UNIT- 1 :-

① CHAPTER- 1 : THE MECHANICAL SYSTEM:-

Equations of motion:-

The differential eqn of motion for a System of N particles can be obtained by using Newton's laws of motion to the particles individually.

consider a single particle of mass m which is subject to a force $\bar{F}$. We obtain from Newton's second law,

$$\bar{F} = m\bar{a} \quad (or) \quad \ldots\ldots >①$$

$$\bar{F} = \frac{d\bar{p}}{dt} = \dot{\bar{p}} \quad \ldots\ldots\ldots\ldots >②$$

Where $\dot{\bar{p}}$ is the ~~m~~ linear momentum given by

$$\bar{p} = m\bar{v} \quad \ldots\ldots >③$$

and the accelaration $\bar{a}$ or $\dot{\bar{v}}$ is measured relative to an inertial frame reference. The existance of an Inertial or Newtonian refrence frame fundamental postulate of Newtonian dynamics.

Let us suppose that we have found such a suitable inertial frame. Let denote the position vector of the i^{th} particle relative to the frame The eqn of motion for the system of N particles can be written as

$$m_i\ddot{\bar{r}} = \bar{F}_i + \bar{R}_i \quad (i = 1,2,\ldots N) \quad \ldots\ldots >④$$

where m_i is the mass of the i^{th} particle the total force acting on the i^{th} been broken into two vector components $\bar{F_i}$ and $\bar{R_i}$, $\bar{F_i}$ is the applied force and $\bar{R_i}$ is the constraints force. That is, $\bar{R_i}$ is that force which ensures that the geometrical constraints are followed in the motion of the ith particle. The applied force $\bar{F_i}$ represents the sum of all other forces acting on the ith particle.

In general, the forces that act on the body may be classified according to their mode of application as

(i) contact forces and (2) body or field forces. contact forces are transmitted to the body by a direct mechanical push (or) pull. On the other hand body forces are associated with action at a distance and are represented by gravitational, electrical or other fields. The forces $\bar{R_i}$ associated with the geometrical constraints are always contact forces. However, the applied forces $\bar{F_i}$ may be other either body or contact or combination of the two.

Let (x_i, y_i, z_i) represent the position of the ith particle then equation (4) can be expressed in the scalar form as follows:

$$m_i\bar{x}_i = F_{ix} + R_{ix}$$

$$m_i\bar{y}_i = F_{iy} + R_{iy}$$

$$m_i\bar{z}_i = F_{iz} + R_{iz} \quad (i=1,2, \quad N) \qquad ...(5)$$

Where F_{ix} and R_{ix} are the x-components of $\bar{F_i}$ & $\bar{R_i}$ respectively. $F_{iy}, R_{iy}, F_{iz}, R_{iz}$ are ||ly defined.

The notation adopted in eqn (5) is unwiel inorder to simplify the writing of equations. Let us denote the cartesian coordinates of the 1st particle by (x_1, x_2, x_3), of the 2nd particle by (x_4, x_5, x_6) & so on.

We note that the mass of the k particle is,

$$m_{3k-2} = m_{3k-1} = m_{3k} \quad \ldots \rightarrow ⑥$$

The eqn of motion in the form,

$$m_i \ddot{x}_i = F_i + R_i \quad (i = 1,2,3 \ldots 3N) \quad \ldots \rightarrow ⑦$$

For the case in which there are no constraints, the force components F_i are expressed as function of position, velocity and time. The forces R_i are zero. Hence the system of N particles is described by 3N second order differential eqn which are in general non linear.

Generalised coordinates :-

Degrees of freedom :-

An important characteristic of a given mechanical system is its number of degrees of freedom.

The number of degrees of freedom is equal to the number of coordinates minus the number of independent equations of constraint

Example :-

The configuration of a system of N particle is represented by 3N Cartesian coordinates. and if there are l independent eqns of constraint relating these coordinates, Then the degrees of freedom of the system $= 3N - l$.

To illustrate the idea of degrees of freedom.

Ex consider the 3 particles are connected by rigid rods to form a traingular body with the particles as its corners.

The configuration of the system is described by 9 cartesian coordinates which specify the location of the 3 particles. Also each rigid ~~is~~ rod is represented mathematically by an independent eqn of constraint. So that the degrees of freedom of the system is,

$$3N - l = 9 - 3 = 6.$$

5 consider the transformation eqn relating the cartesian coordinates $x_1, x_2, \dots x_{3N}$ to the generalized coordinates $q_1, q_2, \cdot q_n$ as follows.

$$\left.\begin{array}{l} x_1 = x_1(q_1, q_2 \cdot \; q_n, t) \\ x_2 = x_2(q_1, q_2 \cdots \; q_n, t) \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_{3N} = x_{3N}(q_1, q_2 \; q_n, t) \end{array}\right\} \quad ⑥$$

It is possible that each system of coordinates may have equations of constraint associated with it. let the x's have l equations of constraint and the q's have m equations of constraint, then, equating the number of degrees of freedom in each system, we have

$3N - l = n - m \quad \rightarrow (7)$

There should be a 1-1 correspondence b/w the points in the allowable domain of the x's and the points in the allowable domain of the q's for each value of time.

The neccessary and sufficient condition that one can solve th for the q's in terms of the x's & 1 is that the Jacobian determinant of the transformation is non-zero.

Suppose that x_i's $(i = 1, 2, \dots 3N)$ have l eqn. of constaint of the form,

$$f_j(x_1, x_2 \cdot \cdot x_{3N}) = \alpha_j, \; (j = 3N - l + 1, \cdots 3N) \rightarrow ⑩$$

let the generalised coordinates be choosen so that they are independent.

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Hence $n = 3N - l$.

Let us define an additional set of l q's and identify them with the l constant's α's.

$$\therefore q_j = \alpha_j \, (j = n+1, \dots 3N) \dots \rightarrow ⑪$$

Then the eqns of transformation (8) can be considered to be of the form.

$$\left.\begin{aligned} x_1 &= x_1(q_1, q_2 \dots q_{3N}, t) \\ x_2 &= x_2(q_1, q_2 \dots q_{3N}, t) \\ &\vdots \\ &\vdots \\ x_{3N} &= x_{3N}(q_1, q_2 \dots q_{3N}, t) \end{aligned}\right\} \dots \rightarrow ⑫$$

If the Jacobian determinant

$$\frac{\partial(x_1, x_2, \dots x_{3N})}{\partial(q_1, q_2 \dots q_{3N})} \neq 0 \quad \dots \rightarrow ⑬$$

Then eqns (8) or (12) can be solved for the q's in term of x's and t.

$$q_j = f_j(x_1, x_2 \dots x_{3N}, t) \; (j = 1, 2 \dots n) .) ⑭$$

The remaining q's for $j = n+1 \dots 3N$ can be obtained by eqns (10) & (11)

Example :-

Let us take a simple example of a transformation from cartesian to generalised coordinates. consider a particle which is constrained to move on a fixed circular path of radius a.

Solution.

$x_1^2+x_2^2=a^2$

* The equation of constraint is $\sqrt{x_1^2+x_2^2}=a$.

Let a single generalised coordinate q_1 (which is the polar angle) represent the one degree of freedom.

This coordinate q_1 can be varried freely without violating the constraint.

let us define ~~+~~ a second generalised coordinate q_2 which is constant i.e) $\boxed{q_2=a}$

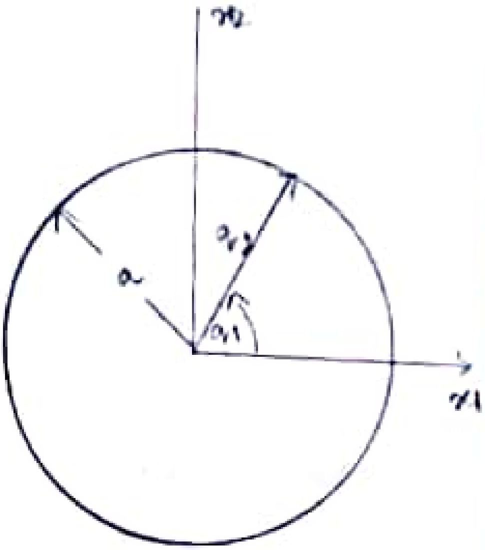


Fig 1.1 a particle on a fixed circular path

The transformation eqns are,

$$x_1 = q_2\cos q_1$$

$$x_2 = q_2\sin q_1$$

The Jacobian of this transformation is,

$$\frac{\partial(x_1,x_2)}{\partial(q_1,q_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix}$$

$$= \begin{vmatrix} -q_2\sin q_1 & \cos q_1 \\ q_2\cos q_1 & \sin q_1 \end{vmatrix}$$

$$= -q_2\sin^2 q_1 - q_2\cos^2 q_1 = -q_2$$

Hence, the q's may be expressed as the functions of the x's except when the Jacobian is zero at $q_2=0$

In this case the radius of the circle is zero and the polar angle q_1 is undefined. The transformation eqns are,

$$q_2 = \sqrt{x_1^2 + x_2^2}$$

$$q_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{q_2 \sin q_1}{q_2 \cos q_1}$$

$$x_2/x_1 = \tan q_1$$

$$q_1 = \tan^{-1}(x_2/x_1)$$

where we arbitrarily take $0 \le q_1 < 2\pi$ and $0 \le q_2 < \infty$. So that q's will be single valued functions of x's.

$$\boxed{q_2 = q} \Rightarrow q_2 = \sqrt{x_1^2 + x_2^2}$$

These transformation eqns are valid at all points on the finite $x_1 x_2$ plane except at the origin.

configuration space :-

We have seen that the configuration of a system of N particles is described by the values of its 3N cartesian coordinates. If the system has l independent equations of constraint of the form $f_j(x_1, x_2, \dots x_{3N}) = \alpha_j$

$$(j = 3N - l + 1, \dots 3N)$$

then it is possible to choose independent generalised coordinates $q_1, q_2, \dots q_n$ when $n = 3N - l$.

constraints :-

We have seen that a system of N particles may have less than 3N degrees of freedom due to the presence of constraints.

The constraints put restrictions

upon the possible motions of the system and result in corresponding forces of constraint types.

1) Holonomic Constraints :-

Suppose the configuration of a system is described by the n generalized coordinates $q_1, q_2 \cdots q_n$ and suppose that there are k independent eqns constraint of the form,

$$\phi_j(q_1, q_2, \ldots q_n, t) = 0 \quad (j = 1, 2 \ldots k) \qquad (15)$$

A constraint which can be expressed in this fashion is known as a holonomic constraint.

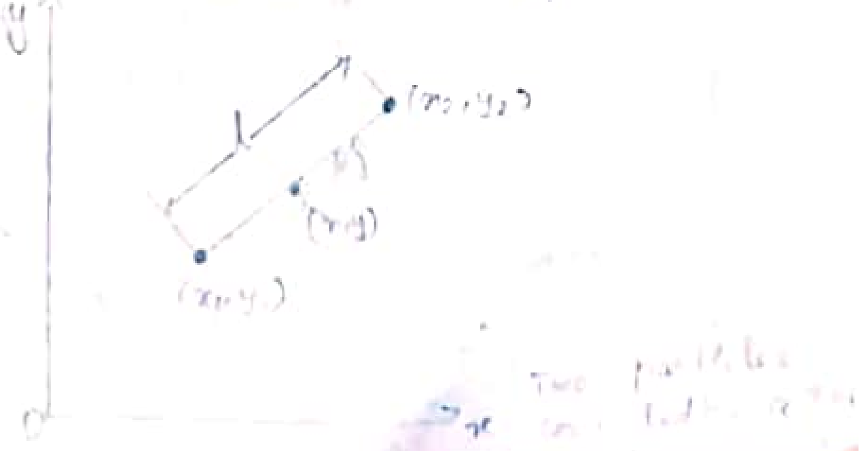
Holonomic System :-

A system whose constraint eqns, if any, are all of the holonomic form given in (15). Is called a holonomic system.

Example :-

There are 4 coordinates & one eqn ~~of constraint~~ yielding 3-degrees of freedom.

Consider the motion in the xy plane of two partition shown in the figure



these particles are connected by a rigid rod of length l. hence the corresponding equation of constraint is,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0$$

In this case there are 4 coordinates & one eqn of constraint yielding three degrees of freedom.

Scleron constraint :-

The length l of the rod is constant, & ∴ the holonomic constraint eqn does not contain time.

Constraints of the sort in which the time t does not appear explicity are known as scleron constraint

Rheonomic constraint :-

On the other hand, if the length l had been given as an explicit action of time, the constraint would have been cllassed as rheonomic, Then it is a moving constraint.

A System is Scleronomic

If (1) none of the constraint eqns contain t explicity (2) the transformation eqns given by (8) give the r's functions of q's.

If any of the constraint eqns or transformation eqns contain t explicity then the system is rheonomic.

2) Non-holonomic constraint :-

Let us consider a system of m constraints which are as non-integrable differential expressions of the form

$$\sum_{i=1}^{n} a_j^i \, dq + a_j = 0 \quad (j = 1, 2, \ldots m) \ldots \rightarrow ⑯$$

where the a's are in general functions of q's and t. constraints of this type are known as non holonomic constraint.

Example :-

Consider the motion in the xy plane of the two particles which are connected by a rigid rod. we assume that the particles can slide on the horizontal xy plane without friction. The system is changed by the addition of non-holonomic constraint in the form of knife-edge supports at two particles These supports move with the system and are oriented ⊥r to the direction of the rod in such way that they do not allow velocity component along the rod at either particle.

Hence, the velocity of the mid point of the rod must be ⊥r to the rod.

Then, $x = -y \tan\theta$

$$\text{(or)} \quad \cos\theta \, dx + \sin\theta \, dy = 0 \ldots \rightarrow (17)$$

The expression in the L.H.S not exact Futher the above eqn cannot multiplied by any integrating factor which makes it integrable

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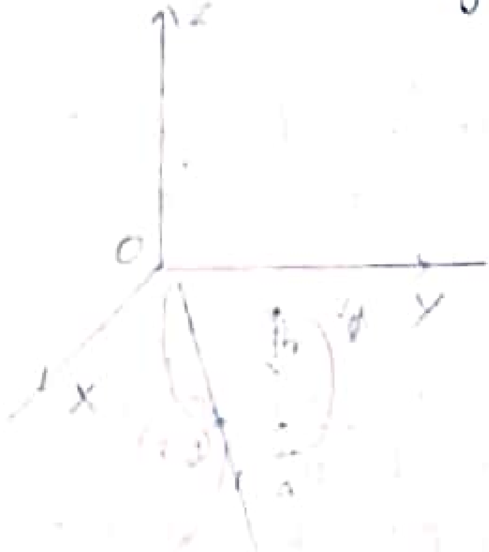
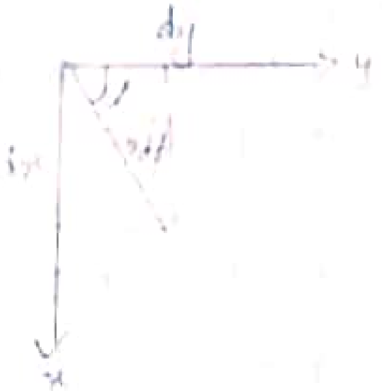
Example:-

A classical example of a non holonomic constraint occurs when there lling contact without slipping. consider a vertical disk of radius r which rod without slipping on the horizontal xy plane.





The point of contact (x,y), the angle of rolation of the disk about a ⊥r axis through its centre, & the angle α b/w the plane of the disk & the yz plane are taken as the generalised co ordinates.

As per conditions of rolling we have

$$\left.\begin{aligned} dx - r\sin\alpha\, d\phi &= 0 \\ dy - r\cos\alpha\, d\phi &= 0 \end{aligned}\right\}$$

Since $r\,d\phi$ is a differential element of displacement along the path traced by the point of contact & α is the angle b/w the tangent to the path & the y axis.

In this example, these 4 coordinates (x, y, α, ϕ) & 2 independent eqns, resulting in two degrees of freedom. it is clear that no integrating factor can be found such that the expressions be exact, hence both constraints are non holonomic.

5) Unilateral constraints

The constraints so for discussed are all bilateral.

That is if one imagines a small allowable displacement from any configuration of the system, the -ve of this displacement is also allowable assume any fixed value of time. Such bilateral constraints are always expressed as an equality.

Consider a constraint written in the form of an inequality involving a function of q's in time such as $f(q_1, q_1 \cdots q_n, t) \leq 0$

This means that the configuration point is restricted to a certain region of an n-dimensional configuration space which may vary with time.

Example:-

Suppose that a free particle is contained within a fixed hollow sphere of radius r whose centre is the origin of a Cartesian coordinate system.

Then using (x,y,z) as the coordinates of a particle, the unilateral constraint is given by $x^2+y^2+z^2-r^2 \leq 0$ ....>(19)

It can be seen that the unilateral (or inequality) form of constraint is holonomic in -nature.

Virtual work:-

The concept of virtual work is very important in the study of analytical mechanics. It is associated with the application of energy methods in the derivation of equations of motion & plays an important role in the study of stability.

Virtual displacement:-

Let us supportse that the configuration of a system of N particles is given by the 3N cartesian coordinates $x_1, x_2 \cdots x_{3N}$ & being measured relative to interial frame and may be subject to constraints.

At any given time, let assume that the coordinates move through infinitesimal displacement $\delta x_1, \delta x_2 \dots \delta x_{3N}$ which are virtual (or imaginary) in the sense that they assumed to have occured without passage of time & do not neccessarily con to the constraints. This small $\overline{\delta x}$ in the configuration of the system is known as a virtual displacement.

A virtual displacement conforms to the instantaneous constraints, any moving constraints are assumed to be stopped during the virtual displacement.

Example,

consider the system which is subjected to k holonomic constraints of the form

$$\phi_j (x_1, x_2 \dots x_{3N}, t) = 0 \quad (j = 1, 2, 3 \dots k)$$

Taking total differential of ϕ_j, we get,

$$d\phi_j = \sum_{i=1}^{3N} \frac{\partial \phi_i}{\partial x_i} dx_i + \frac{\partial \phi_i}{\partial} dt = 0 \, (j = 1, 2, k) \to (21)$$

A virtual displacement consisment with these constraints has the δx's related by the k equations

$$\sum_{i=1}^{3N} \frac{\partial \phi_i}{\partial x_i} \delta x_i = 0 \, ; \, (j = 1, 2, \dots k) \dots\dots \to (22)$$

Here dx's are replaced by δx's & the term dt has been omitted because the time is held fixed during a virtual displacement.

$^{lll}/_{ly}$. Let us suppose that the system has m nonholonomic constraints of the form

$$\sum_{i=1}^{3N} \alpha_{ji} \, dx_i + \alpha_{ji} \, dt = 0 \quad (j = 1, 2, \dots m) \dots \to (23)$$

Any virtual displacement consistent with these constraints must have the connected by the m equations.,

$$\sum_{i=1}^{3N} \alpha_{ji} \, \delta x_i = 0 \quad (j = 1, 2, \dots m) \dots\dots\dots \to (24)$$

Let us see whether a virtual displacement can also be a possible real ' " described by a set of dx's, & assumed to occur during the time displacement dt.

comparing eqns (21) & (22) we get that any holonomic constraint also be scleronomic if

$$\frac{\partial \phi_j}{\partial} = 0 \quad (j = 1, 2, \dots k) \quad \dots \to (25)$$

Illy any nonholonomic constraints must satisfy the condition.

$$\alpha_{ji} = 0 \quad (j = 1, 2, 3 . \quad m) \quad \ldots\ldots\rightarrow (26)$$

Since the above condition cannot be met in the general case, it is clear that a virtual displacement is not in general, a possible real displacement. It is sometimes convenient to assume that a set of δr's confirming to the instantaneous constraints occurs during an interval δt. The corresponding ratios of the form $\frac{\delta r}{\delta t}$ are known as virtual velocities.

In general virtual displacement velocities are not possible velocities for the actual system.

In the above discussion of virtual displacements we used cartesion coordinates.

Now let us consider a system whose configuration has been specified by the minimum num of generalised coordinates.

Thus any constraints which are nonholonomic will be expressed in the form

$$\sum_{i=1}^{n} a_{ji}\, dq_i + a_{ji}\, dt = 0 \quad (j = 1, 2, \ldots m) \ldots\rightarrow (27)$$

(or) $$\sum_{i=1}^{n} a_{ji}\, \dot{q}_i + a_{ji} = 0 \quad (j = 1, 2, \ldots m) \ldots\rightarrow (28)$$

where a's are functions of q's and t.

* Any virtual displacement consistent with the constraints must satisfy the condition.

$$\sum_{i=1}^{n} a_{ji} q_i + a_{ji} = 0 \quad (j=1,2,\ldots m) \quad \ldots\ldots \rightarrow (29)$$

The corresponding generalised ϵ_i has components u_i satisfying

$$\sum_{i=1}^{n} a_{ji} u_i = 0 \quad (j=1,2,\ldots m) \quad \ldots\ldots \rightarrow (30)$$

The necessary condition for the virtual velocity of any point of the system be a possible velocity is $a_{ji} = 0$ $(j=1,2,\ldots m)$ .... $\rightarrow$ (31)

(on comparing eqns (28) & (30)).

In addition, the transformation eqn relating x's and the q's must have the condition l explicity. That is there can be no moving constraints as moving constraints induce actual velocity components other than the allowable virtual velocities v_i.

Virtual Work :-

Consider a system of N particles whose configuration is given by the cartesion coordinates $x_1, x_2, \ldots x_{3N}$. Suppose that force components $F_1, F_2 \ldots F_{3N}$ are applied at the corresponding coordinates in a positive sense. The virtual work δW of these forces in a virtual displacement δx is given by

$$\delta W \, 8_{\bar{w}} : \sum_{j=1}^{3N} F_j \, \delta x_j \quad \ldots > (32)$$

(or) $$\delta W = \sum_{j=1}^{N} \bar{F}, \overline{\delta x_j} \quad \ldots > (33)$$

Where $\bar{F}_i$ is the force applied at i th particle & $\bar{r}_i$ is its position vector.

In the expression for virtual work, it is to be noted that the forces are assumed to remain constant. throughout the virtual displacement. This is true even if the actual force changes drastically as the result of infinitesimal displacement

Another point of importance is that the virtual work expression are defined to be linear. That is virtual work is ||r to a first variation.

Consider a system which is subject to constraints. Let the total force acting on ith particle be separated into an applied force $\bar{F}_i$ & a constraint force $\bar{R}_i$.

The virtual work of the constraint force is

$$\overline{\delta x_i} = \sum_{i=1}^{N} \bar{R}_i \cdot \overline{\delta x_i} \quad \ldots\ldots > (34)$$

Many of the constraints that commonly occur are of a class known as workless constraints.

A workless constraint is any bilateral constraint such that the virtual work of the corresponding constraint forces is zero for any virtual displacement consistent with the constraints.

It is clear that for a system having only workless constraints, the virtual work δr_i is 0.

$$\sum_{i=1}^{N} \bar{R}_i \, \delta \bar{r}_i = 0 \quad \ldots\ldots \rightarrow (35)$$

Example: 1:-

Examples of workless contraints:- are, (1) sliding motion in a frictionless surface (2) Rolling contact without slipping and (3) Rigid inter connections b/w particles.

Example of workless constraints

SOLUTION:-

1st assume that 2 particles (m_1) & (m_2) are connected by a rigid massless rod.

The forces exerted by the rod on the particles are equal & opposite, & collinear by Newton's 3rd law, $\therefore \bar{R}_2 = R_2 \bar{e}_n = -\bar{R}_1 \ \ldots\ldots \rightarrow (16)$

$\bar{e}_n$ being a unit vector directed along the rod as shown in the fig (c). Since the rod is rigid, the displacement components of the particles in the direction of the rod must be the same.

i.e) $\bar{e}_n \cdot \overline{\delta r_1} = \bar{e}_n \cdot \overline{\delta r_2}$ .....→(37)

∴ The virtual work of the constraint force is zero

i.e) $\delta W_c = \bar{R}_1 \cdot \overline{\delta r_1} + \bar{R}_2 \cdot \overline{\delta r_2} = 0$ ....→(38).

Example: 2:-

consider a body which slides without friction on a fixed surface S.S (see fig(a)). The constraint force R is normal to the surface at the point of contact p. Any virtual displacement of p involves sliding in the tangent plane at p. Hence no work is done by the constraint force R in a virtual displacement.

Example: 3:-

consider a vertical circular disk which roles without slipping along a straight horizontal path (see fig (b)). The total force of the surface acting on the disk can be separated as a normal component Rn and a frictional component Rf. Where which is directed along p the tangent to the surface. These force components pass through the instantaneous centre C which does not move as a result of virtual displacement $\delta\theta$. Hence the virtual work of the constraint force is zero.

unilateral constraints are not classified as workless constraints because allowable virtual displacement can be found in which the virtual work of the constraint force is zero

Principle of virtual work:

Suppose we consider a scleronomic system of N particles. If the system is in static equilibrium then $\overline{F_i} + \overline{R_i} = 0$ ----> (39) for each particle.

Therefore the virtual work done by all the forces in moving through an arbitrary virtual displacement consider with the constraints is zero.

ie) $\sum_{i=1}^{N} (\overline{F_i} + \overline{R_i}) \cdot \overline{\delta r_i} = 0$

i.e) $\sum_{i=1}^{N} \overline{F_i} \cdot \overline{\delta r_i} + \sum_{i=1}^{N} \overline{R_i}\, \overline{\delta r_i} = 0$ ---------> (40)

If we assume all the constraints are workless and if δr_i are reversible virtual displacements consistent with the constraints then.

$\sum_{i=1}^{N} \overline{R_i} \cdot \overline{\delta r_i} = 0$ --------------> (41)

From (40) & (41) we defind that,

$\delta_W = \sum_{i=1}^{N} \overline{F_i} \cdot \overline{\delta r_i} = 0$ ----------> (42)

Thus we have shown that if a system of particles with workless constraints is in static equilibrium, then the virtual work of the applied forces is zero in any virtual displacement consistent with the constraints.

Now, let us suppose that the same system of particles is initially motionless, but is not equilibrium.

Then one (or) more of the particles must have a net force applied to it will start to move in the direction of that force according to Newton's law of motion. Since any motion must be compatiable with the constraints which are assumed to be fixed, it is always possible to choose a virtual displacement in the direction of actual motion at each point so that the virtual work is +ve.

i.e) $\sum_{i=1}^{N} F_i \cdot \delta \bar{r}_i + \sum_{i=1}^{N} R_i \delta \bar{r}_i > 0 \cdots\cdots \rightarrow (43)$

Since the constraints are workless using equation (41) we get, $\delta_W = \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i > 0 \cdots\cdots\cdots \rightarrow (44)$

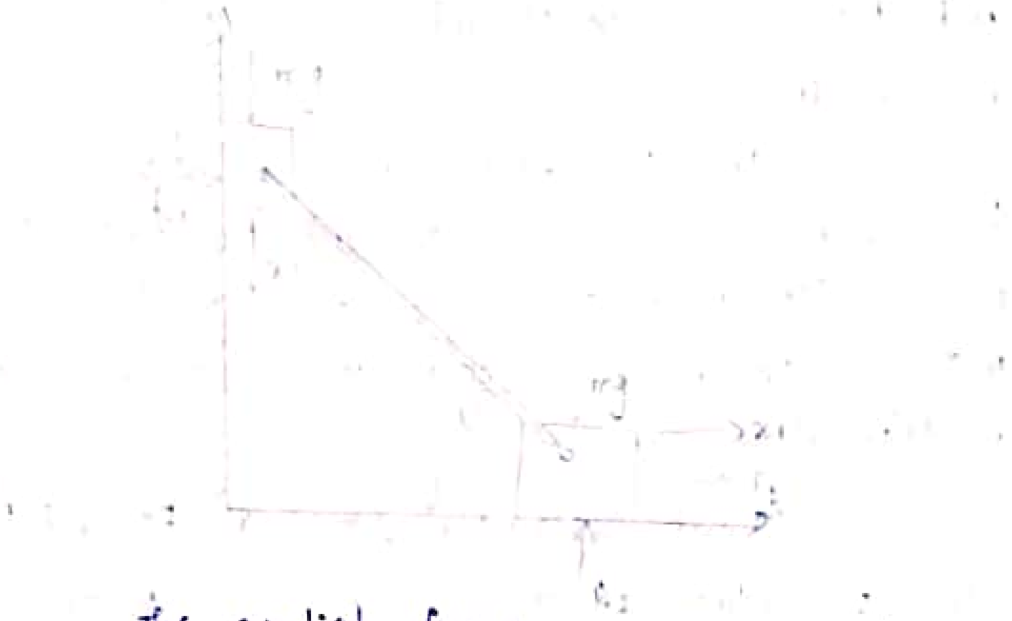
Thus if the system is not in equilibrium. It is always possible to find a set of virtual displacements consist with the constraints such that the virtual work of the applied forces is not zero.

The necessary & sufficient condition for the static equilibrium of an initially motionless scleronomic system which is subject to workless constraints is that zero virtual work be done by the applied forces in moving through an arbitrary virtual displacement consistent with the constraints.

Illustrative examples:-

Example 1:. Two frictionless blocks of equal mass m are connected by a massless as shown in fig using x_1 and x_2 as coordinates solve for the force F_1 if the system is in static equilibrium.

This example shows a scleronomic system with workless constraints. The external constraint forces are the wall and floor reactions R_1 & R_2 the internal constraint forces are the equal and opposite compressive forces in the rod. We note that the total virtual work of these constraint forces is zero.



The applied forces are the gravitional forces acting on the block the external force F_1. Hence, using the principle of virtual work, we seen the required condition for static equilibrium is that

$$mg\,\delta x_1 + F_1 \delta x_2 = 0 \quad \text{①}$$

But δx_1 & δx_2 are related by an eqn of constraint since the displacement components along the rod must be equal at the 2 ends, we have

$$\sin\theta\,\delta x_1 - \cos\theta\,\delta x_2 = 0 \quad (2)$$

Solving (1) & (2), $F_1 = -mg\cot\theta$.

This is the force required to keep the initially motionless system in static equilibrium

D'Alembert's Principle:-

Let us consider again a system of N particles and write the equation of motion for each particle in the form F = ma.

$$F_i + R_i - m_i \ddot{P}_i = 0 \quad (i = 1, 2, \ldots N) \ldots \rightarrow ①$$

where, as before, F_i is the applied force and R_i is the constraint force acting on the ith particle.

The term $-m_i \ddot{P}_i$ has the dimensions of force and is known as the inertial force acting on the ith particle, where m_i is the constant mass & $\ddot{P}_i$ is its acceleration relative to an inertial ~~force~~ frame.

It is customary to call F_i and R_i real or actual forces in contrast to the inertial forces. Hence eqn ① states that the sum of all the forces, real and inertial, acting on each particle of a system is zero. This result is known as d'Alembert's principal

The requirement that the sum of all forces at each particle be 0. ///y to the necessary condition for static equilibrium. Since, the principal of virtual work applies to system in static equilibrium.

Let us use the principal on the force

system, including the inertial forces. The total work done by the forces in an arbitrary virtual displacement is, $\delta W = \sum_{i=1}^{N} (F_i + R_i - m_i \ddot{r}_i) \cdot \delta r_i = 0$ → (1) (1.71)

If we now assume that the R_i are workless constraint forces & if we obtain choose the δr_i to be reversible virtual displacements consistent with the constraints

Then we obtain from (1.41) & (1.62)

that $\sum_{i=1}^{N} (F_i - m_i \ddot{r}_i) \cdot \delta r_i = 0.$ (1.62)

Eqn is the Lagrangian form of d'Alembert's principle & is one of most important eqn of classical dynamics.

Example: 1-3.

A particle of mass m is suspended by a massless wire of $r = a + b\cos\omega t$ $(a > b > 0)$ (1.64) to form a spherical pendulum. Find the eqns of motion.

Let us use the spherical coordinates θ and ϕ, where θ is measured from the upward vertical,

The angle ϕ is measured b/w a vertical reference plane passing through the support point O & the vertical plane containing the pendulum.

Accelaration of a particle: $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)e_r$

$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)e_\theta$

$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)e_\phi$

Displacement $\delta r = r\delta_\theta e_\theta + r\sin\theta\,\delta_\phi e_\phi$

Gravitational force $F = -mg\cos\theta\, e_r + mg\sin\theta\, e_\theta$

$1.65 \Rightarrow \sum_{i=1}^{N} (F_i - m_i\ddot{r}_i)\,\delta r_i = 0$

$\Rightarrow (-mg\cos\theta\, e_r + mg\sin\theta\, e_\theta) - m[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)e_r$

$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)e_\theta + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta$

$+ 2r\dot{\theta}\dot{\phi}\cos\theta)e_\phi]$

$\times (r\delta_\theta e_\theta + r\sin\theta\,\delta_\phi e_\phi) = 0$

$\Rightarrow \{-me_r[g\cos\theta + \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta] + me_\theta[(g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta]$

$- me_\phi[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta] \times (r\delta_\theta e_\theta + r\sin\theta\,\delta_\phi e_\phi) = 0$

$\Rightarrow mr[g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta]\,\delta_\theta$

$- mr\sin\theta\,[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta]\,\delta_\phi = 0.$

$\Rightarrow mr[g\sin\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta]\,\delta_\theta$

$- mr\sin\theta\,[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta]\,\delta_\phi = 0$

$\delta_\phi \Rightarrow 0, \quad \delta_\theta \Rightarrow 0$

$r = a + b\cos\omega t$

$\dot{r} = -b\sin\omega t\;\omega$

$\ddot{r} = -b\cos\omega t \cdot \omega^2$

we get,

$\Rightarrow mr[g\sin\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)]\,\delta_\theta = 0$

$\delta_\theta = 0$

$\Rightarrow m_r = 0$

$g\sin\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta$

$g\sin\theta = (a + b\cos\omega t)\ddot{\theta} + 2(-b\sin\omega t)\omega\dot{\theta} + (a + b\cos\omega t)\dot{\phi}^2\sin\theta\cos\theta$

$g\sin\theta = [a + b\cos\omega t]\ddot{\theta} - 2b\omega\sin\omega t\,\dot{\theta} - (a + b\cos\omega t)\dot{\phi}^2\sin\theta\cos\theta$

$\Rightarrow -mr\sin\theta\ [r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta\ 2r\dot{\theta}\dot{\phi}\cos\theta]\delta\phi = 0$

$\delta\phi = 0$

$\Rightarrow -mr\sin\theta = 0$

$\Rightarrow (a+b\cos\omega t)\ddot{\phi}\sin\theta + 2(-b\sin\omega t\cdot\omega)\dot{\phi}\sin\theta + 2(a+b\cos\omega t)\dot{\theta}\dot{\phi}\cos\theta = 0$

$\Rightarrow (a+b\cos\omega t)\ddot{\phi}\sin\theta - 2b\sin\omega t\sin\theta\dot{\phi} + 2(a+b\cos\omega t)\dot{\theta}\dot{\phi}\cos\theta = 0$

To find out velocity:

$\Rightarrow \quad x = r\cos\theta\sin\phi$

$$\frac{dx}{dt} = \frac{dr}{dt}\cdot\cos\theta\sin\phi + r(-\sin\theta\sin\phi)\dot{\theta} + r\cos\theta(\cos\phi)\dot{\phi}$$

$$\dot{x} = \dot{r}\cos\theta\sin\phi - r\sin\theta\sin\phi\dot{\theta} + r\cos\theta\cos\phi\dot{\phi}$$

$$y = r\sin\theta\sin\phi$$

$$\dot{y} = \frac{dy}{dt} = \dot{r}\sin\theta\sin\phi + r(\cos\theta)\dot{\theta}\sin\phi + r\sin\theta\cos\phi)\dot{\phi}$$

$$z = r\cos\phi$$

$$\frac{dz}{dt} = \dot{r}\cos\phi + r(-\sin\phi)\dot{\phi}$$

$$\dot{z} = \dot{r}\cos\phi - r\sin\phi\dot{\phi}$$

$(\dot{r}\cos\theta\sin\phi - r\sin\theta\sin\phi\dot{\theta} + r\cos\theta\cos\phi\dot{\phi})e_r$

$+ (\dot{r}\sin\theta\sin\phi + \dot{r}\cos\theta\sin\phi\dot{\theta} + r\sin\theta\cos\phi\dot{\phi})e_\theta +$

$(\dot{r}\cos\phi - r\sin\phi\dot{\phi})e_\phi$

for this system.

Generalized force:-

If a given aset of forces $F_1, F_2 \ldots \ldots F_{3N}$ is applied to a system of N particles, the virtual work of these forces is,

$$\delta_W = \delta W = \sum_{j=1}^{3N} F_j \cdot \delta x_j \ldots \to (1.70)$$

Now let us suppose that the 3N ordinary Cartesian coordinates $x_1, x_2, \ldots x_{3N}$ are related to the form of eqn (1.8).

If we differentiate this question and set $\delta t = 0$ (since we are considering a virtual displacement), we obtain the following:

$$\delta x_j = \sum_{i=1}^{n} \frac{\delta x_j}{\delta q_i} \delta q_i \quad (j = 1, 2, \ldots \ldots 3N) \quad (1.7$$

where the coefficients $\partial x_j / \partial q_i$ are, in general functions of the q's and t. Subtituting this expression for δx_j into eqn (1.70), we obtain

$$\delta W = \sum_{j=1}^{3N} \sum_{i=1}^{n} F_j \frac{\partial x_j}{\partial q_i} \delta q_i \quad \ldots (1.72)$$

29 let us define the generalised force Q_i by the eqn. $Q_i = \sum_{j=1}^{3n} F_j \frac{\partial x_j}{\partial q_i}$ $(i = 1, 2, \dots n)$ (1.73)

Subtituting from Eq (1-73) into Eq (1-72) &, changing the order of notation, we obtain $\delta W = \sum_{i=1}^{n} Q_i \delta q_i$ (1.74)

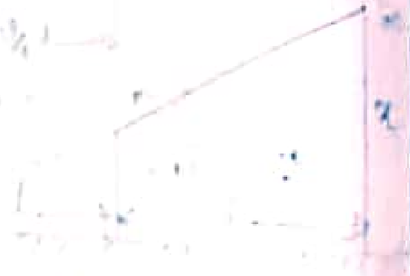
The necessary and sufficient condition for Static equilibrium is that all the Q's due to the applied forces be zero.

Example 1.4:

Three particles are connected by two rigid rods have joint b/w them to form the system shown in figure. A vertical force and a moment M are applied as shown. The configuration of the system given by the ordinary coordinates (x_1, x_2, x_3) or by the generalised coordinates (q_1, q_2, q_3) where

$$x_1 = q_1 + q_2 + q_3$$
$$x_2 = q_1 - q_3$$
$$x_3 = q_1 - q_2 + q_3$$ (1.75)

Find the generalized forces Q_1, Q_2 & Q_3. Assume small motions.



A system with an applied force and moment

First let us check whether the transformation eqn (1.75) yield independent q's. Evaluating the Jacobian of the transformation, we obtain.

$$\frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{vmatrix} 1 & 1 & 1/2 \\ 1 & 0 & -1 \\ 1 & -1 & 1/2 \end{vmatrix} = -3.$$

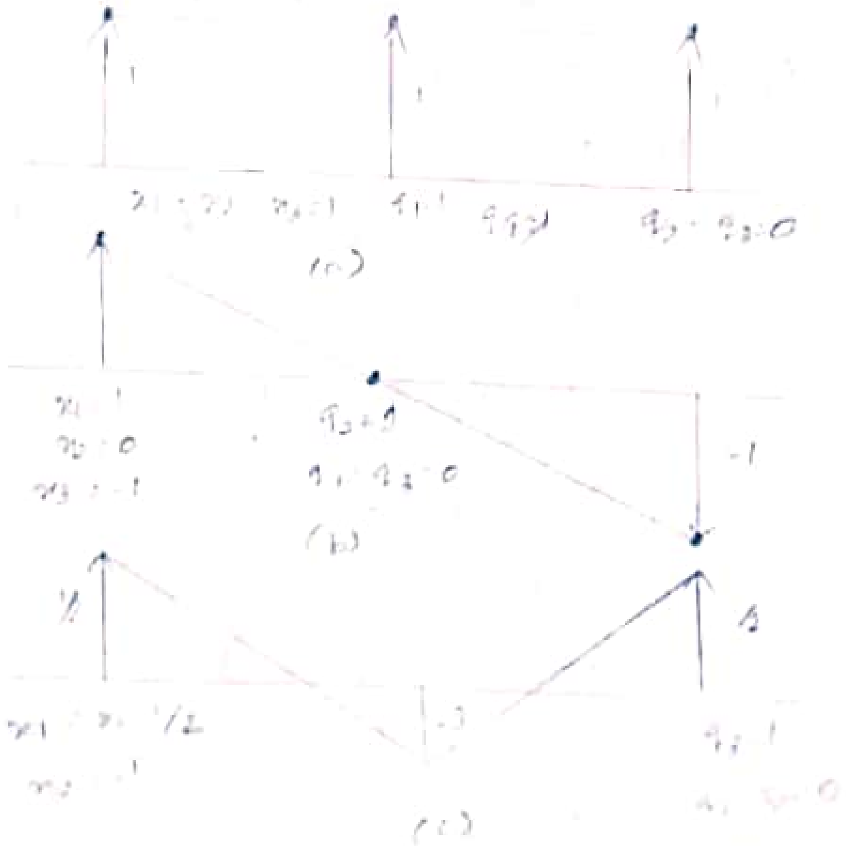
Since this determinant is non zero, we see that the q's are Independent Eq (1-15) can be solved, for the q's as functions of the x's, yielding.

$$q_1 = \tfrac{1}{3}(x_1 + x_2 + x_3)$$
$$q_2 = \tfrac{1}{2}(x_1 - x_3) \qquad (1\text{-}16)$$
$$q_3 = \tfrac{1}{3}(x_1 - 2x_2 + x_3)$$

Thus, for any set of x's, we obtain a corresponding unique set of q's

The generalized forces are obtain by considering small virtual displacements of each of the generalised coordinates, whose deflection forms are shown in figure 1-9.

An increase in q_1 represents a pure translation while q_2 is associated with a rotation about the center & q_3 denotes a deformation or bending of the system. If we consider the translation displacement



Deflection forms corresponding to the generalized coordinates of the pnt of application of the applied force F, & also the rotation of applied moment M, we obtain the following expression for the virtual work,

$$\delta W = F\delta q_1 + \left(\frac{3}{4}F - \frac{M}{l}\right)\delta q_2 + \left(\frac{1}{8}F + \frac{3M}{2l}\right)\delta q_3$$

Comparing Eqs (1.74) & (1.77) we find the generalised forces $Q_1 = F_1\frac{\partial x_1}{\partial q_1} + F_2\frac{\partial x_2}{\partial q_1} + F_3\frac{\partial x_3}{\partial q_1}$

$= (3/4)(1) + (1/4 \; [illegible])(1) + [illegible] = F$

$$Q_1 = F$$

$$Q_2 = 3/4 F - \frac{M}{l}$$

$$Q_3 = 1/8 F + \frac{3M}{2l}$$

Another approach is to obtain the Q's by using Eqn (1.73) directly note that the force F can be replaced by a force 3F/4 at x_1 & a force at x_2.

Also, the moment M can be replaced by equal & opposite force magnitude M/l acting in the directions of $-x_2$ & x_3, with these substitutions of equipollent force system, we find that

$$F_1 = 3/4 F$$

$$F_2 = 1/4 F - \frac{M}{l}$$

$$F_3 = \frac{M}{l}$$

Then using Eq (1.73) & remembering that the partial derivatives have been calculated previously in evaluating the Jacobian, we obtain expressions the Q's which are identical with these given in Eqn (1.78)

1.2 1.6) ENERGY AND MOMENTUM:

2M Potential Energy:

Let us consider a single particle whose position is given by the cartesian coordinates (x, y, z). Suppose that the total force F acting on the particle has the components,

$$F_x = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y} \qquad (1.50)$$
$$F_z = -\frac{\partial V}{\partial z}$$

Where the potential energy function $V(x, y, z)$ is a single valued function of position only. that is not a function of velocity or time. A force F meeting these conditions is known as a conservative force.

Now, let us consider the work dW done by the force F as it moves through an infinitesimal displacement dr. We have

$$dW = F \cdot dr = F_x dx + F_y dy + F_z \cdot dz. \qquad (1.51)$$

Subtituting from Eq (1.50), We obtain

$$dW = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz = -dV(x, y, z) \qquad (1.52)$$

We see that dW is an exact differential. If we consider next the work W then by the force F as the particle moves over a certain path b/w points A & B, we find that

$$W = \int_A^B F \cdot dr = -\int_A^B dV = V_A - V_B$$

Hence, the potential energy is a function of position only. We conclude that the work done on the particle depends upon the initial & final positions, but is independent of the specific path connecting these points. A further conclusion arises if A & B coincide, namely that the work done in moving around any closed path is zero.

Thus we have $\oint F \cdot dr = 0$ for any conservative force F.

Work and kinetic Energy:-

Suppose we define the kinetic energy T of of a particle mass m by

$$T = \frac{1}{2} m v^2$$

where v is the velocity of the particle relative to an inertial reference frame.

let us consider the line integral of eq (1-83) which gives the work done on the particle ~~on~~ by on the particle by the total force F as the particle moves over a certain path from A to B. In accordance with Newton's law of motion, we can replace F by $m\ddot{r}$ & obtain

$$W = m\int_A^B \ddot{r} \cdot dr = \frac{1}{2} m \int_A^B \frac{d}{dt}(\dot{r} \cdot \dot{r}) \cdot dt = \frac{1}{2} m \int_A^B d(v^2)$$

where each integral is evaluated over the same path.

Then using the definition of kinetic energy, we obtain

$$W = \frac{1}{2} m (v_B^2 - v_A^2) = T_B - T_A$$

Eqn (1-86) is the principle of work & kinetic

The increase in the kinetic energy of a particle as moves from the one arbitrary point to another is equal to the work done by the forces acting on the particle during the given interval.

Conservation of Energy :-

If the only forces acting on a given particle are conservative, then eqn (1-83) applies, & with the aid of eqn (1-86), we obtain,

$$V_A - V_B = T_B - T_A$$

or,

$$V_A + T_A = V_B + T_B = E \quad \ldots\ldots > (1\text{-}87)$$

Since, the points A & B are arbitrary, we conclude that the total mechanical energy E remains constant during the motion of particle. This is the principle of conservation of energy.

Equilibrium & Stability :-

consider a system of N particles whose applied forces are conservative and are obtained from a potential energy function of the form $V(x_1, x_2, \ldots x_{3N})$.

From eqns ⇒ (1-35)

$$(1\text{-}88) \Rightarrow F_j = \frac{\partial V}{\partial x_j}$$

we see that, the virtual work of these applied force is, $\delta W = -\sum_{j=1}^{3N} \frac{\partial V}{\partial x_j} \delta x_j = -\delta V$

which we note is linear in the δx's & is ∴ the 1st Variation of the potential energy. Then using the principle of virtual work, we find that the necessary & sufficient condition for the static equilibrium of the system is that

$$\delta V = 0 \quad \cdots > (1\text{-}91)$$

for every virtual displacement consistent with the constraints.

If the potential energy is expressed in terms of the generalized coordinates $q_1, q_2, \ldots q_n$, then $\delta V = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} \delta q_i \quad \cdots > (1\text{-}92)$

For a holonomic system having independent q's, the condition that $\delta V = 0$ for an arbitrary virtual displacement requires that the coefficients be zero at the equilibrium configuration ; that is, $\frac{\partial V}{\partial q_i} = 0 \quad (i = 1, 2, \ldots n) \quad \cdots > (1\text{-}9$

But these conditions imply that the potential energy is at a stationary value. ∴ we conclude that an equilibrium configuration of a conservative holonomic system with workless fixed constraints must occur at a position where the potential energy has a stationary value.

Next, let us consider the ques of the stability of this system at a position of static equilibrium. If we expand the potential energy in a Taylor series about a reference value V_0. we obtain

$$V = V_0 + \left(\frac{\partial V}{\partial q_1}\right)_0 \delta q_1 + \left(\frac{\partial V}{\delta q_2}\right)_0 \delta q_2 + \cdots + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 (\delta q_1)^2 + \cdots$$

$$+ \left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 \delta q_1 \delta q_2 + \cdots$$

$\cdots \rightarrow (1.94)$

where a zero subscript on a function implies that it is to be evaluated at the reference values of the q's. The δq's represent infinitesimal changes from this reference configuration.

Now assume that we choose an equilibrium configuration as the ~~therefore~~ reference position. From eqn, $\frac{\partial V}{\partial q_i} = 0 \ (i = 1, 2, \ldots n)$, we set all the coefficients $(\partial V/\partial q_i)_0$ are zero. Therefore the potential energy expression contains no terms of 1st order in the δq's. Assuming small δq's, we can write

$$\Delta V = V - V_0 = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2}\right)_0 (\delta q_1)^2 + \left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_0 \partial q_1 \partial q_2 + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_2^2}\right)_0 (\partial q_2)^2 + \cdots \quad \dashrightarrow (1\text{-}95)$$

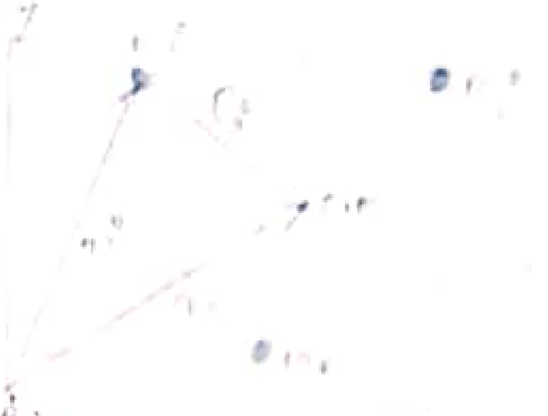
where ΔV is is the change in the potential energy from its value at equilibrium. Here we use ΔV rather than δV to indicate that terms of higher order than δq are included.

If $\Delta V > 0$ for every possible virtual displacement having at least one of δq's nonzero, then the reference position is one of minimum potential energy corresponding to stable equilibrium. On other hand, if a virtual displacement can be found such that $\Delta V < 0$, the equilibrium position is unstable. A 3rd possibility is that $\Delta V \geq 0$ for all the virtual displacements, but $\Delta V = 0$ for some virutal displacements in which the δq's are not all zero. This is the case of netural stability.

Kinetic Energy of a system:-

Consider a system of N particles and let r_i be the position vector of the ith particle relative to a point O fixed in an inertial frame (Fig (1-101)). With respect to this inertial frame, which we shall consider to be fixed, the total kinetic energy of the system is the sum of the individual kinetic energies of the particles, namely,

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2 \qquad \rightarrow (1\text{-}96)$$



To find T

From (1-97)

$\dot{r}_i^2 = (\dot{r}_c + \dot{\rho}_i)^2$

$= \dot{r}_c^2 + \dot{\rho}_i^2 + 2\dot{r}_c\dot{\rho}_i \rightarrow (1)$

(1) Sub in (1-96)

$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{r}_c^2 + \dot{\rho}_i^2 + 2\dot{r}_c\dot{\rho}_i)$

$= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{r}_c^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2 + \frac{1}{2}\sum_{i=1}^{N} m_i \cancel{2}\dot{r}_c\dot{\rho}_i$

$= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{r}_c^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2 + \sum_{i=1}^{N} m_i\dot{r}_c\dot{\rho}_i$

$\sum_{i=1}^{N} m_i\rho_i = 0$,

$T = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{r}_c^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2$

Where we use the notation that,

$$\dot{r}_i^2 = \dot{r}_i \cdot \dot{r}_i = v_i^2$$

From, Fig (1-10) we see that, $r_i = r_c + \rho_i \qquad \rightarrow (1\text{-}97)$

Hence, we obtain, $T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{r}_c^2 + 2\dot{r}_c \cdot \dot{\rho}_i + \dot{\rho}_i^2) \rightarrow (1\text{-}98)$

But $\sum_{i=1}^{N} m_i\rho_i = 0 \quad \dots\dots\rightarrow (1\text{-}99)$

Since ρ_i is measured from the center of mass. Also, $\dot{r}_c$ does not enter into the summation and can be factored out. Therefore, using the notation that m is the total mass of the system, we see that eqn (1-98) reduces to,

$$T = \frac{1}{2}m\dot{r}_c^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2 \quad \dots \rightarrow (1\text{-}100)$$

The last term of eqn (1-100) can be considered as the kinetic energy of the system relative to its mass center; that is, it is the kinetic energy of the system as viewed by an observer translating with the center of mass but not rotating.

Now we can state könig's theorem:-

(1) The kinetic

The total kinetic energy of a system is equal to the sum of

(1) The kinetic energy due to a particle having a mass equal to the sum of (1) the kinetic energy due to a particle having a mass equal to the total mass of the system and moving with the velocity of the center of mass ch

(2) The kinetic energy due to the motion of the system relative to its center of mass.

The limit, each element of the body can be considered as a particle of infinitesimal mass.

and eqn (1-100) can be modified as follows:

$$T = \frac{1}{2} m \dot{r}_c^2 + \frac{1}{2} \int_V \rho \rho'^2 \, dv \quad \ldots\ldots > (1-101)$$

Where ρ is the position vector. of the volume element relative to mass center.

Translational kinetic Energy:-

The first term on the right side is called T.k.E.

Rotational kinetic Energy:-

The Second term is R.k.E.

ROTATONAL KINETIC ENERGY:-

$$\dot{\rho} = \omega \times \rho \quad \to ①$$

$$\dot{\rho}^2 = \dot{\rho} \cdot \dot{\rho}$$

$$= \dot{\rho} \cdot (\omega \times \rho) \quad \text{using } ①$$

$$= (\dot{\rho} \cdot \omega) \times \rho \quad \dashrightarrow \text{Associative law } (a+b)+c = a+(b+c)$$

$$\therefore \dot{\rho} \cdot (\omega \cdot \rho) = (\omega \cdot \rho) \times \dot{\rho} \quad \dots \to \text{Associative property } (a \cdot (b \times c) = (a \times b) \cdot c$$

Here the rotational kinetic energy of the rigid body can be written in the form using (1-101).

$$T_{rot} = \frac{1}{2} \int_V \rho \, \dot{\rho}^2 \, dV$$

$$= \frac{1}{2} \int_V \rho (\omega \cdot \rho \times \dot{\rho}) \, dV$$

$$= \frac{1}{2} \omega \int_V \rho (\rho \times \dot{\rho}) \, dV \quad \dashrightarrow \text{using } ①$$

$$= \frac{1}{2} \omega \int_V \rho (\rho \times (\omega \times \rho)) \, dV \quad \dots \to ②$$

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \qquad \left[\because a = \rho,\ b = \omega,\ c = \rho \right]$$

$$\rho \times (\omega \times \rho) = (\rho \cdot \rho) \omega - (\rho \cdot \omega) \rho$$

$$= \rho^2 \omega - (\rho \cdot \omega) \rho \quad \dots \to ③$$

(3) in ②

$$T_{rot} = \frac{1}{2} \omega \int_V \rho \left[\rho^2 \omega - (\rho \cdot \omega) \rho \right] dV \quad \dots \to ④$$

$$\rho = x\vec{i} + y\vec{j} + z\vec{k} \quad \dots \to ⑤$$

$$\omega = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} \quad \to ⑥$$

⑤ & ⑥ in ④

$= \frac{1}{2}(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \cdot x\hat{i} + y\hat{j} + z\hat{k} \; [\, \rho (x^2 + y^2 + z^2)$

$(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) - (x\omega_x + y\omega_y + z\omega_z)$

$(x\hat{i} + y\hat{j} + z\hat{k})\,]\, dv$

$= \frac{1}{2}[\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}] \int_V \rho\, [\,(x^2\omega_x \hat{i} + y^2\omega_y \hat{j} + z^2\omega_z \hat{k})$

$- (x^2\omega_x \hat{i} + xy\omega_y \hat{i} + xz\omega_z \hat{i})\,]\, dv$

$= \frac{1}{2}[\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}] \int_V \rho\, [\,(y^2\omega_x + z^2\omega_x)\hat{i}$

$- (xy\omega_y + xz\omega_z)\hat{i}\,]\, dv$

~~$= \frac{1}{2}[\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}] \int_V \rho\, [\,(y^2$~~

$= \frac{1}{2}\omega \int_V \rho\, [\,(y^2\omega_x + z^2\omega_x) - (xy\omega_y + xz\omega_z)]\hat{i}\, dv$

$= \frac{1}{2}\omega \int_V \rho\, [\,(y^2 + z^2)\omega_x - (xy\omega_y + xz\omega_z)]\hat{i} + \; dv.$

which reduces to

$T_{rot} = \frac{1}{2} I_{xx}\omega_x^2 + \frac{1}{2} I_{yy}\omega_y^2 + \frac{1}{2} I_{zz}\omega_z^2$

$+ I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z$

$T_{rot} = \frac{1}{2}\sum_i \sum_j I_{ij}\,\omega_i\omega_j$

where moment of inertia

$I_{xx} = \int_V \rho (y^2 + z^2)\, dV$

$I_{yy} = \int_V \rho (x^2 + z^2)\, dV$

$I_{zz} = \int_V \rho (x^2 + y^2)\, dV$

Product of inertia

$I_{xy} = I_{yx} = -\int_V \rho\, xy\, dV$

$I_{xz} = I_{zx} = -\int_V \rho\, xz\, dV$

$I_{yz} = I_{zy} = -\int_V \rho\, yz\, dV$

Rotational kinetic energy

$$T_{rot} = \frac{1}{2} \omega^T I \omega$$

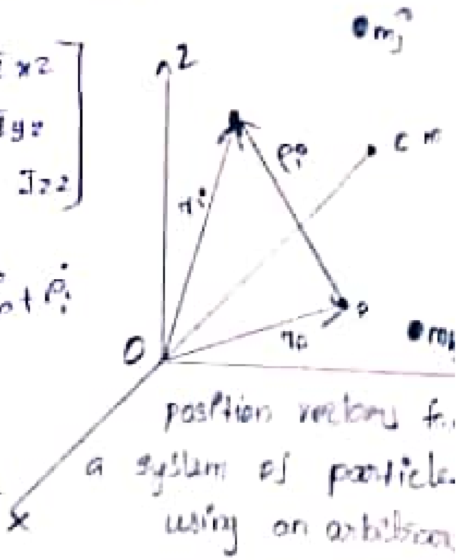
$$T_{rot} = \frac{1}{2} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\omega_x + I_{yx}\omega_y + I_{zx}\omega_z & \omega_x I_{xy} + \omega_y I_{yy} + \omega_z I_{zy} & \omega_x I_{xz} + \omega_y I_{yz} + \omega_z I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\omega_x^2 + I_{xy}\omega_x^2 + I_{xz}\omega_x^2 \\ I_{yx}\omega_y^2 + I_{yy}\omega_y^2 + I_{yz}\omega_y^2 \\ I_{zx}\omega_z^2 + I_{zy}\omega_z^2 + I_{zz}\omega_z^2 \end{bmatrix}$$

$$= \begin{bmatrix} \omega_x^2 \\ \omega_y^2 \\ \omega_z^2 \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$



position vectors for a system of particles using an arbitrary reference point.

$$r_i = r_p + \rho_i \Rightarrow \dot{r}_i = \dot{r}_p + \dot{\rho}_i$$

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_p^2 + \dot{\rho}_i)^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{r}_p^2 + \dot{\rho}_i^2 + 2\dot{r}_p\dot{\rho}_i)$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_p^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\rho}_i^2 + \sum_{i=1}^{N} m_i \dot{r}_p \dot{\rho}_i \quad \text{--} ⑩$$

Total mass is M.

But center of mass ρ_c relative to P

$$\rho_c = \frac{1}{m} \sum_{i=1}^{N} m_i \rho_i$$

$$m\dot{\rho}_c = \sum_{i=1}^{N} m_i \dot{\rho}_i \longrightarrow ⑪$$

⑪ in ⑨,

$$T = \frac{1}{2} m \dot{r}_p^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\rho}_i^2 + \dot{r}_p m \dot{\rho}_c \cdots \rightarrow ⑫$$

$$T_{rot} = \frac{1}{2} m \dot{\rho}_c^2 + \frac{1}{2} \sum_i \sum_j I_{ij}^{o} \omega_i^{o} \omega_j^{o} \cdots ⑬ \quad [\because$$

$$T = \frac{1}{2} m \dot{r}_p^2 + \frac{1}{2} m \dot{\rho}_c^2 + \frac{1}{2} \sum_i \sum_j I_{ij}^{o} \omega_i^{o} \omega_j^{o} + \dot{r}_p m \dot{\rho}_c$$

$$T_{rot} = \frac{1}{2} \sum_i \sum_j I_{ij}^{o} \omega_i \omega_j$$

But the center of location ρ_c relative to p is,

$$\rho_c = \frac{1}{m}\sum_{i=1}^{N} m_i\rho_i \quad \ldots\ldots \to (1\cdot118)$$

Hence, $T = \frac{1}{2} m \dot{r}_p^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\rho}_i^2 + \dot{r}_p \cdot m\dot{\rho}_c \ldots \to (1\cdot119)$

Thus we find the total kinetic energy

is the sum of 3 parts:

(1) the kinetic energy due to a particle having a mass m & moving with the reference pnt P,

(2) The k.E of the system due to its motion relative to p.

(3) The scalar product of the velocity of the reference point & the linear momentum of the system relative to the reference point.

The kinetic energy due to its motion relative to p is, $T_{rot} = \frac{1}{2} m\dot{\rho}_c^2 + \frac{1}{2}\sum_i\sum_j I_{ij}\omega_i\omega_j$

$\to (1\text{-}120)$

Where the moments and products of inertia are taken with respect to the mass center.

Then the total k.E is

$$T = \frac{1}{2} m\dot{r}_p^2 + \frac{1}{2} m\dot{\rho}_c^2 + \frac{1}{2}\sum_i\sum_j I_{ij}\omega_i\omega_j + \dot{r}_p\, m\dot{\rho}_c \to (1\cdot121)$$

THE k.E relative motion can be written in the form.

$$T_{rot} = \frac{1}{2}\sum_i\sum_j I_{ij}\omega_i\omega_j \ldots \to (1\text{-}122)$$

Angular Momentum:-

The total angular momentum H with respect to a fixed point O is,

$$H = \sum_{i=1}^{N} r_i \times m_i \dot{r}_i \quad \text{-----}> (1-123)$$

That is, it is the sum of the moments about O of the individual linear momenta of the particles, assuming that each vector $m_i \dot{r}_i$ has a line of action passing through the corresponding particle.

Now, let us substitute the expression for r_i from eqn (1-97) into eqn (1-123),

Obtaing.

$$H = \sum_{i=1}^{N} (r_c + \rho_i) \times m_i (\dot{r}_c + \dot{\rho}_i)$$

$$= r_c \times m \dot{r}_c + r_c \times \sum_{i=1}^{N} m_i \dot{\rho}_i + \sum_{i=1}^{N} m_i \rho_i \times \dot{r}_c + \sum_{i=1}^{N} \rho_i \times m_i \dot{\rho}_i$$

$$\text{----}> (1-124)$$

using Eq (1-99), the two middle terms are zero & the result can be simplified to,

$$H = r_c \times m \dot{r}_c + \sum_{i=1}^{N} \rho_i \times m_i \dot{\rho}_i \quad \text{----}> (1-125)$$

Where the last term on the right is the angular momentum H_c about the center of mass, that is,

$$H_c = \sum_{i=1}^{N} \rho_i \times m_i \dot{\rho}_i \quad \text{-----}> (1-126)$$

To summarize, the angular momentum of a system of particles of total mass m about a fixed point O is equal to the angular momentum about O of a single particle of mass m which is moving with the center of mass plus the angular momentum of the System about the center of mass.

Generalised Momentum:-

consider a system whose configuration is described by n generalized coordinates. let us define the Lagrangian function $L(q, \dot{q}, t)$ as follows:

$$L = T - V \quad \dashrightarrow (1\text{-}136)$$

The generalised momentum p_i associated with the generalised coordinates q_i is defined by the equation, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ ----> (1-137)

It is general, a function of the q's, $\dot{q}$'s & t. However, that the Lagrangian function is, at most, quadratic in the $\dot{q}$'s. Therefore p_i is a linear function of the $\dot{q}$'s.

In the usual case the potential energy V is not velocity-dependent; hence $\partial V / \partial \dot{q}_i$ equals zero and, $p_i = \frac{\partial T}{\partial \dot{q}_i}$ ----> (1-138)

If the potential energy is of the form $V(q, t)$.

As an example, consider a free particle of mass m whose position is given by the Cartesian coordinates (x, y, z).

The K.E is $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

& using Eq (1-138), we obtain

$$p_x = m\dot{x}$$

Thus, p_x is just the x component of the linear momentum.

In a similar fashion, if the position of the particle is given by the spherical coordinates (r, θ, ϕ), as in fig 1-7, the K.E is

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$$

Using eqn (1-138), we have

$$p_r = m\dot{r}$$

$$p_\theta = mr^2\dot{\theta}$$

$$p_\phi = mr^2\dot{\phi}\sin^2\theta.$$

We see that p_r is the linear momentum component in the radial direction, while p_θ is the horizontal component of the angular momentum That is, it is the an " " about horizontal axis associated with an angular velocity $\dot{\theta}$. ||ly since $\dot{\phi}$ is vertical, p_ϕ is the vertical component of angular momentum.

Example: 1-5

consider once again the system of particles and rods shown in (1-8) fig. Find expressions for the kinetic energy and the generalised momenta.

The total K.E can be obtained easily in terms of x's, namely

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) \quad \dots \rightarrow (1\text{-}139)$$

In order to express T as a function of the $\dot{q}$'s, we first differentiate Eqn (1-75) & obtain

$$\dot{x}_1 = \dot{q}_1 + \dot{q}_2 + 1/2\,\dot{q}_3$$
$$\dot{x}_2 = \dot{q}_1 - \dot{q}_3 \quad \dots \rightarrow (1\text{-}140)$$
$$\dot{x}_3 = \dot{q}_1 - \dot{q}_2 + 1/2\,\dot{q}_3$$

Substituting from eqn (1-140) into (1-139) & simplifying.

$$T = \frac{m}{2}\left[(\dot{q}_1 + \dot{q}_2 + 1/2\,\dot{q}_3)^2 + (\dot{q}_1 - \dot{q}_3)^2 + (\dot{q}_1 - \dot{q}_2 + 1/2\,\dot{q}_3)^2\right]$$

$$= m/2\left[(\dot{q}_1^2 + \dot{q}_2^2 + \frac{1}{4}\dot{q}_3^2 + 2\dot{q}_1\dot{q}_2 + 2\dot{q}_2\cdot 1/2\,\dot{q}_3 + 2\cdot\frac{1}{2}\dot{q}_3\dot{q}_1) + (\dot{q}_1^2 + \dot{q}_3^2 - 2\dot{q}_1\dot{q}_2) + \left[\right.\right.$$

$$= (\dot{q}_1^2 + \dot{q}_2^2 + 1/4\,\dot{q}_3^2 - 2\dot{q}_1\dot{q}_2 - 2\dot{q}_2\,1/2\,\dot{q}_3 + 2\,1/2\,\dot{q}_3\dot{q}_1)\left.\right]$$

using,
$$(a-b)^2 = a^2 + b^2 - 2ab$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca.$$

$$\therefore T = \frac{m}{2}\left(3\dot{q}_1^2 + 2\dot{q}_2^2 + 3/2\,\dot{q}_3^2\right) \quad \dots \rightarrow (1\text{-}141)$$

The momenta are obtained with the aid of eqn (1-138); the result being ,

$$P_1 = \frac{\partial T}{\partial \dot{q}_1} = 3m\dot{q}_1$$

$$P_2 = \frac{\partial T}{\partial \dot{q}_2} = 2m\dot{q}_2 \qquad \ldots\ldots) \; (1-142)$$

$$P_3 = \frac{\partial T}{\partial \dot{q}_3} = 3/_2 \, m\dot{q}_3$$

Inertia coefficients:-

The coefficients of the $\dot{q}$'s expressions for the generalised momenta are known as inertia coefficients.

The " " corresponding to q_3 is

$$m\left[(\tfrac{1}{2})^2 + (-1)^2 + (1/_2)^2 \right] = 3/_2 \, m .$$

——— × ———

UNIT-2

LAGRANGE'S EQUATIONS :-

Sec 2.1 : Derivation of Lagrange's Equations :-

Kinetic energy :-

Let us consider a system of N particles whose positions relative to an inertial frame are given by the Cartesian coordinates $x_1, x_2, \ldots, x_{3N}$. The total kinetic energy of the system is from Eq. (1-96) or, equivalently, from

$$T = \frac{1}{2} \sum_{k=1}^{3N} m_k \dot{x}_k^2 \quad \ldots\ldots\ldots \to (2.1)$$

Where $m_1 = m_2 = m_3$ is the mass of the first particle, and (x_1, x_2, x_3) specifies its position. ||ly $m_4 = m_5 = m_6$ is the mass of the second particle and so forth.

Now let us express the kinetic energy in terms of the generalized coordinates $q_1, q_2, \ldots q_n$. Using transformation equations given the x's as functions of the q's and time,

we have $x_k = x_k(q, t)$, $(k = 1, \ldots, 3N) \ldots \to (2.2)$

where we assume that these functions are twice differentiable with respect to q's and t. We find that

$$(\dot{x}_k \cdot \dot{x}_k(q, \dot{q}, t) = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \dot{q}_i + \frac{\partial x_k}{\partial t} \ldots \to ③$$

Note that $\dot{x}_k$ is linear in the $\dot{q}$'s, and that $\partial x_k / \partial q_i$ and $\partial x_k / \partial t$ are functions of the q's & t.

Note that the coefficients m_{ij} & a_i, as well as T_0, are funcs of the q's and t.

Assuming that $m_k > 0 \ \forall \ k$, we see from (3) that the total kinetic energy T is a positive definite quadratic function of the $\dot{x}$'s. In other words, T is zero only if all the $\dot{x}$'s are zero; if any of the $\dot{x}$'s is nonzero, the kinetic energy is positive.

Now suppose we express T as a function of the q's, $\dot{q}$'s and t. It is still true for any real system that the kinetic energy is zero only if the system is motionless; otherwise it is positive. Since the q's are usually choosen such that the $\dot{q}$'s are all zero iff the system is motionless, T is usually a positive definite function of the $\dot{q}$'s. But this is not always the case, particularly if there are moving constraints.

Let us consider T_2 in more detail. We see from eqns (4), (6) & (7) that T_2 is the total kinetic energy for the case in which all the partial derivatives $\partial x_k/\partial t$ are zero, that is, for a system in which any moving constraints or moving reference frames are held fixed. Then, assuming that one or more non zero $\dot{q}$'s implies the motion of one or more particles of the system, and vice versa, we conclude that T_2 must be positive definite quadratic function of $\dot{q}$'s.

④ The positive definite nature of T_2 restricts the possible values of the inertia coefficients m_{ij}. If we consider the symmetric $n \times n$ generalized inertia matrix m, the necessary and sufficient conditions that T_2 be +ve definite are that

$$m_{11} > 0, \quad \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0 \ldots, \quad \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & & \vdots \\ \vdots & & & \vdots \\ m_{n1} & \cdots & \cdots & m_{nn} \end{vmatrix} > 0 \ldots (11)$$

This is equivalent to the requirment that the determinant of the matrix and the principal minors be positive. One of the consequences is that all the inertia coefficients m_{ii} along the main daigonal must be positive, as can be seen directely from eqn (7)

From eqns (9) & (10), we observe that T_1 & T_0 are nonzero only for the case of rheonomic systems. It follows, then, the kinetic energy T of a scleronomic system is a homogeneous quadratic function of the $\dot{q}$'s. In this case we see from eqn (7) that the inertia coefficients m_{ij} are funcs of the q's but not of time.

Since T_1 is linear in the $\dot{q}$'s, it is apparent that it can be +ve or -ve. On the other hand, T_0 is +ve or zero.

Lagrange's Equations :-

As a starting point for the derivation of Lagrange's eqns, let us consider a system of N particles and write d' Alembert's principle Eqn (1-68), in the form

$$\sum_{k=1}^{3N} (F_k - m_k \ddot{x}_k)\, \delta x_k = 0 \quad \cdots\cdots \rightarrow (12)$$

where F_k is the applied force component associated with x_k, that is, it includes all the real forces acting on the given particle except the workless constraint forces.

Using eqn (2), the virtual displacement δx_k can be expressed in terms of the δq's as follows :

$$\delta x_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\, \delta q_i \quad \cdots\cdots \rightarrow (13)$$

$x_k = x(q,t)$

Hence, from eqns (12) & (13), we obtain

$$\sum_{k=1}^{3N} \sum_{i=1}^{n} \left(F_k \frac{\partial x_k}{\partial q_i} - m_k \ddot{x}_k \frac{\partial x_k}{\partial q_i} \right) \delta q_i = 0 \quad \cdots \rightarrow (14)$$

Now, it is evident from eqn (3) that

$$\frac{\partial \dot{x}_k}{\partial \dot{q}_i} = \frac{\partial x_k}{\partial q_i} \quad \cdots\cdots \rightarrow (15)$$

Also, nothing but that the order of differentiation can be changed, we obtain

$$\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_i}\right) = \sum_{j=1}^{n} \frac{\partial^2 x_k}{\partial q_j \partial q_i} \dot{q}_j + \frac{\partial^2 x_k}{\partial t \partial q_i} = \frac{\partial \dot{x}_k}{\partial q_i} .$$ (1

Then we can write generalised momentum p_i in the form

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial \dot{x}_k}{\partial \dot{q}_i} \quad \dots\to (17)$$

and using eqns (15) & (16), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{k=1}^{3N} m_k \cdot \ddot{x}_k \frac{\partial x_k}{\partial q_i} + \sum_{k=1}^{3N} m_k \cdot \dot{x}_k \frac{\partial \dot{x}_k}{\partial q_i} \quad \dots\to (8)$$

But,

$$\frac{\partial T}{\partial q_i} = \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial \dot{x}_k}{\partial q_i} \quad \dots\to (19)$$

Hence from eqns (18) & (19), we have

$$\sum_{k=1}^{3N} m_k \ddot{x}_k \frac{\partial x_k}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \quad \dots\to (20)$$

The generalized force Q_i was previously defined to be

$$Q_i = \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial q_i} \quad \dots\to (21)$$

Then, using eqns (14), (20), & (21), we obtain

$$\sum_{i=1}^{n} \left[Q_i - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} \right] \delta q_i = 0 \quad \dots\to (22)$$

which is essentially a restatement of the Lagrangian form of d'Alembert's principle in terms of generalized coordinates

Thus far we have not made any restrictions on the δq's, except that they must confirm to the instantaneous constraints. This restriction was necessary in order to neglect the virtual work of the constraint forces. Now, let us make the additional assumptions that the system is holonomic and its configuration is described by a set of independent generalized coordinates. If the δq's are independent, then the coefficient of each δq_i in eqn (22) must be zero. Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i=1,2,\dots n) \qquad (23)$$

These n equations are known as Lagrange's equations and written here in one of their principal forms. As we shall demonstrate, they consist of n second-order nonlinear differential equations.

In attempting to gain some physical insight into the meaning of Lagrange's equations, let us note first that, since the δq's are independent, all the generalized constraint forces are zero. Then we see that generalized applied force Q_i is equal and opposite to a generalized inertia force given by

$$\frac{\partial T}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$$

Where $-\frac{d}{dt}(\partial T/\partial \dot{q}_i)$ represents the negative rate of change of the generalized momentum.

We have been considering a holonomic system whose configuration is given by a set of independent generalized coordinates. Now let us make the additional assumption that all generalized forces are derivable from a potential function $V(q,t)$ as follows:

$$Q_i = -\frac{\partial V}{\partial q_i} \quad \cdots\cdots\rightarrow (24)$$

Then Lagrange's equations can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \quad (i = 1, 2, \ldots n) \ldots \rightarrow (25)$$

Let us recall from eqn (1-136) that the Lagrangian function $L(q, \dot{q}, t)$ is,

$$L = T - V \quad \ldots \rightarrow (26)$$

Then, since V is not a function of the $\dot{q}$'s, we find from eqns (25) & (26), that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2, \ldots n) \ldots \rightarrow (27)$$

This is the standard form of Lagrange's equation for a holonomic system.

Here, we have derived a method of obtaining a complete set of differential equations of motion for the system by operating on a single scalar function $L(q, \dot{q}, t)$. Hence, it can be seen that the necessary Lagrangian function must contain all the necessary information concerning its possible motions. Furthermore, the form of eqn (27) does not depend upon which particular set of generalized coordinates is chosen to describe the system, provided that the q's are independent.

Another form of Lagrange's equations can be written for systems in which the generalized forces are not wholly derivable from a potential function.

Let,

$$Q_i = -\frac{\partial V}{\partial q_i} + Q_i' \quad \ldots \rightarrow (28)$$

Thus we obtain from eqns (23), (26) & (28) that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \quad (i = 1, 2, \ldots n) \rightarrow (29)$$

where the Q'_i are those ~~force~~ generalized forces are not derivable from a potential function. Examples of typical Q' forces are friction forces and time-varying forcing functions.

FORM OF THE EQUATIONS OF MOTION :-

Now let us consider more explicitly the form of the equations of motion which result from the application of Lagrange's equations in the standard form given by eqn (2

we note first that the generalized momentum is linear in the $\dot{q}$'s. Referring to eqns (26) & (8), we have,

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij}\dot{q}_j + a_i \qquad (30)$$

where m_{ij} and a_i are functions of the q's & t

Hence it follows that the equations of motion are linear in the $\ddot{q}$'s, since all the terms containing $\ddot{q}$'s arise from differentiating eqn (3 with respect to time. performing this differentia-

we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} \dot{m}_{ij}\dot{q}_j + \dot{a}_i \quad (3$$

where,

$$\dot{m}_{ij} = \sum_{i=1}^{n} \frac{\partial m_{ij}}{\partial q_i}\dot{q}_i + \frac{\partial m_{ij}}{\partial t} \quad \ldots\to (32)$$

and

$$\dot{a}_i = \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_j}\dot{q}_j + \frac{\partial a_i}{\partial t} \quad \ldots\to (33)$$

Also,

$$\sum_{j=1}^{n} \dot{m}_{ij}\dot{q}_j = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial m_{ij}}{\partial q_i}\dot{q}_i\dot{q}_j + \sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial t}\dot{q}_j$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j}\right)\dot{q}_j \dot{q}_l + \sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial t}\dot{q}_j \longrightarrow (34)$$

$$\frac{\partial T_2}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial m_{lj}}{\partial q_i}\dot{q}_j \dot{q}_l \;\cdots\cdots\rightarrow (35)$$

$$\frac{\partial T_1}{\partial q_i} = \sum_{j=1}^{n}\frac{\partial a_j}{\partial q_i}\dot{q}_j \;\cdots\cdots\rightarrow (36)$$

where we note that several dummy incides have been changed.

Finally, substituting from eqns. (31) through (36) into Lagrange's eqns, we obtain

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{jl}}{\partial q_i}\right)\dot{q}_j \dot{q}_l$$

$$+ \sum_{j=1}^{n}\left(\frac{\partial m_{ij}}{\partial t} + \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right)\dot{q}_j + \frac{\partial a_i}{\partial t} - \frac{\partial T_0}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

$$(i = 1, 2, \ldots n) \cdots\rightarrow (37)$$

The notation can be shortened by using a christoffel symbol of the first kind which is applied here to the quadratic form T_2.

let $$[jl, i] = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{il}}{\partial q_j} - \frac{\partial m_{jl}}{\partial q_i}\right) \cdots\rightarrow (38)$$

The eqn (37) can be written in the form

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{l=1}^{n}[jl, i]\dot{q}_j \dot{q}_l + \sum_{j=1}^{n}\gamma_{ij}\dot{q}_j$$

$$+ \sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial t}\dot{q}_j + \frac{\partial a_i}{\partial t} - \frac{\partial T_0}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

$$(i = 1, 2, \ldots n) \cdots\rightarrow (39)$$

where γ_{ij} is an element of a skew-symmetric matrix γ is given by

$$\gamma_{ij} = -\gamma_{ji} = \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i} \;\cdots\cdots\rightarrow (40)$$

n eqns of eq (37) (or) (39) are the eqns of motion. Although these eqns are nonlinear in general, they are linear in the $\ddot{q}$'s. Furthermore, since T_2 has been shown to be +ve definite quadratic form in the $\dot{q}$'s, the matrix m is also +ve definite & has an inverse. Therefore, it is always possible to solve for the $\ddot{q}$'s in terms of the q's, $\dot{q}$'s & t. If this is done, the resulting eqns of motion are of the form

$$\ddot{q}_i + f_i(q, \dot{q}, t) = 0 \quad (i=1,2,\dots n) \dots\to (4$$

Nonholonomic Systems :-

The derivation of lagrange's eqns for holonomic system required that the generalized coordinates be independent. For a nonholonomic system, however, there must be more generalized coordinates than the number of degrees of freedom. Therefore, the δq's are no longer independent if we assume a virtual displacement consistent with the constraints. For example, if there are m nonholonomic constraint eqns of the form

$$\sum_{i=1}^{n} a_{ji}\, dq_i + a_{jt}\, dt = 0 \qquad (j=1,2,\cdots m) \qquad (42)$$

the δq's must meet the following ~~properties~~. conditions :

$$\sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \qquad (j = 1,2,\dots\dots m) \qquad (43)$$

Now, let us assume once again that each generalized applied force Q_i is obtained from a potential function, as in Eq.(24). The constraints are assumed to be workless so the generalized constraint forces C_i must

If we substitute from eqn (3) into eqn (1), we obtain,

$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{2}\right)^2$$

Also, $\dot{x}_k(q,\dot{q},t) = \frac{1}{1}\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{2}$.

So, we get,

$$T(q,\dot{q},t) = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t}\right)^2 \dots\!> (4)$$

Let us group the terms according to their degree in the $\dot{q}$'s, using the notation

$$T = T_2 + T_1 + T_0 \quad \dots\dots> (5)$$

Where T_2 is a homogeneous quadratic function of the $\dot{q}$'s, T_1 is a homogeneous linear func of the $\dot{q}$'s and T_0 includes the remaining terms which are functions of the q's & t.

More explicitly, we find that T_2 is of the form,

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j \quad \dots\dots\dots\dots> (6)$$

Where $$m_{ij} = m_{ji} = \sum_{k=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\cdot\frac{\partial x_k}{\partial q_j} \quad \dots\dots> (7)$$

Also, $$T_1 = \sum_{i=1}^{n} a_i\dot{q}_i \quad \dots\dots\dots> (8)$$

where $$a_i = \sum_{k=1}^{3N} m_k\cdot\frac{\partial x_k}{\partial q_i}\cdot\frac{\partial x_k}{\partial t} \quad \dots\dots> (9)$$

Finally, $$T_0 = \frac{1}{2}\sum_{k=1}^{3N} m_k\left(\frac{\partial x_k}{\partial t}\right)^2 \quad \dots\dots\dots> (10)$$

meet the condition

$$\sum_{i=1}^{n} c_i \delta q_i = 0 \quad \ldots\ldots\ldots > (44)$$

for any virtual displacement consistent with the constraints.

Now suppose we multiply Eqn (43) by a factor λ_j known as Lagrange multiplier & obtain the m eqns

$$\lambda_j \sum_{i=1}^{n} a_{ji} \, \delta q_i = 0 \quad (j=1,2,\ldots m) \ldots > (45)$$

Next, subtract the sum of these m equations from Eq. (44). Interchanging the order of summation, we obtain

$$\sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{m} \lambda_j a_{ji} \right) \delta q_i = 0 \quad \ldots > (46)$$

Up to this point, the λ's have been considered to be arbitrary, while the δq's must conform to the constraints of Eq (43). But if we choose the λ's such that

$$c_i = \sum_{j=1}^{m} \lambda_j a_{ji} \quad (i=1,2,\ldots n) \ldots > (47)$$

then the coefficients of the δq's are zero, & eqn (46) will apply for any set of δq's. In other words, the δq's can be choosen independently.

With these assumptions, we can equate the generalized force c_i with Q_i', using eqns (39) & (47), we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \quad (i=1,2,\ldots n) \ldots > (48)$$

This is the standard form of Lagrange's eqn for a nonholonomic system.

In addition to these n eqns of motion, we have the m nonholonomic constraint eqns which can be written in the form

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0 \quad (j = 1,2,\ldots m) \dashrightarrow (49)$$

Thus we have total of (n+m) eqns with w to solve for the (n+m) independent Variables, namely, the n q's & the m λ's.

If we consider what has been accomplishe by the Lagrange multiplier method, we see fr Eqs. (47) and (48) that the constraints enter th eqns of motion in the form of constraint forc rather than in geometric terms. Also, we can understand the physical significance of the λ's by noting that they are linearly related to the constraint forces.

The standard form of Lagrange's equatio for a nonholonomic system Eq. (45), can also be applied to a holonomic system in which there are more generalized coordinates than degrees of freedom. For example, suppose there are m holonomic constraint eqns of the form

$$\phi_j(q_1,q_2,\ldots q_n,t) = 0 \quad (j=1,2\ldots m) \dashrightarrow (50)$$

Taking the total differential of ϕ_j we obtain

$$d\phi_j = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i} dq_i + \frac{\partial \phi_j}{\partial t} dt = 0 \dashrightarrow (51)$$

which is of the form of Eq. (49), where we let,

$$a_{ji} = \frac{\partial \phi_j}{\partial q_i}, \quad a_{jt} = \frac{\partial \phi_j}{\partial t} \dashrightarrow (52)$$

⑭ If the Lagrange multiplier method is applied to this system and the resulting differential eqns are completely solved, the result will be that the q's and λ's are expressed as explicit functions of time. Then, with the aid of Eqn (A-1), the generalized constraint forces C_i can also be obtained as explicit functions of time.

In general, however, holonomic systems will be described in terms of independent q's, thereby avoiding any eqns of constraint. The Lagrange multiplier method is normally used with holonomic systems only if one desires to solve for the constraint forces.

Sec 2.2 : Examples :-

Example: 2.1: Find the differential eqns of motion for a spherical pendulum of length l (Fig 2.1)

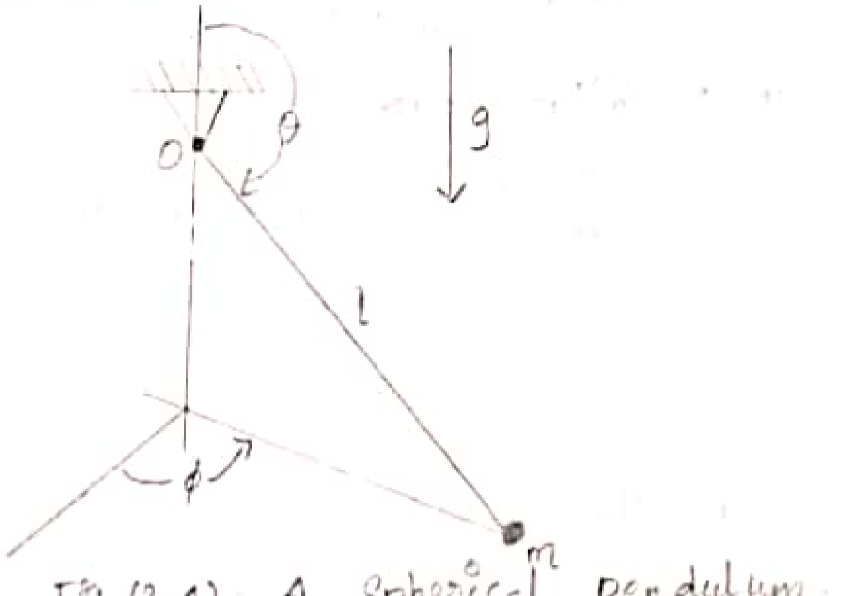


Fig (2.1). A Spherical pendulum.

Using spherical coordinates, we see that the kinetic energy of the particle is

$$T = \tfrac{1}{2} m(l^2 \dot{\theta}^2 + l^2 \phi^2 \sin^2\theta) \dots \rightarrow (53)$$

(15) Let m be the mass of the particle suspended by a massless wire of length l, from a fixed pnt O to form a spherical pendulum. Let us use the spherical coordinates θ and φ where θ is measured from the upward vertical as shown in the fig.

Angle φ is measured b/w a vertical reference plane passing through O and the vertical plane containing the pendulum.

Then the cartesian coordinates w.r to the axes gn in fig are given by

$x = l\sin\theta\cos\theta$; $y = l\sin\theta\sin\phi$; $z = -l\cos\theta$

$\therefore \dot{x} = l(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})$

$\dot{y} = l(\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\sin\phi\,\dot{\phi})$

$\dot{z} = l\sin\theta\,\dot{\theta}$

kinetic energy $T = \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]$

$= \frac{1}{2}m\big[l^2(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})^2$

$+ l^2(\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi})^2$

$+ l^2\sin\theta\,\dot{\theta}\big]$

potential energy $V = -mg(ON)$

$= -mg\,l\cos(180-\theta) = mgl\cos\theta$

$\therefore$ The Lagrangian func L

$= T - V$

$= \frac{1}{2}ml^2[(\dot{\theta})^2 + \sin^2\theta(\dot{\phi})^2] - mgl\cos\theta$

$\therefore \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2[2\dot{\theta}]$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{1}{2}ml^2\ddot{\theta}$

and $\left(\frac{\partial L}{\partial \theta}\right) = \frac{1}{2} ml^2 [2\sin\theta\cos\theta(\dot\phi)^2] + mgl\cos\theta$

||ly $\frac{\partial L}{\partial \dot\phi} = \frac{1}{2} ml^2 \sin^2\theta \, 2\dot\phi$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = ml^2[2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi]$$

and $\frac{\partial L}{\partial \phi} = 0$

The Lagrange's eqns are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow ml^2\ddot\theta - ml^2\sin\theta\cos\theta(\dot\phi)^2 - mgl\cos\theta = 0$$

and $ml^2[2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi] - 0 = 0$

i.e) $l\ddot\theta - l\sin\theta\cos\theta(\dot\phi)^2 - g\cos\theta = 0$

and $2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi = 0$

These 2 eqns are the differential equations of motion.

Example: 2.2:

A double pendulum consists of two particles suspended by massless rods, as shown in fig (2.2). Assuming that all motion takes place in a vertical plane, find the diff eqn of motion-linearize these eqns, assuming small motions.

Solu:-

Let O be the point of suspension. Let the rod connecting the upper particle to O make an angle θ with the vertical and the rod connecting the lower particle to the upper particle make an angle φ

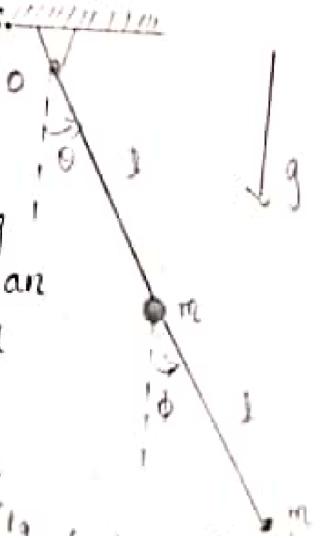


fig (2.2) A Double pendulum

[illegible]

(11)

with the vertical

The absolute velocity of the lower particle

$$l \cdot \sqrt{[(\dot{\theta})^2 + (\dot{\phi})^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)}$$

(by cosine law)

The total kinetic energy

$$T = \frac{1}{2} ml^2 [2(\dot{\theta})^2 + (\dot{\phi})^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)$$

choosing the reference level for potential energy at O, we get

$$V = mgl\cos\theta - mg(l\cos\theta \; l\cos\phi)$$

$$= -mgl(\cos\theta + l\cos\theta + \cos\phi)$$

$$= -mgl(2\cos\theta + \cos\phi)$$

The Lagrangian function $L = T - V$

$$L = \frac{1}{2} ml^2 [2(\dot{\theta})^2 + (\dot{\phi})^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)$$

$$+ mgl(2\cos\theta + \cos\phi)$$

$$\therefore \frac{\partial L}{\partial \dot{\theta}} = ml^2 (2\dot{\theta} + \dot{\phi}\cos(\phi-\theta)]$$

$$= \frac{1}{2} ml^2 [4(\dot{\theta})^2 + 2\dot{\phi}\cos(\phi-\theta)].$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2 [2\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) - \dot{\phi}\sin(\phi-\theta)$$

$$(\dot{\phi} - \dot{\theta})]$$

and $\frac{1}{2} ml^2 [2\dot{\theta}\dot{\phi}\sin(\phi-\theta) + mgl(-2\sin\theta)]$

$$\text{III}^{ly} \; \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2} ml^2 [2\dot{\phi} + 2\dot{\theta}\cos(\phi-\theta)].$$

$$= ml^2 [\dot{\phi} + \dot{\theta}\cos(\phi-\theta)]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ml^2 [2\ddot{\phi} + \ddot{\theta}\cos(\phi-\theta) - \dot{\theta}(\dot{\phi} - \dot{\theta})$$

$$\sin(\phi-\theta)]$$

$$\frac{\partial L}{\partial \phi} = \frac{1}{2} ml^2 [-2\ddot{\phi} + \ddot{\theta}\cos(\phi-\theta) - \dot{\theta}(\dot{\phi} - \dot{\theta})\sin(\phi-\theta)]$$

(8) $$\frac{\partial L}{\partial \phi} = \frac{1}{2} ml^2 [-2\dot\theta\dot\phi \sin(\phi-\theta)] - mgl \sin\phi$$

∴ lagranges eqns of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta} \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi} = 0$$

i.e) $ml^2 [2\ddot\theta + \ddot\phi \cos(\phi-\theta)] - \dot\phi(\dot\phi-\dot\theta)\sin(\phi-\theta)$

$-\frac{1}{2} ml^2 [2\dot\theta\dot\phi \sin(\phi-\theta)] + mgl \sin\theta = 0$

i.e) $l^2 [2\ddot\theta + \ddot\phi \cos(\phi-\theta) - (\dot\phi)^2 \sin(\phi-\theta) + 2g \sin\theta$

and $ml^2 [\ddot\phi + \cos(\phi-\theta)(\dot\phi-\dot\theta)\sin(\phi-\theta)]$

$+\frac{1}{2} ml^2 [2\dot\theta\dot\phi \sin(\phi-\theta)] + mgl \sin\phi = 0$

i.e) $l[\ddot\theta + \ddot\phi \cos(\phi-\theta) - (\dot\phi)^2 \sin(\phi-\theta)] + g\sin\phi = 0$

Equations (1) and (2) are the differential equations of motion of the system

2.3) A block of mass m_2 can slide on another block of mass m_1 which, in turn, slides on a horizontal surface, as shown in fig (2-3) (a) using x_1 and x_2 as coordinates, obtain the diff eqns of motion. Solve for the accelarations of the 2 blocks as they move under the influence of gravity, assuming that all surfaces are frictionless. Find the force of interaction b/w the blocks.

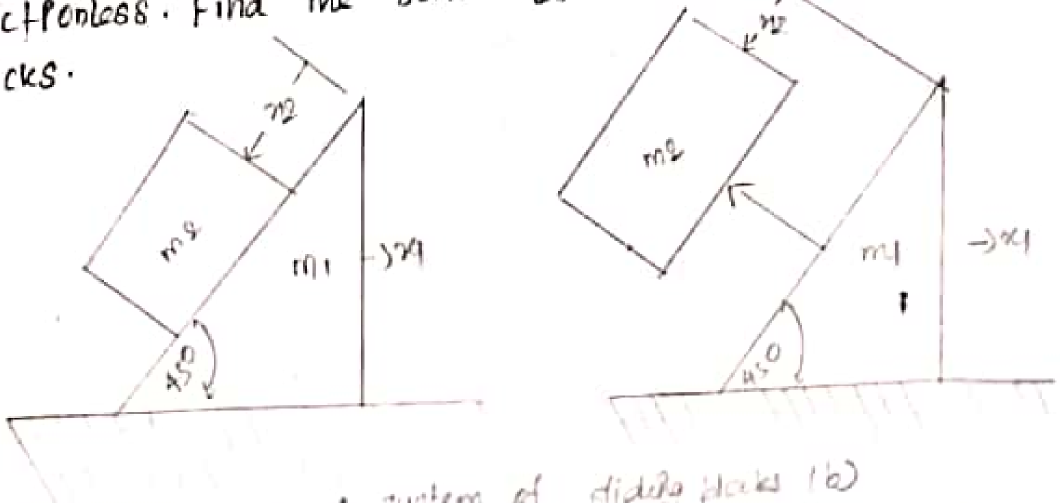


fig (2-3) (a) A system of sliding blocks (b)

(iv) Soln -

The absolute velocity of m2

$$= \sqrt{(\dot{x}_1)^2 + (\dot{x}_2)^2 + 2\dot{x}_1\dot{x}_2 \cos(90+45)}$$

by cosine law

The total kinetic energy is,

$$T = \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 [(\dot{x}_1)^2 + (\dot{x}_2)^2 - 2\dot{x}_1\dot{x}_2 \sin 45^\circ]$$

(or)

$$T = \frac{1}{2} [(m_1+m_2)(\dot{x}_1)^2 + m_2(\dot{x}_2)^2 - \sqrt{2}\, m_2 \dot{x}_1 \dot{x}_2]$$

Any changes in potential energy arise from changes in x_2

$$\therefore V = -m_2 g x_2 \sin 45^\circ = -\frac{1}{\sqrt{2}} m_2 g x_2$$

The Lagrangian function, $L = T - V$

$$\therefore = \frac{1}{2} [(m_1+m_2)(\dot{x}_1)^2 + m_2(\dot{x}_2)^2 - \sqrt{2}\, m_2 \dot{x}_1 \dot{x}_2] + \frac{1}{\sqrt{2}} m_2 g x_2$$

Hence $\frac{\partial L}{\partial x_1} = \frac{1}{2} [(m_1+m_2) 2\dot{x}_1 - \sqrt{2}\, m_2 \dot{x}_2]$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial x_1}\right] = \frac{1}{2}[2(m_1+m_2)2x_1 - \sqrt{2}\, m_2 x_2]$$

and $\frac{\partial L}{\partial x_1} = 0$

III^{ly} $\frac{\partial L}{\partial x_2} = \frac{1}{2} [(m_2 2x_2 - \sqrt{2}\, m_2 x_1]$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_1}\right) = \frac{1}{2}[2m_2 x_2 - \sqrt{2}\, m_2 x_1]$$

and $\frac{\partial L}{\partial x_1} = \frac{1}{\sqrt{2}} m_2 g$

The Lagrange's equations of motion are

$$\frac{d}{dt}\left[\frac{\partial L}{\partial x_1}\right) - \frac{\partial L}{\partial x_1} = \text{and } \frac{d}{dt}\left(\frac{\partial L}{\partial x_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

i.e) $\frac{1}{2}\left[2(m_1+m_2)\ddot{x}_1 - \sqrt{2}m_2\ddot{x}_2\right] = 0$

(or) $\left[(m_1+m_2)\ddot{x}_1 - \frac{1}{\sqrt{2}}m_2\ddot{x}_2\right] = 0$ ----> (1)

and $\frac{1}{2}\left[2(m_2\ddot{x}_2 - \sqrt{2}m_2\ddot{x}_1\right] - \frac{1}{\sqrt{2}}m_2g = 0$

or $m_2\ddot{x}_2 - \frac{1}{\sqrt{2}}m_2\ddot{x}_1 - \frac{1}{\sqrt{2}}m_2g = 0$)(2)

Equations (1) & (2) are the differential equations of motions ~~of~~ for the system

Eqn (1) + $\frac{1}{\sqrt{2}}$ (2) give ;

$$\left(m_1+m_2-\frac{1}{2}m_2\right)\ddot{x}_1 = \frac{1}{2}m_2g$$

(or) $\ddot{x}_1 = \frac{m_2g}{2m_1+m_2}$

Substituting this in equation (1) we get,

$$\ddot{x}_2 = \frac{\sqrt{2}(m_2+m_2g)}{2m_1+m_2}$$

Thus the accelarations of the two blocks are,

$$\ddot{x}_1 = \frac{m_2g}{2m_1+m_2} \quad \text{and} \quad \ddot{x}_2 = \frac{\sqrt{2}(m_1+m_2)g}{2m_1+m_2}$$

Now, to find the force of interaction b/w the two blocks, let us use the Lagrange multiplier method. This interactions force is normal to the surface of contact and may be considered as the generalised constraint force corresponding to a coordinate x_3 which shown in figure. There are only 2 degrees of freedom, since $x_3 = 0$.

Let us write this holonomic constraint equation in the form $x_3 = 0$ which is $\|^{r}$ to an eqn of non-holonomic constraint. comparing

$\dot{x}_3 = 0$ with the nonholonomic constraint equations of the form

$$\sum_{j=1}^{n} a_{kj}\, \dot{q}_j + a_{kt} = 0 , \quad k = 1, 2, \ldots m$$

Here $n = 3$ and $m = 1$

$$\therefore a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + a_{13}\dot{q}_3 + a_{11} + a_{12} + a_{13} = 0$$

we have $a_{11} = 0$; $a_{12} = 0$; $a_{13} = 1$

$\therefore$ From, $C_j = \sum_{k=1}^{m} \lambda_k a_{kj}$, $j = 1, 2, \ldots n$

we get,

$$C_1 = \lambda_1 a_{11} = 0$$

$$C_2 = \lambda_1 a_{12} = 0$$

$$C_3 = \lambda_1 a_{13} = \lambda_1$$

writing the vertical and horizontal velocity components separately we get

$$T = \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 \left[\left(\dot{x}_1 - \frac{\dot{x}_2 + \dot{x}_3}{\sqrt{2}} \right)^2 + \left(\frac{\dot{x}_3 - \dot{x}_2}{\sqrt{2}} \right)^2 \right]$$

$$= \frac{1}{2} (m_1 + m_2) (\dot{x}_1)^2 + \frac{1}{2} m_2 \left[(\dot{x}_2)^2 + (\dot{x}_3)^2 - \sqrt{2}\, \dot{x}_1 (\dot{x}_2 + \dot{x}_3) \right]$$

and $V = \frac{1}{2} m_2 g (x_3 - x_2)$

$$\therefore L = T - V$$

$$= \tfrac{1}{2} (m_1 + m_2) (\dot{x}_1)^2 \frac{1}{2} m_2 \left[(\dot{x}_2)^2 + (\dot{x}_3)^2 - \sqrt{2}\, \dot{x}_1 (\dot{x}_2 + \dot{x}_3) \right] - \frac{1}{\sqrt{2}} m_2 g (x_3 - x_2) .$$

ciency for the blocks to separate.

Example: 2.4:-

A particle of mass m can slide without friction on the inside of a small tube which is bent in the form of a circle of radius r. The tube rotates about a vertical diameter with a constant angular velocity ω, as shown in Fig (2.4). Write the diff eqn of motion.

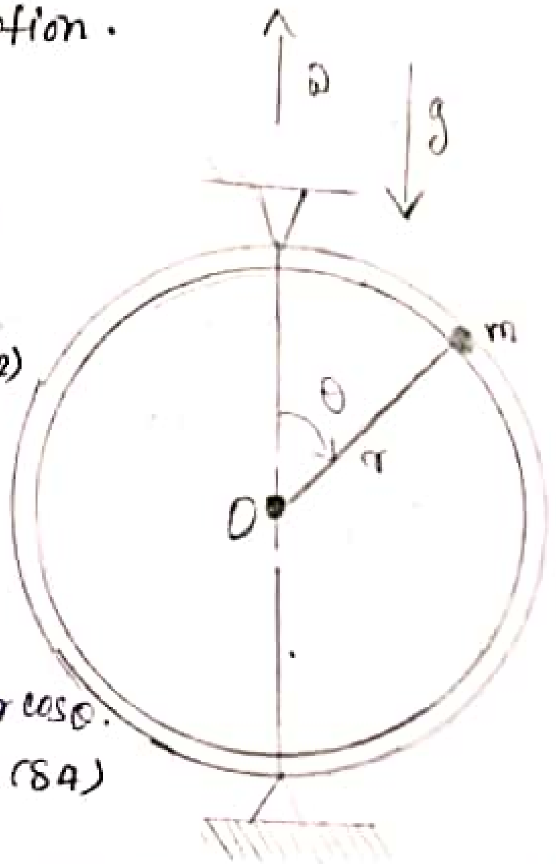


Fig (2.4) - A particle in a whirling tube

The kinetic & potential energies of the particle are

$$T = \frac{1}{2} m r^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta) \ldots \rightarrow (82)$$

$$V = mgr\cos\theta \ldots \rightarrow (83)$$

Resulting in,

$$L = \frac{1}{2} m r^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta) - mgr\cos\theta. \ldots \rightarrow (84)$$

This is a rheonomic system because any set of equations giving the inertial Cartesian coordinates of the particle in terms of its single generalized coordinate θ must involve time explicity. Note, however, that in this case the Lagrangian L is not an explicit function of time.

(23) The tube is a moving constraint which does work on the particle in an actual displacement. Nevertheless, no work is done by the constraint forces in a virtual displacement, and the constraint is classed as a workless constraint.

Since the only generalized force acting on the system is derivable from a potential function, we can use the standard form of Lagrange's equation given by Eq. (57).

We find that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mr^2\omega^2 \sin\theta \cos\theta + mgr\sin\theta$$

and therefore the eqn of motion is

$$mr^2\ddot{\theta} - mr^2\omega^2 \sin\theta\cos\theta - mgr\sin\theta = 0 \quad \text{---} \rightarrow (85)$$

This system will be discussed further in sec (2-3).

~~Example: 2~~.5:-

Example: 8.5 : A particle of mass m is connected by a massless spring of stiffness k and unstressed length r_0 to a point P which is moving along a circular path of radius a at a uniform angular rate ω (fig 8.5). Assuming that the particle moves without friction on a horizontal plane, find the diff eqns of motion.

This is a rheonomic system with two degrees of freedom corresponding to the independent generalized coordinates r and θ. Let us obtain the kinetic energy with the aid of Eqn (1-119) which for this case of a single particle, can be written in the form,

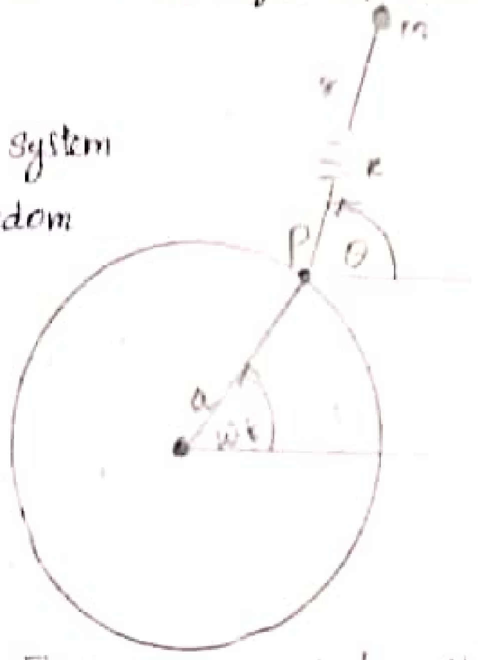


Fig (8.5) A particle attached by a massless spring to a moving point.

$$(119) \Rightarrow T = \frac{1}{2} m \dot{r}_p^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\rho}_i^2 + \dot{r}_p \cdot m \dot{\rho}_c$$

$$T = \tfrac{1}{2} m \dot{r}_p^2 + \tfrac{1}{2} m \dot{\rho}^2 + \dot{r}_p \cdot m\dot{\rho} \quad \text{. . .} > (8.6)$$

Where $\dot{r}_p$ is the absolute velocity of the point P, and $\dot{\rho}$ is the velocity of the particle relative to P. We see that

$$\dot{r}_p^2 \cdot a^2 \omega^2$$

$$\dot{\rho}^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

Hence.

$$T_2 = \frac{1}{2} m \dot{\rho}^2 = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \quad \text{. . . . . .} > 8.7$$

$$T_1 = \dot{r}_p \cdot m \dot{\rho} = m a \omega [\dot{r} \sin(\theta - \omega t) + r \dot{\theta} \cos(\theta - \omega t)] \quad \text{. .} > (8$$

$$T_0 = \frac{1}{2} m \dot{r}_p^2 = \frac{1}{2} m a^2 \omega^2 \quad \to (89)$$

Also,

$$V = \frac{1}{2} k (r - r_0)^2 \quad \to (90)$$

Therefore, the Lagrangian function is

$$L = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2 + 2 a \omega \dot{r} \sin(\theta - \omega t)$$
$$+ 2 a \omega r \dot{\theta} \cos(\theta - \omega t) + a^2 \omega^2] - \frac{1}{2} k (r - r_0)^2$$

We obtain the r equation from $\to (91)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m \ddot{r} -$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{1}{2} m \cdot 2 \dot{r} + \frac{1}{2} m \cdot 2 a \omega \cdot \sin(\theta - \omega t)$$
$$= m \dot{r} + m a \omega \sin(\theta - \omega t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m \ddot{r} + m a \omega^2 \cos(\theta - \omega t) + m a \omega \dot{\theta} \cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 + m a \omega \dot{\theta} \cos(\theta - \omega t) - k (r - r_0)$$

We get the Lagrange's equation

$$m \ddot{r} - m r \theta$$

$$m \ddot{r} - m a \omega^2 \cos(\theta - \omega t) + m a \omega \dot{\theta} \cos(\theta - \omega t)$$
$$- m r \dot{\theta}^2 \mp m a \omega \dot{\theta} \cos(\theta - \omega t) + k (r - r_0)$$

$$\Rightarrow m \ddot{r} - m r \dot{\theta}^2 - m a \omega^2 \cos(\theta - \omega t) + k (r - r_0) = 0$$

Now, we obtain, (92)

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \cdot r^2 \cdot 2 \dot{\theta} + 2 a \omega r \cos(\theta - \omega t) \cdot \frac{1}{2} m$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} + m a \dot{r} \omega \cos(\theta - \omega t)$$
$$- m a r \omega \dot{\theta} \sin(\theta - \omega t) + m a r \omega^2 \sin(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = ma\dot{r}\omega\cos(\theta-\omega t) - mar\omega\dot{\theta}\sin(\theta-\omega t)$$

The θ eqn is

$$\Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + ma\dot{r}\omega\cos(\theta-\omega t)$$

$$- mar\omega\dot{\theta}\sin(\theta-\omega t) + mar\omega^2\sin(\theta-\omega t).$$

$$- ma\dot{r}\omega\cos(\theta-\omega t) + mar\omega\dot{\theta}\sin(\theta-\omega t)$$

$$\Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mar\omega^2\sin(\theta-\omega t).$$

$\cdots) 93.$

Example : 2.6 :- Two particles are connected by a rigid massless rod of length l which rotates in a horizontal plane with a constant angular velocity ω (Fig. 2.6). knife-edge supports at the two particles prevent either particle from having a velocity component along the rod, but the particles can slide without friction in a direction ⊥° to the rod. Find the diff eqns of motion. Solve for x, y, and the constraint force as funcs of time if the center of mass is initially at the origin and has a velocity v_0 in the +ve y direction.

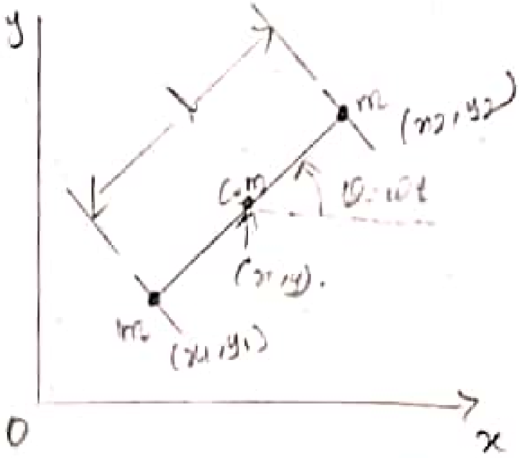


Fig (2.6). A nonholonomic rheonomic System

This is a nonholonomic rheonomic system. let us express the configuration initially in terms of the Cartesian coordinates (x_1, y_1) & (x_2, y_2) of the individual particles. There are two independent eqns of holonomic constraint, namely,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \quad \cdots\cdots \rightarrow \quad (94)$$

and
$$y_2 - y_1 = (x_2 - x_1) \tan \omega t \quad \cdots \rightarrow \quad (95)$$

Expressing the given length and orientation of the rod. In addition, there is the nonholonomic constraint equation

$$(\dot{x}_1 + \dot{x}_2) \cos \omega t + (\dot{y}_1 + \dot{y}_2) \sin \omega t = 0 \quad \cdots\cdots \rightarrow (96)$$

Which restricts the velocity of the center of the rod to a direction which is ⊥r to rod, as shown in the discussion preceding Eq (1.17).

We have been using four cartesian coordinates & 3 independent eqns of constraint, indicating that the system has one degree of freedom. It is convenient, however, to choose generalized coordinates in such a manner that there are no holonomic constraints, and only one ~~holonomic~~ nonholonomic constraint. This can be accomplished by choosing the cartesian coordinates (x, y) of the ~~c~~ center of mass as the generalized coordinates. The transformation equations are

$$x_1 = x - \tfrac{1}{2} l \cos \omega t$$
$$y_1 = y - \tfrac{1}{2} l \sin \omega t$$
$$x_2 = x + \tfrac{1}{2} l \cos \omega t$$
$$y_2 = y + \tfrac{1}{2} l \sin \omega t \quad \cdots \rightarrow \quad (97)$$

From eqns (96) & (97), we see that the

non holonomic constraint equation becomes

$$\dot{x} \cos\omega t + \dot{y} \sin\omega t = 0 \quad \dots\dots > (98)$$

The total kinetic energy of the system is,

$$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \quad \dots\dots > (99)$$

or, substituting from eqn (97)

$$T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} m l^2 \omega^2 \quad \dots > (100)$$

This result can be obtained directly by adding the translational & rotational kinetic energies, nothing that the total mass is 2m and the moment of inertia about the center of mass is $ml^2/2$.

The potential energy V is zero for this system, so we can write Eq. (2-48) in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \lambda_1 a_{1i} \quad \dots\dots > (101)$$

where, from the nonholomic constraint equation of (98), we see that

$$\left.\begin{aligned} a_{11} &= \cos\omega t \\ a_{12} &= \sin\omega t \end{aligned}\right\} (102)$$

The differential eqns of motion are obtained by substituting from Eqns. (100) & (102) into (101) the result being.

$$\left.\begin{aligned} 2m\ddot{x} &= \lambda_1 \cos\omega t \\ 2m\ddot{y} &= \lambda_1 \sin\omega t \end{aligned}\right\} (103)$$

These 2 eqns & the constraint eqn of (98) must now be solved for x, y & λ_1. From Eq. (103) we obtain $\ddot{y} = \ddot{x} \tan\omega t \dashrightarrow (104)$

& substituting for $\tan\omega t$ from (98), we get,

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 0 \quad \dots\dots \rightarrow (105)$$

Next, integrating & using the initial conditions, we see that,

$$\dot{x}^2 + \dot{y}^2 = v_0^2 \quad \dots\dots \rightarrow (106)$$

indicating that the center of mass moves with a constant speed v_0. Since the direction of the motion is always ⊥r to the rod, we have

$$\dot{x} = -v_0 \sin\omega t$$

$$\dot{y} = v_0 \cos\omega t \quad \dots\dots \rightarrow (107)$$

Integrating, again we obtain

$$x = \frac{v_0}{\omega}(\cos\omega t - 1) \quad \dots\dots \rightarrow (108)$$

$$y = \frac{v_0}{\omega}(\sin\omega t)$$

From Eqs (103) & (107), the Lagrange multiplier is found to be, $\lambda_1 = -2mv_0\omega \quad \dots\dots \rightarrow (109)$

It can be seen that the system travels in a circular path of radius v_0/ω at a constant speed v_0. The centripetal constraint force exerted on the system is of magnitude $2mv_0\omega$ & is represented by $-\lambda_1$. The generalized constraint forces, obtained with the aid of Eq. (47) are

$$C_1 = -2mv_0\omega\cos\omega t$$

$$C_2 = -2mv_0\omega\sin\omega t \quad \dots\dots \rightarrow (110)$$

and are directed along the +ve x & y axes, respectively.

Sec: 2.3: Integrals Of The Motion:-

(140) Integrals (or) constants of the motion:-

Any general analytical solution of the diff eqns of motion contains 2n constants of integration which are mo usually evaluated from the 2n initial conditions.

One method ~~of~~ of expressing the general solution is to obtain 2n independent functions of the form

$$f_j(q, \dot{q}, t) = \alpha_j \quad (j=1,2,\dots 2n) \quad \dots\dots\to (111)$$

where the α's are arbitrary constants. These 2n functions are called the integrals on constants of the motion.

Each function f_j maintains a constant value α_j as the motion of the system proceeds, the value of α_j depending upon the initial conditions. In principle, these 2n equations can be solved for the q's & $\dot{q}$'s as functions of the α's and t, that is, it is possible to find

$$\left.\begin{aligned} q_i &= g_i(\alpha_1, \dots\dots, \alpha_{2n}, t) \\ \dot{q}_i &= \dot{g}_i(\alpha_1, \dots\dots, \alpha_{2n}, t) \end{aligned}\right\} \quad (i=1,2,\dots n) \to (112)$$

Such that Eq. (2-111) is satisfied for all j.

Ignorable coordinates:-

Consider a holonomic system which can be described by the standard form of Lagrange's equations, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (i=1,2,\dots\dots n) \dots\to (113)$$

Suppose that $L(q, \dot{q}, t)$ contains all n $\dot{q}$'s, but some of the q's, say $q_1, q_2, \dots q_k$, are missing from the Lagrangian. These k coordinates are called ignorable coordinates.

Since, $\frac{\partial L}{\partial q_i}$ is zero for each ignorable coordinate, it follows that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0 \quad (i=1,2,\dots k) \quad \dots\rightarrow (114)$$

(or)

$$p_i=\frac{\partial L}{\partial \dot{q}_i}=\beta_i \quad (i=1,2,\dots k) \quad \dots\rightarrow (115)$$

where the β's constants evaluated from the initial conditions. Hence, we find that the generalized momentum corresponding to each ignorable coordinate is constant, that is, it is an integral of the motion.

Example: 2.7: Let us consider the kepler problem i.e), the problem of the motion of a particle of unit mass which is attracted by an inverse-square gravitional force to fixed point O (fig 2.7). Using polar coordinates, the kinetic and potential energies are

$$T=\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2) \quad \dots\rightarrow (116)$$

$$V=-\frac{\mu}{r} \quad \dots\rightarrow (117)$$

where μ is a +ve constant known as the gravitional coefficient.

The Lagrangian function is

$$L=\frac{1}{2}(\dot{r}^2+r^2\dot{\theta}^2)+\frac{\mu}{r} \rightarrow (118)$$

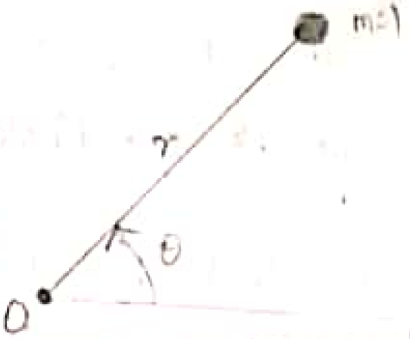


Fig (2.7). The kepler problem terms of polar coordinates.

and using Eq (113), we find that the r equation of motion is,

$$\ddot{r}-r\dot{\theta}^2+\frac{\mu}{r^2}=0. \quad \rightarrow (119)$$

Since θ does not appear explicitly in the Lagrangian function, it is an ignorable coordinate . The θ eqn of motion is

(ii) $$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \dots \to (120)$$

(or) $$r^2\dot{\theta} = \beta \quad \to (121)$$

where β is a constant & is equal to the angular momentum of the particle about the attracting center O.

The Routhian Function :-

We consider a standard holonomic system whose configuration is given by n independent generalized co ordinates of which the first k are ignorable. In other words, the Lagrangian L is a function of $q_{k+1}, \dots q_n, \dot{q}_1, \dots \dot{q}_n, t$.

Now let us define a Routhian function $R(q_{k+1}, \dots q_n, \dot{q}_{k+1}, \dots \dot{q}_n, \beta_1, \dots \beta_k, t)$ as follows :

$$R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \quad \dots \to (122)$$

where we eliminate the $\dot{q}$'s corresponding to the ignorable coordinates by solving the k-equations

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i \quad (i = 1, 2, \dots k) \quad \dots \to (123)$$

for $\dot{q}_1, \dot{q}_2 \dots \dot{q}_k$ as functions of $q_{k+1}, \dots q_n, \dot{q}_{k+1}, \dots, \dot{q}_n, \beta_1, \dots \beta_k, t$. These expressions for the $\dot{q}$'s are linear in the β's.

Next, let us make an arbitrary variation of all the variables in the Routhian function.

We have $$\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i}\delta \beta_i + \frac{\partial R}{\partial t}\delta t \quad \dots \to (124)$$

where we note that the β's are regarded as r.h.s of Variables. A similar variation in the (122) yields

$$\delta L = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{k} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t \quad (125)$$

and using Eq (123),

$$\delta \sum_{i=1}^{k} \beta_i \dot{q}_i = \sum_{i=1}^{k} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \dot{q}_i \delta\beta_i \cdots\rightarrow (126$$

which results in

$$\delta\left(1 - \sum_{i=1}^{k} \beta_i \dot{q}_i\right) = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \sum_{i=1}^{k} \dot{q}_i \delta\beta_i + \frac{\partial L}{\partial t}\delta t \cdots\rightarrow (12$$

We assume that the varied quantities in Eqs (124) & (127) are independent; hence the corresponding coefficients must be equal.

Thus,

$$\frac{\partial L}{\partial q_i} = \frac{\partial R}{\partial q_i} \quad (i = k+1, \cdots n$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial R}{\partial \dot{q}_i} \quad \cdots\rightarrow (128)$$

and

$$\dot{q}_i = -\frac{\partial R}{\partial \beta_i} \quad (i = 1, 2, \ldots k)$$

$$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t} \quad \cdots\rightarrow (129)$$

Now, let us substitute from Eq. (128) into Lagrange's equations and obtain

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0 \quad (i = k+1, \cdots n) \cdots\rightarrow (1$$

these eqns are of the form of Lagrange's eqns with the Routhian function used in place of the Lagrangian function. Note, however, that there are only (n-k) second-order eqns in the non-ignorable variables. Thus, the Routhain procedure has succeeded in eliminating the integrable ignorable coordinates from the eqns of motion. In effect, the number of degrees of freedom has been reduced to (n-k).

Frequently there is no need to solve for the ignorable coordinates. But if Eq-(130) has been solved for the (n-k) non-ignorable coordinates, then we can integrate Eq.(129) to obtain expressions for the ignorable coordinates, that is,

$$q_i = -\int \frac{\partial R}{\partial \beta_i}\, dt \quad (i = 1, 2, \dots k) \quad \dots \to (131)$$

To illustrate the Routhian method, consider again the kepler problem of Example (2-7). From eqn (121), we have,

$$\dot{\theta} = \frac{\beta}{r^2} \quad \dots \to (132)$$

Substituting this expression for $\dot{\theta}$ into the Lagrangian function of Eqn (118), we obtain the following Routhian function:

$$R = L - \beta\dot{\theta} = \frac{1}{2}\dot{r}^2 - \frac{\beta^2}{2r^2} + \frac{\mu}{r} \quad \dots \to (133)$$

Thus, we see that the system has been reduced to a single degree of freedom. The corresponding eqn of motion is obtained from Eq. (130).

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = \ddot{r}$$

$$\frac{\partial R}{\partial r} = \frac{\beta^2}{r^3} - \frac{\mu}{r^2}$$

yielding, $$\ddot{r} - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0 \quad \dots \to (134)$$

This result is identical with that found by substituting from Eq. (132) into eqn (119) that is, into the r eqn obtained by the Lagrangian method.

We have seen that the Routhian func replaces the Lagrangian function for the system after the number of degrees of freedom has been reduced by the ignoration of coordinates. If we look at the Routhian function of Eq. (133) from the viewpoint of an observer rotating with the line drawn from the attracting centre O to the particle, we see that

$$R = T' - V' \quad \text{.....>} \ (135)$$

where $T' = \frac{1}{2}\dot{r}^2$

$$V' = \frac{\beta^2}{2r^2} - \frac{\mu}{r} \quad \text{....>} \ (136)$$

Here T' is the kinetic energy associated with the single degree of freedom. V' is the potential energy arising from the inverse-square gravitional field and from the centrifugal force field due to the angular motion of the particle in its orbit.

Conservative Systems:

A conservative force field has the properties that (1) the generalized force components are obtained from the potential energy function by using

$$Q_i = -\frac{\partial V}{\partial q_i} \quad \text{....>} \ (137)$$

where V(q) is a function of the configuration only, and (2) the integral,

$$W = \int_{A}^{B} Q \cdot dq = \sum_{i=1}^{n} \int_{A_i}^{B_i} Q_i \, dq_i \quad \cdots\to (138)$$

is independent of the path taken b/w the gn end-points in q-space. If no other forces do work on the system, the total mechanical energy is conserved; hence the system is called a conservative system. The total energy $E(q, \dot{q}) = T + V$ is an integral of the motion.

Now let us define a system to be conservative if it meets the following conditions:

1. The standard form of Lagrange's Eqn (holonomic or nonholonomic) applies.

2. The Lagrangian function L is not an explicit function of time.

3. Any constraint eqns can be expressed in the differential form $\sum_{i=1}^{n} a_{ji} \, dq_i = 0 \quad (j = 1, 2, \cdots m) \cdots\to (139)$

i.e) all the coefficients a_{ji} are equal to zero.

In order to show that the three gn conditions are sufficient to ensure the existence so of an energy integral, let us consider the case in which a system is described by the standard nonholonomic form of Lagrange's eqns, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \quad (i = 1, 2 \; n) \cdots\to (140)$$

where $L(q, \dot{q})$ is not an explicit function of time. The actual constraints may be holonomic or nonholonomic; but, in either event, let us write the m eqns of constraints in the form $\sum_{i=1}^{n} a_{ji} \dot{q}_i = 0 \quad (j = 1, 2, \cdots m) \cdots\to (141)$

where the a's are funcs of the q's or possibly time. Notice, however, that any holonomic constraint functions $\phi_j(q)$ cannot be explicit funcs of time because of the assumption that

$$a_{ji} = \frac{\partial \phi_j}{\partial t} = 0 \quad \cdots\rightarrow (142)$$

Now, let us consider the total derivative

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q_i}\ddot q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\dot q_i \quad \cdots\rightarrow (143)$$

But, from Lagrange's Eqns as given in Eq. (140), we have

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji} \quad \cdots\rightarrow (144)$$

Hence, we obtain from Eqs (143) & (144), that

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot q_i}\ddot q_i + \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\dot q_i - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j a_{ji} \dot q_i \quad \cdots\rightarrow (145)$$

The double summation term of Eqn (145) is zero, as a result of Eqn (41)

Therefore, we obtain

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot q_i}\dot q_i\right) \quad \cdots\cdots\rightarrow (146)$$

Which can be integrated to give

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot q_i}\dot q_i - L = h \quad \cdots\rightarrow (147)$$

where h is constant.

Thus, we have obtained a constant of the motion which is known as the Jacobi integral or the energy integral. This integral of the motion exists for all conservative systems

Let us recall that the Lagrangian function can
2) be written in the form,

$$L = T_2 + T_1 + T_0 - V \quad \dots\to (148)$$

where the kinetic energy terms are separated in accordance with their degree in the $\dot{q}$'s as defined by Eqs (2-6), (8) & (10). Assuming that V is not a function of the $\dot{q}$'s, we see that

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = 2T_2 + T_1 \quad \dots\to (149)$$

and therefore Eq. (147) can be written in the form

$$T_2 - T_0 + V = h \quad \dots\to (150)$$

Hence, we confirm that the Jacobi integral has the units of energy. Notice that T_0 and V are both functions of the q's only, for this case of a conservative system. If we group these functions together, we can write

$$T' + V' = h \quad \dots\to (151)$$

where,

$$T' = T_2$$

$$V' = V - T_0 \quad \to (152)$$

We see that, in addition to the original force field represented by V, the potential function V' induces a force field due to T_0. This field is artificial in the sense that it consists of inertial forces arising from the fact that some of the q's are measured relative to a moving reference. T' is the kinetic energy, assuming that any moving constraints or reference frames are held fixed.

In summary, then, the energy $T' + V'$ is constant for any conservative system, but this energy is not always the total energy of the System measured relative to an inertial frame.

Natural systems :-

A natural system is a conservative system which has the additional properties that (1) It is described by the standard holonomic form of Lagrange's Eqn and (2) The kinetic energy is expressed as a homogeneous quadratic function of the $\dot{q}$'s, i.e)

$$T = T_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \quad \cdots\cdots \rightarrow (153)$$

where the inertial coefficients m_{ij} may be functions of the q's but not of time.

The Jacobi integral is particularly simple for a natural system; it is equal to the total energy. We see this by noting first that $T_0 = T_1 = 0$ and therefore Eq.(150) becomes

$$T + V = h \quad \cdots\cdots \rightarrow (154)$$

indicating that the total energy is conserved.

Although the transformation equations relating the x's and the q's may contain t explicitly for certain conservative systems, this is no longer possible for a natural system. Since $T_0 = 0$, we see from (10), that $\partial x_k / \partial t$ must be zero $\forall$ k.

Now let us consider the form of the eqns of motion for a natural system. First, since $T_1 = 0$ & T_2 is not an explicit function of time, we find from Eqns (6) & (8),

(40) that $a_i = 0$, $\frac{\partial m_{ij}^0}{\partial t} = 0$ $(i, j = 1, 2, \dots n)$

then, referring to Eqn (37), we obtain

$$\sum_{j=1}^{n} m_{ij}^0 \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \left(\frac{\partial m_{ij}^0}{\partial q_l} + \frac{\partial m_{il}^0}{\partial q_j} - \frac{\partial m_{jl}^0}{\partial q_i} \right) \dot{q}_j \dot{q}_l + \frac{\partial V}{\partial q_i} = 0$$

$$(i = 1, 2, \dots n) \rightarrow (156)$$

comparing eqns (37) & (156), we see that the diff eqns of motion for a natural system are considerably simpler than those for a more general conservative system. In particular, the eqns describing a natural system contain no linear terms in the $\dot{q}$'s. consequently, the $\dot{q}$'s appear only as quadratic terms.

It is important to notice that a holonomic conservative system with $T_1 = 0$ & $T_0 \neq 0$ has eqns of motion which are very ∥n to Eq. (156) even though it is not considered as a natural system. This ∥lyrity results from the fact that if T_0 is viewed as a part of potential energy $V' = V - T_0$, then the remaining kinetic energy T_2 is quadratic in the $\dot{q}$'s, as in a natural system. Hence, the eqns of motion are those of a natural system V' as the potential energy function & T_2 as its kinetic energy.

A holonomic ~~kinetic~~ conservative system with $T_1 \neq 0$, is, in general, a gyroscopic system. The presence of T_1 results in terms of the form $\gamma_{ij} \dot{q}_j$ in the eqns of motion, where the coefficients γ_{ij} form a skew symmetric matrix, ~~that~~ as was shown in Eq. (40).

Example: 2.8 :- Suppose a mass-spring System is attached to a frame which is translating with a uniform velocity v_0, as shown in Fig (2.8). Let l_0 be the unstressed spring length and use the elongation x as the generalized coordinate. Find the Jacobi integral for the system.

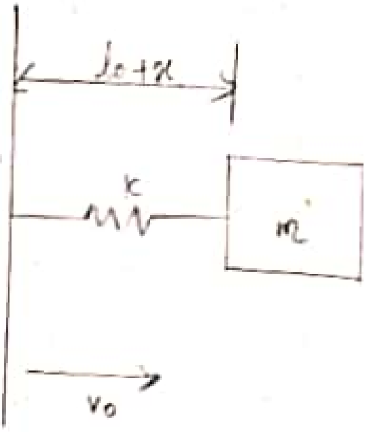


Fig (2.8). A translating mass-spring system.

The kinetic energy is

$$T = \frac{1}{2} m(v_0 + \dot{x})^2 \quad \text{.......} \rightarrow (157)$$

which yields

$$T_2 = \frac{1}{2} m\dot{x}^2$$

$$T_1 = mv_0\dot{x} \quad \text{....} \rightarrow (158)$$

$$T_0 = \frac{1}{2} mv_0^2$$

The potential energy is

$$V = \frac{1}{2} k x^2 \quad \text{......} \rightarrow (159)$$

The mass spring system meets all the conditions of a holonomic conservative system since T & V are not explicit functions of time, and the only generalized force Q_x is derivable from V. Although the moving frame does work on the system, resulting in a changing total energy T+V, the Jacobi integral exists and is equal to

$$T_2 - T_0 + V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} mv_0^2 + \frac{1}{2} kx^2 = h \quad \rightarrow (160)$$

where h is a constant. T_0 is constant in this example, so we see that $T_2 + V$ is also constant.

Another approach is to notice that the moving frame is a valid inertial reference. Relative to this frame, we have

$$T' = \frac{1}{2} m \dot{x}^2 \qquad \rightarrow (161)$$

Since T' is quadratic in $\dot{x}$, we see that we have a natural system relative to this reference frame. Hence, $T' + V'$ is constant, that is, the total energy is conserved. Note that this energy is identical with $T_2 + V$ measured relative to the fixed frame.

Example 2.9:

Let us consider again the system discussed in Ex. 2.4. A small tube, bent in the form of a circle of radius r, rotates about a vertical diameter with a constant angular velocity ω. A particle of mass m can slide without friction inside the tube. At any given time, the configuration of the system is specified by the angle θ which is measured from the upward vertical to the line connecting the center O and the particle. Find the Jacobi integral.

Suppose we assume a fixed Cartesian reference frame with its origin at O and with the z axis vertical. The plane of the tube coincides with the xz plane at t=0. The transformation eqns relating the generalized coordinate θ and the position (x, y, z) of the particle are the following:

$$x = r \sin\theta \cos\omega t$$
$$y = r \sin\theta \sin\omega t \qquad \rightarrow (162)$$
$$z = r \cos\theta$$

Since the transformation eqns contain time

explicitly, this is a rheonomic system.
It is also holonomic & has the same num of degrees of freedom as generalized coordinates, namely, one.

The kinetic & potential energy functions are

$$T = \frac{1}{2} m (a^2 \dot{\theta}^2 + a^2 \omega^2 \sin^2\theta) \quad \dashrightarrow (163)$$

$$V = mga\cos\theta \quad \dashrightarrow (164).$$

∴ The Lagrangian function is,

$$L = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m a^2 \omega^2 \sin^2\theta - mga\cos\theta \quad \dashrightarrow (165)$$

We see that Lagrangian L is not an explicit func of time, even though the system is rheonomic. Hence, the system is conservative. Its Jacobi integral is

$$T_2 - T_0 + V = \frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2\theta + mga\cos\theta = h \quad (166)$$

Now, let us consider the Lagrangian function of Eq. (165) to be of the form

$$L = T' - V' \quad \dashrightarrow (167)$$

where $$T' = T_2 = \frac{1}{2} m a^2 \dot{\theta}^2 \quad \dashrightarrow (168)$$

and $$V' = V - T_0 = mga\cos\theta - \frac{1}{2} m a^2 \omega^2 \sin^2\theta \dashrightarrow (169)$$

We see that T′ is the kinetic energy of the particle relative to a reference frame which is rotating with the circular tube. The potential energy V′ includes the actual gravitional energy plus another term $-T_0$ which accounts for the centrifugal force due to the rotation about the vertical axis.

The result of taking the viewpoint of

a rotating observer is that T' and V' are of the form associated with a natural system: (i.e), neither T' nor V' is an explicit function of time, T' is quadratic in $\dot{\theta}$, & V' is a function of the position θ only. Hence the total energy relative to this rotating frame is conserved, we obtain

$$T' + V' = h \quad \ldots\ (1.10)$$

In agreement with the Jacobi integral of Eq. (166).

V' is plotted as a function of θ in Fig 2.9 for the case where $\omega^2 > g/r$. Equilibrium points relative to the rotating frame occur at those values of θ for which $dV'/d\theta = 0$, that is, at

$$\theta = 0,\ \pi,\ \cos^{-1}\left(\frac{-g}{r\omega^2}\right)$$

The values 0 and π occur at local maxima of V' and $\therefore$ are points of unstable equilibrium, while the values $\cos^{-1}(-g/r\omega^2)$ correspond to stable equilibrium points at local minima of V'.

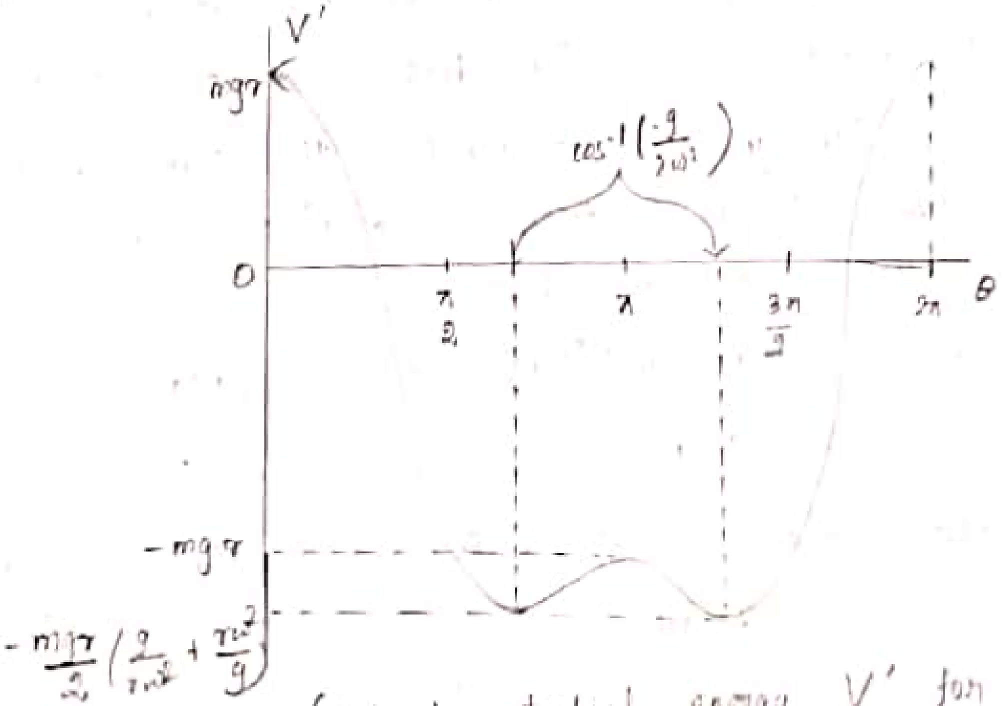


fig (2.9) potential energy V' for the case $\omega^2 > g/r$

If $0 \le \omega^2 \le g/r$, there is no longer a hump in (λ^5) the curve near $\theta = \pi$. In this case, there is a maximum of V' at $\theta = 0$ and a minimum at $\theta = \pi$.

It is interesting to consider the same system using the spherical coordinates θ & ϕ as generalized coordinates. In this case, we obtain

$$T = \frac{1}{2} m (r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) \quad \cdots \rightarrow (171)$$

$$V = mgr \cos\theta \quad \cdots \rightarrow (172)$$

The holonomic constraint can be expressed in the differential form

$$d\phi - \omega dt = 0 \quad \cdots \rightarrow (173)$$

Here, we see that neither T nor V is an explicit func of time & the standard form of Lagrange's eqn applies. Nevertheless, the system is not conservative in this formulation because $a_{\phi} = -\omega f_0$. Furthermore, the Jacobi expression ($T+V$ in this case) is not constant because the moving constraint does work on the system.

Finally, let us consider the same system in terms of the Cartesian coordinates of the particle. Here, we have

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad (174)$$

$$V = mgz \quad (175)$$

There are 2 holonomic constraints, namely,

$$x^2 + y^2 + z^2 = r^2 \quad \rightarrow (176)$$

$$y = x \tan\omega t \quad \rightarrow (177)$$

Written in differential form, these constraints

$$\tan\omega t\, dx - dy + \omega x \sec^2 \omega t\, dt = 0 \quad \ldots (179)$$

Looking at Eq. (179), we see that

$$a_{1t} = \omega x \sec^2 \omega t \neq 0$$

Therefore the system is not conservative. Again, the Jacobi expression is the total energy T+V which is not conserved.

Example: 2.10:-

Let us consider once again the system of (2.6).

Two particles, each of mass m, are connected by a rigid massless rod of length l (Fig 2.6). The particles are supported by knife edges placed ⊥r to the rod. Assuming that all motion is confined to the horizontal ~~p~~ xy plane Find the Jacobi integral.

Since θ is given explicity as a function of time, the system is ~~holonomic~~ rheonomic. Let us take the coordinates (x,y) of the centre of mass as generalized coordinates, we note that the potential energy is zero, So the Lagrangian function is

$$L = T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} m l^2 \omega^2 \quad \ldots\ldots (180)$$

The non holonomic constraint eqn is

$$\cos\omega t\, dx + \sin\omega t\, dy = 0 \quad \ldots\ldots (181)$$

Which specifies that the velocity of the center of mass must be ⊥r to the rod.

We observe that the Lagrangian L is not an explicit function of time, even though the system and the coefficients in the constraint eqn are explicit funcs of time. Furthermore, we see that $\partial L/\partial t = 0$.

Hence, the system is conservative. Its Jacobi integral is, $T_2 - T_0 + V = m(\dot{r}^2 + \dot{y}^2) - \frac{1}{4} m l^2 \omega^2 = h$ --> (18

Note that T_0 is a constant, implying that the velocity of the center of mass is constant.

Another approach to this example is to describe the system configuration by using the Cartesian coordinates of its 2 particles. In this case the Lagrangian func is,

$$L = \tfrac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \quad \text{----}> (183)$$

There are 2 holonomic eqns of constraint,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \quad \text{.....}> (184)$$

$$(x_2 - x_1)\sin\omega t - (y_2 - y_1)\cos\omega t = 0 \quad \text{--}> (185)$$

giving the length and orientation of the rod. There is also a rheonomic non holonomic constraint which can be described by

$$\cos\omega t \, dx_1 + \sin\omega t \, dy_1 = 0 \quad \text{----}> (186)$$

expressing the fact that the velocity at the gn particle is ⊥ to the rod. A similar eqn could be written for the second particle, but it would not be independent of the constraint eqn already gn.

Eqns (184) & (155) can be combined with (186) to yield

$$(x_2 - x_1)\,dx_1 + (y_2 - y_1)\,dy_1 = 0 \quad > (187)$$

$$(x_2 - x_1)\,dx_2 + (y_2 - y_1)\,dy_2 = 0 \quad \text{...}> (188)$$

then we see that Eqs (186)-(186) represent a set of (188) s independent constraint eqns which meet the condition that $a_{fl} = 0$. Hence, the sufficient conditions for a conservative system have been met the existance of the Jacobi integral implies that the total kinetic energy is constant.

It is intersecting to note that if the orientation $\theta(t)$ were given as any continuous func of time, the Lagrangian L would, in general, be an explicit func of time, implying that the system is not conservative. Nevertheless, if we assume that the given orientation is enforced by the application of a couple

$$M = \frac{1}{2} m l^2 \ddot{\theta}$$

then the translational kinetic energy is conserved because no work is done by the constraint forces moving over the path of the center of mass. Of course, the rotational kinetic energy changes as $\dot{\theta}$ changes.

Liouville's system :-

If a conservative holonomic system does not have a sufficient num of ignorable coordinates to guarantee separability, it may still be separable if it is an orthogonal system, i.e), a natural system in ~~in~~ which T contains only $\dot{q}_i^2$ terms and no cross-products in the $\dot{q}$'s.

As an example, suppose that

$$T = \frac{1}{2} f \sum_{i=1}^{n} \dot{q}_i^2 \quad \cdots\cdots\rightarrow (189)$$

$$V = \frac{1}{f}\sum_{i=1}^{n} v_i(q_i) \dashrightarrow (190)$$

(R) Where we define

$$f = \sum_{i=1}^{n} f_i(q_i) > 0 \dashrightarrow (191)$$

We will ~~sh~~ now show that this system that this system is separable.

Let us use Lagrange's eqn in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \dashrightarrow (192)$$

and we obtain

$$\frac{d}{dt}(f\dot{q}_i) - \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{j=1}^{n} \dot{q}_j^2 + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i} = 0 \rightarrow (193)$$

Now, this is a natural system, so it has an energy integral given by

$$T + V = \frac{1}{2} f \sum_{j=1}^{n} \dot{q}_j^2 + V = h \dashrightarrow (194)$$

Hence,

$$\frac{1}{2}\sum_{j=1}^{n}\dot{q}_j^2 = \frac{1}{f}(h - V) \dashrightarrow (195)$$

Sub from Eq. (195) in (193) & simplifying, we have

$$\frac{d}{dt}(f\dot{q}_i) - \frac{h}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \rightarrow (196)$$

Next, we multiply Eqn (196) by $2f\dot{q}_i$ & obtain

$$\frac{d}{dt}(f^2\dot{q}_i^2) - 2h\frac{\partial f_i}{\partial q_i}\dot{q}_i + 2\frac{\partial v_i}{\partial q_i}\dot{q}_i = 0$$

(or)

$$\frac{d}{dt}(f^2\dot{q}_i^2) = 2\frac{d}{dt}(hf_i - v_i) \rightarrow (197)$$

Integrating, the result is,

$$f^2\dot{q}_i^2 = 2[hf_i(q_i) - v_i(q_i) + c_i] \quad (i = 1, 2, \dots n) \rightarrow$$

where the c's are constants of Integration. It can shown from Eqs. (190), (194) & (198) that

$$\sum_{i=1}^{n} c_i = 0 \quad \dashrightarrow (199)$$

Hence the c's are constants of integration. It can be shown from Eqs. (190), (194), (198) that

$$\sum_{i=1}^{n} c_i = 0 \quad \dashrightarrow$$

where the c's and h together comprise n independent constants of the motion.

The remaining n integrals of the motion are obtain by writing Eq. (198) in the form

$$\frac{dq_i}{dt} = \frac{\sqrt{2(hf_i - V_i + c_i)}}{f}$$

which implies that

$$\frac{dq_1}{\sqrt{2(hf_1 - V_1 + c_1)}} = \frac{dq_2}{\sqrt{2(hf_2 - V_2 + c_2)}} = \cdots$$

$$= \frac{dq_n}{\sqrt{2(hf_n - V_n + c_n)}} = \frac{dt}{f} = d\tau \quad \dashrightarrow (200)$$

where τ is a time-like parameter. Each diff expression is a function of a single q_i, so the problem is reduced to quadratures. Integrating these expressions produces the required n additional constants of motion.

This system can be generalized rather easily to become a Liouville system. If we replace dq_i, by $\sqrt{M_i(q_i)}\,dq_i$, we obtain

$$T = \frac{1}{2} f \sum_{i=1}^{n} M_i(q_i)\dot{q}_i^2 \quad \dashrightarrow$$

where we assume that $M_i(q_i) > 0$. As before,

$$V = \frac{1}{f}\sum_{i=1}^{n} V_i(q_i) \quad (202)$$

A natural system having T and V of the form given by Eqs (201) & (202) is called a Leouville system.

Corresponding to Eq. (200), we now have

$$\frac{dq_1}{\sqrt{\phi_1(q_1)}} = \frac{dq_2}{\sqrt{\phi_2(q_2)}} = \cdots\cdots = \frac{dq_n}{\sqrt{\phi_n(q_n)}} = \frac{dt}{f} = d\tau \quad (203)$$

where,

$$\phi_i(q_i) = \frac{2}{M_i}(hf_i - V_i + C_i) \quad (i = 1, 2, \ldots n) \quad (204)$$

Using Eqs (191) & (203), we obtain $\displaystyle\sum_{i=1}^{n} \frac{f_i\, dq_i}{\sqrt{\phi_i(q_i)}} = dt$ (20

(or) $$\sum_{i=1}^{n} \int \frac{f_i\, dq_i}{\sqrt{\phi_i(q_i)}} = \cancel{d}\, t + \beta_1 \quad (206)$$

||ly taking differences of the infinite integrals of Eq. (203), we have,

$$\int \frac{dq_1}{\sqrt{\phi_1(q_1)}} - \int \frac{dq_j}{\sqrt{\phi_j(q_j)}} = \beta_j \quad (j = 1, 2 \ldots n) \quad (207)$$

where the 1st integral is chose arbitrarily as a reference. Thus Eq. (206) & (207) provide n independent constants $\beta_1, \beta_2, \cdots \beta_n$ which, with the previous (n-1) independent c's & the energy constant h, constitute the required 2n independent constants of the motion.

In evaluating the integrals of Eqns. (206) & (207), a ques arises concerning the sign of $\sqrt{\phi_j(q_j)}$. Remembering that f is +ve, we find from (203) that $\sqrt{f_i(q_i)}$ has the same sign as dq_i. This is of particular importance in studying libration motions, i.e, motions in which one or more q's oscillate b/w fixed limiting values.

Examples: 8.11 :

Consider again the spherical pendulum of Example 8.1. Reduce the problem to quadratures and obtain the integrals of the motion.

Method : 1 :- Ignoration of coordinates :-

Using the spherical coordinates θ & ϕ, the expressions for the kinetic & potential energies are

$$T = \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \rightarrow (208)$$

$$V = mgl\cos\theta \rightarrow (209)$$

where m is the mass of a particle which is suspended by a massless string of length l.

Here we have a conservative holonomic system having two degrees of freedom and one ignorable coordinate. Hence it can be solved completely by quadratures.

First, we see that the Lagrangian func is,

$$L = \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mgl\cos\theta \rightarrow (210)$$

since ϕ does not appear explicitly, it is an ignorable coordinate, and we have

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta = \alpha_\phi \rightarrow (211)$$

where we now adopt the notation that α_ϕ is the constant generalized momentum conjugate to ϕ.

In other words, the angular momentum is conserved about a vertical axis through the Support. (5)

The Routhian function is

$$R = L - \alpha_\phi \dot{\phi} = \frac{1}{2} ml^2 \dot{\theta}^2 - \frac{\alpha_\phi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta \dashrightarrow (212)$$

and we note that $\dot{\phi}$ has been eliminated by using Eq.(211). Thus, we see that

$$R = T' - V' \dashrightarrow (213)$$

where

$$T' = \frac{1}{2} ml^2 \dot{\theta}^2 \dashrightarrow (214)$$

$$V' = \frac{\alpha_\phi^2}{2ml^2 \sin^2\theta} + mgl\cos\theta \quad) (215)$$

The form of T' & V' is that of a natural system having one degree of freedom. Hence, we can immediately write the energy integral.

$$T' + V' = \frac{1}{2} ml^2 \dot{\theta}^2 + \frac{\alpha_\phi^2}{2ml^2 \sin^2\theta} + mgl\cos\theta = h \dashrightarrow (216)$$

Solving for $\dot{\theta}$, we obtain

$$\dot{\theta} = \sqrt{\frac{2}{ml^2} \left(h - mgl\cos\theta - \alpha_\phi^2 / 2ml^2 \sin^2\theta \right)}$$

(or)

$$\frac{ml^2 \sin\theta \, d\theta}{\sqrt{2ml^2 \sin^2\theta \left(h - mgl\cos\theta \right) - \alpha_\phi^2}} = dt \quad (217)$$

Integrating, we have

$$\int_{\theta_0}^{\theta} \frac{ml^2 \sin\theta \, d\theta}{\sqrt{2ml^2 \sin^2\theta \left(h - mgl\cos\theta \right) - \alpha_\phi^2}} = t - t_0 \quad (218)$$

where $\theta(t_0) = \theta_0$. The motion in θ is usually a libration with $0<\theta<\pi$. Hence the sign of the square root should be the same as that of $d\theta$ since each indepe increment dt is +ve

From, Eq.(211) we obtain

$$d\phi = \frac{\alpha_\phi \, dt}{ml^2 \sin^2\theta} \quad \cdots \rightarrow (219)$$

and using Eq.(217), we find that

$$\frac{\alpha_\phi \, d\theta}{\sin\theta\sqrt{2ml^2 \sin^2\theta(h-mgl\cos\theta)-\alpha_\phi^2}} = d\phi \; \cancel{d\theta} \rightarrow (220)$$

Hence,

$$\int_{\theta_0}^{\theta} \frac{\alpha_\phi \, d\theta}{\sin\theta\sqrt{2ml^2 \sin^2\theta(h-mgl\cos\theta)-\alpha_\phi^2}} = \phi-\phi_0 \rightarrow (221)$$

where $\phi(t_0)=\phi_0$. Again the sign of the square root is the same as that of $d\phi$. Note that α_ϕ has the same sign as $\dot{\phi}(t_0)$.

Thus, we have obtained the 4 required constants of the motion, namely, the expressions for α_ϕ, h, t_0, and ϕ_0 given by Eqs. (211), (216), (218), (220).

Method : 2 :

Now consider the spherical pendulum as a liouville system. Comparing the expressions for T & V given in Eqs. (208) & (209) with the standard Liouville forms of Eqs (201) & (202), we find that

$$f_\theta = ml^2 \sin^2\theta, \quad M_\theta = \frac{1}{\sin^2\theta}, \quad V_\theta = m^2gl^3 \sin^2\theta\cos\theta \rightarrow (222)$$

and

$$f_\phi = 0, \quad M_\phi = 1, \quad V_\phi = 0 \rightarrow (223)$$

Using Eq. (209), we obtain

$$\Phi_\theta = 2\sin^2\theta\,[\,ml^2\sin^2\theta\,(h - mgl\cos\theta) + C_\theta] \quad \cdots (222)$$

$$\Phi_\phi = 2C_\phi \cdots\rightarrow \quad \Big\} (222)$$

Where, from Eqs. (198), (199) & (211), we have

$$2C_\phi = -2C_\theta = (ml^2\dot\phi\sin^2\theta)^2 = \alpha_\phi^2 \cdots\rightarrow (223)$$

Finally, sub these expressions into Eq.s (206) & (207) & writing the results in the form of definite integrals, we obtain

$$\int_{\theta_0}^{\theta} \frac{ml^2\sin^2\theta\, d\theta}{\sqrt{\Phi_\theta(\theta)}} = t - t_0 \cdots\rightarrow (226)$$

and
$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\Phi_\theta(\theta)}} = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{2C_\phi}}$$

(or)
$$\int_{\theta_0}^{\theta} \frac{\alpha_\phi\, d\theta}{\sqrt{\Phi_\theta(\theta)}} = \phi - \phi_0 \cdots\rightarrow (227)$$

We see that Eqs. (226) & (227) agree with Eqs (215) & (221) obtained previously.

Sec: 2.4: SMALL OSCILLATIONS :-

Equations of Motion :-

Suppose we have a natural system whose configuration is specified by the n independent generalized coordinates $q_1, q_2, \cdots q_n$. Let us assume that the q's are measured from a position of equilibrium, and consider small motions about this equilibrium position.

From eq. (1.95), we find that if we let the reference value V_0 be zero, the potential energy can be written in the form

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right)_0 q_i q_j + \ldots\ldots \quad (228)$$

Neglecting terms of higher order than the second in the q's, we obtain

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\,q_i\,q_j \quad \to (229)$$

where the stiffness coefficients are

$$k_{ij} = k_{ji} = \left(\frac{\partial^2 V}{\partial q_i\,\partial q_j}\right)_0 \quad \ldots\ldots\to (230)$$

Thus, we see that the potential energy V is a homogeneous quadratic function of the q's for all small motions near a position of equilibrium.

Let us assume that the system consists of N particles whose positions are given by the $3N$ cartesian coordinates $x_1, x_2, \ldots\ldots x_{3N}$. We find from Eqs. (7) & (153) that the kinetic energy is of the form.

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\,\dot{q}_i\,\dot{q}_j \quad \ldots\ldots\to (231)$$

where, for small motions,

$$m_{ij} = m_{ji} = \sum_{k=1}^{3N} m_k\left(\frac{\partial x_k}{\partial q_i}\right)_0\left(\frac{\partial x_k}{\partial q_j}\right)_0 \quad \ldots\to (232)$$

Also, for the natural system assumed here, the kinetic energy is a +ve definite quadratic function of the $\dot{q}$'s.

The equations of motion are obtained by using Lagrange's equations. First, we find from Eq. (26) that the Lagrangian func is,

(5.1) $$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}q_iq_j \rightarrow (233)$$

Then using Eq. (14), we obtain the following Eqns :

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij}q_j = 0 \quad (i=1, 2 \ldots n) \ldots \rightarrow (234)$$

Or, in matrix form, $m\ddot{q} + kq = 0 \rightarrow (235)$

Notice that these equations of motion are linear, second-order, ordinary diff eqns. Also, the m & k ~~are~~ matrices are constant and symmetric. In general, if Newton's laws are used in obtaining the eqns of motion for a system of this sort, the m and k matrices, will not be symmetric.

Natural Modes :

Let us consider a system whose diff eqns of motion are given by Eq. (234). Assume solutions of the form

$$q_j = A_j C \cos(\omega t + \phi) \quad (j = 1, 2, \ldots n) \ldots \rightarrow (236)$$

where the amplitude of the oscillation in q_j is the product of the constants A_j & C. Here C acts as an overall scale factor for the q's, whereas the A's indicate their relative magnitudes

If we substitute the trial solutions of Eq. (236) into Eq. (234), we obtain

$$\sum_{j=1}^{n} (-\omega^2 m_{ij} + k_{ij}) A_j C \cos(\omega t + \phi) = 0 \quad (i=1,2,\ldots n) \dashrightarrow (2.37)$$

The factor $C \cos(\omega t + \phi)$ cannot be zero continuously except for the trivial case in which all the q's remain zero. ∴, we conclude that

$$\sum_{j=1}^{n} (k_{ij} - \omega^2 m_{ij}) A_j = 0 \quad (i=1,2,\ldots n) \dashrightarrow (2.38)$$

If the A's are not all zero, then the determinant of their coefficients must vanish, that is,

$$\begin{vmatrix} (k_{11} - \omega^2 m_{11}) & (k_{11} - \omega^2 m_{12}) & \ldots\ldots & (k_{1n} - \omega^2 m_{1n}) \\ (k_{21} - \omega^2 m_{21}) & (k_{22} - \omega^2 m_{22}) & & \vdots \\ \vdots & & & \vdots \\ (k_{n1} - \omega^2 m_{n1}) & \ldots\ldots & \ldots\ldots & (k_{nn} - \omega^2 m_{nn}) \end{vmatrix} = 0 \quad (2.39)$$

The evaluation of this determinant results in an nth-degree alegebric eqn in ω^2 which is called the characteristic eqn.

The nth roots ω_k^2, where $k=1,2,\ldots n$ are known as characteristic values or eigen values, each being the square of a natural frequency which is usually expressed in rad/sec. It is known from matrix theory that the roots ω_k^2 are all real & finite if T is +ve definite and if both m and k are real symmetric matrices.

If in addition, V is +ve definite, then the ω_k^2 are all +ve & the motion occurs about a position of stable equilibrium. If V is +ve semifinite; i.e), if the determinant |k| or any of its principal minors is zero, but none are -ve; then at least one of the ω_k^2 is 0 & the System is in natural equilibrium at the reference configuration. Finally, if |k| or any of its principal minor is -ve, then atleast one of the ω_k^2 is -ve & the reference position is one of unstable equilibrium.

Let us summarize the effect of a given root ω_k^2 on the stability of the system is as follows :

Case(1) :- $\omega_k^2 > 0$. The solution for each coordinate contains a term of the form $C_k \cos(\omega_k t + \phi_k)$ in accordance with the assumed solution given by Eq. (236). This term is stable, but the overall stable depends upon the other roots as well.

Case(2) :- $\omega_k^2 = 0$. Here we have a repeated zero root ω_k, corresponding to a term in the solution of the form $(C_k t + D_k)$. This implies that a steady drift can occur in one or more of the coordinates and is characteristic of a naturally stable system.

Case(3) :- $\omega_k^2 < 0$. The corresponding pair of roots $\omega_k = \pm i \gamma_k$ is imaginary and the resulting terms of the form $(C_k \cos\gamma_k t + D_k \sinh \gamma_k t)$ imply an unstable solution.

Returning now to Eq. (238), we find that for each ω_k^2 we can write a set of n simultaneous algebraic eqns involving the n amplitude coefficients A_j. Because these eqns are homogenous, however, there is no unique soln for the amplitude coefficients, but only for the ratios among them. For convenience, let us solve for the amplitude ratios with respect to A_1, i.e), let us take $A_1 = 1$. If we arbitrarily eliminate the first eqn of (2.38) & use matrix notation in solving the remaining (n-1) eqns for the (n-1) unknown amplitudes, we obtain

$$[(k_{ij} - \omega_r^2 m_{ij})]\{A_j^{(k)}\} = \{(\omega_r^2 m_{i1} - k_{i1})\}$$

$(i,j = 2,3 \dots n)$ ...> (2.40)

where we have shown a typical element of the matrix in parentheses in each case. The subscripts refer to the original rows & columns. Solving Eq. (2.40) for the amplitudes $A_j^{(k)}$ corresponding to the eigen value ω_r^2, we obtain

$$\{A_j^{(k)}\} = [(k_{ij} - \omega_r^2 m_{ij})]^{-1}\{(\omega_r^2 m_{i1} - k_{i1})\}$$

$(i,j = 2,3,\dots n)$ ....> (2.4

where we assume that the eigenvalues are distinct and $A_1^{(k)} \neq 0$, there by ensuring the existance of the inverse matrix. In case, $A_1^{(k)} = 0$, another coordinate should be choosen as a reference. In general, one can use any normalization procedure which preserves the correct relative magnitudes of the coordinates.

A complete set of n amplitude coefficients, including the unit reference coefficient, is known as an eigenvector or modal column. There is an eigenvector $A^{(r)}$ corresponding to each eigenvalue ω_r^2. This set of amplitude coefficients can be considered to define a mode shape associated with the given frequency ω_k. Each natural frequency ω_k with its corresponding eigenvector $A^{(k)}$ defines a natural mode of vibration, sometimes called a principal mode or a normal mode.

(b) For the natural systems that we are considering, the ω_k^2 are real & the corresponding amplitude ratios are also real. If the system is vibrating in a single mode rather than the more general case of a superposition of modes, then all the coordinates execute sinusoidal motion at the same frequency. The relative phase ~~single~~ angle b/w any two coordinates is ~~ne~~ either 0° or 180°, depending upon whether the particular amplitude ratio is +ve or -ve.

The zero frequency modes are somewhat different physically in that no elastic deformation occurs. For this reason, they are known as rigid-body modes. Both the potential & kinetic energies are constant, resulting in uniform translational or rotational motion. The amplitude ratios are calculated in the usual fashion, but are more easily considered as velocity ratios.

Principal Coordinates:-

Using solns of the form of Eq. (2.236), we have

$$q_j = \sum_{k=1}^{n} A_{jk} C_k \cos(\omega_k t + \phi_k) \qquad (2.42)$$

where $A_{jk} = A_j^{(k)}$

Now, let us define the principal coordinate U_k corresponding to the kth mode by the eqn

$$U_k = C_k \cos(\omega_k t + \phi_k) \quad (k = 1, 2, \dots n)$$

Then from Eqs. (242) & (243) we obtain

$$q_j = \sum_{k=1}^{n} A_{jk} U_k \dashrightarrow (244)$$

or, in matrix notation

$$q = AU \dashrightarrow (245)$$

where the modal matrix A is an nxn matrix whose columns are the modal columns for the various modes. The columns may be arranged in any order but, by convention, they are usually placed of increasing frequency of the corresponding modes.

From Eq. (245) we can solve for the principal coordinates. we obtain

$$U = A^{-1}q \dashrightarrow (246)$$

where we assume that the modal columns are linearly independent, a condition which is assured if the corresponding eigenvalues are distinct.

Note that the principal coordinates are generalized coordinates of a particular type. We see from Eq. (244) that if only one U_k is nonzero, the q's are proportional to the corresponding A's for that mode. In this case, the principal coordinate U_k oscillates sinusoidally at the frequency ω_k, assuming, of course, that the given mode is stable. Hence, if only one mode is excited, the entire motion is described by using only one principal coordinate. In general, a description of the motion requires all n principal coordinates. Nevertheless, since we are often interested in the transient response within a certain frequency range, the use of coordinates associated specifically with given frequencies may make possible a reduction in the number of coordinates used in the analysis.

Orthogonality of the Eigenvectors:-

Let us write Eq. (238) for the kth natural mode in the form

$$kA^{(k)} = \omega_k^2 mA^{(k)} \quad \dots\rightarrow (247)$$

|||ly, for the lth mode we have

$$kA^{(l)} = \omega_l^2 mA^{(l)} \quad \dots\rightarrow (248)$$

Now premultiply both sides of Eq. (247) by the row matrix $A^{(l)T}$ and premultiply both sides of Eq. (298) by $A^{(k)T}$. The resulting Eqns are

$$A^{(l)T} k \cdot A^{(k)} = \omega_k^2 A^{(l)T} mA^{(k)} \quad \rightarrow (249)$$

$$A^{k(T)} \cdot kA^{(l)} = \omega_l^2 A^{(k)T} mA^{(l)} \quad \rightarrow (250)$$

Since k and m are symmetric, the modal columns can be interchanged on both sides of either eqn. If we perform this operation on Eq. (249), we find that its r.h.s is identical with that of Eq. (250). Subtracting Eq. (220), we obtain

$$(\omega_k^2 - \omega_l^2) A^{(k)T} mA^{(l)} = 0 \quad \dots\rightarrow (251)$$

If the 2 eigenvalues are distinct, i.e. if $\omega_k^2 \neq \omega_l^2$, then it follows that

$$A^{(k)T} mA^{(l)} = 0 \quad (k \neq l) \quad \dots\rightarrow (252)$$

This is the orthogonality condition for the kth and lth eigenvectors with respect to the inertia matrix m.

Insight into the physical meaning of Eq. [illegible] is obtained by noting that a modal column of amplitude coefficients gives the velocity and the acceleration ratios as well as the displacement ratios for a given mode. So the Orthogonality condition can be interpreted as stating that the Scalar product of the eigenvector for one mode & the generalized inertia force vector for another mode is zero. This indicates that the 2 vectors are orthogonal in n-space. In practical terms, there is no inertial coupling b/w the corresponding principal coordinates.

If one of the modal columns of Eq. (202) corresponds to a ~~zero~~ zero-frequency mode, the Orthogonality condition implies that a corresponding momentum component is 0 ∀ the remaining elastic modes. For examples, if a system can freely translate in the x direction, a zero-frequency mode will exist corresponding to this uniform translation & it will contain the entire x component of translational momentum. All the remaining elastic modes will have no translational momentum in the x direction. More generally, the elastic modes of a body in free space will have zero linear & angular momentum.

Now let us consider the matrix product of Eq. (252) for the case where k=l. In this instance, the product cannot be zero because it is proportional to the kinetic energy in the given mode. So let us write

$$A^{(k)T} m A^{(k)} = M_{kk} \quad (k = 1, 2, \dots n) \cdots \rightarrow (253)$$

where M_{kk} is is a +ve constant. We will show that M_{kk} is the generalized mass or inertia coefficient correspondinding to the principal coordinate U_k.

Eqns (252) & (253) can be summarized in the eqn $A^T m A = M$ $\rightarrow (254)$

where we recall that each column of A is a modal column. Because of the orthogonality property of the eigen vectors, we see that M is a diagonal matrix.

Now let us consider again the expression for the kinetic energy of the system. From eqns (231) & (215) we have

$$T = \frac{1}{2} \dot{q}^T m \dot{q} = \frac{1}{2} \dot{U}^T A^T m A \dot{U} \cdots \rightarrow (255)$$

and using Eq. (254), we obtain

$$T = \frac{1}{2} \dot{U}^T M \dot{U} \quad \cdots \rightarrow (256)$$

Conforming that M is the generalized mass matrix associated with the principal coordinates.

We can show that the transformation to principal coordinates also diagonalizes the stiffness matrix k. Let us write Eqs. (249) & (250) in the form

$$\frac{1}{\omega_k^2} A^{(l)T} k \cdot A^{(k)} = A^{(l)T} m A^{(k)} \quad \rightarrow (257)$$

$$\frac{1}{\omega_l^2} A^{(k)T} \cdot k A^{(l)} = A^{k(T)} \cdot m A^{(l)} \quad \rightarrow (258)$$

where we assume that ω_k^2 & ω_l^2 are not

Again we note that k and m are symmetric; hence the modal columns for the kth & lth modes can be interchanged. performing this interchange in (257) & subtracting (258), we obtain

$$\left(\frac{1}{\omega_k^2} - \frac{1}{\omega_l^2} \right) A^{(k)T} k \cdot A^{(l)} = 0 \quad \ldots \rightarrow (259)$$

If the modes are distinct, we have $\omega_k^2 \neq \omega_l^2$ & it follows that

$$A^{(k)T} k A^{(l)} = 0 \quad (k \neq l) \ldots \rightarrow (260)$$

For the case where either ω_k^2 or ω_l^2 is zero, the same result is obtained directly from Eqs. (249) & (250).

Eqn. (260) represents the orthogonality condition with respect to the stiffness matrix k. It states that the scalar product involving the eigen vector for a given mode & the generalized elastic force vector for another mode is zero. In other words, the principal modes are not elastically coupled because no work is done by the elastic forces of one mode in moving through the displacements of a second mode.

For the case, where k=l, we can write,

$$A^{(k)T} k \cdot A^{(k)} = k_{kk} \quad \ldots\ldots \rightarrow (261)$$

Where k_{kk} is the generalized stiffness coefficient for the kth mode. k_{kk} can be +ve, zero, or -ve according to whether the kth mode is stable, neutrally stable, or unstable. Eqns (260) & (261) are summarized in the single eqn

$$A^T k A = k \quad \ldots \rightarrow (262)$$

Where k is an n×n diagonal matrix.

The potential energy V can be written, using Eqs. (229) & (245), in the form

$$V=\frac{1}{2}q^T kq = \frac{1}{2}U^T A^T k A U \quad \ldots\ldots \rightarrow (263)$$

which can be combined with Eq. (262) to yield

$$V=\frac{1}{2}U^T K U \quad \ldots\ldots \rightarrow (264)$$

Hence K is the generalized stiffness matrix associated with the principal coordinates.

In summary, the transformation to principal coordinates has ~~reduced~~ resulted in the diagonalization of both m & k matrices. This implies that the natural modes have neither inertial nor elastic coupling & ∴ are independent for this case of unforced motion.

In order to emphasize further the independence of the natural modes, let us obtain the differential eqns of motion in terms of the principal coordinates. From Eqns (256) & (264), we find that the Lagrangian function L can be written in the form

$$L = \frac{1}{2}\sum_{k=1}^{n} M_{kk}\dot{U}_k^2 - \frac{1}{2}\sum_{k=1}^{n} K_{kk} U_k^2 \quad \ldots \rightarrow (265)$$

Using Lagrange's eqn, we obtain

$$M_{kk}\ddot{U}_k + K_{kk}U_k = 0 \quad (k=1,2\ldots n) \ldots \rightarrow (266)$$

These diff eqns of motion indicate that the free vibrations of the entire system can be described in terms of n independent undamped second-order systems, each system representing a single mode. We note that the natural frequency of each mode is given by

$$\omega_k^2 = \frac{K_{kk}}{M_{kk}} \quad (k=1,2,\ldots n) \ldots (\ )$$

Repeated Roots:-

Now let us consider a degenerate system in which the eigenvalues are not all distinct. For example, suppose there is a double root such that $\omega_p^2 = \omega_{p+1}^2$. If we use Eq. (241) to solve for the modal columns corresponding to the distinct modes, we find that these modes are us mutually orthogonal. But if we let $\omega_k^2 = \omega_p^2$ in this solution eqn, we find that the inverse matrix $[(k_{ij} - \omega_p^2 m_{ij})]^{-1}$ does not exist, indicating that the corresponding set of (n-1) simultaneous algebraic eqns is not linearly independent.

In order to avoid this problem, we can choose 2 amplitude coefficients arbitrarily when $\omega_k^2 = \omega_p^2$. As an example, we might take $A_1^{(p)} = 1$, $A_2^{(p)} = 0$. Then we can solve for the (n-2) remaining amplitude coefficients from the (n-2) independent eqns contained in Eq. (240). This results in a modal column $A^{(p)}$ which is orthogonal to all the modal columns corresponding to the distinct modes, since the assumptions for the derivation of Eq. (252) are still valid.

The problem remains, however, of finding a final modal column $A^{(p+1)}$ such that it is orthogonal to $A^{(p)}$ and to the other columns. This is accomplished if we set $A_1^{(p+1)} = 1$ & use the (n-2) independent eqns from Eq. (241)

(69) plus the orthogonality condition

$$A^{(p)T} m A^{(p+1)} = 0 \quad \cdots\cdots \rightarrow (60)$$

Thus we have (n-1) eqns from which to solve for the (n-1) remaining amplitude coefficients in $A^{(p+1)}$

For the more general case of m arbitrary repeated roots, a similar procedure is followed. The eigen vector $A^{(p)}$ has m arbitrary components, and the remaining (n-m) components are obtained from the (n-m) independent eqns contained in Eq. (2.40). Each succeeding eigenvector has one fewer arbitrary component, but one more orthogonality condition. Hence, there are sufficient eqns to obtain the required components. In this fashion, a complete set of mutually orthogonal eigen vectors is obtained corresponding to the repeated roots.

The result of this procedure is a modal matrix A which diagonalizes both m & k. Note, however, that the amplitude ratios can now be chosen arbitrarily. In fact, any linear combination of the modal columns corresponding to a repeated root forms another possible modal column for that root; Columns formed in this way, however, are not necessarily mutually orthogonal.

Initial condition:-

The form of the transient solution for a natural system having n degrees of freedom was given in Eq. (2.42) namely,

$$q_j = \sum_{k=1}^{n} A_{jk} C_k \cos(\omega_k t + \phi_k)$$

where we assumed small motions about a position of stable equilibrium. Assuming that the eigenvalues ω_i^2 and the modal matrix A have been found, we must now solve for the n C's and the n ϕ's from the 2n initial conditions. The orthogonality properties of the natural modes can be used to simplify this process.

Let us assume that the initial q's & $\dot{q}$'s are given. From Eq. (242), we see that

$$q_j(0) = \sum_{k=1}^{n} A_{jk} C_k \cos\phi_k \quad (j = 1, 2, \cdots n) \dots \rightarrow (269)$$

and

$$\dot{q}_j(0) = -\sum_{k=1}^{n} A_{jk} C_k \omega_k \sin\phi_k \quad (j = 1, 2, \cdots n) \dots \rightarrow (270)$$

Now multiply each of these eqns by $m_{ij} A_{il}$ & sum over i & j. Becaus of orthogonality condition of Eqs. (252) & (252), namely,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} A_{il} A_{jk} = \begin{cases} 0 , & k \neq l \\ M_{ii} , & k = l \end{cases} \dots \rightarrow (271)$$

we obtain from Eq. (269) that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} q_j(0) m_{ij} A_{il} = M_{ll} C_l \cos\phi_l$$

(or)

$$C_l \cos\phi_l = \frac{1}{M_{ll}} \sum_{i=1}^{n} \sum_{j=1}^{n} q_j(0) m_{ij} A_{il} \dots \rightarrow (272)$$

III^{ly}, we obtain from Eq. (270) that

$$C_l \cos\phi_l = -\frac{1}{\omega_l M_{ll}} \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{q}_j(0) m_{ij} A_{il} \dots \rightarrow (273)$$

Notice that Eqns (272) & (273) enable one to obtain the solns for C_l & ϕ_l directly, rather than being forced to solve 2n simultaneous eqns.

Now, suppose we consider a system having an eigen value $\omega_m^2 = 0$, corresponding to a rigid body mode. we recall that the soln for q_j contains the term $A_{jm}(C_m t + D_m)$ in this case. If we again use the orthogonality properties of the natural modes, we find that

$$C_m = \frac{1}{M_{mm}} \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{q}_j(0) m_{ij} A_{lm} \quad \Big\rangle (2.14)$$

$$D_m = \frac{1}{M_{mm}} \sum_{i=1}^{n} \sum_{j=1}^{n} q_j(0) m_{ij} A_{lm} \cdots\cdots\rightarrow (2.15)$$

Example: 2.12.

In order to illustrate the ~~sto~~ theory of small oscillations, consider a system consisting of a simple pendulum of length l & mass m which is pivoted at a point O on a block of mass 2m (see Fig 2.10). The block can slide without friction on a horizontal surface. Assuming plane motion, & using x & θ as generalized coordinates, obtain the differential eqns of motion & the natural modes. Also obtain the solns for x & θ as funcs of time, assuming the initial conditions are $x(0)=0$, $\dot{x}(0)=1$, $\theta(0)=0.1$, $\dot{\theta}(0)=0$.

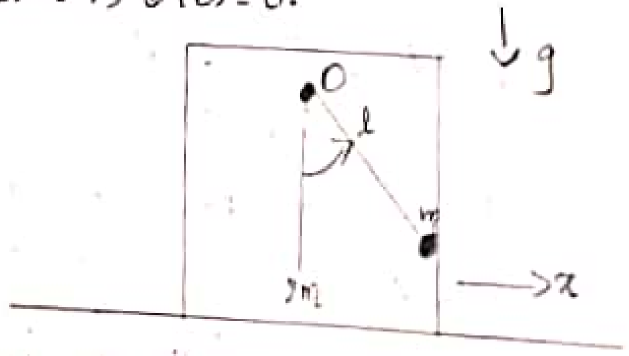


Fig (2.10) - A simple pendulum attached to a sliding block

First let us obtain an expression for the total kinetic energy T. It is the sum of individual kinetic energies of the block and

(13) the pendulum, or

$$T = m\dot{x}^2 + \frac{1}{2} m (\dot{x} + l\dot{\theta})^2 = \frac{3}{2} m\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2} ml^2\dot{\theta}^2 \quad) (276)$$

where we assume $|\theta| \ll 1$..

The potential energy is entirely gravitional & again assuming small motions, we can write

$$V = mgl(1-\cos\theta) \simeq \frac{1}{2} mgl\theta^2 \quad \cdots \rightarrow (277)$$

If we next obtain the Lagrangian function $L = T - V$ & sub into Lagrange's eqn, we obtain the following eqns of motion:

$$3m\ddot{x} + ml\ddot{\theta} = 0$$
$$ml\ddot{x} + ml^2\ddot{\theta} + mgl\theta = 0 \quad \cdots > (278)$$

Now let $q_1 \equiv x$ & $q_2 \equiv \theta$. comparing Eq. (278) with the Standard matrix form of Eq. (235), we get see that

$$m = m\begin{bmatrix} 3 & L \\ L & l^2 \end{bmatrix}, \quad k = mgl\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and note that both matrices are constant & symmetric.

In order to obtain the natural modes, let us first assume solns of the form

$$q_p = A_j C \cos(\omega t + \phi)$$

Sub into Eq. (278) on using Eq. (238), directly, we obtain

$$-3m\omega^2 A_1 - ml\omega^2 A_2 = 0$$
$$-ml\omega^2 A_1 + (mgl - ml^2\omega^2) A_2 = 0 \quad \cdots > (279)$$

The characteristic eqn is obtained by setting the determinant of the coefficients equal to zero, yielding $2l^2\omega^2 - 3gl\omega^2 = 0 \quad \cdots \rightarrow (250)$

23) The eigenvalues are

$$\cancel{2l^2\omega^4 - 3gl\omega^2 = 0} \dashrightarrow$$

$$\omega_1^2 = 0 , \quad \omega_2^2 = \frac{3g}{2l} \dashrightarrow (2$$

From the second eqn of (279) we have

$$\frac{A_2}{A_1} = \frac{\omega^2}{g - l\omega^2} \dashrightarrow (261)$$

Hence we obtain

$$\frac{A_{21}}{A_{11}} = 0 , \quad \frac{A_{22}}{A_{12}} = -\frac{3}{l}$$

and upon setting $A_{11} = A_{12} = 1$, we find that the modal matrix A,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -3/l \end{bmatrix}$$

we see that each column of A is an eigenvector & is associated with a particular ω.

The orthogonality conditions can be checked by using Eqns. (254) & (262) to obtain M & K.

we find that

$$M = m\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} , \quad K = mgl\begin{bmatrix} 0 & 0 \\ 0 & 9/l^2 \end{bmatrix}$$

Furthermore, we note that

$$\omega_1^2 = \frac{K_{11}}{M_{11}} = 0 , \quad \omega_2^2 = \frac{K_{22}}{M_{22}} = \frac{3g}{2l}$$

in agreement with our previous results.

Now, let us obtain the solns for x & θ as funcs of time. These solns are of the form

$$x = A_{11}(C_1 t + D_1) + A_{12} C_2 \cos(\omega_2 t + \phi_2)$$

$$\theta = A_{21}(C_1 t + D_1) + A_{22} C_2 \cos(\omega_2 t + \phi_2)$$

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The constants C_1, C_2, D_1, ϕ_2 are to be determined from the given initial conditions. Instead of finding these constants directly, however, let us use Eqns. (274) & (275), We obtain

$$C_1 = \frac{1}{M_{11}} \dot{x}(0) m_{11} A_{11} = 1$$

$$D_1 = \frac{1}{M_{11}} \theta(0) m_{12} A_{11} = \frac{L}{30}$$

Illy, we find from Eqns (272) & (273) that

$$C_2 \cos\phi_2 = \frac{\theta(0)}{M_{22}} (m_{12} A_{12} + m_{22} A_{22}) = -\frac{L}{30}$$

$$C_2 \sin\phi_2 = -\frac{x(0)}{\omega_2 M_{22}} (m_{11} A_{12} + m_{21} A_{22}) = 0$$

which yields,

$$C_2 = \frac{L}{30}, \quad \phi_2 = \pi$$

Finally, we obtain the solns

$$x = t + \frac{L}{30} \left(1 - \cos\sqrt{\frac{3g}{2L}}\, t\right)$$

$$\theta = \frac{1}{10} \cos\sqrt{\frac{3g}{2L}}\, t$$

Note that the first mode is a rigid-body mode consisting of a uniform translation in the +ve x direction with θ held at 0. The second mode consists of an oscillation of frequency $\sqrt{3g/2L}$ in both x & θ, with the amplitudes & phasing such that the linear momentum in the x-direction is always zero.