PARTIAL DIFFERENTIAL EQUATIONS 18KP3M11

UNIT-I

First Order P.D.E - Curves and surfaces - Genesis of First Order P.D.E. - classification of Integrals - Linear Equations of the First Order - Pfaffian Differential Equations - compatible systems - Charpit's Method - Jacobi's Method

chap 1 : Sec 1.1 to 1.8

UNIT-II

Integral surfaces Through a Given Curve - Quasi - Linear equations - Non Linear First Order P.D.E -

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UNIT-III

Second order P.D.E.: Genesis of second order P.D.E. - classification of second order P.D.E. One-Dimensional Wave Equation - Vibrations of a string of Infinite string - Vibrations of Semi- Infinite string - Vibrations of a String of Finite Length (Methods of separation Variables)

chap 2 : Sec 2.1 to 2.3.5. except 2.3.4

UNIT IV

Laplace Equations - Boundary Value Problems - maximum and minimum principles - The cauchy Problem - The dirichlet problem for the upper half-plane - The neumann problem for the upper half plane - The Dirichlet interior problem for a circle - The Dirichlet Exterior Problem for a circle - The neumann problem for a circle - The Dirichlet problem for a rectangle - Harnack's Theorem - Laplace Equation Greens function.

chap2 : Sec 2.4 to 2.4.11

UNIT V

Heat condition problem - Heat condition - Infinite Rod case - Heat Conduction finite Rod case. Duhamel's Principle Wave equation - Heat conduction equation

chap2 : Sec 2.5 to 2.6.2

Text Book(s)

An Elementary Course in Partial Differential Equations by T. Amarnath, Narosa, 1997

Reference(s)

1. I.C. Evans, P.D.Es, Graduate Studies in Mathematics, Vol.19 AMS, 1998.
2. I.N. Sneddon, Elements of P.D.Es.
3. F. John, P. Prasad, P.D.Es

UNIT : I

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Chapter 1

First Order Partial Differential Equations

1.1 Curves and surfaces

Curves in space

A curve may be specified by means of parametric equations. Suppose f_1, f_2 and f_3 are continuous functions of a continuous variable t which varies in an interval $I \subseteq \mathbb{R}$, then the three equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ represent the parametric equations of a curve in three-dimensional space.

A standard parameter is the length of the curve measured from some fixed point on the curve. In such cases, at times, the symbol s is used instead of t.

Note: The condition that the parameter t is the length of the curve is $f_1'^2 + f_2'^2 + f_3'^2 = 1$, where prime denotes differentiation.

Example 1.1.1:

The simplest example of a curve in space is a straight line with direction cosines (l, m, n) passing through a point (x_0, y_0, z_0) with parametric equations

$$x = x_0 + ls, \quad y = y_0 + ms, \quad z = z_0 + ns, \quad s \in \mathbb{R}$$

Example 1.1.2:

A right circular helix is a space curve lying on a circular cylinder and is given by the following parametric equations

$$x = a\cos\omega t, \quad y = a\sin\omega t, \quad z = kt, \quad t \in \mathbb{R}$$

where a, w and k are constants

Surfaces:

A point (x, y, z) in space is said to lie on a surface if the co-ordinates x, y and z satisfy

$$F(x, y, z) = 0 \rightarrow ①$$

where F is a continuously differentiable function defined on a domain in R^3. Therefore a surface is the locus of a point moving in space with "two degrees of freedom". Consider a set of relations of the form

$$x = F_1(u, v), y = F_2(u, v), z = F_3(u, v) \rightarrow ②$$

Then for each pair of values of u and v there correspond three numbers x, y and z and hence a point (x, y, z) in space. It is however, not true that every point in space corresponds to a pair u and v. If the Jacobian

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial u} \neq 0,$$ then the first two equations of ② can be solved to express u and v as functions of x and y locally. Say, $u = \lambda(x, y)$ and $v = \mu(x, y)$, by the inverse function theorem. Then u and v are determined once x and y are known and the third equation in ② gives a values of z for these values of x and y. That is

$$z = F_3(\lambda(x, y), \mu(x, y)),$$

which is thus a functional relation between the Co-ordinates x, y & z as in ①. Thus any point (x, y, z) determined from Equations ② always lies on a fixed surface. For this reason, equations of the type ② are called Parametric equations of the surface. However, Parametric equations of a surface are not unique as the following example demonstrates.

Ex: 1.1.3

The parametric equations

$x = a\sin u\cos v, \quad y = a\sin u\sin v, \quad z = a\cos u,$

and

$$x = a\frac{(1-v^2)}{(1+v^2)}\cos u, \quad y = a\frac{(1-v^2)}{(1+v^2)}\sin u, \quad z = \frac{2av}{(1+v^2)},$$

where a is a constant, both represent the surface $x^2+y^2+z^2=a^2$ which is a sphere.

Let us now go back to the equation of the curve $x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$

On eliminating t between f_1 and f_2, we obtain a relation $\phi(x,y) = 0$. Similarly, from f_1 and f_3 we get $\psi(x,z) = 0$. Hence a curve can be thought of as the points of intersection of two surfaces.

If the parameter s in the equation of a curve is the length of the curve measured from some fixed point, then $\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$ are the direction cosines of the tangent to the curve at that point s

Suppose that the curve C : $(x(s), y(s), z(s))$ lies on the surface S whose equation is $F(x,y,z) = 0$, Then

$F(x(s), y(s), z(s)) \equiv 0 \quad \forall s$

on differenting with respect to s, we get

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0$$

This is the condition that the tangent to the curve C at the point $P(x,y,z)$ is perpendicular to line with direction ratios $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$. It may be noted that the curve C is arbitrary except for the fact that it passes through the point P and lies on the given surface. Therefore we can conclude that the line whose direction ratios are given by $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is perpendicular to the tangent to every curve lying on S and passing through P, and therefore they give the direction ratios of the normal to the surface S at the point P.

1.2 Genesis of First Order P.D.E

By a partial differential equation (P.d.e), we mean an equation of the form

$$f(x, y, t, \ldots, \theta, \theta_x, \theta_y, \ldots, \theta_{xt}, \ldots) = 0,$$

involving two or more independent variables x, y, t etc one dependent variable $\theta(x, y, t, \ldots) \in C^n$ in some domain D and its partial derivatives $\theta_x, \theta_y, \ldots \theta_{xx \ldots xt}$ where C^n denotes a set of functions possessing continuous partial derivatives of order n. A p.d.e is thus a relation between the dependent variable θ and some of its partial derivatives at every point $(x, y, t, \ldots)$ in D.

Definition 1.2.1: We define the order of a partial differential equation to be the order of the derivative of the highest order n occurring in the equation.

A p.d.e is said to be quasi-linear if the derivatives of the highest order which occur in the equation are linear. A quasi-linear p.d.e is said to be semi-linear if the coefficients of highest order derivatives do not contain either the dependepent variable or its derivatives. A semi linear p.d.e is said to be linear if it is linear in the dependent variables and its derivatives.

A p.d.e. which is not a quasi-linear p.d.e is said to be non-linear.

We are concerned in this chapter with p.d.e of first order with one dependent variable z and mostly two independent variables x and y. We follow the notation $p = z_x$ and $q = z_y$. We can then write the most general first order p.d.e in the form

$$f(x, y, z, p, q) = 0$$

Ex 1.2.1: Surfaces of revolution:

Many surfaces of revolution with z-axis as the axis of revolution are of the form

$$z = F(r), \quad r = (x^2+y^2)^{1/2}$$

Where F is an arbitrary continuously differentiable function.

On differentiating $z = F(r)$ with respect to x & y respectively, we get

$$p = \left(\frac{x}{r}\right) F'(r), \quad q = \left(\frac{y}{r}\right) F'(r)$$

After eliminating the function $F'(r)$, we get

$$yp - xq = 0$$

Which is a p.d.e of first order satisfied by the surfaces of revolution of the form given below.

Ex 1.2.2:

Consider the surfaces of the form

$F(u, v) = 0$ where $u = u(x, y, z)$ and $v = v(x, y, z)$

are known functions of x, y and z, and F is an arbitrary function of u and v having first order partial derivatives with respect to u and v.

On differentiating $F(u,v)=0$ with respect to x and y treating z as a function of x and y, we get respectively,

$$\frac{\partial F}{\partial u}(u_x+pu_z)+\frac{\partial F}{\partial v}(v_x+pv_z)=0,$$

$$\frac{\partial F}{\partial u}(u_y+q u_z)+\frac{\partial F}{\partial v}(v_y+qv_z)=0$$

On Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from these equations, we obtain

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)}$$

which is a p.d.e. of first order satisfied by the surface $F(u,v)=0$, where

$$\frac{\partial(u,v)}{\partial(x,y)}=u_x v_y-u_y v_x$$

is the Jacobian.

Functional dependence:-

Let $v = v(x, y)$ be a known function of x and y
$u = u(x, y)$ be a function of v alone,
That is not involving x and y explicity. Let

$u = H(v)$

where H is orbitary Let $H \in C'$
On differenciating the above equation with respect to x and y respectively, we get

$u_x = H'(v) v_x \rightarrow (1)$

$u_y = H'(v) v_y \rightarrow (2)$

Elemenating $H'(v)$ from these equations

$\frac{(1)}{(2)} \Rightarrow \frac{u_x}{u_y} = \frac{H'(v)}{H'(v)} \cdot \frac{v_x}{v_y}$

$u_x v_y = v_x u_y$

$u_x v_y - v_x u_y = 0$

$u_x v_y - v_y u_y = \frac{\partial(u, v)}{\partial(x, y)} = 0$

Note: Hence if there is a functional relation b/w two functions $u(x,y)$ & $v(x,y)$ not involving x & y explicitly then.

which is the first order P.D.E for u

Euler's equations for a homogeneous function:

A function $f(x, y)$ is said to be a homogeneous function of x and y of degree n if it is satisfies

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Let $z = f(x, y)$ a homogeneous functions of x and y of degree n then by Euler's Theorem the function $f(x, y)$ satisfies the first order P.D.E

$x f_x + y f_y = nf$

Note:

Consider a two parameter family of surfaces $z = F(x, y, a, b)$ on differentiating with respect to x and y.

$\partial z/\partial x = \partial F/\partial x\,(x, y, a, b)$

$p = F_x(x, y, a, b)$

$\partial z/\partial y = \partial F/\partial y\,(x, y, a, b)$

$q = F_y(x, y, a, b)$

we can solve the above 2 equations

To find a and b interms of x, y, p, q

this would be possible to interms of provided

the matrix

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$$

is of rank 2 by the implicit function theorem

x 2.5 Problem 1:

Consider $z = x + ax^2y^2 + b \rightarrow (1)$ Eliminate the parameters

Partial differentiating with respect to x

$\frac{\partial z}{\partial x} = 1 + 2axy^2$

$p = 1 + 2axy^2 \rightarrow (1)$

Partial differentiating with respect to y

$\frac{\partial z}{\partial y} = 2ax^2y$

$q = 2ax^2y \rightarrow (2)$

$(1) \times x \Rightarrow px = x + 2ax^2y^2$

$px = x + qy$

$px - qy - x = 0.$

6 Problem 2:

Consider $(x-a)^2 + (y-b)^2 + z^2 = 1$

P.d w.r.t 'x'

$2(x-a) + 2zp = 0 \rightarrow (2)$

$2[(x-a)+zp]=0$

$(x-a)+zp=0 \rightarrow (1) \Rightarrow zp=-(x-a)$

P.d. w.r.t 'y'

$2(y-b)+2zq=0$

$(y-b)+zq=0 \rightarrow (2) \Rightarrow zq=-(y-b)$

$(1)^2+(2)^2$

$z^2p^2+z^2q^2=(x-a)^2+(y-b)^2$

$z^2p^2+z^2q^2=-z^2+1$

$z^2p^2+z^2q^2+z^2=1$

$z^2(p^2+q^2+1)=1$

$\Rightarrow z^2(1+p^2+q^2)=1$

5m Classification of first order partial Differential Equations :-

1. Linear equation :-

A first order P.D.E is said to be linear equation if it is linear in p, q and z

2m That is if it is of the form

$$P(x,y)p+Q(x,y)q=R(x,y)z+S(x,y)$$

Example: $yp-xq=xyz+x$

2. Semi linear equation :-

A first order P.D.E is said to be semi linear equation if it is linear in p and q and the coeffeients are of p & q are functions of p and q only. That is if it is of the form

$$P(x,y)p+Q(x,y)q=R(x,y,z)$$

Ex: $e^x p+yxq=xz^2$

3. Quasi-linear equation: A first order p.d.e is said to be a quasi-linear equation if it is linear in p and q, i.e, if it is of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z).$$

Ex: $(x^2+z^2)p - xyq = z^3x + y^2$

4. Non-linear equation: Partial differential equations of the form $f(x,y,z,p,q)=0$ which do not come under the above three types are said to be non-linear equations.

2m

Ex: $pq=z$ does not belong to any of the first three types. So it is a non-linear first order p.d.e

Classification of Integrals:-

(i) complete Integral:-

Let us consider the first order p.d.e

$f(x,y,z,p,q)=0 \rightarrow ①$

A two parameter family of solutions

$z = F(x,y,a,b) \rightarrow ②$

is called the complete integral of ① if in the region considered the rank of Matrix

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} \text{ is two}$$

(ii) General Integral:-

$z = F(x,y,a,b)$

we take $b = \phi(a)$

we get a one parameter family of solution ③ which is subfamily of the two parameter family ②

$z = F(x,y,a,\phi(a)) \rightarrow ③$

The envelope of ③ if it exists obtain By eliminating a between ③ and

$F_a + F_b\,\phi'(a) = 0 \rightarrow ④$

$a = a(x, y)$

substituting a in ③

$z = F(x, y, a(x,y), \phi(x,y))$

If the function ϕ which defines this subfamily is arbitrary then such a solution is called a General Integral of ①

Lemma 1.3.1

Let $z = F(x, y, a)$ be a one parameter family of solutions of $f(x, y, z, p, q) = 0$. Then the envelope of this one parameter family, if it exist is also a solution of $f(x, y, z, p, q) = 0$.

Proof:

The envelope is obtained by eliminating a between

$z = F(x, y, a) \rightarrow$ ①

$0 = F_a(x, y, a) \rightarrow$ ②

Hence the envelope will be given by

$z = G(x,y) = F(x, y, a(x,y))$

Where $a(x,y)$ is obtained from ② By solving for a interms of x & y.

The envelope will satisfy the p.d.e $f(x,y,z,p,q) = 0$.

For,

$G_x = F_x + F_a F_x = F_x$

$G_y = F_y + F_a F_y = F_y$

Since $F_a = 0$.

The envelope will have the same partial derivatives as there of member of the family. The P.d.e a every point being only a relation to be satisfied between these derivatives, The envelope will satisfie the P.d.e of $f(x, y, z, p, q) = 0$

Ex: 1.3.2

Consider $f(x,y,z,p,q) = z - px - qy - p^2 - q^2 = 0 \rightarrow ①$

Find the general integral of the above equation

Solution

A two parameter family of

$z = F(x,y,a,b) = px + qy + p^2 + q^2 \rightarrow ②$

Put $p = a$, $q = b$

$z = ax + by + a^2 + b^2 \rightarrow ③$

is the complete integral of ①. since the matrix

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} = \begin{pmatrix} x+2a & 1 & 0 \\ y+2b & 0 & 1 \end{pmatrix}$$

is of rank 2 Let us take $b = \sqrt{1-a^2}$

$z = F(x,y,a,\sqrt{1-a^2}) = ax + \sqrt{1-a^2}\,y + a^2 + 1 - a^2$

$z = ax + \sqrt{1-a^2}\,y + 1.$

$\frac{\partial F}{\partial a} = x + \frac{1}{2\sqrt{1-a^2}}(-2a)y = x - \frac{ay}{\sqrt{1-a^2}} = 0$

$x\sqrt{1-a^2} - ay = 0$

$x\sqrt{1-a^2} = ay$

Squaring on both sides,

$x^2(1-a^2) = a^2y^2$

$x^2 - a^2x^2 = a^2y^2 \Rightarrow x^2 = a^2x^2 + a^2y^2 = a^2(x^2+y^2)$

$a^2 = \frac{x^2}{x^2+y^2} \Rightarrow a = \frac{x}{\sqrt{x^2+y^2}}$

Put 'a' value in the below equation.

$z = ax + \sqrt{1-a^2}\,y + 1$

$= \frac{x^2}{\sqrt{x^2+y^2}} + \sqrt{1 - \frac{x^2}{x^2+y^2}}\,y + 1$

$= \frac{x^2}{\sqrt{x^2+y^2}} + \sqrt{\frac{x^2+y^2-x^2}{x^2+y^2}}\,y + 1$

$= \frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} + 1 = \frac{x^2+y^2+\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

$$z = \frac{x^2+y^2}{\sqrt{x^2+y^2}} + 1 = \sqrt{x^2+y^2}+1$$

$$z-1 = \sqrt{x^2+y^2}$$

$$(z-1)^2 = x^2+y^2$$

On eliminating a on this result

This is the particular solution of given p.d.e

Let us take $b = a$

$$z = F(x,y,a,a) = ax+ay+2a^2$$

$$\frac{\partial F}{\partial a} = x+y+4a = 0$$

$$x+y = -4a$$

$$a = \frac{-1}{4}(x+y)$$

$$z = ax+ay+2a^2$$

$$= \frac{-1}{4}(x+y)x - \frac{1}{4}(x+y)y + 2\cdot\frac{1}{16}(x+y)^2$$

$$= \frac{-1}{4}x^2 - \frac{1}{4}xy - \frac{1}{4}xy - \frac{1}{4}y^2 + \frac{1}{8}(x+y)^2$$

$$8z = -(x+y)^2$$

This is another particular solution of p.d.e

From ②. the condition $F_a = 0$ and $F_b = 0$

$$F_a = x+2a = 0 \rightarrow ③$$

$$F_b = y+2b = 0 \rightarrow ④$$

On Eliminating a and b between ②, ③ & ④

③ → $a = -x/2$

④ → $b = -y/2$

$$z = ax+by+a^2+b^2$$

$$= -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$4z = -(x^2+y^2)$$

This is the singular integral which is Paraboloid of revolution.

Linear equations of the first order.

Ex:1·4·1: Find the general solution of $xp+yq=z$

Sol: The axilary equations are $\frac{dx}{P} = \frac{dy}{Q}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

(i) & (ii)

$\log x = \log y + \log c_1$

$x = y c_1$

$u = c_1 = x/y$

(ii) & (iii)

$\log y = \log z + \log c_2$

$y = z c_2$

$v = c_2 = y/z$

Therefore the general integral is $F(c_1, c_2) = 0$

$$F(x/y, y/z) = 0 \qquad F(u,v)=0 \Rightarrow v = G(u)$$

where F is an arbitrary differencial function

Another form of the general integral is

$$z = y G\left(\frac{x}{y}\right), \qquad y/z = G(x/y)$$

where G is an arbitrary differentiable function.

classifications of Integrals:

Let us consider a first order p.d.e

$$f(x,y,z,p,q) = 0 \rightarrow (1).$$

Essentially a solution of (1) in a region $D \subseteq \mathbb{R} \times \mathbb{R}$ is given by z as a continuously differentiable function of x and y for $(x,y) \in D$. Further if one computes p and q from it and substitues them into (1), then the equation reduces to an identity in x and y. there are different types of solutions (integral surfaces) for the first order p.d.e (1)

Note: A solution $z = z(x,y)$ when interpreted as a surface in three-dimensional space will be called "integral surface of the p.d.e".

Lemma: 1.3.2

The singular solution is obtained by eliminating p and q from the equations

$$\left.\begin{aligned} f(x,y,z,p,q) &= 0 \\ f_p(x,y,z,p,q) &= 0 \\ f_q(x,y,z,p,q) &= 0 \end{aligned}\right\} \to (1)$$

Proof:

Since $z = F(x,y,a,b)$ is a complete integral of (1) the equation

$$f(x,y,F(x,y,a,b), F_x(x,y,a,b), F_y(x,y,a,b)) = 0 \to (2)$$

which holds identically for all a and b can be differentiated with respect to a and b, and hence leads to

$$\left.\begin{aligned} f_z F_a + f_p F_{xa} + f_q F_{ya} &= 0 \\ f_z F_b + f_p F_{xb} + f_q F_{yb} &= 0 \end{aligned}\right\} - (3)$$

On the singular integral, $F_a = 0$ and $F_b = 0$. Therefore the equations in (3) simplify to

$$f_p F_{xa} + f_q F_{ya} = 0$$

$$f_p F_{xb} + f_q F_{yb} = 0$$

On this surface, $F_{xa} F_{yb} - F_{xb} F_{ya} \neq 0$ (Since $F_a = 0, F_b = 0$) and hence $f_p = 0$, $f_q = 0$. otherwise the matrix

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$$

will not have rank two contradicting the fact that $z = F(x,y,a,b)$ is a complete integral. Hence the lemma.

Special integral:-

Usually, the three ~~cases~~ classes (a), (b) & (c) given above include all the integrals of the first order p.d.e. (1). However, there are some solutions of certain first order p.d.e which do not fall under any of the 3 classes (a), (b) or (c). Such solutions are called 'special integrals'.

Ex 1.3.1: $F(x+y, x-\sqrt{z}) = 0$ is the general integral of the equation $p-q = 2\sqrt{z}$. But $z=0$ also satisfies this equation and it cannot be obtained from the general integral. It is a special integral of the equation. A complete integral of the p.d.e is

$$\sqrt{z} = \frac{(ax+y)}{(a-1)} + b.$$

The cauchy Problem (Initial value problem): (IVP)

Given a first order p.d.e and a curve in space, the cauchy problem is to find an integral surface (i.e a solution) of the given p.d.e. which contains the given curve. In other words, given a p.d.e (not necessarily non-linear)

$$f(x, y, z, p, q) = 0$$

and a curve $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$; $s \in [a,b]$. the cauchy problem is to find a solution $z = z(x,y)$ of the p.d.e. such that $z_0(s) = z(x_0(s), y_0(s))\ \forall s \in [a,b]$

1.4 Linear Equations of the First Order

Theorem 1.4.1:

The general solution of the quasi-linear equation (or Lagrange's equation)

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \rightarrow (1)$$

where P, Q & R are continuously differentiable (and not vanishing simultaneously) functions of x, y and z is $F(u,v) = 0 \rightarrow (2)$

where F is an arbitrary differentiable function of u and v and

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2 \rightarrow (3)$$

are two independent solutions of the system

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \rightarrow (4)$$

Note: $u(x,y,z) = c_1$ is a solution of (4), if at every point on the surface, the tangent plane at that point contains the line through that point with direction ratios (P, Q, R). observe that

④. being a system of ordinary differential equations, its solution is actually a curve. The two is nothing but the intersection of the two surfaces $u = c_1$ and $v = c_2$ and the line through each point of the curve with direction ratios (P, Q, R) is actually the tangent to the curve at that point. These curves are called the characteristic curves of the p.d.e (1.4 ①). we will discuss this more. Sometimes Equation ③ is written as $v = G(u)$, where G is an arbitrary differentiable function of u.

Proof.

Since $u(x,y,z) = c_1$ is a solution of ④, we have

$$P u_x + Q u_y + R u_z = 0 \rightarrow ⑤$$

Similarly, since $v(x,y,z) = c_2$ is a solution of ④, we have

$$P v_x + Q v_y + R v_z = 0 \rightarrow ⑥$$

Therefore solving these equations for P, Q & R, we have

$$\frac{P}{\partial(u,v)/\partial(y,z)} = \frac{Q}{\partial(u,v)/\partial(z,x)} = \frac{R}{\partial(u,v)/\partial(x,y)} \rightarrow ⑦$$

This is rendered possible because of the fact that $u = c_1$ and $v = c_2$ are independent. We showed earlier (refer Ex 1.2.2) that the relation ③ leads to the p.d.e.

$$p \frac{\partial(u,v)}{\partial(y,z)} + q \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \rightarrow ⑧$$

On substituting from ⑦ into ⑧, we see that ③ is a solution of the equation ① if $u = c_1$ and $v = c_2$ are independent solution of ④

Ex : 1.4.2

Find the general integral of $yzp + xzq = xy$

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

Sol:

5m The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{xz} \Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

$$x\,dx = y\,dy$$

Integrating on both sides

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = c_1 \Rightarrow x^2 - y^2 = 2c_1$$

$\therefore x^2 - y^2 = c_1 = u$

$$\frac{dy}{xz} = \frac{dz}{xy} \Rightarrow \frac{dy}{z} = \frac{dz}{y}$$

$$y\,dy = z\,dz$$

Integrating on both sides

$$\frac{y^2}{2} = \frac{z^2}{2} + c_2 \Rightarrow \frac{y^2}{2} - \frac{z^2}{2} = c_2 \Rightarrow y^2 - z^2 = 2c_2$$

$\therefore y^2 - z^2 = c_2 = v$

Therefore the general integral is $F(u,v) = 0$

$v = \phi(u) \Rightarrow y^2 - z^2 = \phi(x^2 - y^2)$

$$F(x^2 - y^2, y^2 - z^2) = 0 \quad (\text{or}) \quad z^2 - y^2 + G(x^2 - y^2)$$

where F and G are arbitrary differentiable functions

Ex: 1.4.3

Find the general solution of

$$x(y^2 - z^2)p - y(z^2 + x^2)q = (x^2 + y^2)z$$

Sol: The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = -\frac{dy}{y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)} = \frac{dz}{z(x^2+y^2)}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} = \frac{dz}{z(x^2+y^2)}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

Integrating, we get

$$x^2 + y^2 + z^2 = c_1 = u$$

Also

$$\frac{\frac{dx}{x} - \frac{dy}{y}}{y^2 - z^2 + z^2 + x^2} = \frac{\frac{dz}{z}}{x^2+y^2}$$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} = \frac{dz}{z}$$

Integrating,

$$\log x - \log y = \log z$$

$$\log x = \log z + \log y$$

$$\log x = \log yz + \log c_2$$

$$\log yz - \log x = \log c_2$$

$$\log\left(\frac{yz}{x}\right) = \log c_2 \Rightarrow \frac{yz}{x} = c_2 = v$$

Hence a general solution is

$$v = G(u)$$

$$z = \frac{x}{y} G(x^2+y^2+z^2) \Leftarrow \frac{yz}{x} = G(x^2+y^2+z^2)$$

or $\quad F\left(\frac{yz}{x}, x^2+y^2+z^2\right) = 0 \; / \; F(u,v) = 0$

where G and F are arbitrary differentiable functions

Ex : 1.4.4

Find the general integral of $z_t + 2zz_x = 0$ and verify that it satisfies the equation.

Sol:

The auxiliary equations are

$$\frac{dt}{1} = \frac{dx}{2z} = \frac{dz}{0}$$

The two intermediate integrals are

$u = z = c_1$,

$v = x - c_1 t = c_2$

i.e, $v = x - zt = c_2$

The general integral is

$F(z, x-zt) = 0$ or $z(x,t) = \phi(x-zt) \rightarrow$ (1)

On differentiating (1) with respect to t and x respectively, we obtain

$$z_t = -\frac{z\phi'}{(1+t\phi')}$$

$$z_x = \frac{\phi'}{(1+t\phi')}$$

It can be verified that these expressions satisfy the given equation. so far, ϕ is an arbitrary function. we can determine its form if we prescribe z as a function of x at t=0. suppose $z(x,0) = -x$ then $\phi(x) \rightarrow$ and from (1) we get

$$z(x,t) = \frac{-x}{1-t}$$

Note: The solution becomes unbounded as $t \rightarrow 1$

Theorem 1.4.2

If $u_i(x_1, \ldots x_n, z) = c_i$, $(i=1, \ldots n)$ are independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

where $P_1, P_2, \ldots P_n$ and R are continuously differentiable functions of $x_1, x_2 \ldots x_n$ and z, not simultaneously zero, then the relation $\phi(u_1, u_2 \ldots u_n) = 0$ where ϕ is an arbitrary differentiable function is a general solution of the quasi-linear p.d.e

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \ldots + P_n \frac{\partial z}{\partial x_n} = R$$

Proof: Similar to the proof given in Theorem 1.4.1

5 Pfaffian Differential Equations

By a Pfaffian differential equation, we mean a differential equation of the form

2m $F_1(x_1 \dots x_n)dx_1 + F_2(x_1, \dots x_n)dx_2 + \dots + F_n(x_1, \dots x_n)dx_n = 0$ → (1)

Where F_i's $(i=1, \dots n)$ are continuous functions. The expression on the left hand side of equ (1) is called a Pfaffian differential form

Definition 1.5.1 : A Pfaffian differential form is said to be exact if we can find a continuously differential function $u(x_1, \dots x_n)$ such that

$du = F_1(x_1, \dots x_n)dx_1 + F_2(x_1 \dots x_n)dx_2 + \dots + F_n(x_1 \dots x_n)dx_n$

Definition 1.5.2 : A Pfaffian differential equation is said to be integrable if there exists a non-zero differentiable function $\mu(x_1, \dots x_n)$ such that the Pfaffian differential form

$\mu [F_1(x_1 \dots x_n)dx_1 + \dots + F_n(x_1, \dots x_n)dx_n]$

Note: A Pfaffian differential equation (1) is said to be exact if the Pfaffian differential form on the left hand side of Equ (1) is exact.

Observe that $u(x_1 \dots x_n) = c$ is a surface in R^n such that at every point on it the normal has direction ratios $(F_1, F_2, \dots F_n)$

Theorem 1.5.1

m There always exists an integrating factor for a Pfaffian differential equation in two variables

Proof:

Consider $P(x,y)dx + Q(x,y)dy = 0$

If $Q(x,y) \neq 0$, then $P(x,y)dx = -Q(x,y)dy$

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

From the existence theorem for a first order o.d.e, the above equation has a solution then

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

This exact Pfaffian form differs from $Pdx + Qdy = 0$ by a factor as they have the same solution. This factor is nothing but the integrating factor.

In general, a Pfaffian differential equation in more than two variables may not be integrable. In the next theorem, we shall derive a necessary and sufficient condition for the integrability of a Pfaffian differential equation in three variables. We shall first prove two lemmas which are used in the theorem.

It was shown in Ex 1.2.3 that if there is a functional dependence between $u(x,y)$ and $v(x,y)$ not involving x and y respectively, then $\frac{\partial(u,v)}{\partial(x,y)} = 0$. We shall prove the theorem converse in the following lemma.

Lemma: 1.5.1

Let $u(x,y)$ and $v(x,y)$ be two functions of x and y such that $\frac{\partial v}{\partial y} \neq 0$. If further $\frac{\partial(u,v)}{\partial(x,y)} = 0$ then there exists a relation $F(u,v) = 0$ between u and v not involving x and y explicitly.

Proof:

Given $\frac{\partial v}{\partial y} \neq 0 \rightarrow (1)$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \rightarrow (2)$$

$$f(u,v) = 0 \rightarrow (3)$$

Since $\frac{\partial v}{\partial y} \neq 0$ we can eliminate y between $u = u(x,y)$ and $v = v(x,y)$ and obtain the relation $F(u,v,x) = 0 \rightarrow (4)$

Now we will show that F does not depend on x.

On differentiating (4) with respect to x and y

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \rightarrow (5) \quad F[\text{[illegible]}]$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \rightarrow (6)$$

On Eliminating $\frac{\partial F}{\partial v}$ which is possible if $\frac{\partial v}{\partial x} \neq 0$

we find that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \rightarrow (7)$$

$$\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \rightarrow (8)$$

$$(7) \times \frac{\partial v}{\partial y} \rightarrow \frac{\partial F}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

$$(8) \times \frac{\partial v}{\partial x} \rightarrow \quad \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} = 0$$

$(-) \quad (-)$

$$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial F}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial u}\left[u_x v_y - u_y v_x\right] = 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

$$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial(u,v)}{\partial(x,y)} = 0.$$

$\frac{\partial F}{\partial u} \neq 0$,

$$\frac{\partial F}{\partial x}\cdot\frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial v}{\partial y} \neq 0$, this implies that $\frac{\partial F}{\partial x} = 0$

Suppose $\frac{\partial v}{\partial x} = 0$, then $\frac{\partial u}{\partial x} = 0$. In this case, also

eqn (5) $\rightarrow$ $\frac{\partial F}{\partial x} = 0$. Hence F is independent of x

Lemma: 1.5.2

If $\vec{X} \cdot \text{curl}\,\vec{X} = 0$, where $\vec{X} = P, Q, R$ and μ is an arbitrary differentiable function of x, y, z then

$$\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = 0.$$

Proof:

By $\vec{X} = P, Q, R$

(i.e) $\vec{X} = P\vec{i} + Q\vec{j} + R\vec{k}$

where i, j, k are unit vectors in the positive direction x, y, z.

$$\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = \mu(P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \nabla \times \mu\vec{X}$$

$$= \mu(P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mu P & \mu Q & \mu R \end{vmatrix}$$

$$= \mu(P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \left[\vec{i}\left(\frac{\partial \mu R}{\partial y} - \frac{\partial \mu Q}{\partial z}\right) - \vec{j}\left(\frac{\partial \mu R}{\partial x} - \frac{\partial \mu P}{\partial z}\right) + \vec{k}\left(\frac{\partial \mu Q}{\partial x} - \frac{\partial \mu P}{\partial y}\right)\right]$$

$$= \mu P\left(\frac{\partial \mu R}{\partial y} - \frac{\partial \mu Q}{\partial z}\right) - \mu Q\left(\frac{\partial \mu R}{\partial x} - \frac{\partial \mu P}{\partial z}\right) + \mu R\left(\frac{\partial \mu Q}{\partial x} - \frac{\partial \mu P}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\mu P \mu R - \frac{\partial}{\partial z}\mu P \mu Q - \frac{\partial}{\partial x}\mu Q \mu R + \frac{\partial}{\partial z}\mu Q \mu P + \frac{\partial}{\partial x}\mu R \mu Q - \frac{\partial}{\partial y}\mu R \mu P$$

$$= 0$$

Conversely, if $\mu\vec{X} \cdot \text{curl}\, \mu\vec{X} = 0$, then $\vec{X} \cdot \text{curl}\, \vec{X} = 0$

$$\vec{X} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\vec{X} \cdot \text{curl}\, \vec{X} = (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \left[\vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial y}PR - \frac{\partial}{\partial z}PQ - \frac{\partial}{\partial x}RQ + \frac{\partial}{\partial z}PQ + \frac{\partial}{\partial x}QR - \frac{\partial}{\partial y}PR$$

$$= 0$$

Ex 1.5.1

Show that the following Pfaffian differential equation is integrable and find its integral

$$y\,dx + x\,dy + 2z\,dz = 0$$

$\Rightarrow$ Let $\text{curl}\, \vec{X} = 0$ where $\vec{X} = (y, x, 2z)$ therefore the Pfaffian differential equation is exact

In fact,

It can be written as

$$y\,dx + x\,dy + 2z\,dz = d(xy + z^2) = c$$

Therefore the integral is

$$u(x,y,z) = xy + z^2 = c$$

Theorem 1.5.2:

A necessary and sufficient condition that the Pfaffian differential equation $\vec{X} \cdot d\vec{r} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0 \rightarrow (1)$ be integrable is that the $\vec{X} \cdot \text{curl}\, \vec{X} = 0$

Proof:

Necessary Part:

If equ① is integral then there exist a differentiable function

$\mu(x,y,z)$ and $u(x,y,z)$ such that

$du = \mu(x,y,z)[P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz] \rightarrow ③$

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \rightarrow ④$

Comparing ③ & ④

we get, $\mu P = \frac{\partial u}{\partial x}$, $\mu Q = \frac{\partial u}{\partial y}$, $\mu R = \frac{\partial u}{\partial z}$

(i.e) $\mu \vec{x} = \nabla u$

Since curl $\nabla u = 0$

we have curl $\mu \vec{x} = 0$

$\therefore \mu\vec{x} \cdot \text{curl}\, \mu\vec{x} = 0$

By the converse of the following lemma

we get $\vec{x} \cdot \text{curl}\, \vec{x} = 0$

Sufficient Part:

If z is tricted has a constant The differential equation ① becomes $P(x,y,z)dx + Q(x,y,z)dy = 0$ since pfaffian differential equation is into variable is always integrable

This have a solution of the form $U(x,y,z) = c_1$ where c_1 may involve z In addition there must exist a non-zero differenciable function μ such that $\frac{\partial u}{\partial x} = \mu P$, $\frac{\partial u}{\partial y} = \mu Q$. On multiplying the equ① by μ

$\mu P(x,y,z)dx + \mu Q(x,y,z)dy + \mu R(x,y,z)dz = 0$

$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \mu R\, dz = 0$

Add & subtract $\frac{\partial u}{\partial z} dz$

$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \left(\mu R - \frac{\partial u}{\partial z}\right) dz = 0$

This implies that,

$$dU + K\,dz = 0 \quad \text{—(a)}$$

Where $K = \mu R - \frac{\partial U}{\partial z} \Rightarrow \mu R = \frac{\partial U}{\partial z} + K$

We know $\vec{X} \cdot \text{curl}\,\vec{X} = 0 \Rightarrow$ and $\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = 0$

$$\vec{X} = (P, Q, R)$$

$$\mu\vec{X} = (\mu P, \mu Q, \mu R)$$

$$= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right)$$

$$= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) + (0, 0, K)$$

$$= \nabla U + (0, 0, K)$$

$$\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right) \cdot \text{curl}\,\mu\vec{X}$$

$$\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right) \cdot \left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}, 0\right)$$

$$\mu\vec{X} \cdot \text{curl}\,\mu\vec{X} = \left(\frac{\partial U}{\partial x} \cdot \frac{\partial K}{\partial y} - \frac{\partial U}{\partial y} \cdot \frac{\partial K}{\partial x}\right)$$

$$= \frac{\partial(U, K)}{\partial(x, y)} \quad \text{—(b)}$$

$\mu\vec{X} \cdot \text{curl}\,\vec{X} = 0$;

Hence K can be explicite the function of U and z allown this equ (b) indicates that there is a relation between U and K independent of x and y but not necessarily of the set. Therefore the equ (a) $\Rightarrow \frac{\partial U}{\partial z} + K(U, z) = 0$ it posses a solution $\phi(U, z) = c$ where c is an arbitrary function. This solution can be expressed in the form $U(x, y, z) = c$ by using the expression for U in terms of (x, y, z).

∴ the pfaffian differential equation ④ is integrable

Compatible system of first order p.d.e

The equations

$f(x, y, z, p, q) = 0 \rightarrow ①$

$g(x, y, z, p, q) = 0 \rightarrow ②$

are compatible on a domain D if

(i) $J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0$ on D $\rightarrow ③$

(ii) $p = \phi(x, y, z)$; $q = \psi(x, y, z)$

obtained by solving ① & ②

$dz = \phi(x, y, z)dx + \psi(x, y, z)dy$ — ④

integrable

Theorem 1.6.1

A necessary and sufficient for the integrability of ④ is

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0$$

Proof

④ is integrable if and only if

$\vec{X} \cdot \text{curl}\,\vec{X} = 0$, where $\vec{X} = (\phi, \psi, -1)$

i.e, $\phi(-\psi_z) + \psi(\phi_z) - (\psi_x - \phi_y) = 0$

$$\psi_x + \phi\psi_z = \phi_y + \psi\phi_z \rightarrow ⑤$$

on substituting ϕ and ψ for p and q respectively in ① and differentiating it with respect to x and z, we obtain

$$f_x + f_p\phi_x + f_q\psi_x = 0$$

$$f_z + f_p\phi_z + f_q\psi_z = 0$$

On multiplying the second equation by ϕ and adding it to the first we get

$f_x + \phi f_z + f_p(\phi_x + \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0$

III'y we may deduce from eqn ② that

$g_x + \phi g_z + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0$

Solving these equations, we find that

$$\psi_x + \phi\psi_z = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right\} \to ⑥$$

If we differentiate the given pair of equations with respect to y and z, we obtain, after a similar analysis as above

$$\phi_y + \psi\phi_z = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \to ⑦$$

On substituting from ⑥ & ⑦ into ⑤ and replacing ϕ and ψ by p and q respectively, we see that the condition for integrability of ④ is that

$$[f,g] = 0$$

Note:

A solution of ④ is of the form

$$F(x,y,z,c) = 0 \to ⑧$$

where c is an arbitrary parameter. Hence we assert that if equations ① & ② are compatible then they have a "one-parameter family of common solutions".

Ex 1.6.1

Show that the equations

5m

$$f = xp - yq - x = 0 \to ①$$

$$g = x^2p + q - xz = 0 \to ②$$

are compatible and find a one-parameter family of common solutions.

Sol:

To find $\frac{\partial(f,g)}{\partial(p,q)}$

$$f = xp - yq - x = 0$$

$$\frac{\partial f}{\partial p} = x \qquad \frac{\partial f}{\partial q} = -y$$

$g = x^2 p + q - xz$

$\frac{\partial g}{\partial p} = x^2 \qquad \frac{\partial g}{\partial q} = 1$

$$\frac{\partial(f, g)}{\partial(p, q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{vmatrix}$$

$$= \begin{vmatrix} x & -y \\ x^2 & 1 \end{vmatrix} = x + x^2 y = x(1+xy) \neq 0$$

On the domain D where D does not contain the points (x, y) such that $x = 0$ or $1+xy = 0$ Then on D, we obtain

$xp - yq - x = 0 \quad \Rightarrow \quad xp - yq - x = 0$

$x^2p + q - xz = 0 \;\; \times y \Rightarrow \quad x^2yp + yq - xyz = 0$

$xp + x^2yp - x - xyz = 0$

$p(x + x^2y) = x + xyz$

$x\,p(1+xy) = x(1+yz)$

$\Rightarrow \quad p = \frac{1+yz}{1+xy}$ //

$x\left(\frac{1+yz}{1+xy}\right) - yq - x = 0$

$x\left(\frac{1+yz}{1+xy}\right) - x = yq$

$\frac{x(1+yz) - x - x^2y}{1+xy} = yq \Rightarrow \frac{x + xyz - x - x^2y}{1+xy} = yq$

$\frac{y(xz - x^2)}{1+xy} = yq$

$\Rightarrow q = \frac{x(z-x)}{1+xy}$

Now we obtain

$dz = \phi(x, y, z)\, dx + \psi(x, y, z)\, dy$

$dz = \left(\frac{1+yz}{1+xy}\right) dx + \left(\frac{x(z-x)}{1+xy}\right) dy \rightarrow (3)$

$dz - dx = \left(\frac{1+yz}{1+xy}\right) dx + \left(\frac{x(z-x)}{1+xy}\right) dy - dx$

$= \left(\frac{1+yz}{1+xy}\right) dx - dx + \left(\frac{x(z-x)}{1+xy}\right) dy$

$$= \frac{(1+yz)dx - (1+xy)dx}{1+xy} + \frac{x(z-x)}{1+xy}dy$$

$$= \frac{dx + yz\,dx - dx - xy\,dx}{1+xy} + \frac{x(z-x)}{1+xy}dy$$

$$dz - dx = \frac{y(z-x)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

$$\frac{dz-dx}{z-x} = \frac{y}{1+xy}dx + \frac{x}{1+xy}dy = \frac{y\,dx + x\,dy}{1+xy}$$

$$\frac{d(z-x)}{z-x} = \frac{d(xy)}{1+xy} \qquad \int \frac{dx}{x} = \log x$$

integrating on both sides

$$\log(z-x) = \log(1+xy) + \log c$$

$$z-x = (1+xy)c$$

$$z = x + (1+xy)c$$

$$c = \frac{z-x}{1+xy}$$

$\therefore$ f and g are compatible on D as (3) is integrable and $\frac{z-x}{1+xy} = c$, is a one-parameter family of common solutions.

Note: Compatibility can also be shown by verifying the conditions $\frac{\partial(f,g)}{\partial(p,q)} \neq 0$ and $[f,g] = 0$ on D

charpit's Method

In this section, we present a method to find a complete integral of a first order p.d.e. Let

$$f(x, y, z, p, q) = 0 \rightarrow (1)$$

be the partial differential equation whose complete integral is being sought.

A family of partial differential equations

$$g(x, y, z, p, q, a) = 0 \rightarrow (2)$$

is said to be a one-parameter family of partial differential equation compatible with (1) if (2) is compatible with (1) for each value of a.

Since $f=0$ and $g=0$ are compatible, we have $[f, g] = 0$, i.e.

$$f_p \frac{\partial g}{\partial x} + f_q \frac{\partial g}{\partial y} + (p f_p + q f_q)\frac{\partial g}{\partial z} - (f_x + p f_z)\frac{\partial g}{\partial p} - (f_y + q f_z)\frac{\partial g}{\partial q} = 0 \rightarrow (3)$$

This is a quasi-linear first order p.d.e for g with x, y, z, p and q as the independent variables.

Our problem i.e. finding a one-parameter family of partial differential equations (2) which is such that each member of the family is compatible with the given p.d.e (1) then is to find a solution of this Equ (3), in as simple a form as possible, involving p or q or both and an arbitrary constant a. This we do by finding an integral of the auxiliary equations

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q} = \frac{-dp}{f_x + p f_z} = \frac{-dq}{f_y + q f_z} \rightarrow (4)$$

charpit's method consists in choosing a one-parameter family of partial differential equations (2) which is such that each member of the family is compatible with the given equ (1). Then, solving for p and q from (1) and (2), we get,

$$p = \phi(x, y, z, a), \quad q = \psi(x, y, z, a)$$

Then $\quad dz = \phi\, dx + \psi\, dy \rightarrow (5)$

is integrable by virtue of the fact that ① & ② are compatible. An integral of ⑤ will be of the form $F(x, y, z, a, b) = 0$

As this is a two-parameter family of solutions of Equ ① it will be a complete integral of ①.

Ex 1.3-1

Find a complete integral of $f = z^2 - pqxy = 0$ by Charpit's method.

Sol: Given $z^2 = pqxy$

The auxiliary equations (④) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

$f_p = \frac{\partial f}{\partial p} = -qxy$; $f_q = \frac{\partial f}{\partial q} = -pxy$

$f_x = \frac{\partial f}{\partial x} = pqy$; $f_y = \frac{\partial f}{\partial y} = pqx$; $f_z = \frac{\partial f}{\partial z} = 2z$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy + pqxy} = \frac{-dp}{pqy + 2zp} = \frac{-dq}{pqx + 2zq}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2zp - pqy} = \frac{dq}{2zq - pqx}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{p(2z - qy)} = \frac{dq}{q(2z - px)}$$

so that

$$\frac{\frac{dp}{p} - \frac{dq}{q}}{2z - qy - 2z + px} = \frac{\frac{dy}{y} - \frac{dx}{x}}{px - qy}$$

$$\frac{\frac{dp}{p} - \frac{dq}{q}}{px - qy} = \frac{\frac{dy}{y} - \frac{dx}{x}}{px - qy}$$

$$\frac{dp}{p} - \frac{dq}{q} = \frac{dy}{y} - \frac{dx}{x}$$

$$\frac{dp}{p} + \frac{dx}{x} = \frac{dy}{y} + \frac{dq}{q}$$

Integrating we get

integrating,

$z = b x^{1/c} y^{c}$

Hence $F(x, y, z, b, c) = z - b x^{1/c} y^{c} = 0$

is a complete integral of $f = 0$

It is easily be verified that the matrix

$$\begin{pmatrix} F_b & F_{bx} & F_{by} \\ F_c & F_{cx} & F_{cy} \end{pmatrix}$$

is of rank two.

Ex: 1.72

Find a complete integral of

$$f = (p^2+q^2)y - qz = 0 \quad \text{--- (1)}$$

10m sol

The auxiliary equations are

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \rightarrow (2)$$

$f_x = \frac{\partial f}{\partial x} = 0$; $f_y = \frac{\partial f}{\partial y} = p^2+q^2$

$f_p = \frac{\partial f}{\partial p} = 2py$; $f_q = \frac{\partial f}{\partial q} = 2qy - z$

$f_z = \frac{\partial f}{\partial z} = -q$

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-p(2py) - q(2qy-z)} = \frac{dx}{-2py} = \frac{dy}{z-2qy} \rightarrow (3)$$

(3) $\Rightarrow$ $\frac{dp}{-pq} = \frac{dq}{p^2}$

$-p\,dp = q\,dq$

Integrating on both sides

$-\frac{p^2}{2} = \frac{q^2}{2} + a^2$ $\qquad$ $-\frac{p^2}{2} - \frac{q^2}{2} = a^2$

$p^2 + q^2 = a^2$ (say) $\qquad$ $\frac{-(p^2+q^2)}{2} = a^2$

(1) $\Rightarrow$ $a^2 y - qz = 0$

$a^2 y = qz$

$q = \frac{a^2 y}{z}$

$p = \frac{a}{z}\sqrt{z^2 - a^2y^2}$

$\therefore dz = p\,dx + q\,dy$

$dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$

$z\,dz = a\sqrt{z^2-a^2y^2}\,dx + a^2y\,dy$

$z\,dz - a^2y\,dy = a\sqrt{z^2-a^2y^2}\,dx$

$$\frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

$$\frac{d(z^2-a^2y^2)}{2\sqrt{z^2-a^2y^2}} = a\,dx$$

$\sqrt{z^2-a^2y^2} = ax+b$

$\Rightarrow z^2 - a^2y^2 = (ax+b)^2$

$z^2 = a^2y^2 + (ax+b)^2$

Ex 1.7.3:

Find a complete integral of $f = xpq + yq^2 - 1 = 0 \rightarrow (1)$

Sol:

the auxilliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$$

$f_p = xq$, $f_q = xp + 2yq$, $f_x = pq$, $f_y = q^2$, $f_z = 0$

$$\frac{dx}{xq} = \frac{dy}{xp+2yq} = \frac{dz}{xpq + xpq + 2yq^2} = \frac{-dp}{+pq} = \frac{-dq}{q^2}$$

we get, $\frac{dp}{pq} = \frac{dq}{q^2}$

integrating, $\ln p = \ln q + \ln a$

$\ln p = \ln qa$

$p = qa$

$g = p - qa = 0 \rightarrow (2)$

Solving $f=0$ and $g=0$ for p and q we get

(3) $\Rightarrow p - aq = 0 \Rightarrow p = aq$

(1) $\Rightarrow xpq + yq^2 - 1 = 0$

$xaq^2 + yq^2 - 1 = 0$

$q^2(ax+y) = 1 \Rightarrow q^2 = \frac{1}{ax+y}$

$\therefore q = \frac{1}{\sqrt{ax+y}}$

and $p = \frac{a}{\sqrt{ax+y}}$

Then, $dz = p\,dx + q\,dy$

$= \frac{a}{\sqrt{ax+y}}dx + \frac{1}{\sqrt{ax+y}}dy$

$dz = \frac{a\,dx + dy}{\sqrt{ax+y}}$

(4) $\Rightarrow (z+b)^2 = 4(ax+y)$

is the required complete integral.

Ex 1.7.4

Find a complete integral of the p.d.e

10m $f = x^2p^2 + y^2q^2 - 4 = 0 \rightarrow (1)$

Sol:

$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$

$f_p = 2x^2p,\ f_q = 2y^2q,\ f_x = 2xp^2,\ f_y = 2yq^2,\ f_z = 0.$

$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{2(x^2p^2+y^2q^2)} = \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2}$

suppose we consider the equation

$\frac{dy}{2y^2q} = \frac{dq}{-2yq^2}$

Then, $\frac{dy}{y} = \frac{dq}{-q}$

$\ln y + \ln a = -\ln q + \ln a$

$\ln y + \ln q = \ln a$

$\ln yq = \ln a$

Therefore $g = qy - a = 0$, Hence

$q = \frac{a}{y}$

(1) $\Rightarrow x^2p^2 + y^2\left(\frac{a^2}{y^2}\right) - 4 = 0 \Rightarrow x^2p^2 + a^2 - 4 = 0$

$x^2p^2 = 4 - a^2$

$p^2 = \frac{4-a^2}{x^2} \Rightarrow p = \frac{\sqrt{4-a^2}}{x}$

$dz = p\,dx + q\,dy$

$dz = \frac{\sqrt{4-a^2}}{x}dx + \frac{a}{y}dy$

Integrating,

$\therefore z = \sqrt{4-a^2}\log x + a\log y + b$

is a complete integral of $f = 0$.

Instead, if we consider,

$$\frac{dx}{2x^2p} = \frac{dp}{-2xp^2}$$

$$\frac{dx}{x} = \frac{dp}{-p}$$

Then,

$\log x = -\log p + \log a$

$\log x + \log p = \log a$

$\log xp = \log a$

then, $g = xp - a = 0$ and $p = \frac{a}{x}$

(1) $\Rightarrow x^2\frac{a^2}{x^2} + y^2q^2 - 4 = 0 \rightarrow y^2q^2 = 4 - a^2$

$q^2 = \frac{4-a^2}{y^2} \Rightarrow q = \frac{\sqrt{4-a^2}}{y}$

$dz = p\,dx + q\,dy$

$dz = \frac{a}{x}dx + \frac{\sqrt{4-a^2}}{y}dy$

$z = a\log x + \sqrt{4-a^2}\log y + b$,

is another complete integral of $f = 0$.

Some standard types

Type-I

$f(p,q)=0$ i.e., the given p.d.e. does not involve x, y and z explicitly.

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

Solving the last equation, we get either $p=a$ (or $q=a$)

Then we solve $f(a,q)=0$ (or $f(p,a)=0$) for $q=Q(a)$ (or $p=P(a)$). Then

$dz = a\,dx + Q(a)\,dy \Rightarrow z = ax + Q(a)y + b$

(or $dz = P(a)\,dx + a\,dy \rightarrow z = P(a)x + ay + b$)

Ex 1.7.6:

Find a complete integral of $f(p,q) = p+q-pq=0$

$\Rightarrow$ Put $q=a$. Then $p = \frac{a}{a-1}$

$dz = p\,dx + q\,dy$

$= \frac{a}{a-1}dx + a\,dy$

$\Rightarrow z = \frac{ax}{a-1} + ay + b$

which is the required complete integral.

Type-II

$f(z,p,q)=0$ (i.e., the given p.d.e. does not involve x and y explicitly)

The auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

$\frac{dp}{p} = \frac{dq}{q} \Rightarrow p = aq$

$\therefore f(z, aq, q)=0$ or $q = Q(a,z)$ and also $p = aQ(a,z)$

$dz = p\,dx + q\,dy = Q(a,z)(a\,dx + dy)$

The complete integral is

$$\int \frac{dz}{Q(a,z)} = ax + y + b$$

Ex 1.7.7

Find a complete integral of the equation

$$zpq - p - q = 0$$

⇒ Putting $p = aq$ in the above equation we get

$zaq - a - 1 = 0$, or [$p=0, q=0$ in which case z = constant].

If $zaq - a - 1 = 0$, then

$$q = \frac{1+a}{az} \quad \text{and} \quad p = \frac{1+a}{z}$$

Now $dz = p\,dx + q\,dy = \frac{1+a}{z}\left(dx + \frac{1}{a}dy\right)$, and hence

$$z^2 = \frac{2(1+a)}{a}(ax+y) + b.$$

Type-III

$$g(x,p) = h(y,q), \quad \text{(separable type)}$$

The auxiliary equations are

$$\frac{dx}{g_p} = \frac{dy}{-h_q} = \frac{dz}{pg_p - qh_q} = \frac{dp}{-g_x} = \frac{dq}{h_y}$$

Note that $g_x\,dx + g_p\,dp = 0$

$\therefore g(x,p) = a$. Hence $h(y,q) = a$

From these we can solve for p and q as

$$p = G(a,x) \quad \text{and} \quad q = H(a,y)$$

Then $dz = p\,dx + q\,dy$ implies

$$z = \int G(a,x)\,dx + \int H(a,y)\,dy + b.$$

Ex 1.7.8

Find a complete integral of $p^2 + q^2 = x + y$

⇒ Observe that the given equation can be put in the form

$p^2 - x = -(q^2 - y) = a$. Then

$p = \pm\sqrt{x+a}$ and $q = \pm\sqrt{y-a}$, so that

$$dz = (\sqrt{x+a})\,dx + (\sqrt{y-a})\,dy$$

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

Type-IV (Clairaut Form):

$$z = px + qy + g(p,q)$$

Here a complete integral is given by

$$z = ax + by + g(a,b).$$

for, it is a solution and the matrix

$$\begin{pmatrix} x + g_x & 1 & 0 \\ y + g_y & 0 & 1 \end{pmatrix}$$

is of rank two

2m Ex 1.7.9 Find a complete integral of the p.d.e $z = px + qy + \log pq$

$\Rightarrow$ The complete integral is $z = ax + by + \log ab$.

Ex 1.7.10: Find a complete integral of the p.d.e

$$pqz = p^2(xq + p^2) + q^2(yp + q^2)$$

or

$$z = xp + yq + \frac{p^4 + q^4}{pq}$$

$\Rightarrow$ The c.i is $z = ax + by + \frac{a^4 + b^4}{ab}$

Ex: 1.7.11: Find a c.i of the p.d.e $z = px + qy + pq$

We have $\frac{dp}{0} = \frac{dq}{0} \Rightarrow p = a$

Solving for q, we get $q = \frac{z - ax}{y + a}$. Hence,

$$dz = a\,dx + \frac{z - ax}{y + a}\,dy$$

$$\frac{dz - a\,dx}{z - ax} = \frac{dy}{y + a}$$

which implies that $z - ax = b(y + a)$, i.e.,

$$z = ax + by + ab$$

1.8 Jacobi's Method:

Let us consider the following p.d.e

$$f(x, y, z, u_x, u_y, u_z) = 0 \quad \rightarrow (1)$$

Here x, y & z are the independent variables and the dependent variable u does not appear in the equation.

A function $u = F(x, y, z, a, b, c)$ is said to be a complete integral of (1) if it satisfies the p.d.e and the associated matrix

$$\begin{pmatrix} F_a & F_{ax} & F_{ay} & F_{az} \\ F_b & F_{bx} & F_{by} & F_{bz} \\ F_c & F_{cx} & F_{cy} & F_{cz} \end{pmatrix}$$

is the rank three

Jacobi's method: For a given p.d.e of the type ①, we consider two one-parameter families of partial differential equations

$$h_1(x, y, z, u_x, u_y, u_z, a) = 0 \rightarrow ②$$

$$h_2(x, y, z, u_x, u_y, u_z, b) = 0 \rightarrow ③$$

such that,
$$\frac{\partial(f, h_1, h_2)}{\partial(u_x, u_y, u_z)} \neq 0 \rightarrow ④$$

and the Pfaffian form

$$du = u_x dx + u_y dy + u_z dz \rightarrow ⑤$$

is integrable, where $u_x(x, y, z, a, b)$, $u_y(x, y, z, a, b)$ & $u_z(x, y, z, a, b)$ are obtained by solving ①, ② & ③ by virtue of ④ such $h_1 = 0$ & $h_2 = 0$ are said to be compatible with $f = 0$. Observe that ⑤ is either exact or not integrable at all. The conditions for ⑤ to be exact are

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}, \quad \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y}, \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_x}{\partial z} \rightarrow ⑥$$

The integral of ⑤ gives the complete integral of ①

* Equations ①, ② & ③ can be solved for u_x, u_y, u_z.
  The equation $du = u_x dx + u_y dy + u_z dz \rightarrow ⑤$
is integrable

If $h_1 = 0$ & $h_2 = 0$ are compatible with $f = 0$ then h_1 & h_2 satisfy

$$\frac{\partial(f, h)}{\partial(x, u_x)} + \frac{\partial(f, h)}{\partial(y, u_y)} + \frac{\partial(f, h)}{\partial(z, u_z)} = 0 \text{ --- } ⑥$$

for $h = h_i$, $i = 1, 2$. Equ ⑥ leads to a semi linear PDE of the form

$$f_{u_x}\frac{\partial h}{\partial x} + f_{u_y}\frac{\partial h}{\partial y} + f_{u_z}\frac{\partial h}{\partial z} - f_x\frac{\partial h}{\partial u_x} - f_y\frac{\partial h}{\partial u_y} - f_z\frac{\partial h}{\partial u_z} = 0$$

for $h = h_i$, $i = 1, 2$. The associated auxiliary equ. are given

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{-f_x} = \frac{du_y}{-f_y} = \frac{du_z}{-f_z}$$

The rest of the procedure is the same as in Charpit's method.

Theorem 1.8.1. If $h_1 = 0$ & $h_2 = 0$ are compatible with $f = 0$, then h_1 and h_2 satisfy, Jacobi's method: For a given p.d.e of the type ①, we consider two one-parameter families of partial differential equations

$$h_1(x, y, z, u_x, u_y, u_z, a) = 0 \rightarrow ②$$
$$h_2(x, y, z, u_x, u_y, u_z, b) = 0 \rightarrow ③$$

such that, $\dfrac{\partial(f, h_1, h_2)}{\partial(u_x, u_y, u_z)} \neq 0 \rightarrow ④$

and the Pfaffian form

$$du = u_x dx + u_y dy + u_z dz \rightarrow ⑤$$

is integrable, where $u_x(x, y, z, a, b)$, $u_y(x, y, z, a, b)$ & $u_z(x, y, z, a, b)$ are obtained by solving ①, ② & ③ by virtue of ④ such $h_1 = 0$ & $h_2 = 0$ are said to be compatible with $f = 0$. Observe that ⑤ is either exact or not integrable at all. The conditions for ⑤ to be exact are

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} ; \quad \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} ; \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_x}{\partial z} \rightarrow ⑥$$

The integral of ⑤ gives the complete integral of ①.

* Equations ①, ② & ③ can be solved for u_x, u_y, u_z.

The equation $du = u_x dx + u_y dy + u_z dz \rightarrow ⑤$ is integrable

If $h_1 = 0$ & $h_2 = 0$ are compatible with $f = 0$ then h_1 & h_2 satisfy

$$\frac{\partial(f, h)}{\partial(x, u_x)} + \frac{\partial(f, h)}{\partial(y, u_y)} + \frac{\partial(f, h)}{\partial(z, u_z)} = 0 - ⑥$$

for $h = h_i$, $i = 1, 2$. Equ ⑥ leads to a semi-linear PDE of the form

$$f_{u_x}\frac{\partial h}{\partial x} + f_{u_y}\frac{\partial h}{\partial y} + f_{u_z}\frac{\partial h}{\partial z} - f_x\frac{\partial h}{\partial u_x} - f_y\frac{\partial h}{\partial u_y} - f_z\frac{\partial h}{\partial u_z} = 0$$

for $h = h_i$, $i = 1, 2$. The associated auxiliary equ. are given by

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{-f_x} = \frac{du_y}{-f_y} = \frac{du_z}{-f_z}$$

The rest of the procedure is the same as in Charpit's method.

10m Theorem 1.8.1: If $h_1 = 0$ & $h_2 = 0$ are compatible with $f = 0$, then h_1 and h_2 satisfy,

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0 \rightarrow (7)$$

Where $h = h_i \ (i = 1,2)$

Proof: $f(x,y,z,u_x,u_y,u_z) = 0 \quad (1)$

On differentiating (1) with respect to $x, y \& z$, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_x} u_{xx} + \frac{\partial f}{\partial u_y} u_{yx} + \frac{\partial f}{\partial u_z} u_{zx} = 0 \rightarrow (8)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u_x} u_{xy} + \frac{\partial f}{\partial u_y} u_{yy} + \frac{\partial f}{\partial u_z} u_{zy} = 0 \rightarrow (9)$$

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial u_x} u_{xz} + \frac{\partial f}{\partial u_y} u_{yz} + \frac{\partial f}{\partial u_z} u_{zz} = 0 \rightarrow (10)$$

Consider

$$h(x,y,z,u_x,u_y,u_z) = 0 \rightarrow (11)$$

Where $h = h_i \ (i = 1,2)$

On differentiating (11) with respect to $x, y \& z$, we obtain

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial u_x} u_{xx} + \frac{\partial h}{\partial u_y} u_{yx} + \frac{\partial h}{\partial u_z} u_{zx} = 0 \rightarrow (12)$$

$$\frac{\partial h}{\partial y} + \frac{\partial h}{\partial u_x} u_{xy} + \frac{\partial h}{\partial u_y} u_{yy} + \frac{\partial h}{\partial u_z} u_{zy} = 0 \rightarrow (13)$$

$$\frac{\partial h}{\partial z} + \frac{\partial h}{\partial u_z} u_{xz} + \frac{\partial h}{\partial u_y} u_{yz} + \frac{\partial h}{\partial u_z} u_{zz} = 0 \rightarrow (14)$$

on multiplying (8) with $\frac{\partial h}{\partial u_x}$ & (12) by $\frac{\partial f}{\partial u_x}$ & subtracting

$$(8) \times \frac{\partial h}{\partial u_x} \Rightarrow \frac{\partial f}{\partial x}\frac{\partial h}{\partial u_x} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_x} u_{xx} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x} u_{yx} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x} u_{zx} = 0$$

$$(12) \times \frac{\partial f}{\partial u_x} \Rightarrow \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_x} u_{xx} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y} u_{yx} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z} u_{zx} = 0$$

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$$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial x} + \left(\frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y}\right) u_{xy} + \left(\frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z}\right) u_{xz} = 0$$

$\rightarrow (15)$

Since $u_{xy} = u_{yx}$, $u_{yz} = u_{zy}$, $u_{xz} = u_{zx}$ from (6)

on multiplying (9) with $\partial h/\partial u_y$ & (13) by $\partial f/\partial u_y$ & subtracting

$$(9) \times \frac{\partial h}{\partial u_y} \Rightarrow \frac{\partial f}{\partial y}\frac{\partial h}{\partial u_y} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y}u_{xy} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_y}u_{yy} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y}u_{zy} = 0$$

$$(13) \times \frac{\partial f}{\partial u_y} \Rightarrow \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial y} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x}u_{xy} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_y}u_{yy} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z}u_{zy} = 0$$

$$(-) \qquad (-) \qquad (-) \qquad (-)$$

$$\Rightarrow \frac{\partial f}{\partial y}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial y} + \left(\frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x}\right)u_{xy}$$

$$+ \left(\frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z}\right)u_{zy} = 0 \rightarrow (16)$$

On multiplying (10) with $\partial h/\partial u_z$ & (14) by $\partial f/\partial u_z$ & subtracting,

$$(10) \times \frac{\partial h}{\partial u_z} \Rightarrow \frac{\partial f}{\partial z}\frac{\partial h}{\partial u_z} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z}u_{xz} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z}u_{yz} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_z}u_{zz} = 0$$

$$(14) \times \frac{\partial f}{\partial u_z} \Rightarrow \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial z} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x}u_{xz} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y}u_{yz} + \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_z}u_{zz} = 0$$

$$(-) \qquad (-) \qquad (-) \qquad (-)$$

$$\Rightarrow \frac{\partial f}{\partial z}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial z} + \left(\frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x}\right)u_{xz} +$$

$$\left(\frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y}\right)u_{yz} = 0 \rightarrow (17)$$

On adding (15), (16) & (17) we get

$$\frac{\partial f}{\partial x}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial z}$$

$$+ \left(\frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y} + \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_x}\right)u_{xy} + \left(\frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x}\right)u_{xz}$$

$$+ \left(\frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial u_z}\right)u_{xz} + \left(\frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z} + \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial u_y}\right)u_{yz} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\frac{\partial h}{\partial y}$$

$$+ \frac{\partial f}{\partial z}\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\frac{\partial h}{\partial z} = 0.$$

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0$$

Which is the required result.

The above equation can also be rewritten as

$$f_{u_x}\frac{\partial h}{\partial x} + f_{u_y}\frac{\partial h}{\partial y} + f_{u_z}\frac{\partial h}{\partial z} - f_x\frac{\partial h}{\partial u_x} - f_y\frac{\partial h}{\partial u_y} - f_z\frac{\partial h}{\partial u_z} = 0 \quad \text{—(8)}$$

Which is a semi-linear p.d.e for h. That is, for a given f if h_1 and h_2 satisfy (4) & (18), then $h_i = 0$ $(i=1,2)$ are compatible with $f = 0$.

Ex 1.9.1

By Jacobi's Method, solve the equation

$$z^2 + zu_z - u_x^2 - u_y^2 = 0 \quad \text{—(a)}$$

⇒ this equation is of the form (1) $f(x,y,z,u_x,u_y,u_z)=0$

The equation for h_1 and h_2 (i.e (9)) in this example is

$$f_{u_x}\frac{\partial h}{\partial x} + f_{u_y}\frac{\partial h}{\partial y} + f_{u_z}\frac{\partial h}{\partial z} - f_x\frac{\partial h}{\partial u_x} - f_y\frac{\partial h}{\partial u_y} - f_z\frac{\partial h}{\partial u_z} = 0$$

$$-2u_x\frac{\partial h}{\partial x} - 2u_y\frac{\partial h}{\partial y} + z\frac{\partial h}{\partial z} - (2z+u_z)\frac{\partial h}{\partial u_z} = 0$$

The auxiliary equations are,

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{-f_x} = \frac{du_y}{-f_y} = \frac{du_z}{-f_z}$$

$$\frac{dx}{-2u_x} = \frac{dy}{-2u_y} = \frac{dz}{z} = \frac{du_x}{0} = \frac{du_y}{0} = \frac{du_z}{-2z-u_z}$$

The two independent solutions of the above system of equations are $u_x = a$ & $u_y = b$ & then from the given equation

$\frac{dx}{-2u_x} = \frac{du_x}{0}$

$0 = -2u_x\, du_x$

$du_x = 0$

Integrating,

$u_x = a$

(a) ⇒ $z^2 + zu_z - a^2 - b^2 = 0$

$$z^2 + zu_z = a^2 + b^2$$

$$zu_z = a^2 + b^2 - z^2$$

$$u_z = \frac{a^2+b^2-z^2}{z}$$

Now,

$$du = u_x dx + u_y dy + u_z dz$$

$$du = a\,dx + b\,dy + \left(\frac{a^2+b^2-z^2}{z}\right)dz$$

is exact. On integrating, we get

$$u = ax + by + (a^2+b^2)\log z - \frac{1}{2}z^2 + c$$

Which is a complete integral of the given equation.

Ex 1.8.2:

Show that a complete integral of $f(u_x, u_y, u_z) = 0$ is $u = ax + by + cz + d$ where $f(a, b, c) = 0$. Hence find the complete integral of $u_x + u_y + u_z - u_x u_y u_z = 0$

⇒ The auxiliary equations are

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{0} = \frac{du_y}{0} = \frac{du_z}{0}$$

Therefore $u_x = a$, $u_y = b$, $u_z = c$

$du = u_x dx + u_y dy + u_z dz$
$du = a\,dx + b\,dy + c\,dz$

$$u = ax + by + cz + d \quad \rightarrow (1)$$

where a, b, c and d are constants, will satisfy the given equation if

$$f(a, b, c) = 0$$

This equation determines c in terms of a and b. Hence (1) has three arbitrary constant a, b, d and it is a complete integral. For example, a complete integral of $u_x + u_y + u_z - u_x u_y u_z = 0$ is (1) if

$$a + b + c - abc = 0$$

$$a + b + c(1 - ab) = 0$$

$$c(1 - ab) = -(a + b)$$

$$c = \frac{-(a+b)}{(1 - ab)}$$

i.e., $c = \frac{a+b}{ab-1}$

Module 2: First order P.d.e

We will now apply Jacobi's method to find a complete integral for a first order p.d.e in two independent variables. Consider the following p.d.e

$$f(x, y, z, p, q) = 0 \rightarrow ①$$

Next, we shall show how to transform the equation $f(x, y, z, p, q) = 0$ into the eqn $g(x, y, z, u_x, u_y, u_z) = 0$ so that the above procedure can be applied.

If $u(x, y, z)$ is a relation between x, y and z and satisfies ① then we have

$u_x + u_z z_x = 0 \Rightarrow u_x + u_z p = 0 \Rightarrow p = -u_x/u_z$

$u_y + u_z z_y = 0 \Rightarrow u_y + u_z q = 0 \Rightarrow q = -u_y/u_z$

Substituting

$$p = -u_x/u_z \quad \text{and} \quad q = -u_y/u_z$$

in ① we obtain an equation

$$g(x, y, z, u_x, u_y, u_z) = 0$$

which can be solved by Jacobi's method

Ex 1.8.3

Find a complete integral of the equation $p^2x + q^2y = z$ by Jacobi's method

→ Step 1: (Converting the given PDE into the form $f(x, y, z, u_x, u_y, u_z) = 0$)

we have $p = -u_x/u_z$ & $q = -u_y/u_z$ in the given PDE we obtain

$$\frac{u_x^2}{u_z^2} x + \frac{u_y^2}{u_z^2} y = z$$

$$\Rightarrow x u_x^2 + y u_y^2 - z u_z^2 = 0$$

$$f(x, y, z, u_x, u_y, u_z) = x u_x^2 + y u_y^2 - z u_z^2 = 0 \rightarrow ①$$

Step 2: Solving PDE ① by Jacobi's Method

Step 2(a): Compute f_{u_x}, f_{u_y}, f_{u_z}, f_x, f_y, f_z

$$f_{u_x} = 2x u_x, \quad f_{u_y} = 2y u_y, \quad f_{u_z} = -2z u_z$$

$f_x = u_x^2,\ f_y = u_y^2,\ f_z = -u_z^2$

Step 2(b): writing auxiliary equation and solving for u_x, u_y & u_z.

The auxiliary equations are given by

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{-f_x} = \frac{du_y}{-f_y} = \frac{du_z}{-f_z}$$

$$\Rightarrow \frac{dx}{2xu_x} = \frac{dy}{2yu_y} = \frac{dz}{-2zu_z} = \frac{du_x}{-u_x^2} = \frac{du_y}{-u_y^2} = \frac{du_z}{u_z^2}$$

Now,

$$\frac{dx}{2xu_x} = \frac{du_x}{-u_x^2} \Rightarrow \frac{u_x dx}{2xu_x^2} = \frac{-2x du_x}{2xu_x^2}$$

$$\Rightarrow u_x dx = -2x du_x$$

$$\Rightarrow \frac{dx}{x} = -2\frac{du_x}{u_x}$$

$$\Rightarrow \log x = -2\log(u_x) + \log(a)$$

$$\Rightarrow \log x + \log(u_x^2) = \log(a)$$

$$\Rightarrow xu_x^2 = a \Rightarrow u_x = \left(\frac{a}{x}\right)^{1/2}$$

$$yu_y^2 = b \Rightarrow u_y = \left(\frac{b}{y}\right)^{1/2}$$

and $$u_z = \left[\frac{(a+b)}{z}\right]^{1/2}$$

Step 2(c): solving the equation $du = u_x dx + u_y dy + u_z dz$

$$du = \left(\frac{a}{x}\right)^{1/2} dx + \left(\frac{b}{y}\right)^{1/2} dy + \left(\frac{a+b}{z}\right)^{1/2} dz$$

$$\Rightarrow u = 2(ax)^{1/2} + 2(by)^{1/2} + 2((a+b)z)^{1/2} + c \rightarrow ②$$

Step 3: (Finding solution of the given PDE from the solution of PDE ①)

Writing $u = c$ in ② we get the complete integral of the given PDE as

$$z = \left[\left(\frac{ax}{a+b}\right)^{1/2} + \left(\frac{by}{a+b}\right)^{1/2}\right]$$

UNIT : II

Integral Surfaces Through a Given Curve: The Cauchy Problem

This section deals with finding an integral surface passing through a given curve. Let us first discuss the problem for a quasi-linear p.d.e. Here we will obtain the required integral surface from the general integral.

Let $F(u,v)=0$ (or $v=G(u)$) be the general integral of the p.d.e $Pp+Qq=R$, where F is an arbitrary function of u and v (or G is an arbitrary function of u), where $u(x,y,z)=c_1$, $v(x,y,z)=c_2$ are the solutions of

Equ. $\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$ —(A)

Let C be the given curve whose parametric equations are given by $x=x_0(s)$, $y=y_0(s)$, $z=z_0(s)$, where s is a parameter (not necessarily the arc length from a fixed point.) We want to find a particular F such that the surface $F(u,v)=0$ contains the given curve C. This is done as follows. Consider the equations

$u(x,y,z)=c_1$; $v(x,y,z)=c_2$

Substituting $x=x_0(s)$, $y=y_0(s)$, $z=z_0(s)$ in these equations we get $u(x_0(s),y_0(s),z_0(s))=c_1$ and $v(x_0(s),y_0(s),z_0(s))=c_2$. Eliminating s between them, we get a relation between c_1 and c_2. Let the relation between them be given by $F(c_1,c_2)=0$. Then $F(u,v)=0$ is the required solution as $F(u,v)=0$ is an integral surface. Moreover the initial curve lies on it. For,

$F(u(x_0(s),y_0(s),z_0(s)),v(x_0(s),y_0(s),z_0(s)))=F(c_1,c_2)=0$

Sometimes the solution can also be obtained by assuming $v=G(u)$ and determining G.

Suppose we assume that form of the integral surface containing the given curve C as $v=G(u)$. Since the given curve C lies on the surface, we find that substituting $x=x_0(s)$, $y=y_0(s)$, $z=z_0(s)$ in the relation $v(x,y,z)=G(u(x,y,z))$ may sometimes enable us to find the explicit form of the function G as $G=G(s)$

as illustrated in the following example. However, it may not always be possible to do so

Ex 1.9.1

Find the integral surface of the equation

5m $$(2xy-1)p + (z-2x^2)q = 2(x-yz)$$

Which passes through the line $x_0(s)=1$, $y_0(s)=0$ & $z_0(s)=s$

Sol: The auxiliary equations (B) are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$$

Now that

(i) $z\,dx + dy + x\,dz = 0$

(ii) $2x\,dx + 2y\,dy + dz = 0$

$$= \frac{z\,dx + dy + x\,dz}{0}$$

$$= \frac{2x\,dx + 2y\,dy + dz}{0}$$

$\therefore\ u = y + xz = c_1,\quad v = x^2+y^2+z = c_2.$

∴ Integral surface is $F(xz+y,\ x^2+y^2+z) = 0$

The general integral is

$$x^2+y^2+z = G(y+xz).$$

Where G is arbitrary. We want to choose G such that the given curve lies on the surface $x^2+y^2+z = G(y+xz)$. This happens if $G(s) = 1+s$. Therefore the required integral surface is

Using the data: $x_0(s)=1$, $y_0(s)=0$, $z_0(s)=s$ in the integrals, we obtain

$$x^2+y^2+z = G(y+xz) = 1+y+xz,$$

$s+0=c_1$ & $1+0+s=c_2$

i.e. $x^2+y^2-xz-y+z = 1.$

leads to $c_1 = s$; $1+s=c_2$ or $1+c_1=c_2$

∴ The required integral surface is $1+xz+y = x^2+y^2+z$

Ex 1.9.2

Find the integral surface of the equation

or $x^2+y^2-xz-y+z=1$

$$x^3p + y(3x^2+y)q = z(2x^2+y),$$

5m Which passes through the curve $x_0=1$, $y_0=s$, $z_0=s(1+s)$.

Sol: The auxiliary equations are

$-x^{-1}(x^3) + y^{-1}[y(3x^2+y)] - z^{-1}[z(2x^2+y)]$

$-x^2+3x^2+y-2x^2-y=0$

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} = \frac{dz}{z(2x^2+y)} = \frac{-x^{-1}dx + y^{-1}dy - z^{-1}dz}{0}$$

$$\therefore\ -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0 \Rightarrow u = \frac{y}{xz} = c_1$$

$\frac{dy}{y} - \frac{dx}{x} - \frac{dz}{z} = 0$

$\log y - \log x - \log z = c_1$

$\log y - (\log x + \log z) = c_1$

$\frac{\log y}{\log xz} = c_1$

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2+y)} \Rightarrow \frac{(3x^2+y)\,dx}{x^3} = \frac{dy}{y} = \frac{(3x^2+y)dx + dy}{x^3+y}$$

Hence

$$\frac{(3x^2+y)dx + dy + x\,dy}{x^3 + y + xy} = \frac{dy}{y} \Rightarrow v = \frac{(x^3+y+xy)}{y} = c_2$$

On substituting $x=1,\ y=s,\ z=s(1+s)$ in $u=c_1$ and $v=c_2$, we get $\frac{1}{s+1} = c_1$ and $\frac{1+2s}{s} = c_2$. Eliminating s between them gives $F(c_1, c_2) = c_1c_2 - c_1 - c_2 + 2 = 0$. Therefore $F(u,v)=0$ implies that

$$yz = x^3z + xyz - x^2y - y^2$$
$$= (x^2+y)(xz-y),$$

Which is the required integral surface.

Let us now discuss the same problem for a non-linear P.d.e. Here we will obtain the solution from a complete integral. Let

$$F(x, y, z, a, b) = 0 \rightarrow (1)$$

be a complete integral of

$$f(x, y, z, p, q) = 0 \rightarrow (2)$$

We are interested in finding a solution of (2) which passes through the curve $C:\ x = x_0(s),\ y = y_0(s),\ z = z_0(s)$, s being a parameter.

We expect this solution to be an envelope of a one-parameter subfamily of (1). Let E be this envelope which contains the curve C. Let S be the sub-family. Then E, the envelope of the sub-family, will touch each member of the sub-family. As C lies entirely on E, it will touch each member of the sub-family for some s.

This requires the relation

$$F(x_0(s), y_0(s), z_0(s), a, b) = 0 \rightarrow (3)$$

to be satisfied for each a and b belonging to the corresponding sub-family, for some s (this is the condition for C to intersect a member of the sub-family) and also the relation

$$\frac{\partial F}{\partial s}(x_0(s), y_0(s), z_0(s), a, b) = 0 \rightarrow (4)$$

to be satisfied for some s. (This is the condition that at this point of intersection, the curve is actually a tangent to that member).

On eliminating s between (3) and (4), we get the required relation between a and b. This defines the sub-family we are looking for. However there could be many solutions. Since on eliminating s we get

$$\psi(a,b)=0 \rightarrow ⑥$$

Equ ⑥ may be factored into a set of alternative equations $b=\phi_1(a)$, $b=\phi_2(a)$ etc. Each one of them defines a sub-family and the envelope of each of these sub-families, if it exists, is a solution of the problem. Hence a solution if it exists, may not be unique.

Observe that this does not happen in quasi-linear equations, if the curve C is properly chosen.

Ex 1.9.3

Find a complete integral of the equation

$$(p^2+q^2)x = pz, \quad ①$$

and the integral surface containing the curve C : $x_0 = 0$, $y_0 = s^2$, $z_0 = 2s$

Sol:

The auxiliary equations are

$$\frac{dx}{2px-z} = \frac{dy}{2qx} = \frac{dz}{pz} = \frac{dp}{-q^2} = \frac{dq}{pq}$$

$$\frac{dp}{-q^2} = \frac{dq}{pq} \Rightarrow p^2+q^2=a^2.$$

Therefore

$$p = \frac{a^2x}{z}, \quad q = \pm\frac{a\sqrt{z^2-a^2x^2}}{z},$$

$$dz = \frac{a^2x}{z}dx \pm \frac{a\sqrt{z^2-a^2x^2}}{z}dy.$$

$$\Rightarrow \quad \frac{zdz - a^2xdx}{\sqrt{z^2-a^2x^2}} = \pm a\,dy$$

on integrating, the complete integral is found to be

$$z^2 = a^2x^2 + (ay+b)^2 \rightarrow ⑥$$

Equ ③ in this case becomes

$$4s^2 = (as^2+b)^2 \rightarrow ⑦$$

Then on differentiating ⑦ w.r.t. s, we get

$$2 = a(as^2+b) \rightarrow ⑧$$

Further, on eliminating [illegible] between ⑦ and ⑧, we get ab=1.
Therefore, on substituting $b = 1/a$ in ⑥, the one-parameter sub-family of the complete integral is

$$z^2 = a^2x^2 + \left(ay + \frac{1}{a}\right)^2.$$

or $\quad a^4(x^2+y^2) + a^2(2y - z^2) + 1 = 0 \rightarrow ⑨$

and the envelope is obtained by eliminating a between ⑨ and

$$2a^2(x^2+y^2) + (2y - z^2) = 0 \rightarrow ⑩.$$

The envelope is obtained by eliminating $(2y - z^2)^2 = 4(x^2+y^2)$

i.e., $z^2 = 2(y \pm \sqrt{x^2+y^2})$. Since $\sqrt{x^2+y^2} \geq y$, the minus sign is to be discarded and then

$$z^2 = 2(y + \sqrt{x^2+y^2}),$$

Which is the required integral surface.

Ex 1.9.4

Find the complete integral of the equation

$$p^2x + qy - z = 0$$

and derive the equation of the integral surface containing the line $y = 1$, $x + z = 0$.

Sol:

The auxiliary equations are

$$\frac{dx}{2px} = \frac{dy}{y} = \frac{dz}{2p^2x + qy} = \frac{dp}{-p(1+p)} = \frac{dq}{0}$$

Therefore

$$q = a \quad \text{and} \quad p = \pm\left(\frac{z - ay}{x}\right)^{1/2}$$

On substituting for p and q in $dz = p\,dx + q\,dy$, we obtain

$$\frac{dz - a\,dy}{\sqrt{z - ay}} = \pm\frac{dx}{\sqrt{x}}.$$

On integrating, we obtain

$$\sqrt{z - ay} = \pm\sqrt{x} + \sqrt{b},$$

i.e., $\quad (ay - z + x + b)^2 = 4bx, \rightarrow ⑪$

as the complete integral.

The parametric form of the given line is $x = s$, $y = 1$, $z = -s$. Equ ⑪ in this case takes the following form

$$(a + b + 2s)^2 = 4bs.$$

Then, on differentiating w.r.t s, we get

$2(a+b+2s) \cdot 2 = 4b$

$4(a + b + 2s) = 4b$

Eliminating c between them gives $b^2 + 2ab = 0$ i.e., $b = 0$ or $b = -2a$. The case $b = 0$ does not lead to a solution. On substituting $b = -2a$ in (13), we get the sub-family

$$(ay - z + x - 2a)^2 = -8ax \rightarrow (12)$$

The envelope of (12) is obtained on eliminating a between (12) and

$(ay - z + x - 2a)(y - z) = -$

$$(y - z)(ay - z + x - 2a) = -4x \rightarrow (13)$$

The envelope of (12) is

$$xy = z(y - z),$$

and this is the required integral surface.

We now show that we can derive any other complete integral from a given one in a similar way. Let

$$F(x, y, z, a, b) = 0 \rightarrow (14)$$

be a complete integral of $f(x, y, z, p, q) = 0$. Suppose

$$G(x, y, z, h, k) = 0 \rightarrow (15)$$

be any other complete integral of the same p.d.e involving two arbitrary constants h and k. We will now show that it is possible to derive (15) as an envelope of a one-parameter sub-family of (14). We first choose a curve C on the surface (15) such that the constants h and k are independent parameters in its equations. Then we find the envelope of a one-parameter sub-family of (14) touching the curve C. This solution containing two arbitrary constants h and k, is thus a complete integral.

Ex 1.9.5:

Show that the differential equation

$$2xz + q^2 = x(xp + yq)$$

has a complete integral

$$z + a^2x = axy + bx^2 \rightarrow (16)$$

and deduce that $x(y + hx)^2 = 4(z - kx^2)$ is also a complete integral.

Sol:

It can be verified that Equ (16) satisfies the given p.d.e. The corresponding matrix $\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$

$$\begin{pmatrix} xy-2ax & y-2a & x \\ x^2 & 2x & 0 \end{pmatrix},$$

is of rank two if $x \neq 0$. Hence (16) is a complete integral in any domain that does not contain the line $x=0$.

Let us now take a curve on the surface $x(y+hx)^2 - 4(z-kx^2)$ involving h and k as

$$x_0 = s,\ y_0 = -hs,\ z_0 = ks^2$$

Therefore we want to find a sub-family of (16) which touches this curve. Hence

$$ks^2 + a^2 s = as(-hs) + bs^2,$$

i.e.,
$$ks + a^2 = -ahs + bs$$

on differentiating w.r.t s, we get $k = -ah + b$. Therefore $b = k + ah$. Hence the sub-family of (16) is

$$z + a^2 x = axy + (k+ah)x^2 \rightarrow (17)$$

To find the envelope of (17) we have to eliminate a between (17) and

$$2ax = xy + hx^2 \rightarrow (18)$$

Equ (18) gives $a = (y+hx)/2$. On substituting for a in (17), we get the required envelope as

$$4(z-kx^2) = x(y+hx)^2,$$

which is the required complete integral.

10 Quasi-Linear Equations : Geometry of Solutions

We shall first discuss the semi-linear case

Semi-linear equations:

Consider a semi-linear equations

$$P(x,y)z_x + Q(x,y)z_y = R(x,y,z) \rightarrow (1)$$

where P, Q and R are continuously differentiable functions and both $P(x,y)$ and $Q(x,y)$ do not vanish simultaneously. Note that the expression on the left hand side of (1) is the directional derivative of $z(x,y)$ in the direction $(P(x,y), Q(x,y))$ at the point (x,y).

Let us consider the one-parameter family of curves in the x,y-Plane defined by the ordinary differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

or the system of ordinary differential equations

$$\frac{dx}{dt} = P(x,y) \; , \; \frac{dy}{dt} = Q(x,y) \rightarrow (2)$$

These curves have the property that along them $z(x,y)$ will satisfy the ordinary differential equation

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx} = \frac{P(x,y)z_x + Q(x,y)z_y}{P(x,y)}$$

or

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt} = z_x P(x,y) + z_y Q(x,y) = R(x,y,z) \rightarrow (3)$$

The one-parameter family of curves C_{λ_0} defined by Equ (2) are called the characteristic curves of the P.d.e (1)

Let (x_0, y_0) be a point in x,y-Plane. By the existence and uniqueness of the solution of the initial value problem for the ordinary differential equations, Equ (2) will define a unique characteristic curve

$$x(t) = x(x_0, y_0, t) \; , \; y(t) = y(x_0, y_0, t) \rightarrow (4)$$

(1) Such that $z(x_0, y_0, 0) = z_0$. That is, $z(x,y)$ is uniquely determined along the whole characteristic passing through (x_0, y_0) if we assign a value for z at (x_0, y_0).

(2) Such that $x(0) = x_0$ and $y(0) = y_0$. Suppose we now assign a value z_0 for $z(x,y)$ at (x_0, y_0). Then the Equ (4) determines a unique solution z as $z = z(x_0, y_0, t) \rightarrow (5)$

Hence if $z(x,y)$ is known on a curve Γ_0 in the x,y-Plane and if this curve is such that it intersects the one-parameter family of characteristic curves C_{λ_0}, we can determine $z(x,y)$ uniquely in the region covered by C_{λ_0}.

Observe that the curve Γ_0 cannot be chosen arbitrarily. We will deal with this point later (refer to Theorem 1.10.1)

Ex 1.10.1

Solve $xz_y - yz_x = z$ with the initial condition $z(x,0) = f(x)$, $x \geq 0$

Sol:

The characteristic curves are given by the equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

having the solution $x^2+y^2=c^2$. Along such a curve, z satisfies the o.d.e

$$\frac{dz}{dx} = -\frac{z}{y} = -\frac{z}{\sqrt{c^2-x^2}},$$

i.e.,

$$\frac{dz}{z} = -\frac{dx}{\sqrt{c^2-x^2}},$$

Whose solution is

$$z = k(c)\, e^{-\sin^{-1}(x/c)},$$

Where the arbitrary function k may depend on c. Hence we have the general solution

$$z = k(x^2+y^2)\, e^{-\sin^{-1}(x/c)} \rightarrow (5)$$

On applying the initial conditions, we get

$$f(x) = k(x^2)\, e^{-\pi/2}$$

$$k(x) = f(\sqrt{x})\, e^{\pi/2}$$

$$k(x^2+y^2) = f(\sqrt{x^2+y^2})\, e^{\pi/2}$$

Hence the required solution is (5)

$$z(x,y) = f(\sqrt{x^2+y^2})\, e^{\pi/2 - \sin^{-1}(x/c)}$$

Quasi-linear equations:

We now consider the quasi-linear equation

$$P(x,y,z)\,z_x + Q(x,y,z)\,z_y = R(x,y,z) \rightarrow (6)$$

where P, Q, and R are continuously differentiable functions of x, y and z and P, Q, and R do not vanish simultaneously. Its solution z(x,y) defines an integral surface $z = z(x,y)$ in the x,y,z-space. The normal to this surface has direction ratios $(z_x, z_y, -1)$. Hence Eqn (6) states the condition that the integral surface is such that at each point, the line with direction ratios (P, Q, R) is tangent to the surface at that point. In fact, any surface $z = z(x,y)$ is an integral surface if and only if the tangent plane contains the characteristic direction (P, Q, R) which is a direction field defined by the p.d.e at each point.

We consider below the integral curves of this field called the characteristic curves. They are a family of space curves whose tangent at each point coincides with the characteristic direction (P, Q, R) at that point and are given by the following system of o.d.e

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} = dt \text{ (say)} \rightarrow (7)$$

or

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x,y,z), \\ \frac{dy}{dt} &= Q(x,y,z), \\ \frac{dz}{dt} &= R(x,y,z) \end{aligned}\right\} \rightarrow (8)$$

By the existence and uniqueness of the solution of the initial value problem for a system of ordinary differential equations, there passes a characteristic curve $x = x(x_0, y_0, z_0, t)$, $y = y(x_0, y_0, z_0, t)$, $z = z(x_0, y_0, z_0, t)$ through each point (x_0, y_0, z_0). The system of o.d.e (8) being an autonomous system, there will be a two parameter family of integral curves.

Hence there is a two-parameter family of characteristic curves. Every surface generated by a one-parameter family of characteristics is an integral surface. For, if we consider any point on such a surface, then the tangent to the characteristic curve passing through that point lies on the tangent plane to the surface. Thus the tangent plane to the surface at each point contains the line with direction ratios (P, Q, R). Hence the surface is an integral surface.

In addition, the converse is also true. i.e every integral surface is generated by a family of characteristic curves. For, let us consider an integral surface S given by $z = z(x,y)$. Consider any point (x_0, y_0, z_0) on S. Let the solution of

$$\frac{dx}{dt} = P(x,y,z(x,y)), \quad \frac{dy}{dt} = Q(x,y,z(x,y))$$

with the initial conditions $x = x_0$, $y = y_0$ at $t = 0$ be given by $x = x(t)$ and $y = y(t)$. Consider the corresponding curve in three dimensions

$$x = x(t), \quad y = y(t), \quad z = z(x(t), y(t)).$$

observe that this curve lies on the given Interval surface. Moreover,

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt} = P(x,y,z) z_x + Q(x,y,z) z_y,$$
$$= R(x,y,z)$$

Therefore the curve satisfies Equ ⑧ for characteristic curves and it passes through the point (x_0, y_0, z_0). Therefore S is generated by the characteristic curves as it contains the characteristic curve through each point (x_0, y_0, z_0) on S. Further, if a characteristic curve has one point in common with the integral surface, it lies entirely on the integral surface since only one characteristic curve passes through a given point. In addition, if two integral surfaces intersect along the entire characteristic curve through this point. Hence the curve of intersection of two integral surfaces must be a characteristic curve

Note: The two-parameter family of characteristics is nothing but the curves of intersection of the surfaces $u = c_1$ and $v = c_2$ given by (1.4-3). To find an integral surface passing through a given curve, as in Sec 1.9. we find a relation between c_1 and c_2, which means we are actually choosing a one-parameter sub-family of characteristics, which in turn generates the required integral surface.

Existence and Uniqueness of solution for the cauchy problem:-

Theorem 1.10.1

Consider the first order quasi-linear p.d.e

$$P(x,y,z) z_x + Q(x,y,z) z_y = R(x,y,z) \rightarrow ⑨$$

where P, Q and R have continuous partial derivatives w.r.t x, y and z and they do not vanish simultaneously. Let the value $z = z_0(s)$ be prescribed along the initial curve $\Gamma_0 : x = x_0(s)$, $y = y_0(s)$, x_0, y_0 and z_0 being continuously differentiable functions $(a \le s \le b)$. Further, for $a \le s \le b$ if

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s)) \neq 0 \rightarrow ⑩$$

then there exists a unique solution $z(x,y)$ defined in some neighbourhood of the initial curve Γ_0 which satisfies the p.d.e. and the initial condition

$$z(x_0(s), y_0(s)) = z_0(s),$$

i.e. the integral surface $z = z(x,y)$ contains the initial data curve $C: x = x_0(s),\ y = y_0(s),\ z = z_0(s)$.

Note: the initial curve Γ_0 is the projection of the initial data curve C on the x,y-plane.

Proof:

The ordinary differential equations ⑧ can be solved to obtain a unique family of characteristics.

$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t) = x(s,t), \\ y &= y(x_0, y_0, z_0, t) = y(s,t), \\ z &= z(x_0, y_0, z_0, t) = z(s,t). \end{aligned}\right\} \rightarrow ⑪$$

having continuous derivatives w.r.t the parameters s and t, satisfying the initial conditions $x(s,0) = x_0(s)$, $y(s,0) = y_0(s)$ and $z(s,0) = z_0(s)$. (from the existence and uniqueness theorem for the o.d.e). Observe that the Jacobian

$$\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{t=0} = \left.\begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}\right|_{t=0} = (x_s Q - y_s P)_{t=0} \neq 0$$

from ⑩. Hence we can solve the first two equ. of ⑪ for s and t in terms of x and y in a neighbourhood of the initial curve $t=0$, i.e. $s = s(x,y)$, $t = t(x,y)$.

Let $z = \phi(x,y) = z(s(x,y), t(x,y))$. Note that $\phi(x,y)$ satisfies the initial condition as $\phi(x_0, y_0) \equiv z(s,0) = z_0(s)$. Moreover, $z = \phi(x,y)$ satisfies the p.d.e ⑨. For consider

$$\begin{aligned} P\phi_x + Q\phi_y &= P(z_s s_x + z_t t_x) + Q(z_s s_y + z_t t_y), \\ &= z_s(Ps_x + Qs_y) + z_t(Pt_x + Qt_y) \\ &= z_s(s_x x_t + s_y y_t) + z_t(t_x x_t + t_y y_t) \end{aligned}$$

observe that

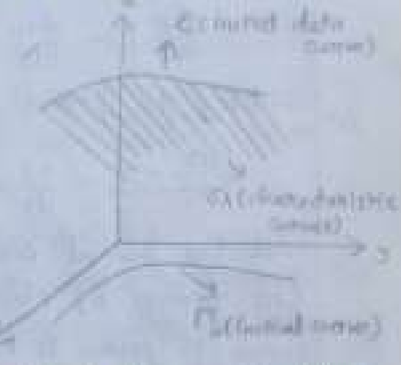
$$s_x x_t + s_y y_t = s_t = 0$$
$$t_x x_t + t_y y_t = t_t = 1$$

Hence $\quad P\phi_x + Q\phi_y = z_t = R.$

The uniqueness of $\phi(x,y)$ follows from the following argument. Suppose $\phi(x,y)$ is not unique, then there are two integral surfaces which intersect along the given initial data curve C. Then through each point on the initial data curve, there passes one and only one characteristic curve. Therefore this characteristic curve has to lie on both the surfaces. Hence the same family of characteristic curves which pass through each point of the initial data curve lie on both the surfaces. Hence both the surfaces must coincide as both are generated by the same family of characteristic curves. Thus $\phi(x,y)$ is unique.

Note: The condition ⑥ is said to the admissibility condition and any initial data curve C satisfying this condition is said to be an admissible curve.

Hence the integral surface (the solution) is generated by the one parameter family of characteristics C_λ issuing from each point of the initial data curve C (refer Fig.) →



Ex 1.10.2: Solve the initial value problem for the quasi-linear equ. $zz_x + z_y = 1$ containing the initial data curve $C: x_0 = s, y_0 = s, z_0 = \frac{1}{2}s$ for $0 \le s \le 1$

Sol: observe that

$$\frac{dy_0}{ds}P - \frac{dx_0}{ds}Q = \frac{1}{2}s - 1 \neq 0 \text{ for } 0 \le s \le 1$$

Solving the following system of o.d.e

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1,$$

with initial conditions $x(s,0) = s$, $y(s,0) = s$, $z(s,0) = \frac{1}{2}s$. the family of characteristics through the initial data curve is found to be

$$x = \frac{1}{2}t^2 + \frac{1}{2}st + s,$$

$$y = t + s,$$

$$z = t + \frac{s}{2}.$$

Solving for s and t in terms of x and y, we obtain

$$s = \frac{x - (y^2/2)}{1 - (y/2)}, \quad t = \frac{y - x}{1 - (y/2)}$$

Hence the solution is

$$z = \frac{2(y-x) + (x - y^2/2)}{2-y} = \frac{2y - x - \frac{y^2}{2}}{2-y} = \frac{4y - 2x - y^2}{2(2-y)}$$

Ex 1.10.3

Find the integral surface of $zz_x + z_y = 0$ containing the initial data curve C : $x_0 = s$, $y_0 = 0$, $z_0 = f(s)$, where

$$f(s) = \begin{cases} 1, & s \le 0, \\ 1-s, & 0 \le s \le 1, \\ 0, & s \ge 1 \end{cases}$$

Here we observe that

$$\frac{dy_0}{ds} P - \frac{dx_0}{ds} Q = -1 \neq 0 \quad \forall s$$

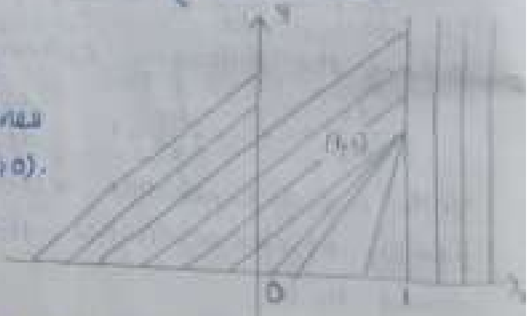
solving the following system of o.d.e

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0$$

with the initial conditions $x(s,0) = s$, $y(s,0) = 0$, $z(s,0) = f(s)$, the family of characteristics through the initial data curve is found to be $x = t f(s) + s$, $y = t$, $z = f(s)$ (Refer Ex 1.4.4). Hence

$$x(s,t) = \begin{cases} s+t, & s \le 0, \\ (1-s)t + s, & 0 \le s \le 1, \\ s, & s \ge 1, \end{cases} \qquad z(s,t) = \begin{cases} 1, & s \le 0, \\ 1-s, & 0 \le s \le 1, \\ 0, & s \ge 1 \end{cases}$$

Here $x = y f(s) + s$. Hence the characteristics are straight lines intersecting the x-axis at $(s,0)$. The slope of the characteristic through $(s,0)$ is $\frac{1}{f(s)}$ and z is a constant on each of them we observe that in the interval $0 \le s \le 1$, $f(s)$ is a decreasing function of s. So the slopes of these characteristics increase with s and hence they intersect in this region.



On a characteristic issuing from a point $(s,0)$, z takes the values $(1-s)$ in $0 \le s \le 1$. Thus z takes a different value on each characteristic and as all these lines intersect at (1, 1), z is multi-valued at (1,1) (refer Fig) Thus the solution cannot be single valued and breaks down at (1,1), i.e, it cannot be defined uniquely. The characteristics issuing from (s,0) with $s<0$ intersect those issuing from (s,0) with $s>1$ and since z has different values on them, the solution is not defined in the quadrant $x \geq 1$, $y \geq 1$

Ex 1.10.4

Solve the Cauchy problem for $2z_x + yz_y = z$ for the initial data curve $C: x_0 = s,\ y_0 = s^2,\ z_0 = s,\ 1 \leq s \leq 2$

Sol:

$P = 2,\ Q = y,\ R = z$

We observe that

$$\frac{dy_0}{ds} P - \frac{dx_0}{ds} Q = 4s - s^2 \neq 0$$

for $1 \leq s \leq 2$.

Therefore C is admissible for $1 \leq s \leq 2$. We solve

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = y,\ \frac{dz}{dt} = z.$$

such that $x(s,0) = s,\ y(s,0) = s^2,\ z(s,0) = s$.

The solutions of the system of O.D.E. which determine the characteristic through the initial data curve are

$$x = s + 2t,\ y = s^2 e^t,\ z = se^t$$

The solution is obtained by eliminating s and t, i.e.,

$$z^2 = y \exp[(xz - y)/2z]$$

Ex 1.10.5

Find the solution of the initial value problem for the quasi-linear equation $z_x - 2zz_y + z = 0$ for all y and $x > 0$, for the initial data curve $C: x_0 = 0,\ y_0 = s,\ z_0 = -2s,\ -\infty < s < \infty$

Sol:

$P = 1,\ Q = -2z,\ R = -z$

The admissibility condition is satisfied since

$$\frac{dy_0}{ds} P - \frac{dx_0}{ds} Q = 1 \neq 0 \quad \forall s.$$

The characteristic curves which generate the surface are obtained from the solutions of

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = -2z,\ \frac{dz}{dt} = -z.$$

satisfying the following conditions

$x(s,0) = 0,\ y(s,0) = s,\ z(s,0) = -2s$

The solutions are found to be

$x = t, \quad y = -2se^{-t} + 3s, \quad z = -2se^{-t}$

The parameters s and t in terms of x and y are found to be

$$s = -\frac{y}{2e^{-x} - 3}, \quad t = x$$

Therefore

$$z = \frac{2y}{2e^{-x} - 3} e^{-x},$$

$$= -\frac{2y}{3e^{x} - 2}, \quad \log(2/3) > x \geq 0$$

Which passes through the x-axis

The solution breaks down at $x = \log(2/3)$.

Note: In the case of quasi-linear equations, the sol. of the cauchy problem is unique if the initial curve is smooth and is not parallel to the characteristic curve (i.e. it satisfies the condition 1.10.10). However, the same is not true in the case of non-linear equations as seen in the following example

Ex 1.10.6: Find the solution of the equation

Imm

$$z = \frac{1}{2}(z_x^2 + z_y^2) + (z_x - x)(z_y - y).$$

Which passes through the x-axis

Ans 1: $z = y(4x - 3y)/2$

Ans 2: $z = y^2/2$

Observe that the above two surfaces pass through the x-axis and satisfy the given equation. Hence this problem has no unique solution. Refer Ex (1.11.2).

Non-linear equations, unlike quasi-linear equations may have many solutions or no solution for the cauchy problem. One needs some more information to have a unique solc for the cauchy problem in the non-linear case. We

.11 Non-linear First order P.D.E.

Consider a non-linear first order P.d.e

$$f(x, y, z, p, q) = 0 \quad \rightarrow (1)$$

We assume that f has continuous second order derivatives with respect to its variables x, y, z, p & q and either f_p or f_q is non-zero at every point. Without loss of generality, let us assume that $f_q \neq 0$, so that we can solve ① at each point (x, y, z) for q as $q = q(x, y, z, p)$. Of course, q need not always be a single valued function of p, but it may be assumed that one branch of a possible set of solutions $q = q(x, y, z, p)$ is chosen.

Monge Cone: Let (x_0, y_0, z_0) be some point in space. We consider a possible integral surface $z = z(x, y)$ and the direction ratios $(p, q, -1)$ of the normal to the tangent plane

$$z - z_0 = p(x - x_0) + q(y - y_0) \rightarrow ②$$

to the integral surface at that point. The equ ① which is equivalent to

$$q = q(x_0, y_0, z_0, p) \rightarrow ③$$

indicates that p and q are not independent at (x_0, y_0, z_0). This implies that the integral surfaces are those surfaces having tangent planes belonging to a one-parameter (p) family.

Consider the family of planes given by Equ ② passing through (x_0, y_0, z_0) as p varies and q is determined by Equ ③. These planes envelope a cone called the Monge cone having its vertex at (x_0, y_0, z_0). Thus if $z = z(x, y)$ is a solution of ① it must be tangent to the Monge cone at each point (x, y, z) on the surface. Thus the differential equ ① characterises a field of cones such that a surface will be an integral surface iff it is tangent to the Monge cone at each point (refer Fig.). The line of contact of the Monge cone with the tangent plane to the integral surface at each point gives a field of directions. These directions are called the characteristic directions at that point and lie along the generators of the Monge cone. The integral curves of this field of directions on the surface define a family of curves called the characteristic curves.

Analytic expression for the Monge cone at (x_0, y_0, z_0):

5m 2m As the Monge cone at (x_0, y_0, z_0) is the envelope of the one-parameter family of planes

$$z - z_0 = p(x - x_0) + q(y - y_0),$$

where q is given by

$$q = q(x_0, y_0, z_0, p) \rightarrow ③$$

it can be obtained by eliminating p from the following equations

$$z - z_0 = p(x - x_0) + q(x_0, y_0, z_0, p)(y - y_0), \rightarrow ④$$

and $\quad 0 = (x - x_0) + (y - y_0)\dfrac{dq}{dp} \quad \rightarrow ⑤$

on differentiating ① w.r.t p, we get

$$\frac{df}{dp} = f_p + f_q \frac{dq}{dp} = 0$$

as q is a function of p. If one eliminates $\frac{dq}{dp}$ from ⑤ & the above eqn., then the equations describing the Monge cone can be written as

$$\left.\begin{aligned} q &= q(x_0, y_0, z_0, p), \\ z - z_0 &= p(x - x_0) + q(y - y_0), \\ \frac{x - x_0}{f_p} &= \frac{y - y_0}{f_q} \end{aligned}\right\} \rightarrow ⑥$$

The last two equations define a generator of the cone for a given p and the corresponding q given by Eqn ③, i.e., the line of contact between the tangent plane & the cone.

Ex 1.11.1:

Consider $p^2 + q^2 = 1$. Let $(x_0, y_0, z_0) = (0, 0, 0)$. Then the Monge cone is obtained by eliminating p and q from

$$q = \sqrt{1 - p^2},$$

$$z = px + qy,$$

$$\frac{x}{2p} = \frac{y}{2q}.$$

which is $x^2 + y^2 = z^2$. This is a cone with vertex at (0,0,0) (as every second degree homogeneous equation in x, y & z represents a cone with vertex at (0,0,0))

Note: observe that in the quasi-linear case $Pp + Qq = R$, every tangent plane ② at (x_0, y_0, z_0) contains the following line (why?)

$$\frac{x - x_0}{P} = \frac{y - y_0}{Q} = \frac{z - z_0}{R}.$$

Hence the Monge cone degenerates into the above straight line at each point

Let us now find the equations which give the characteristic curves. As the line of contact between the tangent plane to an integral surface and the Monge cone at that point is the generator of the Monge cone, it can be written from (6) as

$$\frac{x-x_0}{f_p} = \frac{y-y_0}{f_q} = \frac{z-z_0}{pf_p+qf_q} \rightarrow (7).$$

Hence the characteristic directions which lie along the generators of the Monge cone are given by (f_p, f_q, pf_p+qf_q).

The characteristic curves are thus the solutions of

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q}.$$

or

$$\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q, \quad \frac{dz}{dt} = pf_p+qf_q. \rightarrow (8)$$

Since these three o.d.e for determining the characteristic curve $(x(t), y(t), z(t))$ cannot be solved because they involve the functions p and q, we require more information regarding the variation of p and q along a characteristic curve. Along such a curve on the given integral surface

$$\left.\begin{aligned} \frac{dp}{dt} &= p_x\frac{dx}{dt} + p_y\frac{dy}{dt} = p_x f_p + p_y f_q, \\ \frac{dq}{dt} &= q_x\frac{dx}{dt} + q_y\frac{dy}{dt} = q_x f_p + q_y f_q. \end{aligned}\right\} \rightarrow (9)$$

Further, by differentiating Eqn (1) w.r.t x as well as y, we get respectively

$$f_x + f_z p + f_p p_x + f_q q_x = 0.$$

$$f_y + f_z q + f_p p_y + f_q q_y = 0.$$

Since $p_y = q_x$, Eqn (9) may now be rewritten as

$$\frac{dp}{dt} = -f_x - f_z p, \quad \frac{dq}{dt} = -f_y - f_z q \rightarrow (10)$$

Thus for a given integral surface $z = z(x,y)$, there corresponds a family of characteristic curves on the surface associated with it such that the co-ordinates of the curve $(x(t), y(t), z(t))$ and the numbers $p(t)$ and $q(t)$ along the curve are related by a system of five ordinary differential equations given by (8) & (10). These five ordinary differential equations are called the characteristic differential equations related to the given p.d.e (1).

In order to determine the integral surface, by the p.d.e ① along with previous discussion, we consider the p.d.e ① along with the system of characteristic equ ⑥ & ⑩ as a system of six equations, i.e.,

$$\left.\begin{aligned} & f(x,y,z,p,q)=0, \\ & \frac{dx}{dt}=f_p, \\ & \frac{dy}{dt}=f_q, \\ & \frac{dz}{dt}=pf_p+qf_q, \\ & \frac{dp}{dt}=-f_x-f_z p, \\ & \frac{dq}{dt}=-f_y-f_z q, \end{aligned}\right\} \rightarrow ⑪$$

for the five unknown functions $x(t), y(t), z(t), p(t)$ and $q(t)$. This system is not over-determined. Consider a sol. of the last five equations of ⑪. Along such a sol. we find that

$$\frac{df}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}+f_z\frac{dz}{dt}+f_p\frac{dp}{dt}+f_q\frac{dq}{dt},$$

$$=f_xf_p+f_yf_q+pf_zf_p+qf_zf_q-f_pf_x-f_pf_zp-f_qf_y-f_qf_zq$$

$$=0.$$

Hence $f=$ constant, is an integral of the system of o.d.e. Thus we can conclude that the equation $f(x,y,z,p,q)=0$ is not much of a restriction. Suppose that $f=0$ at (x_0,y_0,z_0,p_0,q_0) at $t=0$. Then the unique solution of the characteristic equations passing through this point is such that $f=0$ will be satisfied for all t along the solution.

Characteristic Strip:

A solution $(x(t), y(t), z(t), p(t), q(t))$ of the system of o.d.e ⑪ can be interpreted as a strip.

The first three functions, viz., $(x(t), y(t), z(t))$ determine a space curve. At each point of the space curve, $p(t)$ and $q(t)$ define a tangent plane with $(p,q,-1)$ as the normal vector

Fig 1.8.2

The curve along with these tangent planes at each point is called a characteristic strip (refer Fig 1.11.2) and the curve is called the characteristic curve.

For a fixed t given by t_0, $(x_0, y_0, z_0, p_0, q_0)$ is said to define an element of the strip. i.e, the point (x_0, y_0, z_0) on the curve and the corresponding tangent plane whose normal is given by $(p_0, q_0, -1)$ at that point.

Note: Any set of five functions $x(t), y(t), z(t), p(t)$ & $q(t)$ can be interpreted as a strip only if the following condition

$$\frac{dz}{dt}(t) = p(t)\frac{dx}{dt}(t) + q(t)\frac{dy}{dt}(t) \rightarrow (12)$$

is satisfied. This is the condition that at each point on the space curve $x = x(t), y = y(t), z = z(t)$, the tangent to the space curve lies on the plane whose normal has direction ratios $(p(t), q(t), -1)$.

Any solution $(x(t), y(t), z(t), p(t), q(t))$ of (11) automatically satisfies the condition (12) by virtue of the second, third & fourth equ. of (11)

Lemma 1.11.1

If an element $(x_0, y_0, z_0, p_0, q_0)$ is common to both an integral surface $z = z(x,y)$ and a characteristic strip, then the corresponding characteristic curve lies completely on the surface.

Proof:

Let $z = z(x,y)$ be an integral surface consider the two o.d.e

$$\frac{dx}{dt}(t) = f_p(x, y, z(x,y), z_x(x,y), z_y(x,y)),$$

$$\frac{dy}{dt}(t) = f_q(x, y, z(x,y), z_x(x,y), z_y(x,y)),$$

for $x(t), y(t)$ with the initial conditions $x(0) = x_0$, $y(0) = y_0$ which uniquely determine a curve $x = x(t)$ and $y = y(t)$ in the x, y-plane. Consider the curve $x = x(t), y = y(t)$, in space on the integral surface $z = z(t) = z(x(t), y(t))$.

Then on such a curve

$$\frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt} = z_x f_p + z_y f_q \rightarrow (13)$$

$$\frac{dz_x}{dt} = z_{xx} f_p + z_{xy} f_q \rightarrow (14)$$

$$\frac{dz_y}{dt} = z_{yx} f_p + z_{yy} f_q \rightarrow (15)$$

Further,

$$z(0) = z(x_0, y_0) = z_0,$$

$$z_x(0) = z_x(x_0, y_0) = p_0,$$

$$z_y(0) = z_y(x_0, y_0) = q_0$$

Since $z = z(x, y)$ is an integral surface of the given p.d.e. we have

$$f(x, y, z(x, y), z_x(x, y), z_y(x, y)) = 0$$

on differentiating the above equ. w.r.t x and y, we get

$$f_x + f_z z_x + f_{z_x} z_{xx} + f_{z_y} z_{yx} = 0,$$

$$f_y + f_z z_y + f_{z_x} z_{xy} + f_{z_y} z_{yy} = 0.$$

Equ (14) & (15) therefore reduce to

$$\frac{dz_x}{dt} = -f_x - f_z z_x. \rightarrow (16)$$

$$\frac{dz_y}{dt} = -f_y - f_z z_y \rightarrow (17)$$

$f_x + f_z z_x + \frac{dz_x}{dt} = 0$

$\frac{dz_x}{dt} = -f_x - f_z z_x$

$f_y + f_z z_y + \frac{dz_y}{dt} = 0$

therefore the five functions $x = x(t)$, $y = y(t)$, $z = z(x(t), y(t))$, $p = z_x(x(t), y(t))$ and $q = z_y(x(t), y(t))$ determine a characteristic strip as they satisfy the characteristic equ (11). In addition, these functions determine the unique characteristic strip with the initial element x_0, y_0, z_0, p_0 and q_0. However, this curve lies on the surface by definition. Hence the result.

We shall make use of this lemma while proving the uniqueness of the solution of the cauchy problem for the non-linear p.d.e.

Initial Strip:

Let $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$ be some arbitrary initial data curve. Suppose that along this curve we can specify the functions $p_0(s)$ and $q_0(s)$ such that together with the initial data curve $(x_0(s), y_0(s), z_0(s))$, they satisfy the equation

$$f(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0 \rightarrow (18)$$

and the strip condition

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} \rightarrow (19)$$

then such an initial element $x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)$ is said to define an initial strip for the initial data curve $(x_0(s), y_0(s), z_0(s))$. The integral surface may be constructed with the help of the characteristic strips issuing from each point of the initial data curve by piecing up the characteristic curves together to form a smooth surface.

As there can be more than one pair of functions p_0 and q_0 satisfying the two equ (18) & (19), there can be several initial strips associated with the same initial data curve. Hence there can be more than one integral surface passing through a given initial data curve.

Note: In the quasi-linear case, we note that both Equs (18) & (19) reduce to

$$P(x_0(s), y_0(s), z_0(s))p_0 + Q(x_0(s), y_0(s), z_0(s))q_0 = R(x_0, y_0, z_0)$$

and

$$\frac{dz_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}$$

which the are linear in p_0 and q_0. Further, if $P\frac{dy_0}{ds} - Q\frac{dx_0}{ds} \neq 0$ then p_0 and q_0 are determined uniquely. (Refer The. 1.10.1)

The above discussion suggests a method of solving the Cauchy problem for the non-linear p.d.e (1). The method involve the following steps.

Step 1: Given the p.d.e (1) and an initial data curve $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$, determine the functions $p_0(s)$ and $q_0(s)$ such that the five functions $x_0(s), y_0(s), z_0(s), p_0(s)$ & $q_0(s)$ satisfy Equs (18) & (19). There could be several choices for $p_0(s)$ and $q_0(s)$. One may expect to find a unique solution for each such choice.

Step 2: Once a choice is made for $p_0(s)$ and $q_0(s)$, i.e., the initial strip is chosen, we can solve the last five equs. of Equ (11) subject to the initial conditions $x = x_0(s), y = y_0(s)$, $z = z_0(s)$, $p = p_0(s)$ & $q = q_0(s)$ at $t = 0$. Hence we have the characteristic strips issuing from each point of the initial data curve. The corresponding characteristic curves generate the required integral surface. We demonstrate the method in the following examples.

10m

Ex: 1.8.2

Find the solution of the equation

$$z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$$ which passes through the x-axis

Sol: Let us find the possible initial strips. The initial data curve is $x_0 = s$, $y_0 = 0$, $z_0 = 0$, and eqns (11) & (13) respectively are

$$0 = \frac{1}{2}(p_0^2+q_0^2) + (p_0 - s)(q_0),$$

$$0 = p_0 \cdot 1 + q_0 \cdot 0$$

The second eqn. implies that $p_0 = 0$.

On substituting in the first eqn. we get

$$\frac{1}{2} q_0^2 + (-s)(q_0) = 0$$

$$q_0 = 0 \quad \text{or} \quad q_0 = 2s$$

Therefore there are two initial strips

case (i) $x_0 = v$, $y_0 = 0$, $z_0 = 0$, $p_0 = 0$, $q_0 = 2v$ → (20)

case (ii) $x_0 = s$, $y_0 = 0$, $z_0 = 0$, $p_0 = 0$, $q_0 = 0$ → (21)

The characteristic equations of this p.d.e are given by

$$\frac{dx}{dt} = p+q-y$$

$$\frac{dy}{dt} = p+q-x$$

$$\frac{dz}{dt} = p(p+q-y) + q(p+q-x)$$

$$\frac{dp}{dt} = p+q-y$$

$$\frac{dq}{dt} = p+q-x$$

Therefore

$$\frac{d}{dt}(x-p) = 0 \quad \text{and} \quad \frac{d}{dt}(y-q) = 0 \rightarrow (22)$$

In case (i) Eqn (22) give $x = v + p$, $y = q - 2v$. Observe that

$$\frac{d}{dt}(p+q-x) = p+q-x$$

$$p+q-x = ve^t$$

Similarly $p+q-y = 2ve^t$ hence

$$x = v(2e^t - 1), \quad y = v(e^t - 1) \rightarrow (23)$$

$$p = 2v(e^t - 1), \quad q = v(e^t + 1) \rightarrow (24)$$

On substituting in the equ for z and integrating, we get

$$z = \frac{5}{2} v^2 (e^{2t} - 1) - 3v^2(e^t - 1) \rightarrow (24)$$

On solving for t and v from (23), we get

$$e^t = \frac{y - x}{2y - x}, \quad v = x - 2y.$$

On substituting for t and v in terms of x and y in (25) we get

$$z = \frac{1}{2} y (4x - 3y)$$

In case (ii), Equ (22) give $x = 0 + t$, $y = q$. Observe that

$$\frac{d}{dt}(p + q - y) = p + q - y$$

$$p + q - x = -se^t$$

Similarly $p + q - y = 0$.

Therefore $p = 0$, $q = y$, $x = s$, $q = y = s - se^t$. Hence

$$s = x, \quad e^t = \frac{x - y}{x},$$

$$\frac{dz}{dt} = y(y - x),$$

$$= s^2 (e^{2t} - e^t)$$

On integrating and using the initial conditions, we get

$$z = \frac{s^2}{2}(e^{2t} - 2e^t + 1)$$

$$= \frac{s^2}{2}(e^t - 1)^2$$

On eliminating s and t, we get

$$z = \frac{x^2}{2}\left(\frac{x - y}{x} - 1\right)^2 = \frac{y^2}{2}$$

Ex 1.11.2:

Find by the method of characteristics, the integral surface of $pq = xy$ which passes through the line $z = x$, $y = 0$.

Sol:

Let us find the initial strips. The initial data curve is

$$x_0(s) = s, \quad y_0(s) = 0, \quad z_0(s) = s$$

In this case, Equ (15) & (19) respectively become

$$p_0(s)\, q_0(s) = 0$$

and $1 = p_0 \cdot 1 + q_0 \cdot 0$

Therefore $p_0 = 1$, $q_0 = 0$. (Unique initial strip)

The characteristic equations are

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p, \quad \frac{dz}{dt} = 2pq, \quad \frac{dp}{dt} = y, \quad \frac{dq}{dt} = x$$

Hence
$$x = Ae^{t} + Be^{-t}, \quad q = Ae^{t} - Be^{-t}$$

Similarly
$$y = Ce^{t} + De^{-t}, \quad p = Ce^{t} - De^{-t}$$

$$\frac{dz}{dt} = 2pq = 2ACe^{2t} + 2BDe^{-2t} - 2(BC + AD),$$

$$z = ACe^{2t} - BDe^{-2t} - 2(BC + AD)t + E.$$

Using the initial conditions

$A + B = s, \quad A - B = 0 \Rightarrow A = B = \frac{s}{2}$

$C + D = 0, \quad C - D = 1 \Rightarrow C = -D = \frac{1}{2}$

$AC - BD + E = s$

Therefore $E = \frac{s}{2}$

Finally, $x = s\cosh t$, $y = \sinh t$, $z = s\cosh^2 t$, $p = \cosh t$, $q = s\sinh t$. Hence the surface passing through the initial data curve is given by $z^2 = x^2(1+y^2)$ (4)

Ex 1.11.4:

Find the characteristic strips of the equ. $xp + yq - pq = 0$ and obtain the equ. of the integral surface through the curve $C: z = x/2, y = 0$.

Sol:

The initial data curve is

$x_0(s) = 2s, \quad y_0(s) = 0, \quad z_0(s) = s$

The Equ (15) & (14) become respectively

$2sp_0 - p_0q_0 = 0$ and $1 = 2p_0$

Which implies that

$p_0 = \frac{1}{2}$ and $q_0 = 2s$

Since only one initial strip is possible through the given initial data curve, we can find only one solution.

The characteristic equations are

$\frac{dx}{dt} = x - q, \quad \frac{dy}{dt} = y - p, \quad \frac{dz}{dt} = -pq$

$\frac{dp}{dt} = -p, \quad \frac{dq}{dt} = -q.$

The characteristic strip issuing from the initial element at 's' on the initial data curve is

$p = \frac{1}{2}e^{-t}, \quad q = 2se^{t}, \quad z = \frac{s}{2}(1+e^{-2t}),$

$x = 2s\cosh t, \quad y = \frac{-1}{2}\sinh t.$

Therefore the integral surface through C is

$x^2 - 4z^2 + 8xyz = 0$

Theorem 1.11.1 — We will now state a theorem that under certain conditions, the solution to the Cauchy problem for a semilinear P.d.e exists & is unique.

Consider the P.d.e

$f(x, y, z, p, q) = 0 \rightarrow (16)$

where f has continuous second order derivatives w.r.t the variables x, y, z, p & q and at every point either $f_p \neq 0$ or $f_q \neq 0$. Suppose that the initial values $z = z_0(s)$ are specified along the initial curve $\Gamma_0 : x = x_0(s), y = y_0(s), a \leq s \leq b$, where $x_0(s), y_0(s)$ & $z_0(s)$ have continuous second order derivatives. Suppose $p_0(s)$ and $q_0(s)$ have been determined such that

$f(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0.$

and $\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}.$

where p_0 and q_0 are continuously differentiable functions of s. If, in addition, the five functions x_0, y_0, z_0, p_0 and q_0 satisfy

$f_q \frac{dx_0}{ds} - f_p \frac{dy_0}{ds} \neq 0$

then in some neighbourhood of each point of the initial curve there exists one and only one solution $z = z(x,y)$ of (16) such that

$z(x_0(s), y_0(s)) = z_0(s),$

$z_x(x_0(s), y_0(s)) = p_0(s),$

$z_y(x_0(s), y_0(s)) = q_0(s).$

That is, the integral surface $z = z(x,y)$ contains the initial strip.

Proof:

Let us consider the system of characteristic equations

$\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q, \quad \frac{dz}{dt} = pf_p + qf_q,$

$\frac{dp}{dt} = -(f_x + pf_z); \quad \frac{dq}{dt} = -(f_y + qf_z).$

With the initial conditions $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$, $p = p_0(s)$, $q = q_0(s)$ at $t = 0$. We can solve this system of o.d.e to obtain (from the existence and uniqueness theorem for the initial value problems for a system of o.d.e)

$x = X(s,t)$, $y = Y(s,t)$, $z = Z(s,t)$, $p = P(s,t)$, $q = Q(s,t)$

Where X, Y, Z, P and Q have continuous derivatives w.r.t s and t and such that they satisfy the initial conditions

$$X(s,0) = x_0(s), \quad Y(s,0) = y_0(s), \quad Z(s,0) = z_0(s),$$

$$P(s,0) = p_0(s), \quad Q(s,0) = q_0(s).$$

Proposition 1.11.1:

The characteristic curves $x = X(s,t)$, $y = Y(s,t)$, $z = Z(s,t)$ indeed form an integral surface. That is, if s and t are solved in terms of x and y from the first two equations, then the last equation gives the s.t. expressed in the form $z = Z(s(x,y), t(x,y)) = z(x,y)$.

Let (x_0, y_0) be a point on the initial curve. Since the Jacobian

$$\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{t=0} = f_q \frac{dx_0}{ds} - f_p \frac{dy_0}{ds} \neq 0$$

at every point on the initial curve there exists a neighbourhood $N(x_0, y_0)$ such that the Jacobian $\frac{\partial(x,y)}{\partial(s,t)} \neq 0$. In this neighbourhood $N(x_0, y_0)$, we have the following well defined functions

$$\left.\begin{aligned}
s &= s(x,y), \quad x = X(s(x,y), t(x,y)), \\
t &= t(x,y), \quad y = Y(s(x,y), t(x,y)). \\
z &= Z(s(x,y), t(x,y)) = z(x,y) \\
p &= P(s(x,y), t(x,y)) = p(x,y) \\
q &= Q(s(x,y), t(x,y)) = q(x,y)
\end{aligned}\right\} \rightarrow (2)$$

We will now show that $z(x,y)$ satisfies

$$f(x,y,z(x,y), z_x(x,y), z_y(x,y)) = 0$$

It suffices to show that $p = z_x$ and $q = z_y$ since $f(x,y,z,p,q) = 0$ as the characteristic strips are essentially soln of ⓑ. Let us consider

$$U(s,t) = z_s - P x_s - Q y_s$$

At $t=0$, $U(s,0) = \frac{dz_0}{ds} - p_0 \frac{dx_0}{ds} - q_0 \frac{dy_0}{ds} = 0$, by the strip condition for the initial elements. We shall show that $U=0$ for all t. Consider

$$\frac{\partial U}{\partial t} = z_{st} - P_t x_s - Q_t y_s - P x_{st} - Q y_{st}.$$

$$= \frac{\partial}{\partial s}(z_t - P x_t - Q y_t) + P_s x_t + Q_s y_t - Q_t y_s - P_t x_s$$

$$= 0 + P_s f_p + Q_s f_q + (f_x + f_z P) x_s + (f_y + f_z Q) y_s,$$

by making use of the strip condition for characteristics. This can be rewritten as

$$\frac{\partial U}{\partial t} = f_x x_s + f_y y_s + f_z z_s + f_p P_s + f_q Q_s - f_z(z_s - P x_s - Q y_s),$$

$$= f_s - f_z U,$$

$$= -f_z U. \quad (\text{since } f \equiv 0 \ \forall s \text{ and } t)$$

Therefore for a fixed s (i.e., along the characteristic curve) U satisfies the following o.d.e.

$$\frac{dU}{dt} = -f_z U$$

which has a solution:

$$U = U(0) \exp\left(-\int_0^t f_z \, dt\right)$$

combined with the condition that $U=0$ for $t=0$, the above expression implies that $U \equiv 0$ for all t. Therefore

$$z_s = P x_s + Q y_s.$$

Consider the four equations

(i) $z_s = P x_s + Q y_s$, (ii) $z_t = P x_t + Q y_t$

(iii) $z_s = z_x x_s + z_y y_s$, (iv) $z_t = z_x x_t + z_y y_t$

(i) It has been proved above. (ii) It is the characteristic eqn. (iii) & (iv) are obtained by differentiating the identities of Equ ㉗.

Observe that (P, Q) satisfy (i) and (ii) and (z_x, z_y) satisfy (iii) and (iv). As a system of simultaneous equ (i) & (ii) are the same as (iii) and (iv). Further, the determinant of the coefficients of these linear system is $\frac{\partial(x,y)}{\partial(s,t)}$

This determinant is non-zero in the neighbourhood $N(x_0, y_0)$. Hence we have

$$P(s,t) = z_x(x(s,t), y(s,t)), \quad Q(s,t) = z_y(x(s,t), y(s,t))$$

or

$$p(x,y) = z_x(x,y), \quad q(x,y) = z_y(x,y)$$

which was to be shown. Hence the result.

Also the sol. $z = z(x,y)$ contains the initial strip. For,

$$z(x_0, y_0) = z(x(s,0), y(s,0)) = Z(s,0) = z_0(s).$$

$$z_x(x_0, y_0) = p(x_0, y_0) = p(x(s,0), y(s,0)) = P(s,0) = p_0(s)$$

$$z_y(x_0, y_0) = q(x_0, y_0) = q(x(s,0), y(s,0)) = Q(s,0) = q_0(s)$$