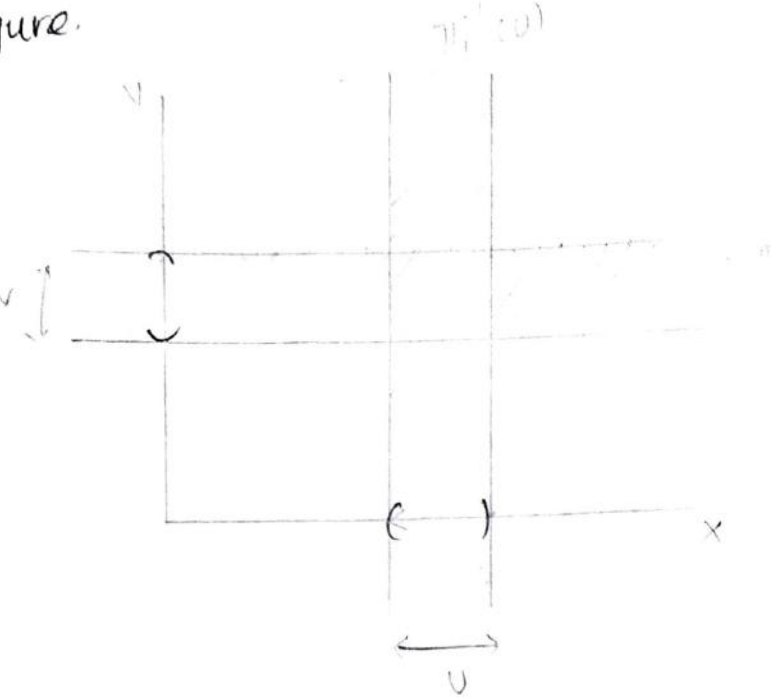
$\pi_1(x, y) = x$;

Let $\pi_2 : X \times Y \to Y$ be defined by $\pi_2(x, y) = y$

The maps π_1 and π_2 are called the projection of $X \times Y$ onto its first and second factor respectively

If U is an open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$.

Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$ as indicated in figure.



Theorem: 15.2

The collection $S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup$ union
$\{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$
is a subbasis for the product topology on $X \times Y$.

Proof:

Let $\mathcal{T}$ denote the product topology on $X \times Y$. Let $\mathcal{T}'$ be the topology generated by S. Because every element of S belongs to $\mathcal{T}$. So do arbitrary unions of finite intersections of elements of S. Thus $\mathcal{T}' \subset \mathcal{T}$. Thus the collection S is a subbasis for the product topology on $X \times Y$.

On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of S. Since $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

Therefore, $U \times V$ belongs to $\mathcal{J}'$, so that $\mathcal{J} \subset \mathcal{J}'$ as well.

16. The subspace Topology:

Definition:

Let X be a topological space with topology $\mathcal{J}$. If Y is a subset of X, the collection $\mathcal{J}_Y = \{Y \cap U \mid U \in$ is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consists of all intersections of open sets of X with Y.

Example:

Hint natural numbers.

The positive integers $1, 2, 3 \ldots$, etc and some times zero as well.

R represent the real numbers with topology. The subspace topology of the natural numbers, as a subspace of $\mathbb{R}$ is the discrete topology.

Lemma: 16.1

If $\mathcal{B}$ is a basis for the topology of X, then the collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Proof:

Given U open in X and given $y \in U \cap Y$. Choose an element B of $\mathcal{B}$ such that $y \in B \subset U$. Then

$y \in B \cap Y \subset U \cap Y$.

It follows from Lemma "let X be a topological space. suppose that $\mathcal{C}$ is the collection of open sets of X such that for each x in X and each set U of X, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X", that $\mathcal{B}_Y$ is a basis for the subspace topology on Y.

Lemma: 16.2

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof:

Since U is open in Y, $U = Y \cap V$ for some set V open in X.

so $Y \cap V$ is open in X

so U is open in X.

Example:

Consider the subset $Y = [0,1]$ of the real line $\mathbb{R}$, in the subspace topology. The subspace topology has as basis all sets of the form $(a,b) \cap Y$, where (a,b) is an open interval in $\mathbb{R}$ such a set is one of the following types.

$$(a,b) \cap Y = \begin{cases} (a,b) & \text{if } a \text{ and } b \text{ are in } Y \\ [0,b) & \text{if only } b \text{ in } Y \\ (a,1] & \text{if only } a \text{ is in } Y. \\ Y \text{ or } \phi & \text{if neither } a \text{ nor } b \text{ is in } Y \end{cases}$$

By definition, each of these sets is open in Y (ie) $(a,b) \cap Y$ is open in Y. but Y is open in real line R. So, $(a,b) \cap Y$ is open in Real line R.

Theorem 16.4

If X is an ordered set in the order topology and if Y is an interval or a ray in the ordered set X, then the subspace topology and the order topology on Y are the same.

Proof:

For any open ray R in X.

$R \cap Y$ open in order topology on Y.

(1) If $a < b$ then $(a,b) \cap Y = ((-\infty, b) \cap Y) \cap ((a,\infty) \cap$

(2) If X has minimal element a_0 then $[a,b) \cap Y =$

$(-\infty, b) \cap Y$

(3) If X has maximal element a_0 then $(a,b] \cap Y =$

$(a,\infty) \cap Y$

Thus every basis element for the subspace topology is open in order topology on Y.

To prove that the basis element for the order topology is open in the subspace topology

The basis elements for the order topology on Y of the form.

$B_1 = \{x \in Y \mid a < x < b\}$

$B_2 = \{x \in Y \mid \bar{a}_0 \le x < b\}$

$B_3 = \{x \in Y \mid a < x \le \bar{b}_0\}$

Where $\bar{a}_0, \bar{b}_0$ are minimal and maximal elements of Y (if exists)

(1) Clearly $B_1 = (a, b) \cap Y$

(2) If $a_0 = \bar{a}_0$ then $B_2 = [a_0, b) \cap Y$

(3) If $a_0 < \bar{a}_0$ then $B_2 = (a_0, b) \cap Y$

(4) If $b_0 = \bar{b}_0$ then $B_3 = (a, b_0] \cap Y$

(5) If $b_0 < \bar{b}_0$ then $B_3 = (a, b_0) \cap Y$

Then B_1, B_2, B_3 belongs to the basis for the subspace topology.

Hence the subspace topology and the order topology on Y are the same.

Theorem: 16.3

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits (receives) as a subspace of $X \times Y$.

Proof:

The set $U \times V$ is the general basis element for $X \times Y$, where U is open in X and V is open in Y.

Therefore $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.

Now $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, respectively, the set $(U \cap A) \times (V \cap B)$ is the general basis element for the product topology on $A \times B$ (i.e) the bases the subspace topology on $A \times B$ and for the product topology on $A \times B$ are the same. Hence the topologies are the same.

17: Closed sets:

2M

A subset A of a topological space X is said to be closed if the set $\boxed{X - A}$ is open.

Example: 1

The subset $[a,b]$ of R is closed because its complement $R - [a,b] = (-\infty, a) \cup (b, \infty)$ is open.

Similarly $(a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open.

Note:

The subset $[a,b)$ of R is neither open nor cla

Example: 2

In the plane R^2, the set $[x \times y \mid x \geq 0$ and $y \geq 0]$ is closed, because its complement is the union of two sets.

$(-\infty, 0) \times R$ and $R \times (-\infty, 0)$ each of which i a product of open sets of R and is therefore open in R^2.

Example: 3

In the discrete topology on the set X, ever set is open, it follows that every set is closed as well.

Theorem 17.1

Let X be a topogical space. Then the following conditions hold.

(1) ϕ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof:

(1) ϕ and X are closed

because $X - \phi = X$ and $X - X = \phi$

the complement X and ϕ are open sets.

(2) Given a collection of closed sets $\{A_a\}_{a \in J}$ we apply Demorgan's law

$$X - \bigcap_{a \in J} A_a = \bigcup_{a \in J} (X - A_a)$$

Since that sets $X - A_a$ are open by definition the right side of this equation represents an arbitrary union of open sets, and is thus open

variant

$\therefore \bigcap A_a$ is closed.

(3) Similarly, If A_i is closed for $(i = 1, 2 \ldots n)$ consider the equation

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$$

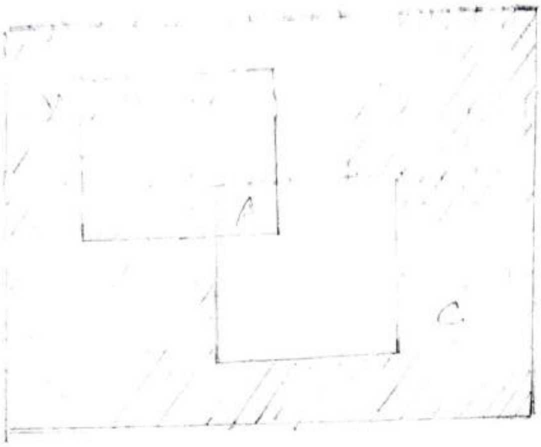
The set on the right side of the equation is a finite intersection of open sets and is therefore open. Hence $\bigcup A_i$ is closed.

Theorem 17.2

Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof:

Assume that $A = C \cap Y$ where C is closed in X



Then $X-C$ is open in X. So $(X-C) \cap Y$ is open in Y (by definition of subspace topology).

But $(X-C) \cap Y = Y-A$

Hence $Y-A$ is open in Y

So, A is closed in Y.

Conversely, Assume that A is closed in Y. Then $Y-A$ is open in Y. So, $Y-A$ equals the intersection of an open set U of X with Y [by definition (i.e) $(U \cap Y)$] (i.e) U is open in X.

The set $X-U$ is closed in X and $A = Y \cap (X-U)$ so that A equals the intersection of a closed set of X with Y.

Theorem 17.3

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Closure and Interior of a set:

A subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A, and the closure of A is defined as the intersection of all closed sets containing A.

Note:

(i) Interior of $A = \tilde{A}$, closure of $A = \bar{A}$

$\tilde{A} \subset A \subset \bar{A}$

(ii) If A is open, $A = \tilde{A}$

(iii) If A is closed, $\bar{A} = A$

(iv) The closure of A in Y and the closure of A in X will in general be different.

Theorem 17.4

Let Y be a subspace of X; Let A be a subset of Y. Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equal $\bar{A} \cap Y$.

A_y

Proof:

Let $\bar{A}_y$ denote the closure of A in Y.

The set $\bar{A}$ is closed in X.

So, $\bar{A} \cap Y$ = Closed set in Y [by theorem 6.2]

[Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.]

So, $A \subset \bar{A} \cap Y$ [since $A \subset A \subset \bar{A}$]

$\bar{A}_y \subset \bar{A} \cap Y$ — (1) by definition of closure

[$\bar{A}_y = \cap$ all closeds

On the other hand,

We know that, $\bar{A}_y$ is closed in Y.

$\bar{A}_y = C \cap Y$ — (2) for some set 'C' closed in X.

Then $A \subset C$

So, $\bar{A} \subset C$

$\bar{A} \cap Y \subset \underset{(2)}{C \cap Y}$ (Intersect on both sides by Y)

$\bar{A} \cap Y \subset \bar{A}_y$ — (3) from (2)

From (1) & (3) $\bar{A}_y = \bar{A} \cap Y$.

Note:

For our convenient, we say that a set A intersects a set B if the intersection $A \cap B$ is not empty.

Theorem: 17.5

Let A be a subset of the topological space X (a). Then $x \in \bar{A}$ iff every open set U containing x intersects A.

(b). supposing the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

Proof:

Transform to contraposite

$x \notin \bar{A} \Leftrightarrow$ There exist an open set U, containing x (i.e $x \in U$) does not intersect.

If $x \notin \bar{A}$, $U = X - \bar{A}$ is an open set containing x does not intersect A.

[$x \notin \bar{A} \Rightarrow x \notin A$, $\because A \subset \bar{A}$, $x \notin A$ does not intersect $x \in X - \bar{A}$]

Conversely, If there exists an open set U containing x which does not intersect A.

then $X - U$ is a closed set containing A.

[$\because A \subset \bar{A}$ (= intersection of all closed set containing A) $\therefore \bar{A} \subset X - U$ but $x \in X - U$

$\therefore x \in \bar{A}$.

Statement (b) If $x \in \bar{A}$ then statement (a) is every open set containing x intersects A.

So every basis element B containing x intersects A (because B is an open set.)

Conversely,

If every basis element containing x intersects A. So every open set U containing x intersects A [because $x \in B \subset U$. by topology $\mathcal{T}$ generated by $\mathcal{B}$.

A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$]

then $x \in \bar{A}$ (from statement (a)).

Note:

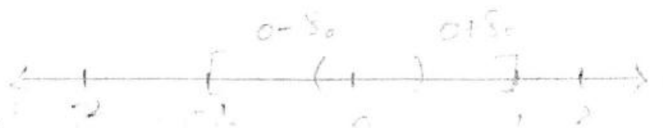
'U is an open set containing x' Using special terminology, one can write 'U is a neighbourhood of x'.

Limit Points:

If A is a subset of the topological space X and if x is a point of X, we say that x is a limit point (or "cluster point") of A if every neighbourhood of x intersects A in some point other than x itself

Example:

In the real line $\mathbb{R}$, 0 is a limit point of $[-1,1]$.



Infinitely many points in the interval $(0-\delta_0, 0+\delta_0)$

LEMMA: 2.3

Let $\mathcal{T}'$ denote the topology on X generated by $\mathcal{C}_0$. Let $\mathcal{T}$ be the topology of X. (The lemma) Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then $\mathcal{T}'$ is finer than $\mathcal{T}$. source that $\mathcal{T}'$ is finer than $\mathcal{T}$.

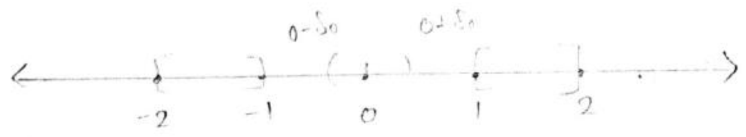
Conversely, since each element of $\mathcal{C}_0$ is an element of $\mathcal{T}$ so are orbitrary unions of elements of $\mathcal{C}_0$.

$\therefore$ by lemma. " let X be a set, let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X, then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$".

$\mathcal{T}'$ is a subset of $\mathcal{T}$. We conclude that $\mathcal{T}'$ equal to $\mathcal{T}$.

Example:

In the real line R, 0 is not a limit point of $[-2,-1] \cup \{0\} \cup [1,2]$.



0 is the only point in the interval $(0-\delta_0, 0+\delta_0)$

Example

If Z_n, is the set of positive integers no point of R is a limit point of Z_n.

All the points are jum points in the set Z_n.

Theorem: 17.6

Let A be a subset of the topological space X, Let A' be the set of all limit point of A. Then $\bar{A} = A \cap A'$.

Proof:

If $x \in A'$ (every neighbourhood of x intersects A) then $x \in \bar{A}$ [by theorem. 17.5] Let A be a subset of the topological space X then $x \in \bar{A}$ if even open set U containing x intersects A.

Hence $A' \subset \bar{A}$

Also $A \subset \bar{A}$ (by closure defin $A \subset \bar{A}$)

$\therefore$ $A \cup A' \subset \bar{A}$ —— (1) ($\bar{A} \cup \bar{A} = \bar{A}$)

Reverse

Let $x \in \bar{A}$

Case (i)

If $x \in A$

then $x \in A \cup A'$ (It is trivial)

case (ii)

Suppose $x \notin A$

(i.e) $x \in A'$ [by theorem 17.5

If $x \in \bar{A}$ then every neighbourhood U of x intersect

A because $x \notin A$, the set U must intersect A in a point different from x then $x \in A'$.

$$\text{So } x \in A \cup A'$$

$$\therefore \bar{A} \subset A \cup A' \quad \text{—— (2)}$$

Hence $A \cup A' = \bar{A}$

Corollary 17.7

A subset of a topological space is closed iff it contains all its limit points.

Proof:

A set A is closed iff $A = \bar{A}$ (by defn)

i.e $A' \subset A$ [$\because \bar{A} = A \cup A'$ by theorem 6.6]

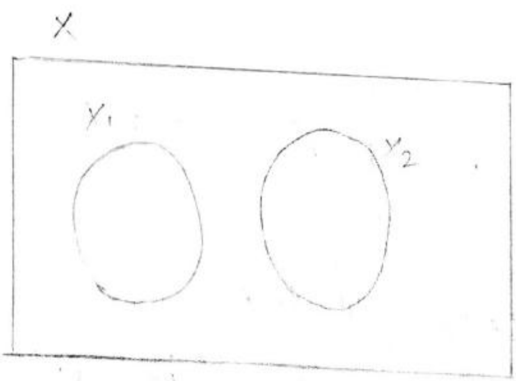
i.e $A = A \cup A'$

Note:

Each one point set $\{x_0\} = A$ is closed in R.

Definition:

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X. there exist neighbourhoods V_1 and U_2 of x_1 and x_2 respectively, that are disjoint (i.e $U_1 \cap U_2 = \phi$)



Theorem: 11.8

Every finite point set in a Hausdorff space X is closed.

Proof:

It suffices to show that every one-point $A = \{x_0\}$ is closed. If x is a point of X different from x_0, then x and x_0 have disjoint neighbourhoods U and V, respectively.

Since U does not intersect $\{x_0\}$.

The point x cannot belong to the $A = \{x_0\}$.

$\therefore x \notin \bar{A}$ ($\because$ contra positive If $x \in \bar{A}$ that every open set containing x intersect A)

$\bar{A} = \{x_0\}$ is closed.

[If A is closed then $A = \bar{A}$]

So, $x \notin A = \{x_0\}$ is closed [It is trivial, only element $x_0 \in \{x_0\}$.]

Note:

$\{x_0\}$ is closed in R.

complement of $\{x_0\} = R - \{x_0\} = (-\infty, x_0) \cup (x_0, \infty)$

Union of open sets are open

$\therefore$ complement of $\{x_0\}$ is open

$\therefore$ $\{x_0\}$ is closed.

Theorem: 17.9

Let X be a Hausdroff space. Let A be a subset of X. Then the point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Proof:

If every neighbourhood of x intersects A in infinitely many points.

(limit point definition)

It centainly intersects A in some point other than x itself.

So, that x is a limit point of A.

conversely,

Suppose that x is a limit point of A.

To prove that, some neighbourhood U of x intersects A at infinitely many points but take some neighbourhood U of x intersects A in only finitely many points. Then U also intersects $A-\{x\}$ infinitely many points.

Let $\{x_1, x_2 \dots x_m\} \in U \cap (A - \{x\})$

The set $X - \{x_1, x_2 \dots x_m\}$ is an open set of X [Since every finite point set in a Hausdroff space is closed]

i.e $\{x_1, x_2, \dots x_m\}$ is closed

Then $U \cap (X - \{x_1, x_2 \dots x_m\})$ is a neighbourhood of x.

[$\because$ Intersection of 2 open sets is open]

So $U' = U \cap (X - \{x_1, x_2 \ldots x_m\})$ is a neighbourhoo of x does not intersect $\cap$ $A - \{x\}$ at all points.

[Since $\{x_1, x_2 \ldots x_n\} \notin U'$

$\{x_1, x_2 \ldots x_m\} \in A - \{x\}$]

no common point for U' and $A - \{x\}$.

This contradicts the assumption that x is a limit point of A.

i.e every neighbourhood of x intersects A in some point other than x itself.

$\therefore$ The correct one is every neighbourhood of x contain infinitely many points of A.

18. Continuity of a function:

Let X and Y be topological space. A function $f: X \to Y$ is said to be continuous if for each ~~pair~~ open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Theorem: 18.1

Let X and Y be topological spaces. Let $f: X \to Y$. Then the following are equivalent

(1) f is continuous

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$

(3) For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.

proof:

$\boxed{(1) \Rightarrow (2)}$

Assume that f is continuous

Let A be a subset of X.

To show that, if $x \in \bar{A}$ then $f(x) \in \overline{f(A)}$

Let V be a neighbourhood of $f(x)$.

Then $f^{-1}(V)$ is an open set of X

containing x (by defin. if continuity) it must

intersect A in some point y (define of limit point).

Then $f(f^{-1}(V))$ intersects $f(A)$ in the point

$f(y) = V$.

So $f(y) \in f(A)$

$f(y) \in \overline{f(A)}$ $\because A \subset \bar{A}$ $\therefore f(A) \subset \overline{f(A)}$

Hence $f(x) \in \overline{f(A)}$ [$f(y) = f$ (function of x) $= f(x)$

$\boxed{(2) \Rightarrow (3)}$

Let B be a closed set in Y and let $f^{-1}(B) = A$

We wish to prove that A is closed in X.

So, we only show that $\bar{A} \subset A$ $(\because \bar{A} = A)$

By elementary set theory $f(A) \subset B$

If $x \in \bar{A}$

$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$

[$\because$ B is closed in Y]

by (2) [$\because f(A) \subset B$, $\overline{f(A)} \subset \bar{B}$]

So that $x \in f^{-1}(B)$

Then $\bar{A} \subset A$

$\boxed{(3) \Rightarrow (1)}$

Let V be an open set in Y.

Let $B = Y - V$.

i.e B is closed in Y.

$\Rightarrow f^{-1}(B)$ is closed in X [by 3]

By elementary set theory

$f^{-1}(V) = f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$

So $f^{-1}(V)$ is open in X.

Hence f is continuous.

HOMEOMORPHISMS:

Let X and Y be topological spaces,

Let $f: X \to Y$ be a bijection. If both the functic

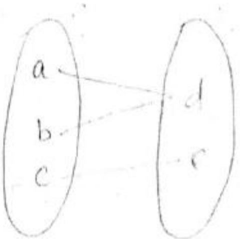
f and the inverse function $f^{-1}: Y \to X$ are continu

then f is called a homeomorphism.

Note:

(1) One to one $\Rightarrow$ each x in the domain has exactly one image in the rang

(2) Onto subjective $\Rightarrow$



(3) Bijective $\Rightarrow$ both Onto + One to one

(4) Isomorphism $\Rightarrow$ homomorphism + Linearity Property

Note:

The condition that f^{-1} be continuous

For each open set U of X, the inverse image of U under the map $f^{-1}: Y \to X$ is open in Y.

But the inverse image of U under the map f^{-1} is the same as the image of U under the map f.

Note:

Define homomorphism.

It is a bijective correspondence $f: X \to Y$ such that $f(U)$ is open iff U is open.

Example: 4

The function $f: R \to R$ given by $f(x) = 3x+1$ is a homomorphism.

If we define $g: R \to R$ by $g(y) = \frac{1}{3}(y-1)$ then $f(g(y)) = y$ and $g(f(x)) = x$ for all real numbers x and y.

It follows that f is bijective and $g = f^{-1}$ the continuity of f

$$[f(g(y))] = f\left(\tfrac{1}{3}(y-1)\right) = \frac{1}{3} f(y-1)$$

$$= \frac{1}{3} f(3x+1-1) = \frac{1}{3} f(3x)$$

$$= f(x)$$

$$= y$$

$$[g(f(x))] = g(3x+1) = g(y)$$

$$= \frac{1}{3}(y-1)$$

$$= \frac{1}{3}(3x+1-1)$$

$$= x$$

f is bijective

Theorem: 18.2

[Rules for constructing continuous function]

Let X, Y and Z be topological spaces.

(a) (Constant function)

If $f: X \to Y$ maps all of X into the single point y of Y, then f is continuous

(b) Inclusion:

If A is a subspace of X, the inclusion function $j: A \to X$ is continuous.

(c) Composites:

If $f: X \to Y$ and $g: Y \to Z$ are continuous then the map $g \circ f: X \to Z$ is continuous.

(d) Restricting the domain:

If $f: X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f(A): A \to Y$ is continuous.

(e) Restricting or expanding the range.

Let $f: X \to Y$ be continuous. If Z is a Subspace of Y containing the image set $f(X)$, Then the function $g: X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a Subspace, then the function $h: X \to Z$ obtained by expanding the range of f is continuous.

(f) Local formulation of continuity

The map $f: X \to Y$ is continuous if X can be written as the union of open sets $\bigcup_{\alpha}$ such that f / U_α is continuous for each α.

(g) Continuity at each point.

The map $f: X \to Y$ is continuous of for each $x \in X$ and each neighbour hood V of $f(x)$, there is a neighbour hood V of $f(x)$. there is a neighbour hood U of x Such that $f(U) \subset V$.

If the condition in (g) holds for a particular point x of X, we say that f is continuous at the point x.

Proof:-

(a) Let $f(x) = y_0$ for every x in X, Let V be open in Y. The set $f^{-1}(V)$ equals X or $\emptyset$, depending on whether V contains y_0 or not. In either case it is open.

(b) If U is open in X, then $f^{-1}(U) = U \cap A$ which is open in A by defin of subspace topology [defin of subspace topology, let X be a topological space with topology $\mathcal{J}$. If Y is a subset of X. the collection $\mathcal{J}_Y = \{ Y \cap U \mid U \in \mathcal{J} \}$ is a topology on Y, called the subspace topology ($\mathcal{J}$) with this topology, Y is called a subspace of X, its open sets consist of all intersection of open set of X with Y.] ($X \cap Y$)

(c) To prove that $(g \circ f)^{-1}$ is continuous.

If given $f : X \to Y$ and $g : Y \to Z$ are continuous

If U is open in Z then $g^{-1}(V)$ is open in Y an $f^{-1}(g^{-1}(U))$ is open in X.

But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ by elementary se theory. Hence $(g \circ f)^{-1}$ is continuous.

(d) To prove that restricted function $f|A : A \to Y$ is continuous.

i.e $f|A = f \circ j : A \to Y$ is continuous

$\because$ f and inclusion function $j : A \to X$ are continuous.

$\therefore$ composites $f \circ j : A \to Y$ is continuous.

(e) Given $f : X \to Y$ be continuous

Z is a subspace of Y, $f(X) \subset Z \subset Y$

To prove that $g : X \to Z$ is continuous.

[obtained by restricting the range of f].

Let B be open in Z

Then $B = Z \cap U$ for some open set U of Y

[by defin of subspace topology]

Because Z contains the entire image set $f(x)$

$f^{-1}(U) = g^{-1}(B)$ by elementary set theory.

$\because f^{-1}(U)$ is open

so $g^{-1}(B)$ is open

Hence $g: X \to Z$ is continuous.

If Z has Y as a subspace to prove that

$h: X \to Z$ is continuous.

$\therefore$ The inclusion map $j: Y \to Z$ is continuous

but $f: X \to Y$ is continuous.

$\therefore h = j \circ f : X \to Z$ is continuous

(composite is continuous)

[h obtained by expanding the range of f.

(f) By hypothesis $X = \cup U_\alpha$ (U_α is an open set)

Such that f / U_α is continuous for each α.

Restricting the domain

$U_\alpha \subset X$

$f: U_\alpha \to Y$

$f|U_\alpha : U_\alpha \to Y$ is con

$(f|U_\alpha)^{-1}(V) = V_\alpha$

Let V be an open set in Y.

Then $f^{-1}(V) \cap U_\alpha = (f | U_\alpha)^{-1}(V)$

$[\because f^{-1}(V) = \cup U_\alpha$, $\cup U_\alpha \cap U_\alpha = U_\alpha$

$[(f / U_\alpha)^{-1}(V) = U_\alpha]$

because both expression represent the set of those points x lying in U_α for which $f(x) \in V$

Since $f|U_\alpha$ is continuous this set is open in U_α and hence open in X.

(g) Let V be an open set of Y. Let x be a point of $f^{-1}(U)$. Then $f(x) \in V$ so there is a neighbourhood U_α of x such that $f(U_\alpha) \subset V$. Then $U_\alpha \subset f^{-1}(V)$. [by hypothesis each $x \in X$} and each neighbourhood V of $f(x)$]. It follows that $f^{-1}(V) =$ union of open set U_α.

so $f^{-1}(V)$ is open.

Theorem: 18.3 The Pasting Lemma

Let $X = A \cup B$ where A and B are closed in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous. If $f(x) = g$ for every $x \in A \cap B$ then f and g combine to give a continuous function $h: X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Proof:

To prove that there is a unique function. $h: A \cup B \to Y$ is continuous.

For this we prove that the inverse image of any closed subset of Y is closed in $A \cup B$. Let C be a closed subset of Y, $h(x) \in C$

iff $f(x) \in C$ or $g(x) \in C$

Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

Since f is continuous, $f^{-1}(C)$ is closed in A and therefore closed in $A \cup B$.

Theorem: 18.4 Maps into Products

Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous iff the function $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. The maps f_1 and f_2 are called the co-ordinate functions of f.

Proof:

Let $\pi_1 : X \times Y \to X$

$\pi_2 : X \times Y \to Y$ be projections onto the 1^{st} and 2^{nd} factors respectively.

These maps are continuous for

$\pi_1^{-1}(U) = U \times Y$

$\pi_2^{-1}(V) = X \times V$ and

these sets are open of U and V are open

[by defin π_1 and π_2 be projections]

Note that for each $a \in A$

$f_1(a) = \pi_1(f(a))$ composite function

$f_2(a) = \pi_2(f(a))$

If the function f is continuous then f_1 and f_2 are composites of continuous function and therefore f_1 and f_2 are continuous.

Conversely

Suppose that f_1 and f_2 are continuous

To prove that $f : A \to X \times Y$ is continuous.

It is enough to prove that for each basis element $U \times V$ for the topology of $X \times Y$.

Its inverse image $f^{-1}(U \times V)$ is open.

A point 'a' is in $f^{-1}(U \times V)$ iff $f(a) \in U \times V$

(i.e) iff $f_1(a) \in U$ and $f_2(a) \in U$

$\therefore f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$

Since both the sets f_1^{-1} and f_2^{-1} are open.

So their intersection is open.

$\therefore f^{-1}(U \times V)$ is open

Hence $f: A \to X \times Y$ is continuous.

UNIT: II

Connectedness and Compactness:

23.) Connected Spaces

Definition:

Let X be a topological space. A separation of X is a pair U, V of disjoint non empty open subset of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Note:

Another way of definition

If X is connected so, any space is homomorph to X.

(i) A Space X is connected iff the only subset of X that are both open and closed in X are empty set and X itself.

(ii) Connectedness is obviously a topological property. Since it is formulated entirely in terms of the collection of open sets of X.

Lemma 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint non empty sets A and B whose union is Y. neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof:

Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\bar{A} \cap Y$.

Since A is closed in Y, $A = \bar{A} \cap Y$ or $\bar{A} \cap B = \phi$

[Since $B \subset Y$] or $A \cap B = \phi$ $(A \cup A') \cap B = \phi$.

Since $\bar{A}$ is the union of A and its limit point [by theorem Let A be a subset of the topological space X, Let A' be the set of all limit points of A, Then $\bar{A} = A \cup A'$]

B contains no limit point of A.

A similar argument shows that A contains no limit points of B.

Conversely, Suppose that A and B are disjoint non empty sets whose union is Y. neither of which contains a limit point of the other. Then

$$\bar{A} \cap B = \phi \text{ and } A \cap \bar{B} = \phi$$

$$(\bar{A} \cap B) \cup A = \phi \cup A$$

$$\bar{A} \cap (A \cup B) = A$$

Hence we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. Then both A and B are closed in Y and since $A = Y - B$ and

$B = Y - A$

They are open in Y as well.

Lemma: 23.2

If the sets C and D form a separations of X and if Y is a connected subset of X, then Y lies entirely within either C or D.

Proof:

Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ open in Y. [by defin of subspace topology, Y is called a subspace of X, its open sets consist of all intersection of open sets of X with Y.]

These two sets are disjoint and their union is Y. If they were both non empty, They would constitute a separation of Y but Y is a connected subset of X.

$\therefore$ One of them is empty. Hence Y must (i.e) entirely in C or in D.

Theorem: 23.3

The union of a collection of connected sets that have a point in common is connected.

Proof:

Let $\{A_\alpha\}$ be a collection of connected subsets of a space X, Let P be a point of $\cap A_\alpha$. To prove that the set $Y = \cup A_\alpha$ is connected.

Suppose that $Y = C \cup D$ is a separation of Y. The point P is one of the sets C or D. Suppose $P \in C$. Since the set A_α is connected.

It must lie entirely in either C or D. and it cannot lie in D because it contains the point P of C. Hence $A_\alpha \subset C$ for every α hence $\cup A_\alpha \subset C$, contradicting the fact that D is non empty. Hence the union of collection of connected sets that have a point P (of be a point of $\cap A_\alpha$) in Common is connecte

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Theorem 23.4

Let A be a connected subset of X. If $A \subset B \subset \bar{A}$ then B is also connected or If B is formed by adjoining to the connected set A some or all of its limits, then B is connected.

Proof:

Let A be connected set and let $A \subset B \subset \bar{A}$. Suppose that $B = C \cup D$ is a separation of B by lemma 23.2.

If the sets C and D form a separation of X and If Y is a connected subset of X then Y lies entirely within either C or D. So A must lies entirely in 'c' or in 'D'. Suppose that

$A \subset \dot{c}$ Then $\bar{A} \subset \bar{\dot{c}}$

Since $\bar{C} \cap D = \phi$

$B \cap D = \phi \quad [\because B = C \cap D]$

That is D is empty $B \cap D = D = \phi$.

This contradicts the fact that D is a non empty subset of B.

$\therefore$ B is connected.

Note:

A function $f: X \to Y$ is continuous if the preimage of every open set of Y is open in X.

Theorem 3.5

The image of a connected space under a continuous map is connected.

Proof:

Let $f: X \to Y$ be continuous map.

Let X be connected.

We wish to prove that the image space $Z = f(X)$ is connected.

Since restricting the range of Z to the space Z is continuous.

i.e $g: X \to Z$ is continuous.

(Note:

$g: X \to Z$ obtained by restricting the range of f is bijective)

We say that $f: X \to Y$ is a topological imbedding.

(i.e) It suffices to consider the case of a continuous surjective map $g: X \to Z$. Suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z. Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X

Then are open in X because g is continuous and nonempty. $g(x) = Z$ so, they are not seperation. It is connected.

Theorem 23.6

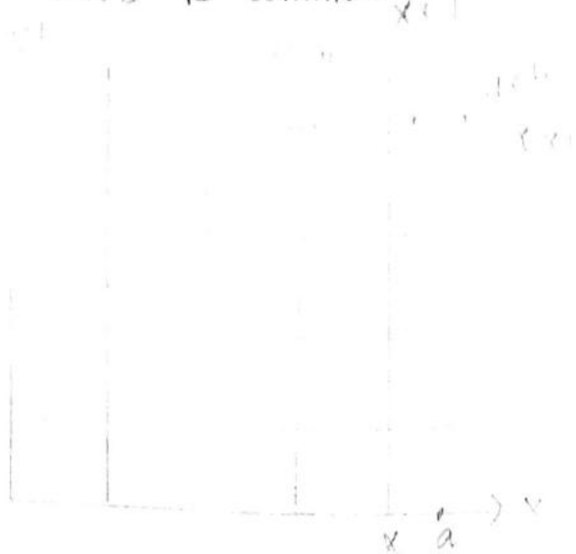
The cartesion product of connected spaces is connected.

Proof:

To prove that the product of two connected spaces X and Y is connected. Choose a base point $a \times b$ in the product $X \times Y$. The horizondal slice $X \times b$ is connected being homomorphic with X and each vertical slice $X \times Y$ is connected being homomorphic with Y.

As a result each T-shaped space.

$T_x = (X \times b) \cup (x \times Y)$ is connected being the union of two connected sets that have the point $x \times b$ is common.



$\bigcup_{x \in X} T_x$ of all these T shaped Spaces.

The union is connected because it is the union of a collection of connected set that have the point $a \times b$ is common.

Since the union equals $X \times Y$, the Space $X \times Y$ is connected.

The proof for any finite product of Connected space followed by induction, using the fact that $X_1 \times X_2 \times \dots \times X_n$ is homomorphic with $(X_1 \times \dots X_{n-1}) \times X_n$

- Totally ordered set

2m

A total ordered set is a set

(4.) plus a relation on the set that satisfies the condition for a partial order plus an additional condition known as the comparability condition.

A relation is a total order on a set. If the following properties hold.

1. Reflexivity for all
2. Antisymmetry and implies
3. Trans°itivity and implies
4. Comparability (trichotomy)

The first three are the axioms of a partial order while addition of the trichotomy defines a total of

- Ordered Set:

2m The ordered pair (X, P) is called a partially ordered set if X is a set and P is a partial order relation in X.

24. Connected Subspaces of the Real Line

Definition:

A simply ordered set L having more than one element is called a linear combination if the following hold

(i) L has the least upper bound property.

(ii) If $x < y$, there exist z such that $x < z < y$.

Note:

least upper bound property

Supremum: LUB upper bounds of S.

A set X has the LUB property iff every non-empty subset of X with an upper bound has a least upper bound in X.

Theorem: 24.1

If L is a linear continuous number in the order topology, then L is connected and so is every interval and ray in L.

Proof:

Let Y be a subset of L that equals either L or an interval or a ray in L. The set Y is 'convex' in the sense that if a and b are any two points of Y and $a < b$ then the entire interval $[a,b]$ of points of L is contained in the set Y.

Let A and B be disjoint non empty sets that are open in Y. We shall show that $Y \neq A \cup B$ (i.e) there exist no separation of Y.

i.e Y is connected

i.e Every interval in L is connected

Choose a point 'a' of A and a point b of B.

Assume $a<b$

(but every point of A is not less than every point of B)

Because Y is convex,

$\therefore [a,b]$ in J.

We shall find a point of $[a,b]$ that belongs to neither A nor B consider the sets.

$A_0 = A \cap [a,b]$ and $B_0 = B \cap [a,b]$.

These sets are open in $[a,b]$ in the subspace topology (which is same as the order topology)

Let $C = \text{Lub } A_0$

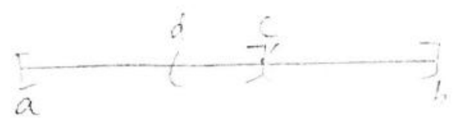
We shall show that c belongs to neither A_0 not B_0

Case (i)

Suppose that $c \in B_0$. Then $c \neq a$ so either

$c=b$ or $a<c<b$.

In either case, it follows from the fact that B_0 is open in $[a,b]$ that there is some interval of the of the form $(d,c]$ contained in B_0.



If $c = b$ we have a contradiction at once for 'd' is smaller upper bound on A_0 than c.

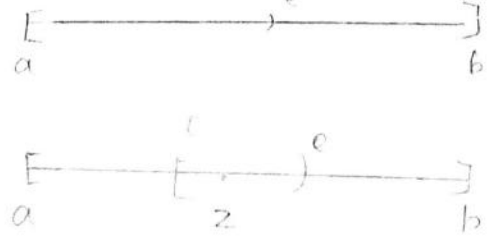
If $c < b$, $(c, d]$ does not intersect A_0 because c is an upper bound on A_0. Then $(d, b] = (d, c] \cup (c, b]$ does not intersect A_0.

Again d is a smaller upper bound on A_0, then 'c' contrary to construction. So 'c' $\notin B_0$

<u>case (ii)</u>

Suppose that $c \in A_0$. Then $c \neq b$

So either $c = a$ or $a < c < b$, because A_0 is open in $[a, b]$ these must be some interval of the form $[c, e]$ contained in A_0, see fig because of order property - Anti symmetric of the linear continum L, we can choose a point $z \in A_0$ contrary to the fact that c is an least upper bound for A_0.



Hence c belongs to neither A_0 nor B_0 hence c belongs to neither A nor B but $c \in [a, b]$ in Y.

Hence Every interval in Y is connected.

Corollary: 24.2

The real line R is connected and so is every interval and ray in R.

Theorem 24.3

(Intermediate value Theorem)

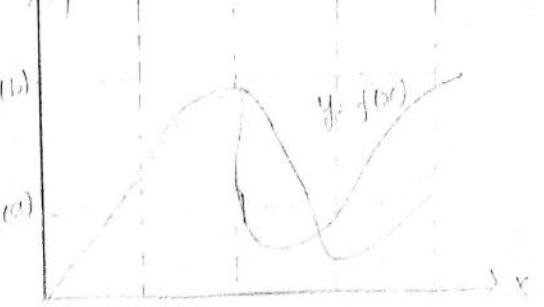
Let $f: X \to Y$ be a continuous map of the connected space X into the ordered set Y, in the order topology. If 'a' and 'b' are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$ then there exist a point 'c' of X such that $f(c) = r$.

Proof:

The intermediate value theorem of calcules is the special case of this theorem that occurs we take X to be a closed interval in R and Y to be R.

Statement: $f: X \to Y$

Suppose $f(x)$ is continuous on the closed interval $[a,b]$. If r is a real number between the value $f(a)$ and $f(b)$ but not equal to either of them, then there exists a number 'c' is the interval $[a,b]$ for which $f(c) = r$.



Here we know that the theorem the image of a connected space under a continuous map is connected.

i.e $f(x)$ is connected.

Assume the set

$A = f(x) \cap (-\infty, r)$ and

$B = f(x) \cap (r, \infty)$ are disjoint and

they are non empty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(x)$ being the intersection of an open ray in Y with $f(x)$ (by def of subspace topology.)

If $c \notin X$ such that $f(c) = r$ then

$A \cup B = [f(x) \cap (-\infty, r)] \cup [f(x) \cap (r, \infty)] \cancel{= f(x)}$

$= f(x) \cap [(-\infty, r) \cup (r, \infty)]$

$= f(x) \cap [(-\infty, \infty) - \{r\}]$

$= f(x) \Rightarrow r \notin f(x)$

$\therefore f(c) \notin f(x)$

i.e separation of $f(x)$.

$f(x)$ would be the union of the set A and B

Then A and B would constitute a separation $f(x)$.

Contradicting the fact that the image of a connected space under a continuous map is connected.

$\therefore c \in X$.

Hence Intermediate value theorem.

Hence proved.

Definition: Path

Given points x and y of the space X, a path in X from x to y is a continuous map $f: [a,b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

Path connected:

A space X is said to be path connected if every pair of point of X can be joined by a path in X.

Eg: $f: [0,1] \to Y$ $\quad 1 \to$

Note:

A path connected space X is necessarity connected.

A connected space need not be path connected.

The components of X can also be described as

The components of X are connected disjoint subsets of X whose union is X.

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25. Components and Local Connectedness

Theorem 25.1

The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one of them.

Proof:

Being equivalance classes, the components of X are disjoint and their union is X.

Each connected set A in X intersects only one of them. For if A intersects the components C_1 and C_2 of X, say in points x_1 and x_2 respectively, then $x_1 \sim x_2$ by defin.

This cannot happen unless $C_1 = C_2$.

To show the component C is connected choose a point x_0 of C for each point x of C, we know that $x_0 \sim x$, so there is a connected set A_x containing x_0 and x.

By the result of the preceeding paragraph $A_x \subset C$ $\therefore$ $C = \bigcup_{x \in C} A_x$

Since the set A_x are connected and have the point x_0 in common their union is connected.

Definition: Path components

We define another equivalance relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalenc classes are called the path components of X.

Example:-

Any interval (closed, open or half-open) is path connected. In particular, it is connected. Hence $X = [0,1) \cup (2,3]$ is a disjoint union of $[0,1)$ and $(2,3]$ each of which is a path connected component.

Theorem 25.2

The path components of X are path connect disjoint subsets of X whose union is X, such tha each path connected subset of X intersects only one of them.

Example:

The component of the subspace $Y = [-1,0) \cup (0,1]$ of the real line $\mathbb{R}$ are two set $[-1,0)$ and $(0,1]$. These are also the path component of Y.