Measure and Integration

UNIT-1

Measure on real line - lebesgue outer measure - measurable sets - Regularity - Measurable function - Borel and lebesgue measurability

chapter 2 : sec 2.1 - 2.5

UNIT-2

Abstract measure space : Measure and outer measure - Extension of a measure - uniqueness of the extension - completion of a measure - measure spaces - Integration with respect to a measure

chapter 5 : sec 5.1 - 5.6

UNIT-3

L space - convex function - Jensen's inequality - Inequalities of holder & Minkowski - completeness of L(μ)

Chapter 6 : sec 6.1 - 6.5

UNIT-4

signed measures - Hahn decomposition, The Jordan decomposition - The Radon - Nikodym theorem

chapter 8 : sec 8.1 - 8.3

UNIT-5

Some application of the Radon - Nikodym theorem - Measurability in a product space - The product measure and Fubini's theorem

chapter 8 : sec 8.4

chapter 10 : sec 10.1 - 10.2.

Measure on real line:

2.1 Lebesgue outer measure

Lebesgue measure:

All the sets considered in the real line R particularly with interval I of the form

$$I = [a,b]$$

where a and b are finite and unless.

Otherwise specified intervals may be suppose to be of this type when $a=b$. I is an empty set ϕ. this is denoted by $l(I)$ for the length of I namely $b-a$

Lebesgue outer measure:

The lebesgue outer measure or outer measure of a set $A \subset R$. It is denoted by $m^*(A)$ and it is defined as.

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} l(I_n) \mid A \subseteq \bigcup_{n=1}^{\infty} I_n\right\}$$

$$= [a,b)$$

and I_n is a countable collection of subinterval of R where $\lambda(I_n)$ denote by length of I

Theorem:1

P.T (i) $m^*(A) \geq 0$ for every subset A of R

(ii) $m^*(A) \leq m^*(B)$ if $A \subset B$

(iii) $m^*(\phi) = 0$

(iv) $m^*([x]) = 0$ for any $x \in R$

where $[x]$ denotes the closed interval containing real line x.

(i) Let $\{I_n \mid n \in N\}$ be a half open covering of A

(i.e) $A \subseteq \bigcup_{n=1}^{\infty} I_n$

$$\sum_{n=1}^{\infty} l(I_n) \geq 0$$

$$\inf \sum_{n=1}^{\infty} l(I_n) \geq 0$$

by the definition

$$m^*(A) = \inf \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A) \geq 0$$

(ii) consider $\phi \subseteq [a,a]$ for any real no 'a'

$$\therefore m^*(\phi) \leq l[a,a]$$

$$\leq l(a-a)$$

$$m^*(\phi) \leq 0 \rightarrow (1)$$

But,

we know that, always $m^*(\phi) \geq 0 \rightarrow (2)$

$$m^*(\phi) = 0$$

(every finite set is countable)

(iii) Let $\{I_n \mid n \in N\}$ be a half open covering of B

since $A \subseteq B$ Every covering of B is also a cover of A

Hence

$$\inf \left\{ \sum_{n=1}^{\infty} l(I_n) \mid A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\leq \inf \left\{ \sum_{n=1}^{\infty} l(I_n) \mid B \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

By the definition

$$m^*(A) \leq m^*(B)$$

(iv) consider the cover

$I_n = [x, x+1/n]$ of closed interval $[x]$

Then $m^*([x]) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) ; [x] \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$

$$= \inf \sum_{n=1}^{\infty} l[x, x+1/n]$$

$$= \inf \sum_{n=1}^{\infty} 1/n$$

$$= \inf (1/1, 1/2, 1/3 \dots 1/\infty)$$

$$= \inf (1, 1/2, 1/3 \dots 0)$$

$$= m^*[x] = 0$$

2) S.T every countable set measure zero.

-x- Let A be a countable subset of R

(i.e) $A = x_1, x \dots$

$$A = \bigcup_{i=1}^{\infty} x_i$$

also $A \subseteq [x, x+1/n)$

by the def $m^*(A) = \inf \sum_{n=1}^{\infty} \{l(I_n) ; A \subseteq \bigcup_{n=1}^{\infty} I_n\}$

$$= \inf \sum_{n=1}^{\infty} [x_i + 1/n - x_i]$$

$$= \inf \sum_{n=1}^{\infty} [1/n]$$

$$= \inf (1/1, 1/2 \dots 0) = 0$$

$m^*(A) = 0$.

3) S.T for any set A, $m^*(A) = m^*(A+x)$ where $A+x = [y+x ; y \in A]$ i.e outer measure is translation invariant

Let A be a set of real numbers $x \in R$ is a countable collection of open intervals

$$A + x \subset \bigcup_{n=1}^{\infty} I_n + x$$

$$m^*(A+x) \leq \sum_{n=1}^{\infty} l(I_n + x)$$

$$m^*(A+x) \leq \sum_{n=1}^{\infty} l(I_n)$$

$\therefore m^*(A+x) \le m^*(A)$

For the reverse inequality

$$A+x \subset \bigcup_{n=1}^{\infty} I_n+x$$

$$A \subset \bigcup_{n=1}^{\infty} I_n+x-x$$

$$m^*(A) \subset \sum_{n=1}^{\infty} l(I_n+x-x)$$

$$m^*(A) \le \sum_{n=1}^{\infty} l(I_n+x)$$

$$m^*(A) \le m^*(A+x) \rightarrow ②$$

From ① and ② we get

$$m^*(A) = m^*(A+x)$$

outer measure is translation invariant

Theorem 2 :

10m The outer measures of a interval equal its length

case 1 :

suppose that I is a closed interval $[a,b]$

Then for any $\varepsilon > 0$ then $\exists$ an open interval $[a, b+\varepsilon]$ contains $[a,b]$

$$\therefore [a,b] \subseteq [a, b+\varepsilon] \text{ contains } [a,b]$$

$$\le b+\varepsilon-a$$

$$\le b-a-\varepsilon$$

$$m^*(I) \le b-a \rightarrow ①$$

ε is arbitrary

$$m^*(I) \le b-a \rightarrow ①$$

To obtain the opposite inequality to the $\varepsilon > 0$. I may be covered by a collection of intervals (i.e) $I \supseteq \bigcup_{n=1} I_n$

$\exists\, m^*(I) \geq m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$

$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \varepsilon \rightarrow (3)$$

where $I_n = [a_n, b_n]$

for each n let $I_n = (a_n - 1/2^n, b_n)$

$$\Rightarrow \bigcup_{n=1}^{\infty} I_n' \supseteq I$$

A finite subcollection I_n' say $I_1, I_2 \ldots I_n$ where $I_n = (c_n, d_n)$ covers I

Then suppose that I_n is contained in any other we have supposing that

$$c_1 < c_2 < \cdots < c_N.$$

$$d_N - c_N = \sum_{K=1}^{\infty} (d_K - c_K) - \sum_{K=n+1}^{\infty} (d_K - c_K)$$

$$\therefore d_N - c_1 \leq \sum_{K=1}^{N} l(x_K)$$

so we have $m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \varepsilon$ from (2)

$$\geq \sum_{n=1}^{\infty} l(I_n') - 2\varepsilon$$

$$\geq \sum_{n=1}^{\infty} l(I_K) - 2\varepsilon$$

$$\geq d_n - c_1 - 2\varepsilon$$

$$\geq b - a - 2\varepsilon$$

$$m^*(I) \geq b - a \rightarrow (3)$$

From (1) & (2) we get

$$m^*(I) = b - a$$

Case 2:

Suppose that $I = [a, b]$ and $a > -\infty$

If $a = b$ then $m^*(\phi) = 0$ gives the result

If $a<b$; $0<\epsilon<b-a$ & write

$I' = [a+\epsilon, b]$ then $m^*(I) \geq m^*(I')$

$m^*(I) \geq b-a-\epsilon$

$m^*(I) \geq b-a \rightarrow ④$

But $I \subseteq I'' = [a, b+\epsilon)$

so $m^*(I) \leq m^*(I'')$

$\leq b+\epsilon-a$

$m^*(I) \leq b-a+\epsilon$

$m^*(I) \leq b-a \rightarrow ⑤$ [$\because \epsilon$ is arbitrary]

From ④ & ⑤ we get

$m^*(I) = b-a$

lll'y we consider the interval $(a,b]$ & $[a,b)$

case 3 :-

suppose that I is infinite

A types of intervals occur

Suppose that $I = [-\infty, a]$ the other cases be in similar

For $M>0$ $\exists$ K the finite interval I_m, where

$I_m = [K, K+M) \subset I$

$m^*(I) > m^*(I_m)$

$> K+M-K$

$m^*(I) > M$

$m^*(I) = \infty = l(I) = b-a$

$m^*(I) = b-a$

Thus the outer measure of an interval equal its length

For any sequence of sets $\{K_i\}$.

$m^*(\bigcup_{i=1}^{\infty} K_i) \le \sum_{i=1}^{\infty} m^*(K_i)$

For each i, and for any $\varepsilon > 0$ there exists sequence of intervals $\{I_{i,j}, j, \ldots\}$ such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j} \text{ and}$$

$$m^*(E_i) \le m^*(\bigcup_{j=1}^{\infty} I_{i,j})$$

$$= \sum_{j=1}^{\infty} l(I_{i,j})$$

$$m^*(E_i) \ge \sum_{j=1}^{\infty} l(I_{i,j}) - \frac{\varepsilon}{2^i} \rightarrow ①$$

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$$

$$m^*(\bigcup_{i=1}^{\infty} E_i) \le m^*(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j})$$

$$\le \sum_{i=1}^{\infty} m^*(\bigcup_{j=1}^{\infty} I_{i,j})$$

$$\le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} l(I_{ij})$$

$$\le \sum_{i=1}^{\infty} [m^*(E_i) + \varepsilon/2^i] \text{ by } ①$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty} \varepsilon/2^i$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon\,[1/2 + 1/2 + 1/2^2 + \ldots]$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon/2\,[1 + 1/2 + 1/2^2 + \ldots]$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon/2\,(1 - 1/2)^{-1}$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon/2\,(2)$$

$$m^*(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} m^*(E_i) + \varepsilon$$

Thus,

$$m^*(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} m^*(K_i)$$

since ε is arbitrary

Example:

S.T for any set A and any $\epsilon > 0$ there is an open set O containing A and B. $m^*(O) \leq m^*(A) + \epsilon$.

Given any set A and any $\epsilon > 0$ there is an countable collection of open interval I_n covering of A such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$m^*(A) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$\leq \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon/2^n$$

$$\sum_{n=1}^{\infty} l(I_n) - \epsilon/2^n \leq m^*(A) \rightarrow (1).$$

If $I_n = [a_n, b_n)$ and let $I_n' = (a_n - \epsilon/2^{n+1}, b_n)$ so that $A \subseteq \bigcup_{n=1}^{\infty} I_n'$.

Let $O = \bigcup_{n=1}^{\infty} I_n'$ since any union of open set is open, O is open

$$m^*(O) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$m^*(O) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\leq \sum_{n=1}^{\infty} l(I_n')$$

$$\leq \sum_{n=1}^{\infty} (b_n - a_n + \epsilon/2^{n+1})$$

$$\leq \sum_{n=1}^{\infty} (b_n - a_n + \epsilon/2^{n+1} + \epsilon/2^n - \epsilon/2^n)$$

$$\leq m^*(A) + \epsilon/2^n (1/2 + 1)$$

$$\leq m^*(A) + \epsilon/2^n (3/2)$$

$$m^*(O) \leq m^*(A) + \epsilon$$

$$m^*(O) \leq m^*(A)$$

$m^*(E) = \inf\{\sum_{n=1}^{\infty} l(I_n) \mid E \subseteq R\}$ prove that

(i) I_n is open

(ii) $I_n = [a_n, b_n)$

(iii) $I_n = (a_n, b_n]$

(iv) I_n is closed

(v) Mixture values allowed for different n of the various type of intervals so that same m^* is obtained

proof:

In case (i) by the def of

$$m^*(A) = \inf\{\sum_{n=1}^{\infty} l(I_n) \mid A \subseteq \bigcup_{n=1}^{\infty} I_n\} \rightarrow (1)$$

and we obtain

$$m^*(K) = \inf\{\sum_{n=1}^{\infty} l(I_n) \mid E \subseteq \bigcup_{n=1}^{\infty} I_n\}$$

In case ( ) we obtain $m^*(E)$ of (1) write the corresponding m^* as m_o^* in case (i) m_{oc}^* in case (iii), m_c^* in case (iv), m^*_m in case (v)

To show that $m_o^* = m_c^* = m_{oc}^* = m_m^*$

If U is any type of interval then there exists an open interval 'v' we have

$$m^*_m(E) \leq m_o^*(E) \rightarrow (2)$$

For the reverse inequality $\{I_n\}$ be a countable cover, choose an open interval I_n' such that $I_n \subset I_n'$ for each $I_n \in I_n'$ and

$$l(I_n') = l(I_n) + \varepsilon\, l(I_n)$$

$$l(I_n') = l(I_n)[1+\varepsilon]$$

$$\therefore l(I_n) = \frac{l(I_n')}{1+\varepsilon} \rightarrow (3)$$

for every $\varepsilon > 0$ since $E \subseteq \bigcup_{n=1}^{\infty} I_n$

$$m_m^*(E) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$\geq \sum_{n=1}^{} l(I_n) - \varepsilon$$

$$\varepsilon + m_m^*(E) \geq \sum_{n=1}^{\infty} l(I_n)$$

$$m_m^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} \frac{l(I_n')}{1+\varepsilon}$$

$$(1+\varepsilon)(m_m^*(E) + \varepsilon) \geq \sum_{n=1}^{\infty} l(I_n')$$

But $E \subseteq \bigcup_{n=1}^{\infty} I_n'$ and each I_n' is open

$$m_o^m(E) \leq m_o^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\leq \sum l(I_n')$$

$$m_o^*(E) \leq (1+\varepsilon)\, m_m^*(E) + \varepsilon$$

For every $\varepsilon \geq 0$, since ε is arbitrary

$$m_o^*(E) \leq m_m^*(E) \rightarrow (4)$$

from (3) & (4) we get

$$m_m^*(E) = m_o^*(E)$$

lll'ly $m^* = m_o^* = m_c^* = m_{oc}^* = m_m^*$

2.2 Lebesgue measurable

A set E is said to be lebesgue measurable if for each set A

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

for every subset

Note:

The above inequality has combination of in two following inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

proof:

we have $A = A \cap R \quad \therefore A = A \cap (E \cup E^c)$

$$A = (A \cap E) \cup (A \cap E^c$$

$m^*(A) \le m^*(A\cap E) + m^*(A \cap E^c) \rightarrow ①$

This shows that ① inequality is always true for any subset A and E of R

$\therefore$ the subset E of R is measurable which implies

$m^*(A) \ge m^*(A\cap E) + m^*(A\cap E^c) \rightarrow ②$

From ① and ②

$m^*(A) = m^*(A\cap E) + m^*(A\cap E^c)$

2. 5m S.T the union of two measurable sets is also measurable

Let A be any subset of R and E_1 & E_2 be the measurable subset of A

To prove that given E_1 & E_2 is measurable

(i.e) $m^*(A) = m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(E_1\cup E_2)^c)$.

Since E_1 is measurable then

$m^*(A) = m^*(A\cap E_1) + m^*(A\cap E_1^c) \rightarrow ①$

Since E_2 is measurable

$\Rightarrow m^*(A) = m^*(A\cap E_2) + m^*(A\cap E_2^c) \rightarrow ②$

Replacing ② A by $A\cap E_1^c$ we get

$m^*(A\cap E_1^c) = m^*((A\cap E_1^c)\cap E_2) + m^*((A\cap E_1^c)\cap E_2^c) \rightarrow ③$

Sub these in ①

$① \Rightarrow m^*(A) = m^*(A\cap E_1) + m^*((A\cap E_1^c)\cap E_2) + m^*((A\cap E_1^c)\cap E_2^c)$

$m^*(A) \ge m^*(A\cap(E_1\cup(E_1^c\cap E_2)) + m^*((A\cap(E_1^c\cap E_2^c$

$m^*(A) \ge m^*(A\cap(E_1\cup E_1^c)\cap(E_1\cup E_2)) + m^*(A\cap(E_1\cup E$

$\ge m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(E_1\cup E_2)^c) \rightarrow ④$

But always

$m^*(A) \le m^*(A\cap(E_1\cup E_2)) + m^*(A\cap(E_1\cup E_2)^c) \rightarrow ⑤$

From ④ & ⑤ we get

$m^*(A) = m^* \cancel{(A\cap(E_1\cup E_2))} + \cancel{m^*(A\cap(E_1\cup E_2)^c)}$ -

$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$

$E_1 \cup E_2$ is measurable

the union of two measurable set is measurable

S.T the intersection of two measurable sets is measurable

Given E_1 and E_2 are two measurable subset of R

E_1^c & E_2^c are also measurable

$\Rightarrow E_1^c \cup E_2^c$ is measurable

$\Rightarrow (E_1 \cap E_2)^c$ is measurable

$\Rightarrow E_1 \cap E_2$ is measurable

$\Rightarrow m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \cap E_2)^c)$

$E_1 \cap E_2$ is measurable

σ- Algebra:

A class $\mathcal{A}$ of subset of x is said to be a σ-algebra if the following conditions are satisfied

i) $X \in \mathcal{A}$

ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$

iii) the countable union of member of $\mathcal{A}$ is in $\mathcal{A}$

(i.e) $A_1, A_2, \dots A_n \in \mathcal{A}$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$

Theorem 4.

P.T the collection of measurable sets is a σ-algebra (or) the class M is a σ-algebra

i) Let A be any subset of x (i.e. R)

$\therefore m^*(A \cap R) + m^*(A \cap R^c) = m^*(A) + m^*(\phi)$

$m^*(A) = m^*(A \cap R) + m^*(A \cap R^c)$

$\Rightarrow R$ is measurable

$R \in M$

(ii) let $E \subseteq M$

$\Rightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

Replace E by E^c

$m^*(A) = m^*(A \cap E^c) + m^*(A \cap (E^c)^c)$

$= m^*(A \cap E^c) + m^*(A \cap E)$

E^c is also measurable

E is measurable

$E^c \in M$

(iii) Let $E_1, E_2 \ldots E_n \in M$

To prove $\bigcup_{i=1}^{n} E_i \in M$

To prove the union of two measurable set is measurable

let A be any subset of R and E_1, E_2 measurable subset of A

To prove that the given $E_1 \cup E_2$ is measurable

(i.e) $m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$

Since E_1 is measurable then

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \rightarrow (1)$

Since E_2 is measurable then

$m^*(A) = m^*(A \cap E_2) + m^*(A \cap E_2^c) \rightarrow (2)$

Replace A by $A \cap E_1^c$ in (2)

$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \rightarrow (3)$

sub (3) in (1)

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$

$\geq m^*[A \cap (E_1 \cup (E_1^c \cap E_2))] + m^*(A \cap (E_1 \cup E_2^c))$

$\geq m^*[A \cap ((E_1 \cup E_1^c) \cap (E_1 \cup E_2))] + m^*(A \cap (E_1 \cup$

$\geq m^*[A \cap (R \cap (E_1 \cup E_2))] + m^*(A \cap (E_1 \cup E_2)^c)$

$m^*(A) \geq m^*[A \cap [E_1 \cup E_2]] + m^*[A \cap [E_1 \cup E_2]^c] \rightarrow (4)$

But always

$m^*(A) \leq m^*[A \cap (A \cap (E_1 \cup E_2))] + m^*[A \cap (E_1 \cup E_2)^c]$ → (5)

From (4) & (5)

$$m^*(A) = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

$E_1 \cup E_2$ is measurable

∴ The union of any two measurable sets is measurable

By the induction gives the finite union of numbers of $E_1, E_2 \ldots E_n$ in M

Let $E_1, E_2 \ldots E_n \in M$

Define pointwise disjoint set is

$B_1 = E_1$

$B_2 = E_2 - E_1$

$B_3 = E_3 - (E_1 \cup E_2)$;

$B_n = E_n - (E_1 \cup E_2 \cup \ldots \cup E_{n-1})$

clearly

$B_1 \cap B_2 = E_1 \cap (E_2 - E_1)$

$= E_1 \cap (E_2 \cap E_1^c)$

$B_1 \cap B_2 = \emptyset$

$\Rightarrow B_m \cap B_n = \emptyset \quad \forall m, n \ (m \neq n)$

The collection $\{B_i\}$ is a pointwise disjoint sets since finite intersection of measurable sets is measurable and the complement of measurable set is measurable

∴ $B_1, B_2 \ldots$ is also measurable sets

Then $B_1 \cup B_2 = E_1 \cup (E_2 - E_1)$

$= E_1 \cup [E_2 \cap E_1^c]$

$= (E_1 \cup E_2) \cap (E_1 \cup E_1^c)$

$= (E_1 \cup E_2) \cap R$

$B_1 \cup B_2 = E_1 \cup E_2$

In general

$m^*(A) = m^* \ldots$

$E_1 \cup E_2$ is measurable

the union of two measurable set is measurable

S.T the intersection of two measurable sets is measurable

Given E_1 and E_2 are two measurable subset of R

E_1^c & E_2^c are also measurable

$\Rightarrow E_1^c \cup E_2^c$ is measurable

$\Rightarrow (E_1 \cap E_2)^c$ is measurable

$\Rightarrow E_1 \cap E_2$ is measurable

$\Rightarrow m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \cap (E_1 \cap E_2)^c)$

$E_1 \cap E_2$ is measurable

σ- Algebra:

A class '$\mathcal{A}$' of subset of x is said to be a σ-algebra if the following conditions are satisfied

(i) $X \in \mathcal{A}$

(ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$

(iii) the countable union of member of $\mathcal{A}$ is in $\mathcal{A}$

(i.e) $A_1, A_2, \ldots A_n \in \mathcal{A}$

$$\Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{A}$$

Theorem. 4.

P.T the collection of measurable sets is a σ-algebra (or) the class M is a σ-algebra

(i) Let A be any subset of x (i.e. R)

$\therefore m^*(A \cap R) + m^*(A \cap R^c) = m^*(A) + m^*(\phi)$

$m^*(A) = m^*(A \cap R) + m^*(A \cap R^c)$

$\Rightarrow R$ is measurable

$R \in M$

$\Rightarrow A\cap C^c \subseteq A\cap C_n^c$

$m^*(A\cap C^c) \le m^*(A\cap C_n^c)$

i.e $m^*(A\cap C_n^c) \ge m^*(A\cap C^c)$

$① \Rightarrow m^*(A) \ge m^*(A\cap C_n) + m^*(A\cap C^c)$

$m^*(A) \ge \sum_{i=1}^{\infty} m^*(A\cap B_i) + m^*(A\cap C^c)$

$\ge m^*(A\cap(\bigcup_{i=1}^{\infty} B_i)) + m^*(A\cap C^c)$

$m^*(A) \le m^*(A\cap C) + m^*(A\cap C^c)$

From the above two inequalities we have

$m^*(A) = m^*(A\cap C) + m^*(A\cap C^c)$

C is measurable

$C\in M$

$\bigcup_{i=1}^{\infty} B_i \in M$

$\bigcup_{i=1}^{\infty} E_i \in M$

$\therefore$ M is σ-algebra

Theorem: 5

$\{E_i\}$ is any sequence disjoint measurable set then union of $m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{n=1}^{\infty} m^*(E_i)$ i.e m^* is countably additive disjoint of M.

Let E_1 and E_2 be any two measurable sets

Let A be any sets since E_1 is measurable

$m^*(A) = m^*(A\cap E_1) + m^*(A\cap E_1^c)$

since E_2 is measurable

$m^*(A) = m^*(A\cap E_2) + m^*(A\cap E_2^c)$

Replace A by $A\cap E_1^c$ in ②

$m^*(A\cap E_1^c) = m^*((A\cap E_1^c)\cap E_2) + m^*((A\cap E_1^c)\cap E_2^c)$

$= m^*(A\cap(E_1^c\cap E_2)) + m^*(A\cap(E_1^c\cap E_2)^c)$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap (E_1^c \cap E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2) + m^*(A \cap (E_1 \cup E_2)^c)$$

lll'y

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2) + m^*(A \cap E_3) + m^*(A \cap (E_1 \cup E_2 \cup E_3)^c)$$

Any three disjoint measurable sets E_1, E_2, E_3

In general

$$m^*(A) = \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap (\bigcup_{i=1}^{n} E_i)^c) \rightarrow ③$$

For any n disjoint measurable sets $E_1, E_2 \dots E_n$

Clearly, $\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i$

$$(\bigcup_{i=1}^{\infty} E_i)^c \subseteq (\bigcup_{i=1}^{n} E_i)^c$$

$$A \cap (\bigcup_{i=1}^{\infty} E_i)^c \subset A \cap (\bigcup_{i=1}^{n} E_i)^c$$

$$m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c) \le m^*(A \cap (\bigcup_{i=1}^{n} E_i)^c)$$

$$m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c) \le m^*(A \cap (\bigcup_{i=1}^{n} E_i)^c)$$

$$m^*(A \cap (\bigcup_{i=1}^{n} E_i)^c) \ge m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)$$

$$③ \Rightarrow m^*(A) \ge \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)$$

But always

$$m^*(A) \le \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)^c)$$

Taking limit as $n \to \infty$

$$m^*(A) \ge \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap (\bigcup_{i=1}^{\infty} A_i)^c)$$

and setting $A = \bigcup_{i=1}^{\infty} E_i$

$$m^*(\bigcup_{i=1}^{\infty} E_i) \ge \sum_{i=1}^{\infty} m^*(\bigcup_{i=1}^{\infty} E_i \cap E_i) + m^*(\bigcup_{i=1}^{\infty} E_i \cap (\bigcup_{i=1}^{\infty} E_i)$$

$$m^*(\bigcup_{i=1}^{\infty} E_i) \ge \sum_{i=1}^{\infty} m^*(E_i) + m^*(\phi)$$

$$m^*(\sum_{i=1}^{\infty} E_i) \ge \sum_{i=1}^{\infty} m^*(E_i)$$

$\bigcup_{i=1} B_i = \bigcup_{i=1} E_i$

Let $C_n = \bigcup_{i=1}^{\infty} B_i \Rightarrow C = \bigcup_{i=1}^{\infty} B_i$

Since B_n is measurable

$m^*(A \cap C_n) = m^*[(A \cap C_n) \cap B_n] + m^*[(A \cap C_n) \cap B_n^c]$

$m^*[A \cap (C_n \cap B_n)] + m^*[A \cap (C_n \cap B_n^c)]$

$C_n \cap B_n = (B_1 \cup B_2 \ldots \cup B_n) \cap B_n$

$= (B_1 \cap B_n) \cup (B_2 \cap B_n) \cup \ldots \cup (B_n \cap B_n)$

$= B_1 \cup B_2 \cup \ldots \cup B_n$

$= \bigcup_{i=1}^{n} B_i$

$C_n \cap B_n = C_n \Rightarrow C_n \cap B_n = B_n$

$C_n \cap B_n^c = (B_1 \cup B_2 \ldots \cup B_n) \cap B_n^c$

$= (B_1 \cap B_n^c) \cup (B_2 \cap B_n^c) \cup \ldots \cup (B_n \cap B_n^c)$

$= B_1 \cup B_2 \ldots \cup B_{n-1}$

$= \bigcup_{i=1}^{n-1} B_i$

$C_n \cap B_n^c = C_{n-1}$

$\therefore m^*(A \cap C_n) = m^*(A \cap B_n) + m^*(A \cap C_{n-1})$

$= m^*(A \cap B_n) + m^*(A \cap B_{n-1}) + m^*(A \cap C_{n-2})$

$= m^*(A \cap B_n) + m^*(A \cap B_{n-1}) + m^*(A \cap B_{n-2}) + m^*(A \cap B_{n-3})$

$= m^*(A \cap B_n) + m^*(A \cap B_{n-1}) + m^*(A \cap B_{n-2}) + m^*(A \cap B_{n-3}) + \ldots + m^*(A \cap C_1)$

$= m^*(A \cap B_n) + m^*(A \cap B_{n-1} + \ldots + m^*(A \cap B_1)$

$m^*(A \cap C_n) = \sum_{i=1}^{\infty} m^*(A \cap B_i)$

Since C_n is measurable

(i.e) $m^*(A) = m^*(A \cap C_n) + m^*(A \cap C_n^c)$ holds any subset A.

But $C_n \subset C$

$\Rightarrow C^c \subset C_n^c$

$\therefore F \cap G$ & $G - F$ are measurable

G is measurable

Every interval is measurable

Suppose that the interval to be of the form (a, ∞)

The proof of the other type of interval is similar

Let A_1, A_2 be any subsets of R we show that

$$m^*(A) \geq m^*(A \cap (a,\infty)) + m^*(A \cap (a,\infty)^c)$$

$$m^*(A) \geq m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a))$$

Let $A_1 = A \cap (a, \infty)$ & $A_2 = A \cap (-\infty, a)$

$$m^*(A) \geq m^*(A_1) + m^*(A_2) \rightarrow ①$$

If $m^*(A) = \infty$ there is nothing to prove Assume that $m^*(A) < \infty$ from the def of m^* given $\varepsilon > 0$ there is a sequence $\{I_n\}$ are open interval

$$A \subset \bigcup_{n=1}^{\infty} I_n$$

$$m^*(A) < \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A) + \varepsilon > \sum_{n=1}^{\infty} l(I_n) \rightarrow ②$$

Let $J_n = I_n \cap (a, \infty)$

$$J_n' = I_n \cap (-\infty, a]$$

J_n and J_n' are disjoint intervals and

$I_n \cup J_n' = I_n$

$$m^*(I_n) = m^*(J_n) + m^*(J_n')$$

$$l(I_n) = l(J_n) + l(J_n')$$

Since $A_1 \subseteq \bigcup_{n=1}^{\infty} I_n$ & $A_2 \subseteq \bigcup_{n=1}^{\infty} I_n'$

$$m^*(A_1) \leq \sum_{n=1}^{\infty} l(J_n)$$

IIIly $m^*(A_2) \leq \sum_{n=1}^{\infty} l(J_n')$

Adding we get

$$m^*(A_1) + m^*(A_2) = \sum_{n=1}^{\infty} l(J_n) + \sum_{n=1}^{\infty} l(J_n')$$

$$\leq \sum_{n=1} l(I_n)$$

$$m^*(A_1)+m^*(A_2) \leq m^*(A)$$

$$m^*(A) \geq m^*(A_1)+m^*(A_2)$$

But always

$$m^*(A) \leq m^*(A_1)+m^*(A_2) \rightarrow (4)$$

From (3) & (4).

$$m^*(A) = m^*(A_1) + m^*(A_2)$$

$$m^*(A) = m^*(A \cap (a,\infty)) + m^*(A \cap (-\infty, a])$$

$$m^*(A) = m^*(A \cap (a,\infty)) + m^*(A \cap (a,\infty)^c)$$

The interval (a,∞) is measurable

$\Rightarrow$ Every interval is measurable.

Example : 6

For any set A, there exists a measurable set E containing A and E

$$m^*(A) = m(E)$$

For any set A then there exists an open set O containing A (i.e) $O \supseteq A$

$$\Rightarrow m^*(O) \geq m^*(A)$$

$$m^*(O) \leq m^*(A)+\epsilon$$

Given any set A, given $\epsilon > 0$ there is a countable collection $\{I_n\}_0^\infty$ of open interval covering of A

(i.e) $\{I_n\} \supset A$

$$m^*(I_n) > m^*(A)$$

$$\leq l(I_n) \leq m^*(A)+\epsilon$$

Let $O = \bigcup_{n=1}^{\infty} I_n$ since any union of open set is open $\therefore O$ is open

clearly $A \subseteq \bigcup_{n=1}^{\infty} I_n = O$

$$m^*(O) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$m^*(O) = \sum_{n=1}^{\infty} l(I_n)$$

$m^*(O) < \sum_{n=1} l(I_n)+\epsilon$

choose $\epsilon = 1/n$ for each n there is an open set E_n

$A \subseteq E_n$

$m^*(E_n) \le m^*(A)+1/n \to ①$

Let $E = \bigcap_{n=1}^{\infty} E_n \subset E_n$

$\Rightarrow A \subseteq E$

$m^*(A) \le m^*(E) \le m^*(E_n)$

But $m^*(A) \le m^*(A)+1/n$ by ①

But $1/n \to 0$ as $n \to \infty$.

$m^*(A) \le m^*(E) \le m^*(E_n) \le m^*(A)+1/n$

$\Rightarrow m^*(A) = m^*(E)$.

Since A is measurable, $m^*(E)$ can be written as $m(E)$

$m^*(A) = m(E)$

Limit supremum & limit infimum

For any sequence subsets $\{E_i\}$ be defined as

$\lim\sup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i\ge n} E_i$

$\lim\inf E_i = \bigcup_{n=1}^{\infty} \bigcap_{i\ge n} E_i$

Note: 1

P.T $\lim\sup E_i \ge \lim\inf E_i$

we have $\lim\sup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i\ge n} E_i$

$= \left(\bigcup_{i\ge 1} E_1\right) \cap \left(\bigcup_{i\ge 2} E_2\right) \cap \dots \left(\bigcup_{i\ge n} E_n\right)$

$= (E_1 \cup E_2 \cup \dots) \cap (E_2 \cup E_3 \dots) \cap \dots$

$\lim\sup E_i \ge \lim\inf E_i$

Note: 2.

$x \in \lim\sup E_i \Rightarrow x \in \bigcap_{i=1}^{\infty} \bigcap_{i\ge n} E_i$

$\Rightarrow x \in \bigcap_{i\ge n} E_i \ \forall n$.

It is clear that the point x ... finitely many of the set E

Note 4:

S.T if $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ then $\limsup E_i = \liminf E_i = \bigcup_{i=1}^{\infty} E_i$

W.K.T $\limsup E_i = \bigcap_{i=1}^{\infty} \bigcup_{i \geq n} E_i$

$= (E_1 \cup E_2 \cup \ldots) \cap (E_2 \cup E_3 \cup \ldots) \cap (E_3 \cup E_4 \ldots)$

$\limsup E_i = \bigcup_{i=1}^{\infty} E_i$

W.K.T

$\liminf E_i = \bigcup_{i=1}^{\infty} \bigcap_{i \geq n} E_i$

$= (E_1 \cap E_2 \cap \ldots) \cup (E_2 \cap E_3 \cap \ldots) \cup \ldots$

$= E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_\infty$

$\liminf E_i = \bigcup_{i=1}^{\infty} E_i$

~~W.K.T~~

$\limsup E_i = \liminf E_i = \bigcup_{i=1}^{\infty} E_i$

Note 5:

If $E_1 \supseteq E_2 \supseteq E_3$ then P.T $\limsup E_i = \liminf E_i = \bigcap_{i=1}^{\infty} E_i$

W.K.T $\limsup E_i = \bigcap_{i=1}^{\infty} \bigcup_{i \geq n} E_i$

$= (E_1 \cup E_2 \cup \ldots) \cap (E_2 \cup E_3 \cup \ldots) \cap (E_3 \cup E_4 \cup \ldots)$

$= E_1 \cap E_2 \cap \ldots E_\infty$

$\limsup E_i = \bigcap_{i=1}^{\infty} E_i$

W.K.T

$\liminf E_i = \bigcup_{i=1}^{\infty} \bigcap_{i \geq n} E_i$

$= (E_1 \cap E_2 \cap \ldots) \cup (E_2 \cap E_3 \cap \ldots) \cup \ldots$

$= E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_\infty$

$\liminf E_i = \bigcap_{i=1}^{\infty} E_i$

Hence

$\lim\sup E_i = \lim\inf E_i = \bigcap_{i=1}^{\infty} E_i$

(i.e) $\lim E_i = \bigcap_{i=1}^{\infty} E_i$

Example:

P.T there exists uncountable sets of zero measure.

The cantor set P is constructed by defining a sequence of sets

$P_0 = [0,1]$

$P_1 = [0, 1/3] \cup [2/3, 1]$

$P_2 = [0, 1/9] \cup [2/9, 3/9] \cup [2/3, 7/9] \cup [8/9, 1]$

So P_n is constructed by removing the middle third open intervals from P_{n-1}

The cantor set is $P = \bigcap_{i=1}^{\infty} P_i$

Since each P_i is union of disjoint closed intervals

P_n is measurable which implies

$\bigcap_{i=1}^{\infty} P_i$ is measurable

P is measurable but the cantor set is uncountable

Hence P is uncountable measure

Claim:

$m(P) = 0$

clearly $P_0 \supset P_1 \supset P_2$

$\lim P_i = \bigcap_{i=1}^{\infty} P_i = P$

Now, $m(\lim P_i) = \lim m(P_i)$

$\Rightarrow m(P) = \lim (2/3) = 0$

$\therefore m(P) = 0$

$\therefore$ P is cantor set is an uncountable measurable set with zero measure.

lim na
∪ = n

Theorem: 9

Let $\{E_i\}$ be a sequence of measurable sets (i) If $E_1 \subseteq E_2 \subseteq$ then $m(\lim E_i) = \lim m(E_i)$

(ii) if $E_1 \supseteq E_2 \supseteq E_3$ then $m(E_i) < \infty$ for each i then $m(\lim E_i) = \lim m(E_i)$

Given that $E_1 \subseteq E_2 \subseteq \dots$ we can express pointwise disjoint sets define

$$B_1 = E_1$$
$$B_2 = E_2 - E_1$$
$$B_3 = E_3 - (E_1 \cup E_2)$$
$$B_i = E_i - (E_{i-1})$$

clearly

$B_m \cap B_n = \phi$ for all m, n and also

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i$$

since complement of measurable set is measurable and union of measurable set is measurable

$\therefore$ each E_i is measurable.

The collection $\{B_i\}$ disjoint measurable set

Now $m(\lim E_i) = m\left(\bigcup_{i=1}^{\infty} E_i\right)$

$$= m\left(\bigcup_{i=1}^{\infty} B_i\right)$$
$$= \sum_{i=1}^{\infty} m(B_i)$$
$$= \lim \sum_{i=1}^{\infty} m(B_i)$$
$$= \lim m\left(\sum_{i=1}^{\infty} B_i\right)$$
$$= \lim m(B_1 \cup B_2 \cup \dots B_\infty)$$
$$= \lim m\left(\bigcup_{i=1}^{\infty} B_i\right).$$
$$= \lim m\left(\bigcup_{i=1}^{\infty} E_i\right)$$
$$= \lim m(E_i)$$

Given that $E_1 \supseteq E_2 \supseteq E_3 \dots \supseteq E_n$

$\Rightarrow E_1 \subset E_2 \subset E_3 \dots E_n$

$m(E_i) < \infty$ for each i

Now we have to prove by the hypothesis

$E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \dots \subseteq E_1 - E_n$

by (i) $m(\lim E_i) = \lim m(E_i)$

$$m(\lim (E_1 - E_i)) = \lim m(E_1 - E_i)$$

$$m(\bigcup_{i=1}^{\infty} (E_1 - E_i)) = \lim m(E_1 - E_i)$$

$$m(\bigcup_{i=1}^{\infty} E_1) - m(\bigcup_{i=1}^{\infty} E_i) = \lim m(E_1) - \lim m(E_i)$$

$$m(E_1) - m(\bigcup_{i=1}^{\infty} E_i) = m(E_1) - \lim m(E_i)$$

$$m(\bigcup_{i=1}^{\infty} E_i) = \lim m(E_i)$$

S.T (i) every non empty open set has positive measure

(ii) the rational Q are enumerated as $q_1, q_2 \ldots$ the set G is defined by

$$G = \bigcup_{n=1}^{\infty} (q_n - 1/n^2, q_n + 1/n^2)$$

prove that for any closed set F, $m(G \Delta F) > 0$.

Let E be any non empty open subset of R then by the definition of open set for every $x \in E$

Given $\epsilon > 0$ there exists a neighbourhood $(x-\epsilon, x+\epsilon)$ of x such that

$$(x-\epsilon, x+\epsilon) \subseteq E$$

$$m^*(x-\epsilon, x+\epsilon) \le m^*(E)$$

$$l(x+\epsilon - x+\epsilon) \le m^*(E)$$

$$2\epsilon \le m^*(E)$$

Since ϵ is arbitrary and $\epsilon > 0$

$$m^*(E) > 0$$

every non empty open set has positive measure

(ii) Given that $Q = \{q_1, q_2 \ldots q_n\}$ the set has rational number and also given that

$$G = \bigcup_{n=1}^{\infty} (q_n - 1/n^2, q_n + 1/n^2)$$

To prove every closed set F, $m(G \Delta F) > 0$

For this by def $\qquad A \Delta B = (A-B) \cup (B-A)$

$$m(G \Delta F) = m(G-F) \cup m(F-G)$$

$$= m(G-F) + m(F-G)$$

[$\because$ $G-F$ and $F-G$ are disjoint sets]

case (i):

If $m(G-F)>0$ then $m(G\Delta F)>0$

case (ii):

If $m(G-F)=0$, $m(G\Delta F)=m(F-G)$

since G and F^c are open sets

$G\cap F^c$ is also open in R

$\Rightarrow G-F$ is open

Since the measure of any non empty set and open set positive.

$G-F\neq\phi$

If $G-F\neq 0$, then $m(G-F)>0$

But in this case $m(G-F)=0$

Hence $G-F=\phi$ $\quad [A\cap B=\phi \Leftrightarrow A\subset B^c]$

$G\cap F^c=\phi$

$G\cap(F^c)^c=F$

$\overline{G}\subseteq\overline{F}$

$\overline{G}\subseteq F$ $\quad$ [$F\times\overline{F}$ and F is closed]

Also $Q\in G$

$\overline{Q}\subset\overline{G}\subseteq F$ $\quad(\overline{Q}=R)$

$R\subseteq F\subseteq R$

$F=R$

$m(F)=\infty$

But $m(G)=m\left[\bigcup_{n=1}^{\infty}(a_n-1/n^2, a_n+1/n^2)\right]$

$\leq\sum_{n=1}^{\infty}m(a_n-1/n^2, a_n+1/n^2)$

$\leq\sum_{n=1}^{\infty}2/n^2$

$=2\sum_{n=1}^{\infty}1/n^2$

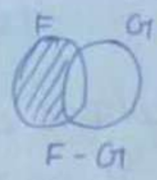
since $\sum_{n=1}^{\infty}1/n^2<\infty$, $m(G)<\infty$

$m(F-G) = m(F \cup G) - m(G)$

$m(F-G) = m(R \cup G) - m(G)$

$\geq m(R) - m(G) = \infty$

> 0



$\therefore m(G \Delta F) = m(F-G) > 0$

Hence proved

2.3 Regularity

F_σ set :

A set that is the union of countable collection of closed set is called F_σ set.

G_δ set :

A set that is the intersection of countable collection of open sets is called G_δ set

Theorem 10

P.T every F_σ set is measurable.

Let U be any $F_\sigma \subset R$

Then $U = \bigcap_{i=1}^{\infty} F_i$

where each F_i is open in R

F_i, $i = 1, 2 \ldots \infty$

To prove that U is measurable

since every open set is measurable

$\Rightarrow F_i^c$ is measurable $\forall i \in N$

since union of measurable set is measurable

$\therefore \bigcup_{i=1}^{\infty} F_i$ is also measurable

$\Rightarrow \left(\bigcup_{i=1}^{\infty} F_i^c\right)^c$ is measurable

$\Rightarrow \bigcap_{i=1}^{\infty} F_i$ is measurable

U is measurable

Hence F_σ is a measurable set

Let F be any $G_\delta \subset R$

Then $F = \bigcup_{i=1}^{\infty} G_i$

where each G_i is closed in R, $i = 1, 2 \ldots \infty$

To prove that F is measurable

Clearly G_i^c is closed in R $\forall i \in N$

$\Rightarrow G_i^c$ is measurable in R $\forall i \in N$

$\Rightarrow \bigcap_{i=1}^{\infty} G_i^c$ is measurable

$\Rightarrow (\bigcap_{i=1}^{\infty} G_i^c)^c$ is measurable

$\Rightarrow \bigcup_{i=1}^{\infty} G_i$ is measurable

F is measurable in R, hence every G_δ set is measurable

10(m) *

Regularity theorem:

Theorem 13:

For a subset E of R the following statement are equivalent

(i) E is measurable

(ii) $\varepsilon > 0$, $O \supseteq E$ $\ni m^*(O-E) \leq \varepsilon$

where O is open set

(iii) $\exists$: G_δ say G, $G \supseteq E$ $\ni m^*(G-E) = 0$.

(iv) $\forall$ $\varepsilon > 0$ $\exists$: F_σ set say F is a closed set, $F \subseteq E$ $\ni$ $m^*(E-F) \leq \varepsilon$

(v) $\exists$ a F_σ set say F, $F \subseteq E$ $\ni m^*(E-F) = 0$

case (i):

(i) $\Leftrightarrow$ (ii)

Assume that E is measurable in R

suppose if $m^*(E) < \infty$ and given $\varepsilon > 0$ $\exists$ an open set O $E \subseteq O$

$\ni m^*(O) \leq m^*(E) + \varepsilon$

$m^*(O) - m^*(E) \leq \varepsilon$

$m^*(O-E) \leq \varepsilon$.

Suppose $m^*(E)=\infty$

Let $\{I_n\}$ be a sequence of disjoint intervals whose endpoints are finite

$\therefore \bigcup_{n=1}^{\infty} I_n = R$

Define $E_n = E \cap I_n$

clearly $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n)$

$= E \cap (\bigcup_{n=1}^{\infty} I_n)$

$\bigcup_{n=1}^{\infty} E_n = E \cap R$

The endpoints each I_n is finite

$\Rightarrow$ The endpoint of each E_n, is finite

$\therefore$ E_n is measurable and every open interval is measurable

$E \cap E_n$ is measurable

E_n is measurable

E_n is measurable and the endpoints are finite

$\therefore m^*(E_n) < \infty$

there exists an open set $O_n \ni E_n \subseteq O_n$

$m^*(O_n) \leq m^*(E_n) + \varepsilon/2^n$

$m^*(O_n) - m^*(E_n) \leq \varepsilon/2^n$

$m^*(O_n - E_n) \leq \varepsilon/2^n$

Define $O = \bigcup_{n=1}^{\infty} O_n$

$\Rightarrow O$ is an open set in $R \ni E \subseteq O$

$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n$

$m^*(O-E) = m^*\left(\bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n\right)$

$m^*(O-E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n)$

$\leq \sum_{n=1}^{\infty} \varepsilon/2^n$

$\leq \varepsilon \left[\sum_{n=1}^{\infty} 1/2^n\right]$

$\leq \varepsilon \left[1/2 + 1/2^2 + \cdots\right]$

$\leq \varepsilon/2 \left[1 + 1/2 + \cdots\right] \leq \varepsilon/2 (1/2)^{-1}$

$m^*(O-E) \le \varepsilon$

(ii) $\Leftrightarrow$ (iii)

For each $n \in N$ let O_n be an open set such that $E \subseteq O_n$

$m^*(O_n - E) \le 1/n$

Let $G = \bigcap_{n=1}^{\infty} O_n$, G is a G_δ set $\ni E \subseteq G$ then for each $n \in N$.

$\therefore m^*(G-E) \le m^*(\bigcap_{n=1}^{\infty} O_n - E)$

$\le m^*(O_n - E)$

$\le 1/n \;\; \forall n \in N$

$\therefore m^*(G-E) = 0$ as $n \to \infty$

(iii) $\Leftrightarrow$ (i)

We can write $E = G-(G-E)$ by (iii)

$m^*(G-E)=0$

$\Rightarrow G-E$ is measurable

Since G is G_δ set, G is measurable

$\Rightarrow G$ and $G-E$ is measurable

$G-(G-E)$ is measurable

$G \cap (G-E)^c$ is measurable

E is measurable

A-B
$A \cap B^c$

(i) $\Leftrightarrow$ (iv)

Suppose that E is measurable which implies E^c is measurable

Since (i) $\Leftrightarrow$ (iii) we have (ii) there is an open set $O \ni E^c \subseteq O$ and

$m^*(O-E^c) \le \varepsilon \rightarrow (9)$

But $O - E^c = O \cap (E^c)^c$

$= O \cap E$

$= E \cap O$

$O - E^c = E - O^c$

Taking $F = O^c$

F is closed set $\ni F \subseteq E$

$m^*(E-F) = m^*(E-O^c)$

$= m^*(O-E^c)$

By sub $\to$ ⓐ

$m^*(E-F) \leq \epsilon$

$(iv) \Leftarrow (v)$

For each $n \in N$, F_n be a closed set $\ni F_n \subseteq E$

$m^*(E-F_n) \leq 1/n$

Let $F = \bigcup_{n=1}^{\infty} F_n$

$\Rightarrow F$ is an F_σ-set $\ni F \subseteq E \ \forall n \in N$

$m^*(E-F) = m^*(E - \bigcup_{n=1}^{\infty} F_n)$

$= m^*(E \cap (\bigcup_{n=1}^{\infty} F_n)^c)$

$= m^*(E \cap (\bigcap_{n=1}^{\infty} F_n^c))$

$\leq m^*(E \cap F_n^c)$

$\leq m^*(E-F_n)$

$\leq 1/n$

$m^*(E-F) = 0$ as $n \to \infty$

$(v) \to (i)$.

Since F is F_σ set it is measurable since $m^*(E-F)=0$

$\Rightarrow E-F$ is measurable

clearly $(E-F) \cup F = E$

Since union of measurable set is also measurable

$\therefore$ E is measurable.

nx. Theorem: 11

$m^*(E) < \infty$ then E is measurable iff $\forall \epsilon > 0$, $\exists$ disjoint finite intervals $I_1, I_2 .. I_n \ni m^*(E \Delta \bigcup_{i=1}^{n} I_i$ the intervals I_i is open closed or left open

Suppose $m^*(E)=\infty$

Let $\{I_n\}$ be a sequence of disjoint intervals whose endpoints are finite

$\therefore \bigcup_{n=1}^{\infty} I_n = R$

Define $E_n = E \cap I_n$

clearly $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n)$

$= E \cap (\bigcup_{n=1}^{\infty} I_n)$

$\bigcup_{n=1}^{\infty} E_n = E \cap R$

the endpoints each I_n is finite

$\Rightarrow$ The endpoint of each E_n, is finite

$\therefore$ E_n is measurable and every open interval is measurable

$E \cap E_n$ is measurable

E_n is measurable

E_n is measurable and the endpoints are finite

$\therefore m^*(E_n) < \infty$

there exists an open set $O_n \ni E_n \subseteq O_n$

$m^*(O_n) \leq m^*(E_n) + \varepsilon/2^n$

$m^*(O_n) - m^*(E_n) \leq \varepsilon/2^n$

$m^*(O_n - E_n) \leq \varepsilon/2^n$

Define $O = \bigcup_{n=1}^{\infty} O_n$

$\Rightarrow O$ is an open set in $R \ni E \subseteq O$

$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n$

$m^*(O - E) = m^*\left(\bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n\right)$

$m^*(O - E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n)$

$\leq \sum_{n=1}^{\infty} \varepsilon/2^n$

$\leq \varepsilon \left[\sum_{n=1}^{\infty} 1/2^n\right]$

$\leq \varepsilon \left[1/2 + 1/2^2 + \cdots\right]$

$\leq \varepsilon/2 \left[1 + 1/2 + \cdots\right] \leq \varepsilon/2 (1/2)^{-1}$

$m^*(O-U) < \epsilon$

sub in ② we get

$m^*(E \Delta U) \leq \epsilon + \epsilon \leq 2\epsilon$

$m^*(E \Delta U) < \epsilon$

$m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$

converse part :

For any set E and $\forall \epsilon > 0$ $\exists$ an open set $O \supseteq E$ $\ni$

$m^*(O-E) < \epsilon$ may be arbitrary

Then we have to prove E is measurable

write when $J = \bigcup_{i=1}^{n} I_i$ and $U = O \cap J$

Then by sub-additivity

$m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E) \rightarrow ④$

since $U \subseteq J$ we have $U - E \subseteq J - E$ and

since $E \subseteq O$ we have $E - U = E - J$ so

$U \Delta E \subseteq J \Delta E$

and $m^*(U \Delta E) < \epsilon$

but $E \subseteq U \cup (U \Delta E)$ so

$m^*(E) < m^*(U) + \epsilon$ from (iii)

$m^*(O \Delta U) = m^*(O-U)$

$= m^*(O) - m^*(U)$

$\leq m^*(O) - m^*(E) + m^*(E) - m^*(U)$

$\leq \epsilon + \epsilon$

$\leq 2\epsilon$

$< \epsilon$

by ④ $m^*(O-E) = m^*(O \Delta E) \leq \epsilon$

E is measurable.

Definition

Measurable function:

An extended real value function f defined on a measurable set E is said to be a lebesgue measurable function or simply measurable function if the set $\{x : f(x) > \alpha\}$ is measurable for each $\alpha \in R$.

Note theorem:

Every constant function is measurable.

Let $f(x) = c$, a constant for all $x \in R$ clearly

$$\{x : f(x) > \alpha\} = \begin{cases} R & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases}$$

since both R and ϕ are measurable sets, $\{x : f(x) > \alpha\}$ on its measurable for all $\alpha \in R$.

$\therefore$ the given constant function is measurable

Theorem:

Every continuous function is measurable.

Let f be a continuous function defined on

clearly we have

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : \alpha < f(x) < \alpha + n\}$$

$$= \bigcup_{n=1}^{\infty} \{x : f(x) \in (\alpha, \alpha + n)\}$$

By definition,

$$f^{-1}(A) = \{x : f(x) \in A\}$$

since f is continuous and $(\alpha, \alpha+n)$ is open in

$\therefore f^{-1}(\alpha, \alpha+n)$ is also open in R.

since any union of open set is open

$\therefore \bigcup_{n=1}^{\infty} f^{-1}(\alpha, \alpha+n)$ is also open in R.

since every interval is measurable

$\therefore \{x : f(x) > \alpha\}$ is measurable

Hence a continuous function f is measurable

Definition:

Characteristic function:

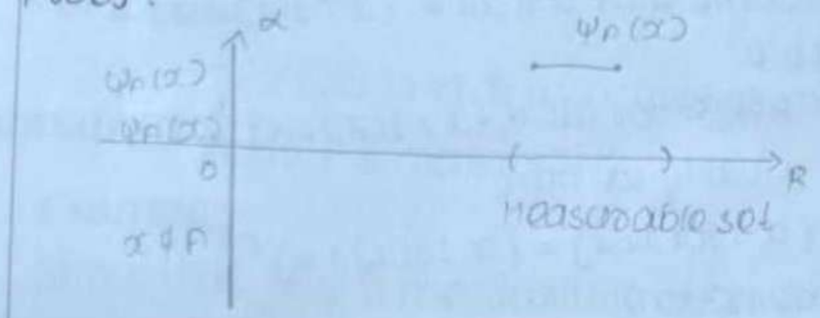
For any subset E of R the characteristic function ψ_E is defined by

$$\psi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Theorem:

Prove that the characteristic function ψ_A of the set A is measurable iff A is measurable

Proof:



For any α, we have

$$\begin{aligned}\{x : f(x) > \alpha\} &= \phi \text{ if } \alpha \geq 1 \\ &= A \text{ if } 0 \leq \alpha < 1 \\ &= R \text{ if } \alpha < 0\end{aligned}$$

Clearly $\{x : f(x) > \alpha\}$ is measurable iff A is measurable

Hence the characteristic function ψ_A of the set A is measurable iff A is measurable.

Theorem:

P.T the following statement are equivalent.

(i) f is a measurable function

(ii) for all α, $\{x : f(x) \geq \alpha\}$ is measurable

(iii) for all α, $\{x : f(x) < \alpha\}$ is measurable

(iv) for all α, $\{x : f(x) \leq \alpha\}$ is measurable

Proof:

(i) ⇒ (ii)

Assume that f is measurable function.

To prove for all α, $\{x : f(x) \geq \alpha\}$ is measurable

For this we have any α

$$\{x : f(x) \geq \alpha\} = \{x : f(x) > \alpha - 1\}$$

countable

$$\therefore \{x : f(x) > \alpha - 1\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$$

is a measurable set

$\therefore \{x : f(x) \geq \alpha\}$ is a measurable set for any α.

(ii) ⇒ (iii)

Assume that for all α $\{x : f(x) \geq \alpha\}$ is measurable

To prove for all α, $\{x : f(x) < \alpha\}$ is measurable

clearly we have

$$\{x : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}^c$$

By assumption,

$\{x : f(x) \geq \alpha\}$ is a measurable set

since the complement of a measurable set is measurable.

$\therefore \{x : f(x) < \alpha\}$ is also a measurable set

(iii) ⇒ (iv)

Assume that $\{x : f(x) < \alpha\}$ is measurable

To prove that $\{x : f(x) \leq \alpha\}$ is measurable

$$\{x : f(x) \leq \alpha\} = \{x : f(x) < \alpha + 1\} \cap \{x : f(x) < \alpha + 1/n\}.$$

$$\{x : f(x) < \ldots \cap \{x : f(x) < \alpha + 1/n\}$$

$$= \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

By assumption,

$f_n\}$ is measurable for all $n \in N$.

$\therefore$ the countable intersection of measurable set is measurable

(iv)$\Rightarrow$(i) :

Assume that for all $\{x : f(x) \leq \alpha\}$ is measurable

To prove that f is measurable

(i.e) $\{x : f(x) > \alpha\}$ is measurable set

clearly we have.

$$\{x : f(x) > \alpha\} = \{x : f(x) \leq \alpha\}^c$$

we know that $\{x : f(x) \leq \alpha\}$ is measurable set

Since the complement of a measurable set is measurable.

$\therefore \{x : f(x) > \alpha\}$ is measurable.

Hence f is measurable.

Example:

Show that if f is measurable then $\{x : f(x) = \alpha\}$ is measurable for each extended real number α.

proof:

Given that f is a measurable function

To prove that a set $\{x : f(x) = \alpha\}$ is measurable for each extended real number α.

case 1 :

If α is finite, then

$$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$$

Since $\{x : f(x) \geq \alpha\}$ & $\{x : f(x) \leq \alpha\}$ are measurable set is measurable.

$\therefore \{x : f(x) = \alpha\}$ is measurable set

case 2 :

If $\alpha = +\infty$ clearly we have

$$\{x : f(x) = \alpha\} = \{x : f(x) > 1\} \cap \{x : f(x) > 2\} \cap \{x : f(x) > 3\} \cdots$$

$= \bigcap_{n=1} \{x : f(x) > n\}$ is measurable

since the intersection of measurable set is measurable.

case 3 :

If $\alpha = -\infty$ clearly we have

$\{x : f(x) = \alpha\} = \{x : f(x) < 1\} \cap \{x : f(x) < -1\}$

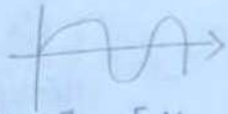
$= \bigcap_{n=1}^{\infty} \{x : f(x) < n\}$ is measurable.

Problem:

If f is a function defined on $[0,1]$ by $f(0) = 0$, $f(x) = x \sin 1/x$ for $x > 0$ find the measure of the set $\{x : f(x) \geq 0\}$

Let $f(x) = 0$ if $x = 0$ $= x \sin 1/x$ if $0 < x \leq 1$ to find $m^* \{x : f(x) \geq 0\}$.

From the diagram



$\{x : f(x) \geq 0\} = \{0\} \cup [1/\pi \cdot 1] \cup [1/3\pi \cdot 1/2\pi] \cup [1/5\pi \cdot \frac{1}{4\pi}] \cup \ldots$

$m^* \{x : f(x) \geq 0\} = m^*(\{0\}) + m^*([1/\pi \cdot 1]) + m^*([\frac{1}{3\pi} \cdot \frac{1}{2\pi}])$

$= 0 + 1 - 1/\pi + 1/2\pi - 1/3\pi + 1/4\pi \ldots$

$= 1 - 1/\pi (1 - \frac{1}{2} + \frac{1}{3} + 1/4 + \ldots)$

$m^* \{x : f(x) \geq 0\} = 1 - 1/\pi \log 2.$

Theorem:

Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E then $f+c$, cf, $f+g$, $f-g$ and fg are also measurable.

Given the f and g are measurable function

To prove:

$(f+c)$ is measurable function

where c is a constant for any α

$\{x : f(x) > \alpha\} = \{x : f(x) + c > \alpha\}$

$\{x : f(x) > \alpha - c\}$ which is measurable set

$\therefore$ The function $(f+c)$ is measurable.

TO prove:

cf is measurable

for this suppose $c > 0$ for any α

$$\{x : (cf)(x) > \alpha\} = \{x : cf(x) > \alpha\}$$
$$= \{x : f(x) > \alpha/c\}$$

which is measurable set

$\Rightarrow$ The function cf is measurable if $c > 0$

suppose let $c = 0$ for any α

$$\{x : (cf)(x) > \alpha\} = \{x : 0 > \alpha\}$$
$$= \begin{cases} \phi & \text{if } \alpha \geq 0 \\ E & \text{if } \alpha < 0 \end{cases}$$

Since E and ϕ are measurable set

The set $\{x : cf(x) > \alpha\}$ is measurable

$\therefore$ cf is measurable if $c = 0$.

TO prove:

$f+g$ is measurable.

for any α

$$\{x : (f+g)x > \alpha\} = \{x : f(x) + g(x) > \alpha\}$$
$$= \{x : f(x) > \alpha - g(x)\}$$

that iff their exist a rational number r_i such that

$f(x) > r_i > \alpha - g(x)$

$$= \{x : f(x) > r_i > \alpha - g(x)\}$$

where r_i is the rational number $(i = 1, 2 \ldots)$

$$= \{x : f(x) > r_i \text{ and } r_i > \alpha - g(x)\}_{i=1}^{\infty}$$
$$= \{x : f(x) > r_i\}_{i=1}^{\infty} \cap \{x : g(x) > \alpha - r_i\}_{i=1}^{\infty}$$
$$= \bigcup_{i=1}^{\infty} \left[\{x : f(x) > r_i\}_{i=1}^{\infty} \cap \{x : g(x) < \alpha - r_i\}_{i=1}^{\infty}\right]$$

The function $(f+g)$ is measurable.

To prove

$(f-g)$ is measurable

since g is measurable, $-g$ is measurable

sing f and $-g$ are measurable and sum of two measurable function is measurable

$\Rightarrow f+(-g)$ is measurable

$f-g$ is measurable.

To prove.

fg is measurable

we can write,

$$fg = \frac{1}{4}[(f+g)^2-(f-g)^2]$$

since f and g are measurable, $(f+g)$ and $(f-g)$ are measurable.

Now we have to prove f^2 is measurable whenever f is measurable for this for any α

$\{x : f^2(x) > \alpha\} = R$ if $\alpha > 0$

$\Rightarrow f^2$ is measurable

for any α,

$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f^2(x) < -\sqrt{\alpha}\}$

If $\alpha > 0$

since f is measurable the set $\{x : f(x) > \sqrt{\alpha}\}$ and $\{x : f(x) < -\sqrt{\alpha}\}$ are measurable

since union of measurable set is measurable.

$\{x : f^2(x) < \alpha\}$ is measurable

f^2 is measurable

It follows that $(f+g)^2$ and $(f-g)^2$ are measurable

$\Rightarrow \frac{1}{4}[(f+g)^2-(f-g)^2]$ is measurable.

$\therefore fg$ is measurable.

Theorem:

If f is measurable then $|f|$ is measurable.

Given that f is measurable.

To prove $|f|$ is measurable.

For any α such that

$\{x : |f|x > \alpha\} = \{x : |f(x)| > \alpha\} = R$

$= \{x : f(x) > \alpha\} \cup \{x : f(x) > -\alpha\}$

We know that $\{x : f(x) > \alpha\}$ and $\{x : f(x) > -\alpha\}$ are measurable

$\therefore |f|$ is measurable.

Theorem:

If f is measurable that $f^{+} = \max(f, 0)$ and $f^{-} = -\min(f, 0)$ are measurable.

Given that f is measurable, clearly

$f^{+} = \max(f, 0)$

$= \frac{1}{2}[f + |f|]$

Since f is measurable, $|f|$ is also measurable.

$\Rightarrow f + |f|$ is measurable

$\Rightarrow \frac{1}{2}(f + |f|)$ is measurable.

$\therefore f^{+}$ is measurable.

Now, $f^{-} = -\min(f, 0)$

$= \max(-f, 0)$

$= \frac{1}{2}[-f + |-f|]$

$f^{-} = \frac{1}{2}[|f| - f]$

Since f is measurable, $|f|$ is measurable

$\Rightarrow |f| - f$ is measurable.

$\Rightarrow \frac{1}{2}[|f| - f]$ is measurable.

f^{-} is measurable.

~~(f-g) is measurable~~

s Theorem:

If f and g are measurable function then max(f, g) and min(f, g) are measurable

For any α, if f and g are measurable then two sets $\{x : f(x) < \alpha\}$ and $\{x : g(x) < \alpha\}$ is measurable

$\{x : \max(f(x), g(x)) < \alpha\} = \{x : f(x) < \alpha\} \cap \{x : g(x) < \alpha\}$

The intersection of countable set is measurable so

$\{x : f(x), g(x)\}$ are measurable.

Hence $\max(f(x), g(x))$ are measurable.

for any α. if f and g are measurable then two sets $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ is measurable.

$\cdot \{x : \min[f(x), g(x)] > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$

The union of countable set is measurable sets are measurable.

Theorem:

Let $\{f_n\}$ be a sequence of measurable functions defined on the same measurable set then

(i) sup f_i is measurable for each n, $1 \leq i \leq n$

(ii) inf f_i is measurable for each n, $1 \leq i \leq n$.

(iii) sup f_n is measurable

(iv) inf f_n is measurable

(v) lim sup f_n is measurable

(vi) lim inf f_n is measurable.

Given that $f_1, f_2 \ldots$ are measurable functions

(i) To prove:

sup f_i is measurable for each n $1 \leq i \leq n$

for any α

$\{x : \sup_{1\le i\le n} f_i > \alpha\} = \bigcup_{i=1}^{n} \{x_i : f_i > \alpha\}$

since,

$f_i \{i = 1,2 \dots n\}$ is measurable

$\{x : f_i > \alpha\}$ is measurable for each n

$\Rightarrow \bigcup_{i=1}^{n} \{x : f_i > \alpha\}$ is measurable

$\therefore \sup_{1\le i\le n} f_i$ is measurable for each n

(ii) To prove

$\inf_{1\le i\le n} f_i$ is measurable for each n

for any α,

$$\{x : \inf_{1\le i\le n} f_i < \alpha\} = \bigcup_{i=1}^{n} \{x : f_i < \alpha\}$$

since,

$f_i \{i = 1,2 \dots n\}$ is measurable,

$\{x : f_i < \alpha\}$ is measurable for each n

$\Rightarrow \bigcup_{i=1}^{n} \{x : f_i < \alpha\}$ is measurable

$\therefore \inf_{1\le i\le n} f_i$ is measurable for each n.

(iii) To prove:

$\sup f_n$ is measurable

$$\{x : \sup f_i > \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i > \alpha\}$$

since, $f_i \{i = 1,2 \dots \infty\}$ is measurable for every n

$\{x : f_i > \alpha\}$ is measurable

$\Rightarrow \bigcup_{i=1}^{\infty} \{x : f_i > \alpha\}$ is measurable.

$\therefore \sup f_n$ is measurable

(iv) To prove.

$\inf f_n$ is measurable

$$\{x : \inf f_i < \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i < \alpha\}$$

since $f_i \{i = 1,2 \dots \infty\}$ is measurable for every n

$\Rightarrow \bigcup_{i=1}^{\infty} \{x : f_i < \alpha\}$ is measurable

$\therefore \inf f_n$ is measurable

(v) To prove :

lim sup f_n is measurable

$\limsup f_n = \inf \left(\sup_{i \geq n} f_i \right)$,

a measurable function by (iii) and (iv)

$\therefore$ lim sup f_n is measurable

(vi) To prove :

lim inf f_n is measurable

$\liminf f_n = -\limsup (-f_n)$

and so is measurable.

Definition:

Borel function:

A function f is said to be a Borel measurable or a borel function if $\{x : f(x) > \alpha\}$ is a borel set for any

Borel set :

If the σ- algebra generated by the class of intervals of the form $[a, b)$ its members are called Borel sets of R. It is denoted by $\mathcal{B}$.

Definition

almost everywhere:

If a property holds except on a set of measure zero we say that it holds almost everywhere.

Equivalently a property is said to hold almost everywhere if the set of points where it fails to hold in a set of measure zero

Example :

Let $f(x) = 1$ if $x \in Q$

$= 0$ if $x \notin Q = R - Q$

except on the set of measure zero
i.e, $f=0$ is almost everywhere

Theorem:

If f is a measurable function and let $f=g$ almost everywhere then g is measurable.

given that f is a measurable function and g any function such that $f=g$ almost everywhere.

To prove g is measurable

It is enough to prove that

$$\{x : f(x)>\alpha\} - \{x : g(x)>\alpha\} = \{x : f(x)>\alpha\} \cap \{x : g(x)<\alpha\}$$
$$\subseteq \{x : f(x) \neq g(x)\} \rightarrow (1)$$

we have,

$$\{x : f(x)>\alpha\} - \{x : g(x)>\alpha\} = \{x : f(x) \leq \alpha\} \cap \{x : g(x)>\alpha\}$$
$$\subseteq \{x : f(x) \neq g(x)\} \rightarrow (2)$$

From (1) & (2) we get

$$\{x : f(x)>\alpha\} \Delta \{x : g(x)>\alpha\} \subseteq \{x : f(x) \neq g(x)\}$$

$$\Rightarrow m^* [\{x : f(x)>\alpha\} \Delta \{x : g(x)>\alpha\}] \leq m^* \{x : f(x) \neq g(x)\}.$$
$$= 0$$

$\Rightarrow \{x : g(x)>\alpha\}$ is measurable.

$m^* (F \Delta G) = 0$. then it is measurable.

g is measurable.

Definition:

Essential supremum of f

If f is a measurable function then $\inf \{\alpha : f(x) \leq \alpha$ almost everywhere$\}$ is called the essential supremum of f and is denoted by ess sup f

Example:

Show that $f \leq$ ess sup f. a.e

Proof:

Case (i):

suppose that ess sup $f = \infty$

$\Rightarrow \bigcup_{i=1}^{\infty} \{x : f_i < \alpha\}$ is measurable

$\therefore \inf f_n$ is measurable

(v) To prove:

lim sup f_n is measurable

$\limsup f_n = \inf \left(\sup_{i \geq n} f_i \right)$,

a measurable function by (iii) and (iv)

$\therefore$ lim sup f_n is measurable

(vi) To prove:

lim inf f_n is measurable

$\liminf f_n = -\limsup (-f_n)$

and so is measurable.

Definition:

Borel function:

A function f is said to be a Borel measurable or a borel function if $\{x : f(x) > \alpha\}$ is a borel set for any

Borel set:

If the σ- algebra generated by the class of intervals of the form $[a, b)$ its members are called Borel sets of R. It is denoted by $\mathcal{B}$.

Definition

almost everwhere:

If a property holds except on a set of measure zero we say that it holds almost everwhere.

Equivalently a property is said to hold almost everywhere if the set of points where it fails to hold in a set of measure zero

Example:

Let $f(x) = 1$ if $x \in Q$

$= 0$ if $x \notin Q = R - Q$

$\sup f = \inf\{\alpha : f(x) \le \alpha \text{ a.e}\}$
$= \inf\{[1, \alpha]\}$

similarly,

we can find ess sup $g = 1$

$\therefore$ ess sup f + ess sup $g = 1$.

But ess sup $(f+g) = 0$

$\therefore$ ess sup $(f+g) <$ ess sup f + ess sup g.

Definition:

If f is a measurable function then $\sup\{\alpha : f \ge \alpha \text{ a.e}\}$ is called essential infimum of f and its denoted by ess inf f.

Example:

P.T ess sup f = $-$ess inf$(-f)$.

ess sup $f = \inf\{\alpha : f(x) \le \alpha \text{ a.e}\}$
$= \inf\{\alpha : -f(x) \ge -\alpha \text{ a.e}\}$.
$= -\sup\{-\alpha : -f(x) \ge -\alpha \text{ a.e}\}$.

Let $-\alpha = \beta$ then
$= -\sup\{\beta : -f(x) \ge \beta \text{ a.e}\}$

ess sup f = $-$ess inf $(-f)$.

Example:

P.T ess inf $f \le f$ a.e

Recall that

$f \le$ ess sup f a.e. and

ess sup f = $-$ess inf $(-f) \to$ 2

To prove. By 1

$f \le$ ess sup f a.e

since $-f$ is measurable

$-f \le$ ess sup $(-f)$ a.e

$-f \le$ ess sup $(-f)$ a.e.

$f \ge$ ess inf f a.e.

Definition:

essentially bounded:

If f is a measurable function and ess sup$|f|<\infty$ then f is said to be essentially bounded.

Theorem:

If f is a measurable function of B a borel set then $f^{-1}(B)$ is a measurable set

Proof:

consider $\mathcal{A} = \{A : f^{-1}(A) \text{ is measurable}\}$

claim:

$\mathcal{A}$ is a σ-algebra

consider a function $f : E \to [-\infty, \infty]$ where E is a measurable set

$\Rightarrow f^{-1}(-\infty, \infty) = E$ which is measurable.

$\Rightarrow$ The whole set lies in $\mathcal{A}$

Let $A \in \mathcal{A}$ of then implies $f^{-1}(A)$ is measurable

$\Rightarrow (f^{-1}(A))^c$ is measurable

$\Rightarrow f^{-1}(A^c)$ is measurable

$\Rightarrow A^c$ is measurable

$\forall A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

Let $A_1, A_2 \ldots \in \mathcal{A}$

$\Rightarrow f^{-1}(A_i)$ is measurable $\forall i \in N$

$\Rightarrow \bigcup_{i=1}^{\infty} f^{-1}(A_i)$ is measurable

$\Rightarrow f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)$ is measurable.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

$\mathcal{A}$ is a σ-algebra

Since the inverse image of every interval, under the function f is measurable $\mathcal{A}$ contains all intervals

$\mathcal{B} \subseteq \mathcal{A}$

Let $B \in \mathcal{B} \subset \mathcal{A}$

$B \in \mathcal{A}$

By definition,

$f^{-1}(B)$ is measurable.

2.5 Borel and lebesque measurability

Theorem: 2.5

Let E be a measurable set then for each y the set $E+y = \{x+y : x \in E\}$ is measurable and the measure are the same

Given that E is measurable set

To prove

i) $E+y$ is measurable

ii)

(i) since E is measurable $\forall\ \varepsilon > 0$ exists an open set say O such that $E \subseteq O$ and $m^*(O-E) < \varepsilon \rightarrow (1)$

clearly,

$E+y \subseteq O+y\ \forall y$

since O is open set and $O+y$ is also open set

Now,

$$m^*[(O+y)-(E+y)] = m^*[(O-E)+y]$$
$$= m^*[O-E]$$
$$< \varepsilon \text{ by } (1)$$

By eqn (1) [since measure is transition under invariant]

This means that the open set $O+y$ containing $E+y$ such that $m^*[(O+y)-(E+y)] < \varepsilon$

The set $E+y$ is measurable

ii) For any set A we have

$m^*(A) = m^*(A+y)\ \forall y.$

clearly $m^*(E) = m^*(E+y)\ \forall y.$

Hence the measure are same

Theorem: 26

P.T there exists a non-measurable set

we shall prove that there exists non measurable set in the interval $[0,1]$

First of all we define the relation $\sim$ in the set $[0,1]$ as follows.

$(x,y) \in [0,1]$, $x \sim y \Rightarrow x-y$ is a rational number

To prove the $\sim$ equivalence relation

(i) $x-x=0$, rational number

$\Rightarrow x \sim x \ \forall x$

$\Rightarrow \sim$ is reflexive.

Let $(x,y) \in [0,1]$ and $x-y$ is a rational number

$\Rightarrow -(x-y)$ is a rational no.

$\Rightarrow y-x$ is a rational no

$\Rightarrow y \sim x$

$\Rightarrow \sim$ is symmetric

(iii) Let $(x,y,z) \in [0,1]$ and $x-y$ and $y \sim x$

$\Rightarrow x-y$ and $y-z$ are rational no

$\Rightarrow (x-y)+(y-z)$ is a rational no

$\Rightarrow (x-z)$ is a rational no

$\Rightarrow x \sim z$

$\Rightarrow \sim$ is transitive

The relation $\sim$ defined on $[0,1]$ is an equivalence relation.

This relation partitions in the set $[0,1]$ into disjoint equivalence classes

This $\sim$ is equivalence relation in $[0,1]$

If $(x,y) \in [0,1]$ let $x \sim y$ if $y-x \in Q$

$\Rightarrow y-x \in Q = Q \cap [-1,1]$

Let $[0,1] = \cup E_\alpha$. E_α is a disjoint set such the x and y are same in E_α iff $x \sim y$

Since Q_1 is countable each E_α is a countable set since $[0,1]$ is uncountable there are uncountable many sets E_α consider a set V in $[0,1]$ containing just one element x_α from each E_α.

Let $\{r_i\}$ be a enumeration of Q_1 and for each n. write $V_n = V + r_n$

If $y \in V_n \cap V_m$ there exists $x_\alpha, x_\beta \in V \ni$

$y, x_\alpha + r_n$

$y . x_\beta + r_m$

But then $x_\beta - x_\alpha \in Q$, so $x_\beta = x_\alpha$ by definition of v and we have n, m so $V_n \cap V_m = \phi$ for $n \neq m$

Also $[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2]$

Since for all $x \in [0,1]$, $x \in E_\alpha$ for some α and then $x = x_\alpha + r_n$ given $x \in V_n$

If V is measurable then V_n is also measurable and $m(V) = m(V_n)$

Then using the measurability of the set V_n we have

$$1 = m([0,1])$$
$$\leq \sum_{1}^{\infty} m(V_n)$$
$$= m(V_1) + m(V_2) + \cdots$$
$$\leq 3$$

But this sum can only b 0 or ∞. so V is not measurable there exists a non measurable set.

Theorem: 27.

Not every measurable set is a borel set

write each $x \in [0,1]$ in binary form

$$x = \sum_{n=1}^{\infty} \epsilon_n / 2^n$$

with $\epsilon_n = 0$ or 1 choosing a non expansion for each $x > 0$ define the function f by

$$f(x) = \sum_{n=1}^{\infty} \frac{2\epsilon_n}{3^n}$$

Then the value of f which is known as cantor's function. since ϵ_n is a measurable function of x, f is measurable.

Also f is 1-1 mapping from $[0,1]$ onto its range, since the value $f(x)$ define the sequence $\{\epsilon_n\}$ in the expansion $\sum_{n=1}^{\infty} \frac{2\epsilon_n}{3^n}$ uniquely

so x is determined uniquely.

If $\mathcal{B}$ and M are same then $f^{-1}(B)$ would be measurable for any measurable set B and any measurable function f.

let f be a cantor function and V a non-measurable set in $[0,1]$

Then $B, f(V)$ lies in P and so has measure zero

so $\mathcal{B}$ is measurable But since f is 1-1.

$f^{-1}(B) = V$.

which is non measurable we conclude that $\mathcal{B}$ is strictly contained in M but not every measurable set is a borel set.

Example:

let T be a measurable set of positive measure and let $T^* = [x-y ; x \in T, y \in T]$ show that T^* contains an interval $(-\alpha, \alpha)$ for some $\alpha > 0$.

There exist T for all $\epsilon > 0$. T contains a closed set C of positive measure.

since $m(C) = \lim [m(C \cap [-n, n])]$

we may assume that C is bounded set

for all $\epsilon > 0$ there exists an open set U

$U \supset C$ such that

$m(U - C) < m(C)$

Define the distance between two sets A and B to be

$d(A,B) = \inf [|x-y|, x \in A, y \in B]$

Since $|x-y|$ is a continuous function of x and y the distance between A and B are disjoint closed set one of which is bounded

Let α be the distance between the closed set C and U so that $\alpha > 0$

Let x be any point of $(-\alpha, \alpha)$ we wish to show

$C \cap (C-x) \neq \phi$.

For then $z' = z + x \in C$ and so $x = z' - z \in C$.

Since $|x| < \alpha$ we have $C - x \subset U$ from

$$m(C-(C-x)) \leq m(U-(C-x))$$
$$= m(U) - m(C-x).$$
$$= m(U) - m(C)$$
$$< m(C)$$

Hence $m(C \cap (C-x)) > 0$ and so we must have $C \cap (C-x) \neq \phi$ are required.

Example:

suppose that f is any extended real value function which for every x and y satisfy

$f(x) + f(y) = f(x+y)$

(i) S.T f is either every whose finite or everywhere infinite

(ii) S.T If f is measurable and finite then $f(x) > x f(1)$ for each x.

(i) f cannot both values $-\infty, \infty$ for the $f(x)+f(y) = f(x+y)$ would be meaningless for some pair x, y

suppose that $f(x) = \infty$ for some x

Then $f(x+y) = \alpha + f(y) = \infty$ for all y and

so $f = \infty$ everywhere.

IIIly if $f(x) = -\infty$ for some x

(ii) By definition

$f(x)+f(y)=f(x+y)$ gives

$f(nx)=nf(x)$ for each x and each positive integer n, so

$f^{-1}\left(\frac{x}{n}\right)=n^{-1}f(x)$ and hence

$f\left(\frac{mx}{n}\right)=mn^{-1}f(x)$

In particular,

$f(x).\ rf(1)$ for each $r\in Q$

since f is finite those exists a measurable set E such that $m(E)>0$ and $|f|<u$ say on E

Let $z\in E^*$, $z=x-y$ where $x,y\in E$ then

$|f(z)|=|f(x-y)|$

$=|f(x)-f(y)|$

$\leq 2M$

E^* contains an interval $(-\alpha,\alpha)$ with $\alpha>0$ so if $|x|<\alpha/n$ we have $|f(nx)|\leq 2M$ and so $|f(x)|<2M/n$ for each n

Let x be real and let r be a rational such that $|r-x|<\alpha/n$ Then since $f(x)=rf(1)$ we have,

$|f(x)-xf(1)|=|f(x)-f(r)+(r-x)f(1)|$

$=|f(x-r)+(r-x)f(1)|$

$\leq 2M/n+\alpha/n\,|f(1)|$

for each n. so

$f(x)=xf(1)$

Example : 1

The class of finite union of intervals of the form $[a, b]$ is a ring.

σ-ring:

A ring is called a σ-ring if it is closed under the formulation of countable unions.

Example : 2.

S.T every algebra is a ring & every σ-algebra is also a σ-ring but the converse is not true.

The first part is as follows.

A σ-field, x belongs to the class & the class is closed under the formulation of countable unions and of complements

we consider any finite unions then we obtain an algebra

Every algebra is a ring.

Let $E, F \in R$ then $E \cup F$ & $E - F \in R$

$\therefore E - F = C(C E \cup F)$

$\therefore$ Every algebra is a ring

For the second consider the σ-ring of an subset of $[0,1]$ which are almost countable

If $\cup A_\alpha \in$ & ~~may~~ where A_α are the set of on the space $\cup A_\alpha$

σ-ring is not a σ-algebra

generated ring:

There exist a smallest ring containing class of subsets of space it is called generated ring

generated σ-ring:

There exists a smallest σ-ring containing a class of subsets of a space it is called generated σ-ring.

*. 2m Measure:

A set function μ defined on a ring R is called a measure.

(i) if $\mu(A) \geq 0 \ \forall A \in \mathcal{R}$.

(ii) $\mu(\phi) = 0$

(iii) for any sequence $\{A_n\}$ of disjoint set of $\mathcal{R}$ $\ni$ $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Complete measure:

A measure μ on script $\mathcal{R}$ is said to be complete if whenever $E \in \mathcal{R}$, $F \subseteq E$ and $\mu(E)=0$ then $F \in \mathcal{R}$.

σ-finite:

A measure μ on R is said to be σ-finite if for every set $E \in \mathcal{R}$ we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ $\ni$ $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.

Example: 3

S.T Lebesgue measure m defined on $\mathcal{X}$ the class of measurable sets of R is a σ-finite & complete.

$\in \mathcal{M}$ defined $E_n = E \cap (-n, n) \subset (-n, n)$

clearly, $E = \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow m(E_n) \leq m(-n, n)$

$\leq 2n < \infty$

$m(E_n) < \infty$

Hence M is σ-finite.

Let $E \in \mathcal{M}$, $F \subseteq E$ and $\mu(E) = 0$

To prove $\mu(F) = 0$

Hence $F \subseteq E$

$\mu(F) \leq \mu(E)$

$\mu(F) \leq 0 \ldots (1)$

W.K.T $\mu(F) \geq 0$ always (2)

Hence $\mu(F) = 0$

m is complete

hereditary class:

The class of sets is said to be hereditary if every subsets of one of its members belongs to the class and the class is denoted by $H \in \mathcal{H}(R)$.

outer measure:

A set function μ^* defined on the close $\mathcal{H}(R)$ is said to be outer measure

(i) μ^* is non negative (≥ 0) for every $A \in \mathcal{H}(R)$ (i.e) $\mu^*(A) \geq 0$

(ii) If $A \leq B$ then $\mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*(\phi) = 0$

(iv) for any sequence $\{A_n\}$ of subset $\mathcal{H}(R)$, $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ (i.e) μ^* is countably sub additive.

Note:

clearly lebesgue measure m^* is an outer measure.

But there exists a σ-ring containing $\mathcal{A}$ namely the class of subsets of X so taking the intersection of the σ-ring containing $\mathcal{A}$ we get a σ-ring containing $\mathcal{A}$

5.2 Extension of measure:

Theorem: 2

Let $\{A_i\}$ be a sequence of Ring R then there is a sequence $\{B_i\}$ of disjoint set of R such that $B_i \subset A_i$ for each i, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ for each n show that union of $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} B_i$

Construct a sequence $\{B_i\}$ as follows

Let us take $B_1 = A_1$

$B_2 = A_2 - A_1 \Rightarrow A_2 - B_1$

$B_3 = A_3 - (B_1 \cup B_2)$

$\vdots$

$B_n = A_n - \left(\bigcup_{i=1}^{n-1} B_i\right)$

clearly,

$B_i \in \mathcal{H}$ and $B_i \subseteq A_i \ \forall i$

claim(i):

$\{B_n\}$ is a sequence of disjoint set. let $m > n$

Now, $B_m \cap B_n = \left(A_m - \bigcup_{i=1}^{m-1} B_i\right) \cap B_n$

$$B_m \cap B_n = \left(A_m - \bigcup_{i=1}^{m-1} B_i\right) \cap B_n$$

$$= \bigcap_{i=1}^{m-1} (A_m - B_i) \cap B_n$$

$$\subset (A_m - B_n) \cap B_n$$

$$= (A_m \cap B_n^c) \cap B_n$$

$$= \phi$$

$\{B_n\}$ is sequence of disjoint set

claim(ii)

$$\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$$

we shall prove the above result by using induction of N.

Let $N = 1$

$A = B$

Let $N = 2$

$$B_1 \cup B_2 = A_1 \cup B_2$$

$$= A_1 \cup (A_2 - A_1) \quad B - A = B \cap A^c$$

$$= A_1 \cup (A_2 \cap A_1^c)$$

$$= (A_1 \cup A_2) \cap (A_1 \cup A_1^c)$$

$$= (A_1 \cup A_2) \cap \phi$$

$$= A_1 \cup A_2$$

Assume that ① is true for $N = n$

$$\bigcup_{i=1}^{N=n} B_i = B_1 \cup B_2 \cup \ldots \cup B_n$$

$= A_1 \cup (A_2 - B_1) \cup (A_3 - B_1 \cup B_2) \cup \dots$

$= A_1 \cup A_2 \cup \dots$ $\qquad A_2 - B_1$

$A_2 \cap B_1^c$

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

Let $N = n+1$

$$\bigcup_{i=1}^{n+1} B_i = B_{n+1} \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \left(A_{n+1} - \bigcup_{i=1}^{n} B_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \cap (A_{n+1} - B_i) \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \cap (A_{n+1} \cap B_i^c) \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \cap (A_{n+1} \cap B_i^c) \cup \left(\bigcup_{i=1}^{n} B_i\right)$$

$$= A_{n+1} \cup \left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \bigcup_{i=1}^{n+1} A_i = \bigcup_{i=1}^{n+1} B_i$$

$\Rightarrow$ (1) is true when $N = n+1$

$\therefore \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ as $n \to \infty$

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Hence proved

Example:

S.T $\mathcal{H}(R) = E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathcal{R}$.

R.H.S defined other class of set which is hereditary contains R & is a σ-ring so that it contains $\mathcal{H}(R)$ but if $E_n \in \mathcal{R}$ for each n we have

$$\bigcup_{n=1}^{\infty} E_n \in S(R)$$

and each subset belongs to $\mathcal{H}(\mathcal{R})$

Theorem: 8

If μ is a measure a ring $\mathcal{R}$ and if the set function μ^* is defined on $\mathcal{H}(R)$ by $\mu^*(E) = \inf \left[\sum_{n=1}^{\infty} \mu(E_n) : E_n \in R, n = 1 \dots \right.$

then $E \subseteq \bigcup_{n=1} E_n]$

i) for $E \in \mathcal{R}$, $\mu^*(E) = \mu(E)$

ii) μ^* is an outer measure on $\mathcal{H}(\mathcal{R})$.

TO prove that $\mu^*(E) = \mu(E)$ $\forall E \in \mathcal{R}$

If $E \in R$, $E \subset \bigcup_{n=1}^{\infty} E_n$ ..... E is a own covering

$\Rightarrow \mu^*(E) \leq \mu(E)$

i) claim

TO prove $\mu(E) \leq \mu^*(E)$

By the definition of inf

$\forall \epsilon > 0$ there exists a covering say $\{E_n\}$ of E

$\therefore E \subseteq \bigcup_{n=1}^{\infty} E_n$ where $E \in \mathcal{R}$

$\mu^*(E) \leq \mu(E_n)$ $\qquad \mu^*(E) \leq \mu^*(E_n)$

$\mu(E_n) \geq \mu^*(E).$ $\qquad \mu^*(E_n) \leq \mu(E_n)$

$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu^*(E) + \epsilon \rightarrow$ 2 $\qquad \therefore \mu^*(E) \leq \mu(E_n)$

Note that

$$E = E \cap \left(\bigcup_{n=1}^{\infty} E_n\right)$$
$$= \bigcup_{n=1}^{\infty} (E \cap E_n)$$

$$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} (E \cap E_n)\right)$$
$$\leq \sum_{n=1}^{\infty} \mu(E \cap E_n)$$
$$\leq \sum_{n=1}^{\infty} \mu(E_n)$$

$\mu(E) < \mu^*(E) + \epsilon$ by 2

$\mu(E) \leq \mu^*(E) \rightarrow$ 3 $\qquad \epsilon$ is arbitrary.

From 1 & 3

$\mu^*(E) = \mu(E).$

ii) TO prove

μ^* is an outer measure.

claim (i)

$\mu^*(E)$ is non-negative $\forall E \in \mathcal{H}(R)$

μ is a measure, μ is non negative

$\mu^*(E)$ is also non negative

claim (ii)

$A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

Let $B \subseteq \bigcup_{n=1}^{\infty} E_n$ where $E_n \subset R$

$\Rightarrow A \subseteq \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow \mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ ($\because \mu$ is motone)

$\Rightarrow \mu^*(A) \leq \mu^*(B)$

claim (iii)

$\mu^*(\phi) = 0$

By the definition,

$\mu^*(\phi) = \inf \left[\sum \mu(E_n)\ E_n \in R,\ \phi \subset \bigcup_{n=1}^{\infty} E_n\right]$

$= 0$

$\mu^*(\phi) = 0$

claim (iv)

μ^* is subadditive

consider $\{A_n\}$ be a sequence of set in a σ-ring R

Define $B_n = A_n - \bigcup_{i=1}^{n-1} B_i$

$B_1 = A_1$

$B_2 = A_2 - A_1$

$B_3 = A_3 - (A_1 \cup A_2)$

clearly $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ also $B_n \subset A_n\ \forall n$

$B_m \cap B_n = \phi \quad \cdots ①$

whenever $m \neq n$ (disjoint)

Now $\mu^*\left(\bigcup_{n=1}^{\infty} A_i\right) = \mu^*\left(\bigcup_{i=1}^{\infty} B_i\right)$

$\leq \sum_{n=1}^{\infty} \mu^*(B_i)$

$$\leq \sum_{i=1} \mu^*(A_i)$$

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

Hence μ^* is an outer measure on $H(R)$.

μ^* measurable:

2m Let μ^* be an outer measure on $H(R)$ then $E \in H(R)$ is said to be μ^* measurable if for each $A \in H(R)$

$\therefore \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Theorem: 4

10m Extension theorem:

2m Let μ^* be an outer measure of $H(R)$ and Let s^* denote the class of μ^* measurable sets then s^* is a σ-ring & μ^* is restricted to s^* is a complete measure.

Let p be an outer measure on $H(R)$ and s^* denote the class of μ^* measurable and s^* is closed under countable unions

To prove s^* is a σ-algebra

claim (i)

If $E, F \in s^*$, $E - F \in s^*$

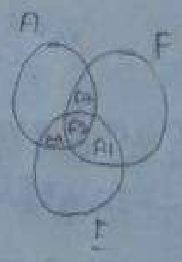
Let $A \in H(R)$ & A as ~~under~~ union of few disjoint set A_1, A_2, A_3, A_4

$A_1 = A - (F \cup E)$

$A_2 = A \cap (F \cap E)$

$A_3 = A \cap (F - E)$

$A_4 = A \cap (E - F)$



Since F is measurable by the definition of

$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c) \longrightarrow (1)$

$= \mu^*(A_2 \cup A_3) + \mu^*(A_1 \cup A_4) \to (2)$

using the fact that E is measurable which gives

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Replacing in this eqn A by $A_1 \cup A_4$

$\mu^*(A_1 \cup A_4) = \mu^*((A_1 \cup A_4) \cap E) + (\mu^*((A_1 \cup A_2) \cap E^c)$

$= \mu^*(A_4) + \mu^*(A_1) \rightarrow ③$

Replacing A by $A_1 \cup A_2 \cup A_3$ & using the fact that F is measurable which gives

$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$

$\mu^*(A_1 \cup A_2 \cup A_3) = \mu^*((A_1 \cup A_2 \cup A_3) \cap F + \mu^*((A_1 \cup A_2 \cup A_3) \cap F^c)$

$= \mu^*(A_2 \cup A_3) + \mu^*(A_1)$

$\mu^*(A_2 \cup A_3) = \mu^*(A_1 \cup A_2 \cup A_3) - \mu^*(A_1) \rightarrow ④$

sub ③ and ④ in ② we get

$\mu^*(A) = \mu^*(A_1 \cup A_2 \cup A_3) - \mu^*(A_1) + \mu^*(A_4) + \mu^*(A_1)$

$= \mu^*(A_1 \cup A_2 \cup A_3) + \mu^*(A_4)$

$= \mu^*(A \cap (E-F)) + \mu^*(A \cap (E-F)^c)$

E-F is measurable

$\therefore E, F \in S^* \rightarrow E-F \in S^*$

claim (ii)

suppose that $\{E_i\}$ is a sequence of disjoint set in S^* then countable union of disjoint set

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i)$$

so μ^* is a measurable on σ-ring S^*

$\therefore S'$ is a σ-ring $S^* \in \mu^*$

Let every set $E \in \mathcal{H}(R)$ such that $\mu^*(E) = 0$

$\Rightarrow \mu^*$ is measurable for if $A \in \mathcal{H}(R)$

Then $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

$\leq \mu^*(E) + \mu^*(A)$

$\mu^*(A) = \mu^*(A)$

shows that $E \subseteq S^*$ In particular if

$$E \in S^* \text{ \& } \mu^*(E)=0 \text{ \& } F \subseteq E \text{ then}$$

$$\mu^*(F) \leq \mu^*(E) = 0$$

$$\mu^*(F) = 0$$

$$\Rightarrow F \in S^*$$

so μ^* is a complete measure of S^*

Theorem: 5

Let μ^* be a outer measure on $H(R)$ defined by μ on R then S^* contains $S(R)$ the σ-ring generated by R.

since S^* is a σ-ring. It is sufficient to show that $R \subseteq S^*$

If $E \in R$, $A \in H(R)$ and $\epsilon > 0$ then by the definition of μ^*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

there exists a sequence $\{E_n\}$ of sets of R such that $A \subseteq \bigcup_{n=1}^{\infty} E_n$ and

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

$$\mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \mu(E_n)$$

$$= \sum_{n=1}^{\infty} \mu(E_n \cap E) + \sum_{n=1}^{\infty} \mu(E_n \cap E^c)$$

so

$$\mu^*(A) + \epsilon \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Since ϵ is arbitrary

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \rightarrow (1)$$

But always

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c) \rightarrow (2)$$

From (1) and (2)

Let $E \in S^*$ then by def of $\bar{u}$ there is a sequence $\{E_n\}$ of sets of R such that

$$\bar{u}(E) \le \sum_{n=1}^{\infty} u(E_n)$$

By the hypothesis, E_n is a union of sequence $\{E_{n=1,2,3}\ldots\}$ of sets of R such that

$$u(E_{n,i}) < \infty \text{ for each } n \text{ and } i$$

$$\text{Then } p(E) = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} u(E_{n,i})$$

$$= \bar{u}(E) < \infty$$

$\bar{u}$ is a finite measure.

So E is the union of a countable collection of sets of finite $\bar{u}$ measure.

SECTION: 5.3

Uniqueness of extension:

m^* Theorem: 6

The outer measure u^* on $H(R)$ defined by u on the R as $u^*(E) = \inf\left[\sum_{n=1}^{\infty} u(E_n) : E_n \in R, E \subseteq \bigcup_{n=1}^{\infty} E_n\right]$ and the corresponding outer measure defined by $S(R)$ & $\bar{u}$ on S^* are the same.

we first observe that the outer measure β^* defined by a measure β on a σ-ring satisfied for $E \in \mathcal{H}(R)$

$$\beta^*(E) = \inf \left[\beta(F) : E \subseteq F \in \mathcal{J} \right] \rightarrow (1)$$

This is the same case,

$$\beta^*(E) = \inf \left[\sum_{n=1}^{\infty} \beta(E_n) \; E \subseteq \bigcup_{n=1}^{\infty} E_n , \; E_n \in \mathcal{J} \right]$$

and replacing the set E_n by disjoint sets

$F_n \in \mathcal{J}$ such that $F_n \subseteq E_n$ & $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$

$$\sum_{n=1}^{\infty} \beta(E_n) \geq \sum_{n=1}^{\infty} \beta(F_n)$$

$$= \beta\left(\bigcup_{n=1}^{\infty} F_n\right)$$

$$= \beta\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$\sum_{n=1}^{\infty} \beta(E_n) \geq \beta^*(E)$$

(1) follows

since $\mathcal{H}(R) = \mathcal{H}(S(R)) = \mathcal{H}(S^*)$

The outer measure to be consider some domain of definition as $\mu = \bar{\mu}$ on R.

$$\mu^*(E) = \inf \left[\sum_{n=1}^{\infty} \mu(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n , \; E_n \in R \right]$$

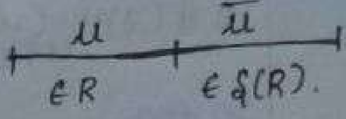
$$\geq \inf \left[\sum_{n=1}^{\infty} \bar{\mu}(E_n) : E \subset \bigcup_{n=1}^{\infty} F_n , \; F_n \in (R) \right.$$

$$= \inf \left[\bar{\mu}(F) : E \subseteq F \in S(R) \right] \text{ by } (1)$$

$$\geq \inf \left[\bar{\mu}(F) : E \subseteq F \subseteq S^* \right] \text{ as } S^* \supseteq S(R)$$

$$\geq \mu^*(E)$$

The outer measure are equal in as the measurable spaces.



... is a measure on $S(R)$ such that $\mu = \nu$ on R.

we wish to show that $\bar{\mu} = \nu$ on $S(R)$ If $E \in S(R)$ and $\epsilon > 0$ there exists $\{E_n\}$ $E_n \in R$.

$$E \subseteq \bigcup_{n=1}^{\infty} E_n \ni$$

$$\bar{\mu}(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu(E_n)$$

Let $A = \bigcup_{n=1}^{\infty} E_n$, A can be written as union of disjoint sets F_n, $F_n \subseteq E_n$, $F_n \in R$

so we get

$$\bar{\mu}(E) + \epsilon \geq \sum_{n=1}^{\infty} \mu(E_n)$$

$$= \sum_{n=1}^{\infty} \nu(E_n)$$

$$\geq \sum_{n=1}^{\infty} \nu(A)$$

$$\bar{\mu}(E) + \epsilon \geq \nu(A)$$

$$\Rightarrow \bar{\mu}(E) + \epsilon \geq \nu(E) \qquad \epsilon \text{ is arbitrary}$$

$$\bar{\mu}(E) \geq \nu(E) \ \forall \ E \in S(R) \cdots\cdots (1)$$

suppose that

$$E \in S(R),\ \bar{\mu}(E) < \infty \text{ and } \epsilon > 0$$

there exists a $A \supseteq E$ such that

$$\bar{\mu}(A) \leq \bar{\mu}(E) + \epsilon$$

$\therefore \bar{\mu}(E) = \gamma(E) + \gamma(A-E)$

But by the 1st part

$\gamma(A-E) \le \bar{\mu}(A-E)$

Also since $\bar{\mu}(E) < \infty$ we have

$\bar{\mu}(A-E) < \epsilon$

$\bar{\mu}(E) \le \gamma(E) + \epsilon$

$\bar{\mu}(E) \le \gamma(E) \cdots (2)$ $\quad \epsilon$ is arbitrary

From (1) & (2) we get

$\bar{\mu}(E) = \gamma(E)$

Since μ is a σ-finite for each $E \in S(R)$ we have $E \subseteq \bigcup_{n=1}^{\infty} E_n$ for each n $E_n \in R$ and $\mu(E_n) < \infty$

write $E = \bigcup_{n=1}^{\infty} F_n$ where F_n are diffornt sets of R & $\mu(F_n) \le \infty$

$\bar{\mu}(E) = \sum_{n=1}^{\infty} \mu(F_n)$

$= \sum_{n=1}^{\infty} \gamma(F_n)$

$\bar{\mu}(E) = \gamma(E)$

$\bar{\mu} = \gamma$

section: 5.4

completion of a measure:

Theorem 8

If μ is a measure on a σ-ring S then the class $\bar{S}$ of each set of the form $E \Delta N$ for any set E, N such that $E \in S$ which N is contained in some set in S of zero measure is a σ ring and the set of function $\bar{\mu}$ is defined by $\bar{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\bar{S}$

It is convenient ... the sets of $\tilde{s}$ we have ...
descriptions of the sets of $\tilde{s}$ we have ...
theoretic identity

$$E \Delta N = (E-M) \cup (M \cap (E \Delta N)) \rightarrow (1)$$

for any set E, M, N such that $M \supseteq N$

let $x \in E \Delta N$ then if $x \in M$ we have

$x \in M \cap (E \Delta N)$,

if $x \in M^c$ we have $x \in N^c$

so $x \in E-N$ and has $x \in E-M$. To get the opposite inclusion in (1) suppose that x belongs to R.H.S

If $x \in M \cap (E \Delta N)$ then $x \in E \Delta N$ if $x \in E-M$

we have $x \in E-N \subseteq E \Delta N$

let $D \in \tilde{s}$, $D = E \Delta N$ as above with

$N \subseteq M \in s$ where $\mu(M)=0$

Then by (1) $D = F \cup A$ where $F \cap A = \phi$ and

$F \in s$ and $A \subseteq M \subseteq s$ where $\mu(M)=0$ and since for

F, A disjoint we have

$$F \cup A = F \Delta A$$

The two characteristic equation of the sets of $\tilde{s}$ are equivalent

Now if $D_i \in \tilde{s}$, $i=1,2,3$

on writing $D_i = F_i \cup A_i$ we see that

$$\bigcup_{i=1}^{\infty} D_i \in \tilde{s}.$$

$D_1 \Delta D_2 \in \bar{\mathcal{S}}$ and so

$D_1 - D_2 = (D_1 \cup D_2) \Delta D_2 \in \bar{\mathcal{S}}$

so $\bar{\mathcal{S}}$ is a σ-ring

Also $D_1 \Delta D_2 = \phi$ only if

$E_1 \Delta E_2 = N_1 \Delta N_2$ so if

$E_1 \Delta N_1 = E_2 \Delta N_2$ we have

$\mu(E_1 \Delta E_2) = 0$ and have

$\mu(E_1) = \mu(E_2)$

so $\bar{\mu}$ is defined

Also $\bar{\mu}$ is a measure for clearly $\bar{\mu}(\phi) = 0$ and if $\{D_i\}$ is a sequence of disjoint set of $\bar{\mathcal{S}}$

$D_i = F_i \cup A_i$

say in the notation used above so that

$F_i \cap A_j = \phi \ \forall i$ and j

Then

$$\bar{\mu}(\cup D_i) = \bar{\mu}[(\cup F_i) \cup (\cup A_i)]$$

$$= \bar{\mu}[\cup F_i \Delta \cup A_i]$$

$$= \mu(\cup F_i)$$

$$= \Sigma \mu(F_i)$$

$$= \Sigma \bar{\mu}(F_i \cup A_i)$$

$$\bar{\mu}(\cup D_i) = \Sigma \bar{\mu}(D_i)$$

so $\bar{\mu}$ is countably additive

Finally μ is complete for let $D \subset D_0 \in \bar{\mathcal{S}}$ where $\bar{\mu}(D_0) = 0$

so $D_0 = E_0 \Delta N_0$

where $N_0 \subseteq M_0$, $M_0 \in \mathcal{S}$, $\mu(E_0) = \mu(M_0) = 0$

and so $D_0 \subseteq M_0' = E_0 \cup M_0 \in \mathcal{S}$ and

$\mu(M_0') = 0$ then

$D = E \Delta N$ with $E = \phi$

$N = D \subseteq E_0 \cup M_0$ and so $D \in \bar{\mathcal{S}}$

$\Rightarrow \bar{\mu}(E)=0$

$\bar{\mu}$ is complete

Example: 9.

S.T the extension $\bar{\mu}$ is unique in the sense that if μ' is a complete measure on a σ-ring ~~S is~~ $S' \supseteq S$ and $\mu'=\mu$ on S then $\mu'=\mu$ on $\bar{S}$

since μ' is complete it is easily seen that $S' \supseteq \bar{S}$

For $D \in S$ we have as above $D = F \cup A$.

F, A disjoint sets with $F \in S$, $A \subseteq M \in S$ with $\mu(M)=0$

$$\text{so } \mu'(D) = \mu'(F \cup A)$$
$$= \mu'(F) + \mu'(A)$$
$$= \mu(F)$$

$$\mu'(D) = \bar{\mu}(D)$$

$\therefore \bar{\mu}$ on $\bar{S}$ the completion of μ on S.

Theorem: 9

The completion of a σ-finite measure is a σ-finite

let $D \in \bar{S}$, $D = F \cup A$ where $F \in S$ and $\bar{\mu}(A)=0$

so $F = \bigcup_{i=1}^{\infty} F_i$ where $\mu(F_i) < \infty$

and hence

$D = A \cup \bigcup_{i=1}^{\infty} F_i$ is a countable union of sets of finite $\bar{\mu}$-measurable.

5.5 section.

Measure space

$S \subseteq X$

A path $[X, S]$ where S is a σ-algebra of subset of a space X is called measurable space.

The set of S are measurable so.

A triple $[X, S, \mu]$ is called a measure space if $[X, S]$ is a measurable space and μ is a measure on S.

Example:10

[[R, M, m]] and [[R, B, m]] are measurable space where $\mathcal{B}$ denotes Borel set

Example:11

let S, $\mathcal{S}$ be a measurable space & let y belongs to $\mathcal{S}$ (i.e) $y \in \mathcal{S}$ then if $\mathcal{S}' = [B \cap y]$: [x, $\mathcal{S}'$] is also measurable space

Theorem:10

let $\{E_i\}$ be a sequence of measurable set we have (i) if $E_1 \subseteq E_2 \subseteq \ldots$ then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim \mu(E_n)$ (ii) if $E_1 \supseteq E_2 \supseteq$ and $\mu(E_1) < \infty$ then $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim \mu(E_n)$

If x be a extended real value function defined on x then f is said to be measurable if $\forall \alpha$ $[x : f(x) > \alpha] \in \mathcal{S}$. It is also called measurability fn.

Example:12

Let [[X, $\mathcal{S}$]] be a measurable space and $X = \bigcup_{n=1}^{\infty} X_n$ where for each n, $X_n \in \mathcal{S}$ and $X_n \cap X_m = \phi$ for $n \neq m$ write $S_n = [B \cap X_n ; B \in \mathcal{S}]$ S.T f is measurable w.r to [[X, $\mathcal{S}$]] $\Leftrightarrow$ for each n f_n to X_n is measurable w.r to [[X_n, S_n]] & conversely for each n, the function f_n are measurable w.r to [[X_n, S_n]] and f is defined by $f(x) = f_n(x)$ when $x \in X_n$, f is measurable w.r to [[S, $\mathcal{S}$]]

for each α $[x : f_n(x) > \alpha] = [x : f(x) > \alpha] \cap X_n$

so f_n is measurable w.r to measurable space [[X_n, S_n]]

The converse follows from $[x : f(x) > \alpha]$

$$= \bigcup_{n=1}^{\infty} [x : f_n(x) > \alpha]$$

Theorem: 11

The measurability of f equivalent to

(i) f is a measurable function.

(ii) $\forall \alpha$, $[f(x) \geq \alpha] \in \mathcal{S}$

(iii) $\forall \alpha$, $[x : f(x) < \alpha] \in \mathcal{S}$

(iv) $\forall \alpha$, $[x : f(x) \leq \alpha] \in \mathcal{S}$

Assume that f is measurable function

To prove (i) $\Rightarrow$ (ii)

$\forall \alpha, \{x : f(x) \geq \alpha\}$ is measurable.

for this we have any α

$\{x : f(x) \geq \alpha\} = \{x : f(x) > \alpha - 1\}$

$\{x : f(x) > \alpha - 1\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$ is measurable set

$\{x : f(x) \geq \alpha\}$ is a measurable set for α.

(ii) $\Rightarrow$ (iii)

Assume that $\forall \alpha$

$\{x : f(x) \geq \alpha\}$ is measurable.

To prove for any α, $\{x : f(x) < \alpha\}$ is measurable.

clearly we have

$\{x : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}^c$

By assumption,

$\{x : f(x) \geq \alpha\}$ is measurable set

the complement of measurable set is measurable

$\{x : f(x) \geq \alpha\}^c$ is measurable

$\{x : f(x) < \alpha\}$ is also measurable.

(iii) $\Rightarrow$ (iv)

Assume that $\{x : f(x) < \alpha\}$ is measurable

To prove $\{x : f(x) \leq \alpha\}$ is measurable

$\{x : f(x) \leq \alpha\} = \{x : f(x) \leq \alpha\}$ is measurable

$\{x : f(x) \leq \alpha\} = \{x : f(x) < \alpha + 1\} \cap \{x : f(x) < \alpha + 1/2\}$

$= \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$

By assumption $\{x : f(x) < \alpha + 1/n\}$ is measurable $\forall n \in N$
since the countable intersection of measurable set is measurable $\{x : f(x) \leq \alpha\}$ is measurable

(iv) $\Rightarrow$ (i)

$\forall \alpha$, $\{x : f(x) \leq \alpha\}$ is measurable

To prove: f is measurable function

$\{x : f(x) < \alpha\}$ is measurable

clearly we have

$$\{x : f(x) > \alpha\} = \{x : f(x) \leq \alpha\}^c$$

By assumption

$\{x : f(x) \leq \alpha\}$ is a measurable set

$\therefore$ The complement of measurable set is measurable

$\therefore \{x : f(x) > \alpha\}$ is measurable.

f is a measurable function.

Example: 13.

(i) f is measurable then $\{x : f(x) = \alpha\}$ is measurable

(ii) the constant function are measurable

(iii) the characteristic function χ_A is measurable $\Leftrightarrow$ A

(iv) continuous function of measurable function is measurable.

(i) Given that f is measurable function

To prove $\{x : f(x) = \alpha\}$ is measurable for each extended real no α.

case (i)

x is finite

Then $f(x) = \alpha$

$$\{x : f(x) = \alpha\} = \{x : f(x) \leq \alpha\} \cap \{x : f(x) \geq \alpha\}$$

are measurable and intersection of measurable set is measurable

$\therefore \{x : f(x) = \alpha\}$ is measurable.

... is also measurable.

(ii) Let $f(x) = c$ is a constant $\forall x \in R$

clearly $\{x : f(x) > \alpha\} = \begin{cases} R & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases}$

since both R and ϕ are measurable sets, $\{x : f(x) > \alpha\}$ is also measurable $\forall x \in R$ $\therefore$ the given constant function is measurable

(iii) For any α, we have

$$\{x : f(x) > \alpha\} = \phi \text{ if } \alpha \geq 1$$
$$= A \text{ if } 0 \leq \alpha < 1$$
$$= R \text{ if } \alpha < 0$$

clearly

$\{x : f(x) > \alpha\}$ is measurable iff $A \in$ & Hence the characteristic function χ_A of the set A is measurable iff A is measurable.

(iv) Let f be a continuous function defined on R. clearly,

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : \alpha < f(x) < \alpha + n\}$$

$$= \bigcup_{n=1}^{\infty} \{x : f(x) \in (\alpha, \alpha + n)\}$$

By the definition

$f^{-1}(A) = \{x : f(x) \in A\}$

Since f is continuous and $(\alpha, \alpha+n)$ is open in R

$\therefore f^{-1}(\alpha, \alpha+n)$ is also open in R.

Since any union of open set is open

$\bigcup_{n=1}^{\infty} f^{-1}(\alpha, \alpha+n)$ is also open in R

Since every interval is measurable

Hence every continuous function is measurable

Theorem

Let c be any real no and let $f+g$ be real valued measurable function defined on the same measurable set E then $f+c$, cf, $f+g$, $f-g$, fg are also measurable.

Given that f and g are measurable function

To prove

$f+c$ is a measurable function where c is a constant

for any $\alpha \ni \{x : (f+c)x > \alpha\}$

$= \{x : f(x) + c > \alpha\}$

$= \{x : f(x) > \alpha - c\}$

The function $f+c$ is measurable

To prove cf is measurable

For this suppose $c > 0$

For any $\alpha \ni \{x : (cf)x > \alpha\}$

$= \{x : f(x) . c > \alpha\}$

$= \{x : f(x) > \alpha / c\}$

which is measurable

$\Rightarrow$ the function cf is measurable if $c > 0$.

let $c < 0$ for any α, such that

$\{x : (cf)x > \alpha\} = \{x : f(x) . c > \alpha\}$

$\{x:(f+g)x > \alpha\} = \{x: f(x)+g(x) > \alpha\}$

$= \{x: f(x) > \alpha - g(x)\}$

(i.e) only if $\exists$ a rational r_i such that $f(x) > r_i \geq \alpha - g(x)$ where $\{r_i : r_i = 1, 2 \ldots\}$ is enumerated of Q

$\therefore \{x: f(x)+g(x) > \alpha\} = \{x: f(x) > r_i\}$ and $\{r_i > \alpha - g(x)\}_{i=1}^{\infty}$

$= \{x: f(x) > r_i\}_{i=1}^{\infty} \cap \{x: r_i > \alpha - g(x)\}_{i=1}^{\infty}$

$= \bigcup_{i=1}^{\infty} \{x: f(x) > r_i\} \cap \{x: g(x) > \alpha - r_i\}$

The intersection of two measurable function is measurable. The function $f+g$ is measurable.

(iv) To proove $f-g$ is measurable:

Given f and g are measurable

Since g is measurable, $-g$ is also measurable

f and $-g$ are measurable and some of the two measurable function is measurable

$f+(-g)$ is measurable

$f-g$ is measurable.

To proove fg is measurable.

$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

Since f and g are measurable, $(f+g)$ & $(f-g)$ are measurable. To prove f^2 is measurable whenever f is measurable for this for any α

$\{x : f^2(x) > \alpha\} = R$ if $\alpha < 0$

$\Rightarrow f^2$ is measurable

For any α $\{x : f^2(x) > \alpha\}$

$= \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$ if $\alpha > 0$

Since f is measurable

The sets $\{x : f(x) > \sqrt{\alpha}\}$ & $\{x : f(x) < -\sqrt{\alpha}\}$ are measurable the union of two measurable set is measurable

$\therefore \{x : f^2(x) > \alpha\}$ is measurable

$\Rightarrow f^2$ is measurable it follows that $(f+g)^2$ and $(f-g)^2$ is measurable

$\Rightarrow fg$ is measurable.

Section: 5.6

Integration with respect to a measure

A measurable simple function ϕ is taking a finite number of non-negative values, each on a measurable sets so if $a_1, a_2 \ldots a_n$ are the discrete values of ϕ we have

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

where $A_i = [x : \phi(x) = a_i]$ then the integral of ϕ w.r to μ is given by

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i)$$

Theorem 14. Fatou's Lemma:

Let $\{f_n\}$ be a sequence of measurable function. $f_n : X \to [0, \infty]$ then

$$\lim\inf \int f_n \, du \le \int \lim\inf f_n \, du.$$

proof:

Let $f = \lim\inf f_n$, then f is a non negative measurable function and f is integrable is given by

$$\int f \, du = \sup \int \phi \, du$$

where the sup is taken by over all measurable simple function ϕ with $\phi \le f$ we have,

$$\int \phi \, du \le \int f \, du$$

$$\int \phi \, du \le \int \lim\inf f_n \, du$$

$$\int \phi \, du \le \lim\inf \int f_n \, du \to (1)$$

case (i):

$$\int \phi \, du = \infty$$

Then ϕ simple measurable w.r to u is given by

$\int \phi \, du = \sum_{i=1}^{n} a_i \, u(A_i)$ for some measurable set A

we have $u(A) = \infty$ and $\phi > a > 0$ on A

write $g_k(x) = \inf f_i(x)$ and

$$A_n = [x = g_k(x) > a, \; k \ge n]$$

a measurable set.

Then $A_n \subseteq A_{n+1}$ for each n

But for each n, $\{g_k(x)\}$ is monotonic increasing and

$$\lim_{k \to \infty} g_k(x) = f(x) \ge \phi(x)$$

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

Hence $\lim u(A_n) = \infty$

But for each n. $\int f_n \, du = \int g_n \, du > a \, u(A_n)$

so $\lim\inf \int f_n \, du = \infty$ and (1) hold.

case (ii)

$\int \phi \, du < \infty$

write $B = [x : \phi(x) > 0]$ then $u(B) < \infty$, ϕ and if $0 < \varepsilon < 1$ write $B_n \subseteq B_{n+1}$ $\forall n$ and

$\bigcup_{n=1}^{\infty} B_n \supseteq B$ so $\{B - B_n\}$ is a decreasing sequence of sets $\bigcap_{n=1}^{\infty} (B - B_n) = \phi$.

As $u(B) < \infty$ then there exists N such that $u(B - B_n) < \varepsilon$ for all $n \in N$ so if $n \geq N$

$$\int g_n du \geq \int_{B_n} g_n du$$

$$\geq (1-\varepsilon) \int_{B_n} \phi du \quad [\text{By case i}]$$

$$\geq (1-\varepsilon) \left(\int_B \phi du - \int_{B_n} \phi du \right)$$

$$\geq (1-\varepsilon) \int_B \phi du - \int_{B_n} \phi du$$

since ε is arbitrary $\int g_n du = \int \phi du$

$\lim \inf \int g_n du \geq \int \phi du$ and

since $f_n \geq g_n$

$\lim \inf \int f_n du \geq \int \phi du$

$\lim \inf \int f_n du = \int \lim \inf f_n du$

Almost everywhere:

If f is measurable function let $f = g$ is almost everywhere (a.e) then g is measurable.

Monotone convergence theorem:

statement:

Let $\{f_n\}$ be a sequence of measurable function let $f_n : X \to [0, \infty]$ such that $f_n(x) \uparrow$ for each x and $f = \lim f_n$ then

$$\int f du = \lim \int f_n du$$

let $f = \lim f_n$

$$\int f \, d\mu = \int \lim f_n \, d\mu$$

By using Fatou's lemma is given

$$\int f \, d\mu = \int \lim \inf f_n \, d\mu$$

$$\int f \, d\mu \leq \lim \inf \int f_n \, d\mu \rightarrow ①$$

. $f = f_n$ by hypothesis

$$\int f \, d\mu \geq \int f_n d\mu$$

since $\int f \, d\mu \geq \int \lim \inf f_n \, d\mu$

$$\int f \, d\mu \geq \lim \inf \int f_n d\mu \rightarrow ②$$

From ① & ② we get

$$\int d\mu \, f = \lim \inf \int f_n d\mu$$

$$\int f \, d\mu = \lim \int f_n d\mu.$$

Theorem : 16

Let f be a measurable function $f : X \rightarrow [0, \infty]$ then there exists a sequence $\{\phi_n\}$ of measurable simple function such that for each x $\phi_n(x) \uparrow f(x)$.

By construction for each n

$$E_{nk} = \left\{x : \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n} \cdots \right\} \; k = 1, 2 \ldots n$$

$$F_n = [x : f(x) > n]$$

Put $\phi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{nk}} + n\chi_{nk}.$

Then the function ϕ_n are measurable simple function the range of f given

ϕ_{n+1} is a refinement of ϕ_n, it is easily seen tha

$$\phi_{n+1}(x) \geq \phi_n(x) \; \forall x$$

— x —

If f is measurable function such that atleast one of $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$ is finite.

Then $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu$

If $f(x)$ is finite, $x \in F_n$ $\forall$ large n then $|f(x)-\phi_n(x)| \leq 2^{-n}$ so $\phi_n(x) \uparrow f(x)$

If $f(x)=\infty$ then $x \in \bigcap_{n=1}^{\infty} F_n$ so $\phi_n(x)=n$ $\forall n$ and again $\phi_n(x) \uparrow f(x)$

Theorem:17

$\{f_n\}$ be a sequence of measurable function $f: X \to [0,\infty]$ then $\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$

Let f and g be a non negative measurable fn :-

$$\int f \, d\mu + \int g \, d\mu = \int (f+g) \, d\mu \to (1)$$

(1) applying to a sum of fn so

$$S_n = \sum_{i=1}^{n} f_i \text{ then}$$

$$\int S_n \, d\mu = \int \sum_{i=1}^{n} f_i \, d\mu$$

$$= \sum_{i=1}^{n} \int f_i \, d\mu$$

$$\text{But } S_n \uparrow f = \sum_{i=1}^{n} f_i$$

$$\therefore \int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

Theorem: 18

Let $[X, S, \mu]$ be a measure space & f be a non negative measurable function $\phi(E) = \int_E f \, d\mu$ be a measure on a measurable space $[X, S]$ if in addition $\int f \, d\mu < \infty$ then $\forall \, \varepsilon > 0$ such that $A \in S$ and $\mu(A) < \delta$ then $\phi(A) < \varepsilon$.

The function ϕ is countably additive, since if $\{E_n\}$ is a sequence of degree disjoint set of S.

$$\phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \int \chi_{\cup A} f \, d\mu = \sum_{n=1}^{\infty} \int \chi_{E_n} f \, d\mu$$

with $f_n = \min(f, n)$. then f_n is measurable. $f_n \uparrow f$ and $\lim \int f_n \, d\mu = \int f \, d\mu$

so if $\int f \, d\mu < \infty$ then for all $\varepsilon > 0$ there exists N such that

$\int f du < \int f_N du + \epsilon/2$

If $A \in \mathcal{S}$ and $\mu(A) < \epsilon/2n$ we have $\int_A f_N du < \epsilon/2$

so take $\delta = \epsilon/2N$ to get

$$\int_A f du = \int_A (f - f_n) du + \int_A f_N du$$

$$\leq \int (f - f_N) du + \epsilon/2 \leq \epsilon$$

$$\int f du < \epsilon$$

$$\phi(A) = \int_A f du < \epsilon$$

$$\phi(A) < \epsilon$$

Integrable:

If f is measurable and both $\int f^+ du$ and $\int f^- du$ are finite, then f is said to be integrable and the integrable of f is $\int f^+ du - \int f^- du$

$$\therefore \int f du = \int f^+ du - \int f^- du$$

If f is measurable function such that atleast one of $\int f du - \int f^+ du - \int f^- du$ is finite

Then $\int f du = \int f^+ du - \int f^- du$.

Theorem : 19

let f and g be integrable function and let a and b be constant then $af + bg$ is integrable & $\int (af + bg) du = a \int f du + b \int g du$ If $f = g$ a.e then $\int f du = \int g du$

suppose that $a \geq 0$, $b > 0$ then

$$(af)^+ = af^+, \ (bg^+) = bg^+$$

$$(af)^- = af^-, \ (bg)^- = bg^-$$

so $\int (af^+) du < \infty$ and $\int (bg)^+ du < \infty$

$$\int (af + bg)^+ < \infty$$

$$\int (af + bg)^- < \infty$$

$\therefore af+bg$ is measurable.

$\therefore \int (af+bg)\,du = \int (af+bg)^+\,du - \int (af+bg)^-\,du$

$= \int (af^+ + bg^+)\,du - \int (af^- + bg^-)\,du$

$= \int (af^+ - af^-)\,du + \int (bg^+ - bg^-)\,du$

$= a\int (f^+ - f^-)\,du + b(g^+ - g^-)\,du$

$\int (af+bg)\,du = a\int f\,du + b \int g\,du$

Theorem: 20

Let f be integrable then $|\int f\,du| \leq \int |f|\,du$ with equality iff $f \geq 0$ or $f \leq 0$ a.e.

sufficient condition for equality

$|f| - f \geq 0 \Rightarrow |f| \geq f$

$\int |f|\,du \geq |f|\,du$

Also $|f| + f \geq 0$

$|f| \geq -f$

$\int |f|\,du \geq -\int f\,du$

Hence $\int |f|\,du \geq |\int f\,du|$

Necessary condition for equality

If $\int f\,du \geq 0$, then $\int |f|\,du = \int f\,du$

(i.e) $\int (|f| - f)\,du = 0$

we know that f is non negative measurable function, then $f=0$ a.e iff $\int f\,du = 0$

$|f| - f = 0$ a.e

$|f| = f$ a.e

If $\int f\,du < 0$, then $\int |f|\,du = \int (-f)\,du$

(i.e) $\int (|f| + f)\,du = 0$ and $|f| = f$ a.e

Hence $f \geq 0$ a.e (or)

$f \leq 0$ a.e is a necessary condition.

Theorem: 21

Lebesgue's dominated convergence theorem:

Statement:

Let $\{f_n\}$ be a sequence of measurable function such that $|f_n| \leq g$ where g is an integrable function and $\lim f_n = f$ a.e. Then f is integrable

$$\lim \int f_n \, du = \int f \, du \text{ and}$$

$$\lim \int |f_n - f| \, du = 0$$

Proof:

Since for each n, $|f_n| \leq g$ we have $|f| \leq g$ (a.e) $\Rightarrow$ f_n and f are integrable. Also $\{g + f_n\}$ is a sequence of non-negative measurable functions

By Fatou's lemma,

$$\lim \inf \int (g + f_n) du \geq \int \lim \inf (g + f_n) du$$

so,

$$\int g \, du + \lim \inf \int f_n \, du \geq \int g \, du + \int f \, du$$

But

$\int g \, du$ is a finite

so

$$\lim \inf \int f_n \, du \geq \int f \, du \quad \dots (1)$$

Again

$\{g - f_n\}$ is also a sequence of non-negative measurable function

$$\lim \inf \int (g - f_n) du \geq \int (g - f_n) du$$

$$\Rightarrow \lim \inf \int (g - f_n) du \geq \int g \, du - \int f \, du.$$

so $\lim \sup \int f_n \, du \leq \int f \, du \leq \lim \inf \int f_n \, du$

$$\therefore \lim \int f_n \, du = \int f \, du.$$

Then

$|f_n - f| \leq 2g$ for each n, by known statement

$$\lim \int f_n \, du = \int f \, du$$

$\{f_n - f\}$ gives the result

Theorem: 22

Let $\{f_n\}$ be a sequence of integrable function such that $\sum_{n=1}^{\infty} \int |f| d\mu < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges its sum f is integrable and $\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$

$$\text{Let } \phi(x) = \sum_{n=1}^{\infty} |f_n| \rightarrow (1)$$

Let $\{f_n\}$ be sequence of non-negative measurable function then

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$$

If $\{f_n\}$ be a sequence of integrable function such that

$$\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$$

From (1)

$$\int \phi d\mu < \infty$$

so if ϕ is finite valued a.e