Core Course - VII

Advanced Operation Research

18KP2307

Unit - I

Linear Programming Problem (LPP) - Graphical Method, Algebraic Solutions, Simplex Method, Two - phase Simplex, Duality in linear Programming, Dual Simplex Method and revised Simplex method;

Unit - II

Non - Linear Programming Problem (NLLP) - Formulating a Non-Linear Programming Problem, Kuhn - Tucker conditions for non - linear Programming, Quadratic Programming - Wolfe's Method and Beale's Method.

Unit - III

Integer Programming - Gomory's fractional cut Method for all integer Fractional cut Method for mixed integer and Branch and Bound Method.

Unit - IV

Inventory control - deterministic Inventory Problems with no Shortages and Projecting Scheduling by PERT and CPM - Network, Critical Path method and PERT Calculations.

Unit - V

Dynamic Programming - Recursive equation - characteristics of Dynamic programming - Dynamic Programming Algorithm - Solution of Discrete D.P.P - Some Applications - Solutions of L.P.P by Dynamic Programming.

Unit - I

Linear Programming Problem :

Mathematical formulation of the Problem :

Step : 1. Study the given Situation to find the Key decision to be made.

Step : 2. Identify the variables involved and designate them by Symbols x_j ($j = 1, 2 \cdots$)

Step : 3. State the feasible alternative which generally are :

$x_j \geq 0$, for all j.

Step : 4. Identify the Constrains in the Problems and express them as linear inequalities or equations. LHS of which linear functions of the decision Variables.

Step : 5. Identify the objective function and express it as a linear function of the decisions Variables.

Graphical Solution Method :

Step : 1. Identify the problem - the decision variables, the objective and the restrictions.

Step : 2 Set up the Mathematical formulation of the Problem.

Step : 3 Plot a graph representing all the Constraints of the problem and identify the feasible region (Solution Space). The feasible region is the intersection of all the regions represented by constraints of the Problem and is restricted to the first quadrant only.

Step 4 : The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner Points of the feasible region.

Step 5 : Find out the value of the objective function at each corner (solution) point determined in Step 4.

Step 6 : Selection the corner point the optimizes (maximum or minimizes) the value of the objective function. It gives the optimum feasible solution.

Simplex Algorithm :

The Solution for any L.P.P by Simplex algorithm. The Steps for the computation of an optimum solution are as follows:

Step 1: Check whether the objective function of the given L.P.P is to be maximized or minimized. If it is to be minimized then we convert into a Problem of maximizing it by using the result.

$$\text{Minimum } z = \text{Maximum}(-z).$$

Step 2 : check whether all b_i $(i = 1, 2 \dots m)$ are non-negative. If any one of b_i is negative then multiply the corresponding inequation of the Constrains by -1 Soas toget all b_i $(i = 1, 2 \dots m)$ non-negative.

Step 3 : Convert all the inequalities of the Constrains into equations by introducing slack and/or surplus Variables in the Constrains. Put the costs of these Variables equal to Zero.

Step 4: obtain an initial basic feasible Solution to the form $X_B = B^{-1} b$ and put it in the first column of the Simplex table.

Step: 5 Compute the net evaluations $z_j - c_j$ ($j = 1, 2 .. n$) by using the relation $z_j - c_j = c_B y_j - c_j$, where $y_j = B^{-1} a_j$.

Examine sign $z_j - c_j$,

(i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution x_B is an optimum basic solution.

(ii) If atleast one $(z_j - c_j) < 0$, Proceed on to the next step

Step: 6. If there more than one negative $z_j - c_j$ then choose the most negative of them. Let it be, $z_r - c_r$ for the some $j = r$.

(i) If all $y_{ir} \leq 0$ ($i = 1, 2 .. m$) then there is an unbounded solution to given Problem.

(ii) If atleast one $y_r > 0$ ($i = 1, 2 .. m$) then the corresponding vector y_r enter the basis y_B.

Step 7: Compute the ratios $\left\{ \frac{x_{Bi}}{y_{ir}}, y_{ir} > 0, i = 1, 2 .. m \right\}$ and choose the minimum of these ratios be x_{BK} / y_{Kr}. Then the vector y_K will leave the basis y_B. The common element y_{Kr} which is in the Kth row and the rth column is known as the leading element (or pivotal element) of the table.

Step 8: Convert the leading element to unity by dividing is row by the leading element it self and all other elements in its column to zeros by making use of the relations:

$$\hat{y}_{ij} = y_{ij} - \frac{y_{Kj}}{y_{Kr}} y_{ir} \qquad i = 1, 2, \dots m+1;\ i \neq K$$

$$\hat{y}_{Kj} = \frac{y_{Kj}}{y_{Kr}} \qquad j = 0, 1, 2 \dots n$$

Step 9: Go to step 5 and repeat the computational Procedure until either an optimum Solution is obtained or there is an column to zero by making the unbounded solution.

Two - Phase Method :

The Iterative Procedure of the algorithm may be Summarise as below :

Step 1: Write the given L.P.P into its Standard form and check whether there exists a Starting basic feasible Solution.

Step 2 :

(a) If there is a ready Starting basic feasible Solution goes to Phase 2.

(b) If there does not exist a ready Starting basic feasible solution go on to the next step.

Phase - 1

Step 2 : Add the artificial Variable to the left side of the each equation that lacks the needed Starting basic Variables. Construct an auxillary objective function aimed at minimising the sum of all artificial Variables.

Thus, the new objective is to,

$$\text{Minimize } Z = A_1 + A_2 + \cdots + A_n$$

$$\text{Maximize } Z^* = A_1 - A_2 - \cdots - A_n$$

Where A_i $(i = 1,2 \ldots m)$ are the non-negative artificial Variables.

Step 3 : Apply Simplex algorithm to the Specially Constructed L.P.P. The following three cases May arise at the least interaction

(a) $\max Z^* < 0$ and at least one artifical Variables is Present in the bais with Positive value. In Such a case the original L.P.P. does not any feasible Solution.

(b) $\max Z^* = 0$ and at least one artifical Variable is Present in the bais at zero value. In Such a case, the original L.P.P ~~doesnot~~ Possess the feasible solution.

(c) Maximum $Z^* = 0$ and no artificial Variables is Present in the basis. In such a Case, a basic feasible Solution to the Original L.P.P has been found. Goto Phase 2.

Phase - 2

Step 4 : Consider the optimum basic feasible Solution of phase I as a Starting basic feasible Solution for the original L.P.P. Assign actual Co-efficients to the Variables in the objective function and a value zero to the artificial variables that appear at zero value in the final Simplex table of Phase - 1.

Apply usual Simplex algorithm to the modified Simplex table to get the optimum Solution of the Original Problem.

Formulating a Dual Problem :

Various Steps involved in the formulation of a Primal - dual pair are ;

Step 1 :

Put the given linear Programming Problem into its Standard form. Consider it as the primal Problem.

Step 2 :

Identify the Variables to be Used in the dual problem. The number of these Variables equals the number of constraints equation in the Primal.

Step 3 :

Write down the objective function of the dual, Using the right hand side constants of the primal constraints.

Step 4 :

If the Primal Problem is of Maximazation type, the dual will be a minimization problem and Vice - Versa.

Step : 5

Making use of dual Variable identified in the Step : 2 Write the Constraints for the dual problem.

(a) If the Primal is a maximazation problem the Prob'm dual constraints must be all of '≥' type of the problem. If the primal is a minimization problem, the dual Constraints must be all of '≤' type.

(b) The column co-efficients of the primal constraints become the row co-effcients of the dual constraints.

(c) The co-efficients of the Primal objectives function becomes the right - hand side Constants of the dual constraints.

(d) The dual Variables are defined to be unrestricted in a sign.

Step :6

Using Steps 3 and 4, Write down the dual of the given L.P.P.

Dual Simplex Method :

Dual Simplex Method is appluable to those linear Programming problems that start with infeasible but otherwise optimum solution. The Method may be Summarized as follows :

Step 1 :

Write the given linear Programming problem in its Standard form and obtain a Starting basic solution.

Step 2 :

(a) If the current bias Solution is feasible, use Simplex method to obtain an optimum Solution.

(b) If the current basic Solution is infeasible, i.e, values of basic variables are ≤ 0, go to the next Step.

Step 3 : Check the whether the solution is optimum,

(a) If the Solution is not optimum, add an artifiial constraints in such way that the conditions of the optimality is statisfied.

(b) If the solution is the optimum to the problem, go to the next step.

Step 4 :

Select the basic Variables having the most negative value. This basic Variable becomes the leaving Variables and the row corresponding to it becomes the key row.

Step 5 :

Obtain the ratios of the net evaluations to the corresponding co-efficients in the key row. Ignore ratios associated the with Positive and zero denominators. The entering Vector is the one with the smallest absolute value of the ratios. column corresponding to the entering vector becomes the key column.

Step 6 :

Reduce the leading element into Unity and all other entires of the Key column to zero by elementary row operations.

Step 7 :

Go to the step 2 and repeat the Procedure until an optimum basic feasible solution is attained

Revised Simplex Algorithm :

Step 1 : Introduce Slack and Surplus Variables, if needed and restate the given L.P.P in maximum Standard form.

Step 2 : Begin with an initial basis $B = I_m$ and form the auxillary matrix $\hat{B}$ and writedown $\hat{B}^{-1}$.

Step 3 : State the objective relation $z = cx$ as an addition constraint and form $\hat{A}$ and $\hat{b}$, Where $\hat{A} = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $\hat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$

Step 4 : Compute the net evaluations $(z_j - c_j)$ $j = 1, 2 \dots n$ by multiplying the Successive columns of $\hat{A}$ with the last row $\hat{B}^{-1}$ that is by using the relation, $(z_j - c_j) = (C_B B^{-1}, 1)\begin{pmatrix} A \\ -C^T \end{pmatrix}$

If all $(z_j - c_j)$ are non-negative, determine the most negative of them say $(z_k - c_k)$ the corresponding vector Y_k enter the basis. go to Step 5.

Step 5 : Compute the $\hat{Y_k} = \hat{B}^{-1}, \hat{a}_k$. If all the $Y_{ik} \leq 0$, there exists an unbounded optimum Solution to the given problem If at least one $Y_{ik} > 0$, the current X_B and deporting vector.

Step 6 : The results of the Obtained in Step 2 through Step 5 in the tabular form known as the revised Simplex table.

Step 7 : Convert the leading element to unity and all the other elements of the entering Column to zero by suitable row operations and update the current basic feasible solution.

Step 8 : Go to step 4 and repeat the Procedure until an optimum basic feasible solution is obtained Of there is an indication of an unbounded Solution.

UNIT - II

Non - linear Programming :-

The optimum Solution lies at one of the extreme Points of the convex feasible region. But in a non-linear Programming Problem (NLPP), the optimum Solution can be found anywhere on the boundary of the feasible region and even at Some interior Point of it.

Kuhn - Tucker Conditions with Non-negative Constraints :

We obtained the necessary conditions for a Point $x^{0} \in R^{n}$ to be relative maximum of $f(x)$ subject to the Constraints $h^{i}(x) \leq 0$, $i = 1, 2 \ldots m$ $x \geq 0$. These Conditions called Kuhn-Tucker Conditions, were found by converting each inequality Constraint to an equation through the addition of Squared Slack variables. S_i^2, imposing the first-order Conditions for maxima, on the first Partial derivatives of the Lagragian functions, and then Simplyfying the Out Come. The following conditions resulted :

(a) $f_j = \sum_{i=1}^{m} \lambda_i h^i_j$ $\quad j = 1, 2, \ldots n$

(b) $-\lambda_i h^i(x) = 0$ $\quad i = 1, 2 \ldots m$

(c) $h^i(x) \leq 0$ $\quad i = 1, 2 \ldots m$

(d) $\lambda_i \geq 0$ $\quad i = 1, 2 \ldots m.$

The reader may have observed that in obtaining these conditions, the non-negativity constraints $x \geq 0$

were completely ignored. However, we always had in the mind to discard all such solutions of (a) to (d) that violate $x \geq 0$.
We shall consider, the non-negativity constraint $x \geq 0$ as one of the constraint $h(x) \geq 0$, $h(x) = x$, and derive the Kuhn-Tucker conditions for the result problem.

Maximize $z = f(x)$. $\quad x \in R^n$ Subject to the constraints;

$h^i(x) \leq 0, \; -x \leq 0;$ $\quad (i = 1, 2 .. m)$

there are $m+n$ inequality constraints, and then we add the squares of $(m+n)$ slack variables $S_1, \ldots S_m, S_{m+1}, \ldots S_{m+n}$ in the inequalities so as to convert them into equations;

$$h^i(x) + S_i^2 = 0 \qquad \text{for } i = 1, 2 .. m$$

$$-x_j + S_{m+j}^2 = 0 \qquad \text{for } j = 1, 2 .. n.$$

To find the necessary conditions for maximum of $f(x)$, we construct the associated Lagrangian function,

$$L(x, S, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i \left[(h^i(x)) + S_i^2 \right] - \sum_{j=1}^{m} \lambda_{m+j} \left[-x_j + S_{m+j}^2 \right]$$

Where $S = (S_1, S_2, \ldots S_{m+n})$ and $\lambda = (\lambda_1 \ldots, \lambda_{m+n})$ are the Lagrangian multipliers. The Kuhn-Tucker conditions are;

$$\frac{\partial L}{\partial x_j} = f_j - \sum_{i=1}^{m} \lambda_i h^i_j + \lambda_{m+j} = 0 \qquad \text{for } j = 1, 2 \ldots n$$

$$\frac{\partial L}{\partial S_i} = -2 \lambda_i S_i = 0 \qquad \text{for } i = 1, 2 \ldots m$$

$$\frac{\partial L}{\partial S_{m+j}} = -2 S_{m+j} \lambda_{m+j} = 0 \qquad \text{for } j = 1, 2 \ldots n$$

$$\frac{\partial L}{\partial \lambda_i} = -(h^i(x) + S_i^2) = 0 \qquad \text{for } i = 1, 2 \cdots m$$

$$\frac{\partial L}{\partial \lambda_{m+j}} = -[-x_j + S_{m+j}^2] = 0 \qquad \text{for } j = 1, 2 \cdots n.$$

and,

The Kuhn - Tucker conditions from these upon, Simplification are ;

(a) $f_j = \sum_{i=1}^{m} \lambda_i h^i_j - \lambda_{m+j}$ $\quad (j = 1, 2 \cdots n)$

(b) $\lambda_i \, [h^i(x)] = 0$ $\quad (i = 1, 2 \cdots m)$

(c) $-\lambda_{m+j} x_j = 0$ $\quad (j = 1, 2 \cdots n)$

(d) $h^i(x) \leq 0$ $\quad (i = 1, 2 \cdots m)$

(e) $\lambda_j, \lambda_{m+j}, x_j \geq 0$ $\quad (i = 1, 2 \cdots m;\ j = 1, 2 \cdots n)$

Maximize of subject to ; $h_i(x) \leq 0, x \geq 0.$

Quadratic Programming:

Quadratic Programming is concered with the NLPP of maximizing (or minimizing) the Quadratic objective function, subject to a set of linear inequality constraints.

Let X^T and $C \in R^n$ Let Q be a symmetric $n \times n$ real matrix. Then the problem of maximizing,

$$f(x) = cx + \tfrac{1}{2} X^T Q X$$

Subject to the constraints ; $AX \leq b$ and $x \geq 0$

Where $b^T \in R^m$ and A is an $m \times n$ real matrix is called a general programming problem.

1. If $x^T Q x$ is positive-semi-definite (negative-semi-definite) then it is convex (concave) in X over all of R^n and.

2. If $x^T Q x$ is positive-definite (negative-definite) and its (strictly concave) in X over all of R^n.

Let us consider a QPP written as follows,

Maximize $z = f(x_1, x_2 \ldots x_n) = \sum_{j=1}^{n} c_j x_j + 1/2 \sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k.$

Subject to constraints : $\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2 \ldots m,$

$x_j \ge 0 \qquad j = 1,2 \ldots n.$

and $d_{jk} = d_{kj}$ for all j and k; and $b_i \ge 0$;

The Kuhn-Tucker Conditions for the maximum are :

(i) $f_j - \sum_{i=1}^{n} \lambda_i h^i_j + \lambda_{m+j} = 0 \quad$ for $(j = 1,2 \ldots n)$ (or)

$\{c_j + 1/2 [2 \sum_{k=1}^{n} d_{jk} x_k]\} - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0 \quad (j = 1,2 \ldots n)$

(ii) $\lambda_i h^i(x) = 0$ (or) $\lambda_i [\sum_{i=1}^{n} a_{ij} x_j - b_j] = 0 \quad (i = 1,2 \ldots m)$

(iii) $\lambda_{m+j}(-x_j) = 0$ (or) $-\lambda_{m+j} x_j = 0 \quad (j = 0,2 \ldots n)$

(iv) $h^i(x) \le 0 \quad (i = 1,2 \ldots n)$ (or) $\sum_{j=1}^{n} (a_{ij} x_j - b_j) \le 0$

(v) $\lambda_i, \lambda_{m+j}, x_j \ge 0 \quad (i = 1,2 \ldots m; \; j = 1,2 \ldots n).$

Thus the Kuhn-Tucker necessary for an optimum solution to the QPP are :

(a) $c_j + \sum_{k=1}^{m} d_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0 \quad (j = 1,2 \ldots n)$

(b) $\lambda_i [\sum_{j=1}^{n} a_{ij} x_j - b_i] = 0 \quad [i = 1,2 \ldots m]$

(c) $(-x_j)\,\lambda_{m+j} = 0$ $\quad (j = 1,2 \cdots m)$

(d) $\sum_{j=1}^{n} a_{ij}\, x_j \le b_i$ $\quad (i = 1,2 \cdots m)$ $\quad$ (e) $\lambda_i, \lambda_{m+j}, x_j \ge 0$

$(i = 1,2 \cdots m;\ j = 1,2 \cdots n)$

Let $S_i^2 \ge 0$ be the slack variables introduced in the ith constraint (d),

(d') $\sum_{j=1}^{n} a_{ij}x_j + S_i^2 = b_i$ $\quad$ for $(i = 1,2 \cdots m)$

and let $u_j = \lambda_{m+j}$ for $j = 1,2 \cdots n$.

(b') $\lambda_j S_i^2 = 0$ $\quad$ for $i = 1,2 \cdots m$

(c') $x_j u_j = 0$ $\quad$ for $j = 1,2 \cdots n$.

further since (e) $S_i^2, \lambda_i, x_i, u_j$ must be all non-negative for $i = 1,2 \cdots m$; $j = 1,2 \cdots n$ (b') and (c') can be expressed by a single constraint:

(b)'' $\sum_{i=1}^{n} \lambda_i S_i^2 \lambda_i S_i^2 + \sum_{j=1}^{n} x_j u_j = 0$ $\quad$ [We can rewrite (a) as]

(a') $\sum_{K=1}^{n} d_{jK} x_K - \sum_{i=1}^{m} \lambda_i a_{ij} + u_j = -c_j,$ $\quad$ for $j = 1,2 \cdots n$.

If we assume that the quadratic form $\frac{1}{2} \sum \sum d_{jK} x_j x_K$ is negative-semi definite then $f(x)$ is concave in x and hence the conditions (a) to (e) above become necessary and sufficient conditions for an optimal solution to the QPP. Thus under such assumption, if we are able to find non-negative λ_j, S_i^2, x_j and u_j such that (d'), (b)'' and (a') are statisfied - then such x_j determines an optimal solution to the given QPP.

WOLFE'S MODIFIED SIMPLEX METHOD :

Step : 1

Convert the inequality constrains into equations by introducing the slack variables s_i^2 in the i^{th} constraint $i=1,2 \dots m$ and the slack variables s_{m+j}^2 in the j^{th} non-negativity constraint, $j=1,2 \dots n$.

Step 2 : Construct the lagrangian function,

$$L(x,s,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij} x_j - b_i + s_i^2 \right] - \sum_{j=1}^{n} \lambda_{m+j} (-x_j + s_{m+j}^2).$$

Where $x=(x_1, x_2, \dots x_n)$, $s=(s_1, \dots, s_{m+n})$, $\lambda=(\lambda_1, \dots, \lambda_{m+n})$.

Differentiate $L(x,s,\lambda)$ partially with respect to the components of x, s and λ and equate the first order partial derivatives equal to zero.

Step 3 : Introduce the non-negative artifical variables A_j, $j=1,2 \dots n$ in the Kuhn-Tucker condition,

$$c_j + \sum_{k=1}^{m} d_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0.$$

for $j=1,2 \dots n$ and construct an objective function,

$$Z = A_1 + A_2 + \dots + A_n$$

Step 4 :

Obtain an initial basic feasible solution to the L.P.P :

Minimize $Z = A_1 + A_2 + \dots + A_n$

Subject to the constraints :

$$\sum_{K=1}^{n} d_{jK} x_K - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} + A_j = -c_j \quad (j=1,2..n)$$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad (i=1,2...m)$$

$$A_j, \lambda_i, \lambda_{m+j}, x_j \geq 0 \quad (i=1,2..m ; j=1,2..n).$$

Where $x_{n+1} = S_i^2$; $i=1,2..m$ and statisfy the Complementary Slackness conditions

$$\sum_{j=1}^{n} \lambda_{m+j} x_j + \sum_{i=1}^{m} x_{n+i} \lambda_i = 0 .$$

Step 5 :

Use two Phase simplex method to obtain an optimum solution to the Lpp of step 4 , the solution statisfy the complementary slack ness condition .

Step 6 :

The optimum solution obtained in step 5 in an optimum solution to the given Qpp also.

Bale's Algorithm for Qpp :

Let the general quadratic programming problem be to maximize $f(x) = c^T x + 1/2\, x^T Q x$ subject to the constraints :

$Ax \{\leq \geq or =\} b$ and $x \geq 0$; Where $x \in R^n$, A is $m \times n$, b is $m \times 1$, c is $n \times 1$ and Q is a $n \times n$ symmetric matrix.

The Bale's iterative Procedure of solving this Problem can be summarized as follows:

Step 1 :

Convert the minimization $f(x)$ into that of maximization if required. Introduce slack and/or surplus variables in the inequality constraints if any and put the QPP in the standard form.

Step 2 :

Choose an arbitary any m variables as the basic variables so, that the remaining $n-m$ variables become a non-basic. Denote the basic and non-basic variables by,

$$X_B = (X_{B_1}, X_{B_2} \cdots X_{Bm}) \text{ ; and } X_{NB} = (X_{NB_1}, X_{NB_2}, \cdots X_{NB_{n-m}})$$

respectively.

Step 3 :

Express the each basic variables x_{Bi} entirely in terms of the non-basic variables by,

$$X_B = (X_{B_1}, X_{B_2}, \cdots X_{Bm}) \text{ and } X_{NB} = (X_{NB_1}, X_{NB_2} \cdots X_{NB_{n-m}})$$

Step 4 :

Express the objective function f also entirely in terms of the non-Basic X_{NB_i}'s.

Step 5 :

Examine the Partial derivatives of $f(x)$ formulated above w.r.t the non-basic variables at the Point $X_{NB} = 0$ (and $u=0$);

(i) If $\left(\dfrac{\partial f(x)}{\partial X_{NBK}}\right)_{\substack{X_{NB}=0 \\ u=0}} = 0$ for each $K = 1, 2 \cdots n-m$

(ii) If $\left(\frac{\partial f(x)}{\partial NBK}\right)_{\substack{XNB=0 \\ u=0}} \geq 0$ for each at least on K.

(iii) If $\left(\frac{\partial f(x)}{\partial XNBK}\right)_{\substack{XNB=0 \\ u=0}} < 0$ for each $K = 1, 2 \ldots n-m$

but, $\left(\frac{\partial f(x)}{\partial u_i}\right)_{\substack{XNB=0 \\ u=0}} \neq 0$ for some $i = r$.

Step 6:

(i) If the minimum ratios occurs for some $\frac{\alpha_{ho}}{|\alpha_{ho}|}$, the corresponding basic variables X_h, will leave the basis.

(ii) If the minimum ratios occurs for some $\frac{\gamma_{Ko}}{|\gamma_{KK}|}$, the exist criterion corresponds to a non-basic variables, In this case, Introduce additional non-basic variables called a free-variable defined by,

$$u_i = 1/2 \frac{\partial f}{\partial XK}.$$

Which relation becomes an additional constraint equations.

Step 7:

Go to step 3 and repeat the procedure until an optimal basic solution is reached.

Step 8:

Determine the optimal values of X_B and $f(x)$ by setting $X_{NB}=0$ in their expressions obtained in step 3 and 4.