STOCHASTIC PROCESSES

UNIT - I

S: 2.1

Stochastic processes :

Families of random variables which are functions of say, time, are known as stochastic processes (or random processes, or random functions).

example :

Consider a simple experiment like throwing a true die. (i) suppose that X_n is the outcome of the nth throw, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a family of random variables such that for a distinct value of n $(= 1, 2, \ldots)$, one gets a distinct random variable X_n, $\{X_n, n \geq 1\}$ constitutes a stochastic process, known as Bernoulli process.

S: 2.2

(i) Discrete time, discrete state space

(ii) Discrete time, continuous state space

(iii) continuous time, discrete state space

(iv) continuous time, continuous state space.

Processes with independent increments :

If for all $t_1, \ldots t_n$, $t_1 < t_2 < \ldots < t_n$, the random variables

$$X(t_2) - X(t_1),\ X(t_3) - X(t_2),\ \ldots,\ X(t_n) - X(t_{n-1})$$

are independent, then $\{X(t), t \in T\}$ is said to be a process with independent increments.

suppose that we wish to consider the discrete Parameter case. consider a process in discrete time with independent increments. writing

$$T = \{0,1,2,\dots\}, t_i = i-1, X(t_i) = X_{i-1}$$

$$Z_i = X_i - X_{i-1}, \ i=1,2,\dots \text{ and } Z_0 = X_0$$

we have a sequence of independent random Variable $\{Z_n, n \geq 0\}$.

Markov process:

If $\{X(t), t \in T\}$ is a stochastic process such that, given the value $X(s)$, the values of $X(t), t > s$, do not depend on the values of $X(u), u < s$, than the Process is said to be a markov process.

If for, $t_1 < t_2 < \dots < t_n < t$

$$Pr\{a \leq X(t) \leq b \mid X(t_1) = x_1, \dots X(t_n) = x_n\}$$

$$= Pr\{a \leq X(t) \leq b \mid X(t_n) = x_n\}$$

the process $\{X(t), t \in T\}$ is a markov process.

A discrete parameter markov process is known as a markov chain.

S: 2.3

Second-order processes:

we consider here real-valued stochastic processes in continuous time. A stochastic process $\{X(t), t \in T\}$ is called a second-order process if $E\{X(t)\}^2 < \infty$.

If it is collection of second-order random variables. The mean function is defined by

$$m(t) = E\{x(t)\}$$

and the covariance function is defined by

$$c(s,t) = cov\{x(s), x(t)\}$$
$$= E\{x(s)\,x(t)\} - E\{x(s)\}\,E\{x(t)\}.$$

~~The~~

A convariance function satisfies the following properties:

(1) It is symmetric in t and s,

(ie) $c(s,t) = c(t,s)$, $s,t \in T$

(2) Application of schwarz inequality yields

$$c(s,t) \le \sqrt{\{c(s,s)\,c(t,t)\}}$$

(3) It is non-negative definite, that is

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k\, c(t_j, t_k) = E\left\{\sum_{j=1}^{n} a_j x_j\right\}^2 \ge 0$$

where $a_1, a_2, \ldots a_n$ is a set of real numbers and $t_i \in T$.

(4) closure properties. The sum and the product of two covariance function are covariance function.

stationarity:

A second order process is called covariance stationary or weakly stationary or wide sense stationary, if its mean function $m(t)$ is independent of t and its covariance function $c(s,t)$ is a function only of the time difference $|t-s|$, for all t,s that is,

$$c(s,t) = f(t-s)$$

This also implies that for any t_0,

$$c(s+t_0, t+t_0) = \text{cov}\{x(s+t_0), x(t+t_0)\}$$

$$= \text{cov}\{x(s), x(t)\}$$

$$= c(s,t).$$

If for arbitrary $t_1, \ldots t_n$ the joint distribution of the vector random variables

$\{x(t_1), x(t_2), \ldots x(t_n)\}$ and

$\{x(t_1+h) \ldots, x(t_n+h)\}$ are the same for all $h>0$, then the stochastic process $\{x(t), t \in T\}$ is said to be stationary of order n. It is strictly stationary if it is stationary of order n for any integer n.

Gaussian processes :

If the distribution of $\{x(t_1), \ldots, x(t_n)\}$ for all $t_1, \ldots, t_n$ is multivariate normal, when $\{x(t), t \in T\}$ is said to be a Gaussian process.

If a Gaussian process $x(t)$ is covariance stationary, then it is strictly stationary.

The multivatriate normal ~~defin~~ distribution of $x(t_1), \ldots, x(t_n)$ is completely determined by its mean vector $(\mu_1, \ldots \mu_n)$ where $\mu_i = E\{x(t_i)\}$ and the variance-covariance matrix $(c(i,j))$, whose elements are ~~c(i,j)x cov~~

$c(i,j) = \text{cov}\{x(t_i), x(t_j)\}$ $\quad i,j = 1,2,\dots n.$

If the Gaussian process is covariance stationary, then $E\{x^2(t_i)\}$, $E\{x^2(t_j)\}$ are finite and the covariance function $c(i,j)$ is a function only of the difference $i-j$.

Example 3(a)

Let $\{x_n, n \geq 1\}$ be uncorrelated r.v's with mean 0 and variance 1. Then $c(n,m) = \text{cov}\{x_n, x_m\} = E\{x_n, x_m\} = 0 \quad m \neq n$

$= 1 \quad m = n$

and so $\{x_n, n \geq 1\}$ is covariance stationary.

If x_n are also identically distributed, then $\{x_n, n \geq 1\}$ is strictly stationary.

Given $c(n,m) = \text{cov}(x_n, x_m)$

$$E(x_n, x_m) = 0 \quad m \neq n$$

$$= 1 \quad m = n$$

$$\text{cov}(x_n, x_n) = E(x_n x_n) - E(x_n)E(x_n)$$

$$= E(x_n^2) - E(x_n)E(x_n)$$

$$\text{var}\{x_n\} = E\{x_n^2\} - [E(x_n)]^2 \qquad [\because \text{var} = 1 \text{ and mean} x]$$

$$1 = E(x_n^2) - 0$$

$$E(x_n^2) = 1$$

$$\text{cov}(x_n, x_n) = 1 - 0$$

$$\text{cov}(x_n, x_n) = 1 \quad (\text{covariance stationary})$$

Example 3(b)

Poisson process. consider the process $\{x(t), t \in T\}$ with $\Pr\{x(t) = n\} = \exp(-at)(at)^n/n! \quad a > 0, n = 0,1,2,\dots$. We see that $m(t) = at$, $\text{var}\{x(t)\} = at$ are functions of t.

Given that $\Pr\{x(t) = n\} = \dfrac{e^{-at}\, at^n}{n!}$

To Find mean :

$$m(t) = E\left[x(t) - \sum_{n=0}^{\infty} n \, pr(x(t))\right]$$

$$= \sum_{n=0}^{\infty} n \left[e^{-at} \frac{(at)^n}{n!}\right]$$

$$= 0 + \frac{e^{-at}(at)}{1!} + \frac{2e^{-at}(at)^2}{2!} + \cdots$$

$$= e^{-at} at \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots\right]$$

$$= e^{-at} at (e^{at})$$

$$m(t) = at$$

$$E(x(t)) = at$$

To Find variance :

$$var(x(t)) = E(x^2(t) - [E(x(t)]^2$$

$$E\{x^2(t)\} = \sum_{n=0}^{\infty} n^2 pr\{x(t)\}$$

$$= \sum_{n=0}^{\infty} [n(n-1)+n] \frac{e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{n(n-1)e^{-at}(at)^n}{n!} + \sum_{n=0}^{\infty} \frac{n e^{-at}(at)^n}{n!}$$

$$= \left[0 + 0 + 2\frac{e^{-at}(at)^2}{2!} + \frac{3.2 \, e^{-at}(at)^3}{3!}\right] + at$$

$$= e^{-at}(at)^2 \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots\right] + at$$

$$= e^{-at}(at)^2 e^{at} | at | (nt)^2 + at$$

$$E(A^2(t)) = (at)^2 + at$$

$$var(x(t)) = E(x^2(t)) - (E\,x(t))^2$$

$$= (at)^2 + at - (at)^2 + at$$

$$var(x(t)) = at$$

$$m(t) = at, \quad var(x(t)) = at$$

are functions of t. The process is evolutionary. The distribution of the process functionally dependent on t.

Example 3(c)

Consider the process

$$x(t) = A_1 + A_2 t,$$

where A_1, A_2 are independent r.v's with $E(A_i) = a_i$, $Var(A_i) = \sigma_i^2$, $i = 1, 2$. ~~We have~~

Given that $x(t) = A_1 + A_2 t$

Given $E(A_i) = a_i$ and $var(A_i) = \sigma^2$.

To prove Evolutionary:

(i) To Find mean:

$$E(x(t)) = E(A_1 + A_2 t)$$

$$= E(A_1) + E(A_2 t)$$

$$E(x(t)) = a_1 + a_2 t$$

Similarly, $E(x(s)) = a_1 + a_2 s$

(ii) $cov(s,t) = cov(x(s), x(t))$

$$= E[x(s)x(t)] - E(x(s)) \; E(x(t))$$

$$E(x(s) \cdot x(t)) = E(A_1 + A_2 s)[A_1 + A_2(t)]$$

$$= E(A_1 A_1 + A_1 A_2 t + A_2 A_1 s + A_2 A_2 st)$$

$$= E(A_1 A_1) + E(A_1 A_2)t + E(A_1 A_2)s + E(A_2 A_2) st$$

$$= E(A_1^2) + E(A_1 A_2)t + E(A_1 A_2)s + E(A_2^2) st$$

$\Big[\because$ using variance $V(A_1^2) = E(A_1^2) - [E(A_1)]^2$

$$\sigma_1^2 = E(A_1^2) - a_1^2$$

$$E(A_1^2) = \sigma_1^2 + a_1^2$$

similarly, $E(A_2^2) = \sigma_2^2 + a_2^2$.]

$$= \sigma_1^2 + a_1^2 + F(A_1)E(A_2)t + E(A_1)E(A_2)s + (\sigma_2^2 + a_2^2)st$$

$$= \sigma_1^2 + a_1^2 + a_1a_2t + a_1a_2s + \sigma_2^2 st + a_2^2 st$$

$$E(x(s) \cdot x(t)) = \sigma_1^2 + a_1^2 + a_1a_2t + a_1a_2st + \sigma_2^2 st + a_2^2 st$$

$$\text{cov}(s,t) = E[x(s)\ x(t)] - E[x(s)] \cdot E(x(t))$$

$$= \sigma_1^2 + a_1^2 + a_1a_2t + a_1a_2s + \sigma_2^2 st + a_2^2 st - (a_1 + a_2s)(a_1 + a_2t)$$

$$= \sigma_1^2 + a_1^2 + a_1a_2t + a_1a_2s + \sigma_2^2 st + a_2^2 st - (a_1^2 + a_1a_2t + a_1a_2s + a_2^2 st)$$

$$= \sigma_1^2 + a_1^2 + a_1a_2t + a_1a_2s + \sigma_2^2 st + a_2^2 st - a_1^2 - a_1a_2t - a_1a_2s - a_2^2 st$$

$$= \sigma_1^2 + \sigma_2^2 st$$

$$\text{cov}(s,t) = \sigma_1^2 + \sigma_2^2 st$$

∴ ~~X(t) is stationary~~. The process is evolutionary.

Example 3(d)

consider the process

$x(t) = A\cos\omega t + B\sin\omega t$ where A, B are uncorrelated r.v's each with mean 0 and variance 1 and ω is a positive constant. is covariance stationary.

Given that, mean $E(A) = E(B) = 0$

Variance $V(A) = V(B) = 1$

To Find mean:

$$E(X(t)) = E(A\cos\omega t + B\sin\omega t)$$

$$= E(A\cos\omega t) + E(B\sin\omega t)$$

$$= E(A)\cdot\cos\omega t + \sin\omega t\, E(B)$$

$$= 0$$

To Find covariance:

$$c(s,t) = \mathrm{cov}(X(s), X(t))$$

$$= E(X(s)\cdot X(t)) - E(X(s)\cdot E(X(t))$$

$$= E(A\cos\omega s + B\sin\omega s)\,(A\cos\omega t + B\sin\omega t) - 0\cdot 0$$

$$= E(A^2\cos\omega s\cos\omega t + AB\cos\omega s\sin\omega t + AB\sin\omega s\cos\omega t + B^2\sin\omega s\sin\omega t)$$

$$= E(A^2\cos\omega s\cos\omega t) + E(AB\cos\omega s\sin\omega st) + E(AB\sin\omega s\cos\omega t) + E(B^2\sin\omega s\sin\omega t)$$

$$c(s,t) = \cos\omega s\cos\omega t\, E(A^2) + \cos\omega s\sin\omega t\, E(AB) + \sin\omega s\cos\omega t\, E(AB) + \sin\omega s\sin\omega t\, E(B^2) \quad (1)$$

W.K.T $V(A) = E(A^2) - [E(A)]^2$

$$1 = E(A^2) - 0$$

$$E(A^2) = 1$$

|||ly, $E(B^2) = 1$

Apply this in eqn (1) we get,

$$c(s,t) = \cos\omega s\cos\omega t + \cos\omega s\sin\omega t\, E(A)\cdot E(B) + \sin\omega s\cos\omega t\, E(A)\cdot E(B) + \sin\omega s\sin\omega t$$

$$= \cos\omega s\cos\omega t + \sin\omega s\sin\omega t$$

$$= \cos(s-t)\omega$$

The process is stationary, because the covariance function is function of $(s-t)$. Thus the process is covariance stationary.

Example : 3 (e)

consider the process $\{X(t), t \in T\}$ whose probability distribution, inder a certain condition, is given by

$$Pr\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}} \quad n = 1, 2, \dots$$

$$= \frac{at}{1+at} \quad n = 0.$$

Given that $Pr\{X(t)=n\} = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}} & , n = 1, 2, \dots \\ \frac{at}{1+at} & , n = 0 \end{cases}$

To Find mean :

$$E(X(t)) = \sum_{n=0}^{\infty} n \cdot Pr\{X(t) = n\}$$

$$= \sum_{n=0}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}} \Rightarrow \sum_{n=0}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1+1-1}}$$

$$= \sum_{n=1}^{\infty} \frac{n}{(1+at)^2} \left(\frac{at}{1+at}\right)^{n-1}$$

$$= \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} n \left(\frac{at}{1+at}\right)^{n-1} \quad — (1)$$

Expand : $\sum_{n=1}^{\infty} n \left(\frac{at}{1+at}\right)^{n-1}$

$$x = \frac{at}{1+at}$$

$$\sum_{n=1}^{\infty} n(x)^{n-1} = 1 + 2(x) + 3(x^2) + \dots$$

$$= (1-x)^{-2}$$

$$\sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1} = \left(\frac{1-at}{1+at}\right)^{-2}$$

Apply this in eqn ① we get,

$$E(x(t)) = \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left(\frac{1}{1+at}\right)^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2 \Rightarrow 1$$

To Find variance:

$$V(x(t)) = E(x^2(t)) - [E(x(t))]^2 \quad \text{——} \ ②$$

$$E(x^2(t)) = \sum_{n=0}^{\infty} n^2 \, Pr\{x(t) - n\}$$

$$= \sum_{n=2}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=2}^{\infty} [n(n-1)+n] \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=2}^{\infty} n(n-1)\frac{(at)^{n-1}}{(1+at)^{n-1}} + \sum_{n=1}^{\infty} \frac{n(at)^{n-1}}{(1+at)^{n-1}}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-1)(at)^{n-1+1-1}}{(1+at)^{n+1+2-2}} + \sum_{n=1}^{\infty} \frac{n(at)^{n+1}}{(1+at)^{n+1}}$$

$$= \frac{at}{(1+at)^3} \sum_{n=2}^{\infty} n(n-1) \left(\frac{at}{1+at}\right)^{n-2} + 1 \quad \text{—— (3)}$$

Expand $\sum_{n=2}^{\infty} n(n-1) \left(\frac{at}{1+at}\right)^{n-2}$

Let $\left(\frac{at}{1+at}\right) = x$

$$\sum_{n=2}^{\infty} n(n-1) \left(\frac{at}{1+at}\right)^{n-2} = \sum_{n=2}^{\infty} n(n-1)(x)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)(x)^{n-2} = 2\cdot 1 + 3\cdot 2x + 4\cdot 3x^2 + \dots$$

$$= 1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \dots$$

$$= 2(1-x)^{-3}$$

$$\left[\because (1-x)^{-3} = \frac{1}{2!}(1\cdot 2 + 2\cdot 3x + \dots)\right]$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) \left(\frac{at}{1+at}\right)^{n-2} = 2\left(\frac{1-at}{1+at}\right)^{3}$$

Apply this in eqn (3) we get

$$E(x^2(t)) = \frac{at}{(1+at)^3}\left[(2)\left(\frac{1-at}{1+at}\right)^{-3}\right] + 1$$

$$= \frac{at}{(1+at)^3}\, 2\left(\frac{1}{1+at}\right)^{-3} + 1 \Rightarrow 2at + 1$$

using this in eqn (1)

$$\operatorname{var}\{x(t)\} = 2at + 1 - (1)^2$$

$$= 2at$$

The process is not stationary.

S : 3.1

Example 1 (a) Polya's urn model

An urn contains b black and r ~~b~~ red balls. A ball is drawn at random and is replaced after the drawing, (ie) drawing is with replacement. The outcomes at the n^{th} drawing is either a black ball or a red ball.

Let the random variable X_n be defined as

$X_n = 1$, if nth drawing result in a black ball and

$X_n = 0$, if it result in a red ball.

There are two possible outcomes of X_n with

$$Pr\{X_n = 1\} = \frac{b}{b+r}$$

$$Pr\{X_n = 0\} = \frac{r}{b+r} \quad \text{for all } n \geq 1$$

We have

$$Pr\{X_1 = j, \ldots, X_n = k\} = Pr\{X_1 = j\} \ldots Pr\{X_n = k\}$$

$$j, k = 0, 1$$

Because of independence of $X_1, \ldots X_n$.

Polya's urn model is such that after each drawing not only that the ball drawn is replaced but $c\,(>0)$ balls of the colour down are added to the urn so that the number of balls of the colour drawn increases, The number of balls of the other colour remains unchanged as the drawing.

we have,

$$Pr\{x_2=1\} = Pr\{x_2=1, x_1=0\} + Pr\{x_2=1, x_1=1\}$$

$$Pr\{x_2=1\} = Pr\{x_2=1 \mid x_1=0\}\, Pr\{x_1=0\} + Pr\{x_2=1 \mid x_1=1\}\, Pr\{x_1=1\}$$

$$Pr\{x_2=1\} = \frac{b}{b+r+c}\,\frac{r}{b+r} + \frac{b+c}{b+r+c}\,\frac{b}{b+r}$$

$$= \frac{br+b^2+bc}{(b+r)(b+r+c)} = \frac{b\cancel{(b+r+c)}}{(b+r)\cancel{(b+r+c)}}$$

$$Pr\{x_2=1\} = \frac{b}{b+r}$$

It can be shown that,

$$Pr\{x_n=1\} = b/b+r$$

$$Pr\{x_n=0\} = r/b+r \quad \text{for all } n$$

Again, $Pr\{x_3=1, x_2=1\} = Pr\{x_3=1, x_2=1, x_1=0\} + Pr\{x_3=1, x_2=1, x_1=1\}$

$$Pr\{x_3=1, x_2=1\} = Pr\{x_3=1 \mid x_2=1, x_1=0\} \cdot Pr\{x_2=1 \mid x_1=0\} \cdot Pr\{x_1=0\} + Pr\{x_3=1 \mid x_2=1, x_1=1\} \cdot Pr\{x_2=1 \mid x_1=1\}\, Pr\{x_1=1\}.$$

$$Pr\{x_3=1, x_2=1\} = \frac{b+c}{b+r+2c}\,\frac{b}{b+r+c}\,\frac{r}{b+r} + \frac{b+2c}{b+r+2c}\,\frac{b+c}{b+r+c}\,\frac{b}{b+r}$$

$$= \frac{(b+c)br + (b+2c)(b+c)(b)}{(b+r)(b+r+c)(b+r+2c)}$$

$$= \frac{b(b+c)(r+\cancel{b+2c})}{(b+r)(b+r+c)(\cancel{b+r+2c})} \Rightarrow \frac{b(b+c)}{(b+r)(b+r+c)}$$

Hence, $Pr\{X_3=1 \mid X_2=1\} = \dfrac{Pr\{X_3=1, X_2=1\}}{Pr\{X_2=1\}}$

$$Pr\{X_3=1 \mid X_2=1\} = \frac{\cancel{b}(b+c)}{(\cancel{b+r})(b+r+c)} \cdot \frac{\cancel{b+r}}{\cancel{b}}$$

$$= \frac{b+c}{b+r+c}$$

$$Pr\{X_3=1 \mid X_2=1\} = \frac{b+c}{b+r+c} \quad \text{— (1)}$$

But,

$$Pr\{X_3=1 \mid X_2=1, X_1=1\} = \frac{b+2c}{b+r+2c} \quad \text{— (2)}$$

and $Pr\{X_3=1 \mid X_2=1, X_1=0\} = \dfrac{b+c}{b+r+2c}$ — (3)

Combine (2) and (3)

Hence, $Pr\{X_3=1 \mid X_2=1\} \neq Pr\{X_3=1 \mid X_2=1, X_1=k\}$,

$k=0,1$.

Here the conditional probability of X_3 given X_2 is not equal to that of X_3 given X_2 and X_1.

Chapter : 3

S: 3.1

Example 1(a) polya's urn model

Definition:

The stochastic process $\{X_n, n=0,1,2,\dots\}$ is called a Markov chain, if for $j, k, j_1, \dots j_{n-1} \in N$

$$\Pr\{X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, \dots X_0 = j_{n-1}\}$$
$$= \Pr\{X_n = k \mid X_{n-1} = j\} = P_{jk}$$

whenever the 1st member is defined. The outcomes are called the states of the markov chain.

The transition probability may or may not be independant of n. If the transition probability P_{jk} is independent of n, the markov chain is said to be $\boxed{\text{homogeneous}}$

If ~~it~~ is dependent of n, the chain is said to be $\boxed{\text{non-homogeneous}}$

The transition probability P_{jk} refers to the states (j, k) at two successive trails, the transition is one-step and P_{jk} is called $\boxed{\text{one - step transition probability}}$

Transition matrix :

The transition probabilities P_{jk} satisfy

$$P_{jk} \geq 0, \quad \sum_k P_{jk} = 1 \text{ for all } j$$

These probabilities may be written in the matrix form

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called the transition probability matrix or matrix of transition probabilities of the markov chain.

Example 1(b)

A particle perform a random walk with absorbing barries say at 0 and 4. whenever it is at any position r $(0 < r < 4)$, it moves to $r+1$ with probability p or to $(r-1)$ with probability q, $p+q=1$.

Let X_n be the position of the particle after n moves. The different states of X_n are the different position of the particle.

$\{X_n\}$ is a markov chain, unit-step transition probabilities are given by

$$P_r\{X_n = r+1 \mid X_{n-1} = r\} = p$$

$$P_r\{X_n = r-1 \mid X_{n-1} = r\} = q$$

$$P_r\{X_n = 0 \mid X_{n-1} = 0\} = 1$$

$$P_r\{X_n = 4 \mid X_{n-1} = 4\} = 1.$$

The transition matrix is given by

$$\begin{array}{cc} & \text{States of } X_n \\ & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \text{States of } X_{n-1} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 1 (c)

$0 \to 0 = 1$
$4 \to 4 = 1$

Random walk between two barriers : General case.

Consider that a particle may be any position r, $r = 0, 1, \dots k$ of the x axis. From state r it moves to state $r+1$, $1 \le r \le k-1$ with probability p and to state $r-1$ with probability q. As soon as it reaches state 0 it remains there with probability a and is reflected to state 1 with probability $1-a$ $(0 < a < 1)$.

If it reached state k it remains there with Probability b and is reflected to $k-1$ with probability $1-b$ $(0 < b < 1)$. $\{X_n\}$ when X_n is the position of the Particle after n steps or moves, is a markov chain with state space $S = \{0, 1, \dots k\}$.

The transition matrix is

$$\begin{array}{cc} & X_n \\ \begin{array}{c} X_{n-1} \\ P = \end{array} \begin{array}{c} 0 \\ 1 \\ \\ k-1 \\ k \end{array} & \begin{pmatrix} a & 1-a & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & q & 0 & p \\ 0 & 0 & 0 & \dots & 0 & 1-b & b \end{pmatrix} \end{array}$$

If $a=1$, then 0 is an absorbing barrier and if $a=0$, then 0 is a reflecting barrier, if $0<a<1$, 0 is an elastic barrier.

III[14] is the case with state k. The case when both 0 and k are absorbing barrier corresponds to the familiar gambler's ruin problem.

Example 1 (d)

suppose that a coin with probability p for a head is tossed indefinitely. (Infinite margov chain)

Let X_n the outcome of the n^{th} trial, be k, where k $(=0, 1, \dots n)$ denotes that there is a run of k successes.

(i) The Length of the uninterrupted block of heads is k. $\{X_n, n \geq 0\}$ constitutes a Markov chain. with unit step transition probabilities.

$$P_{jk} = P_r\{X_n = k \mid X_{n-1} = j\} = \begin{cases} p, & k = j+1 \\ q, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

The transition matrix is given by

$$\begin{array}{cc} & \text{States of } X_n \\ \text{States of } X_{n-1} & \left[\begin{array}{c|cccccccc} & 0 & 1 & 2 & \cdots & k & k+1 & \cdots \\ 0 & q & p & 0 & \cdots & 0 & 0 & \cdots \\ 1 & q & 0 & p & \cdots & 0 & 0 & \cdots \\ 2 & \cdots & & & & \cdots & \cdots & \cdots \\ \vdots & & & & & & & \\ k & q & 0 & 0 & \cdots & 0 & p & \cdots \\ \vdots & & & & & & & \end{array}\right] \end{array}$$

[∵ Infinite number of states]

Probability Distribution :

The probability distribution of $X_r, X_{r+1}, \ldots X_{r+n}$ can be computed in terms of the transition probability P_{jk} and the initial distribution of X_r.

$$Pr\{X_0=a, X_1=b, \ldots, X_{n-2}=i, X_{n-1}=j, X_n=k\}$$

$$= Pr\{X_n=k \mid X_{n-1}=j, \ldots, X_0=a\} \, Pr\{X_{n-1}=j, \ldots X_0=a\}$$

$$= Pr\{X_n=k \mid X_{n-1}=j\} \, Pr\{X_{n-1}=j \mid X_{n-2}=i\} \, Pr\{X_{n-2}=i, \ldots X_0=a\}$$

$$= Pr\{X_n=k \mid X_{n-1}=j\} \, Pr\{X_{n-1}=j \mid X_{n-2}=i\} \ldots \{Pr\{X_1=b \mid X_0=a\} \, Pr\{X_0=a\}$$

$$= \{Pr(X_0=a)\} \, P_{ab} \ldots P_{ij} \, P_{jk}.$$

Thus,

$$Pr\{X_r=a, X_{r+1}=b, \ldots X_{r+n-2}=i, X_{r+n-1}=j, \ldots X_{r+k}=k\}$$

$$= \{Pr(X_r=a)\} \, P_{ab} \ldots P_{ij} \, P_{jk}.$$

Example : 1(g)

Let $\{X_n, n \geq 0\}$ be a markov chain with three states 0, 1, 2 and with transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

and the initial distribution $Pr\{X_0=i\}=1/3$, $i=0,1,2$. Find $Pr\{X_3=1, X_2=2, X_1=1, X_0=2\}$

Given $Pr\{X_0=i\} = 1/3$, $i=0,1,2$

$Pr\{X_0=2\} = 1/3$

$$Pr\{X_1 = 1 \mid X_0 = 2\} = 3/4$$

$$Pr\{X_2 = 2 \mid X_1 = 1\} = 1/4$$

$$Pr\{X_2 = 2, X_1 = 1 \mid X_0 = 2\} = Pr\{X_2 = 2 \mid X_1 = 1\} . Pr\{X_1 = 1 \mid X_0 = 2\}$$

$$Pr\{X_2 = 2, X_1 = 1 \mid X_0 = 2\} = 1/4 \cdot 3/4 = 3/16$$

$$Pr\{X_2 = 2, X_1 = 1, X_0 = 2\} = Pr\{X_2 = 2 \mid X_1 = 1\} \; Pr\{X_1 = 1 \mid X_0 = 2\} \; Pr\{X_0 = 2\}$$

$$Pr\{X_2 = 2, X_1 = 1, X_0 = 2\} = 3/16 \cdot 1/3 = 1/16$$

$$Pr\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\} = Pr\{X_3 = 1 \mid X_2 = 2\}$$
$$Pr\{X_2 = 2 \mid X_1 = 1\} . Pr\{X_1 = 1 \mid X_0 = 2\} \; Pr\{X_0 = 2\}$$

$$Pr\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\} = 3/4 \cdot 1/16 = 3/64$$

Strong markov property :

A sequence of r.v's $\{X_n\}$ is a random variable.

Let N be a stopping time for a markov chain $\{X_n, n > 0\}$ and Let A and B be two events prior and posterior respectively to N. Then

$$Pr\{B \mid X_N = i, A\} = Pr\{B \mid X_N = i\}.$$

This is called the strong markov property.

Order of a markov chain :

A markov chain $\{X_n\}$ is said to be of order $s (s = 1, 2, 3, \ldots)$ if for all n,

$$Pr\{X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, \ldots X_{n-s} = j_{s-1} \ldots\}$$
$$= Pr\{X_n = k \mid X_{n-1} = j, \ldots X_{n-s} = j_{s-1}\}.$$

A Markov chain $\{X_n\}$ is said to be of $\boxed{\text{order one}}$ if $Pr\{X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, \dots\} = Pr\{X_n = k \mid X_{n-1} = j\}$

$$= P_{jk}$$

whenever $Pr\{X_{n-1} = j, X_{n-2} = j_1, \dots\} > 0$.

A chain is said to be of $\boxed{\text{order zero}}$ if $P_{jk} = P_k$ for all j.

If $S = \{1, 2, \dots m\}$ is the set of vertices corresponding to the state space of the chain and a is the set of directed arcs between these vertices, then the graph $G = \{S, a\}$ is the $\boxed{\text{directed graph}}$ or $\boxed{\text{digraph}}$ or $\boxed{\text{transition graph}}$ of the chain.

A digraph such that its arc weights are positive and sum of the arc weights of the arcs from each node is unity is called a $\boxed{\text{stochastic graph}}$, the digraph of a markov chain is a stochastic graph.

Chapman - Kolmogorov equation:

The m-step transition probability is denoted by

$$Pr\{X_{m+n} = k \mid X_n = j\} = P_{jk}^{(m)}. \quad \text{--- (1)}$$

$P_{jk}^{(m)}$ gives the probability that from the state j at n^{th} trial, the state k is reached at trial in m steps.

(i) The probability of transition from the state j to the state k in exactly m steps. The number n does not occur in the r.h.s of the eqn (1) and the chain is homogeneous.

The one-step transition probabilities $P_{jk}^{(1)}$ are denoted by P_{jk} for simplicity.

Consider,

$$P_{jk}^{(2)} = Pr\{X_{n+2}=k \mid X_n=j\} \quad \text{—— } (2)$$

The state k can be reached from the state j in two steps through some intermediate state r. Consider a fixed value of r, we have

$$Pr\{X_{n+2}=k, X_{n+1}=r \mid X_n=j\}$$

$$= Pr\{X_{n+2}=k \mid X_{n+1}=r, X_n=j\}\, Pr\{X_{n+1}=r \mid X_n=j\}$$

$$= P_{rk}^{(1)} P_{jr}^{(1)} = P_{jr} P_{rk}.$$

$\because$ These intermediate states r can assume values $r=1,2,\ldots$ we have

$$P_{jk}^{(2)} = Pr\{X_{n+2}=k \mid X_n=j\} = \sum_r Pr\{X_{n+2}=k, X_{n+1}=r \mid X_n=j\}$$

$$\cancel{\neq P_{rk}^{(1)} P_{jr}^{(1)} \neq P_{jr} P_{rk}}.$$

$$= \sum_r P_{jr} P_{rk} \quad \text{—— } (3)$$

By induction, we have

$$P_{jk}^{(m+1)} = Pr\{X_{n+m+1}=k \mid X_n=j\}$$

$$= \sum_r Pr\{X_{n+m+1}=k \mid X_{n+m}=r\}\, Pr\{X_{n+m}=r \mid X_n=j\}$$

$$= \sum_r P_{rk} P_{jr}^{(m)}$$

$|||^{ly}$, we get

$$P_{jk}^{(m+1)} = \sum_r P_{jr} P_{rk}^{(m)}$$

In general, we have :

$$P_{jk}^{(m+n)} = \sum_r P_{rk}^{(n)} P_{jr}^{(m)} = \sum_r P_{jr}^{(n)} P_{rk}^{(m)} \quad \text{---} (4)$$

This equation is a special case of chapman-kolmogorov equation, which is satisfied by the transition probabilities of a markov chain.

From (4) we get

$$P_{jk}^{(m+n)} \geq P_{jr}^{(m)} P_{rk}^{(n)} \quad \text{for any } r.$$

Let $P = (P_{jk})$ denote the transition matrix of the unit-step transitions and $P^{(m)} = P_{jk}^{(m)}$ denote the transition matrix of the m-step transition.

For $m=2$, we have the matrix $P^{(2)}$ given in (2).

$$P^{(2)} = P.P = P^2$$

$|||^{ly}$, $P^{(m+1)} = P^m.P = P.P^m$ and

$$P^{(m+n)} = P^m.P^n = P^n.P^m \quad \text{---} (5)$$

It should be noted that there exist non-markovian chains whose transition probabilities satisfy chapman-kolmogorov equation.

Example 2(a)

consider the markov chain of Example 1(g). The two-step transition matrix is given by

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

Proof:

The two-step transition matrix is given below

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 9/16 + 1/16 + 0 & 3/16 + 1/8 + 0 & 0 + 1/16 + 0 \\ 3/16 + 1/8 + 0 & 1/16 + 1/4 + 3/16 & 0 + 1/8 + 1/16 \\ 0 + 3/16 + 0 & 0 + 3/8 + 3/16 & 0 + 3/16 + 1/16 \end{pmatrix}$$

$$= \begin{pmatrix} 10/16 & 5/16 & 1/16 \\ 5/16 & 8/16 & 3/16 \\ 3/16 & 9/16 & 4/16 \end{pmatrix} \Rightarrow \begin{array}{c|c} & \begin{matrix} 0 & \quad 1 & \quad 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{pmatrix} \end{array}$$

$$P_{01}^{(2)} = Pr\{X_{n+2}=1 \mid X_n=0\} = 5/10 \quad , \; n \geq 0$$

$$\therefore Pr\{X_2=1 \mid X_0=0\} = 5/10$$

$$\therefore Pr\{X_2=1, X_0=0\} = Pr\{X_2=1 \mid X_0=0\} \cdot Pr\{X_0=0\}$$

$$= 5/16 \cdot 1/3 \Rightarrow 5/48$$

$$\left[\because Pr\{X_i=0\} = 1/3 \;, \; i=0,1,2,\ldots\right]$$

$$\therefore Pr\{X_2=1, X_0=0\} = 5/48$$

Example 2(b)

Two-state markov chain.

suppose that the probability of a dry day (state 0) following a rainy day (state 1) is 1/3 and that the Probability of a rainy day following a dry day is 1/2. we have a two-state markov chain such that $P_{10} = 1/3$ and $P_{01} = 1/2$ and t.p.m

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix} \end{array}$$

Given $P_{10} = 1/3$, $P_{01} = 1/2$

we have to consider two state markov chain

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 + 1/6 & 1/4 + 2/6 \\ 1/6 + 2/9 & 1/6 + 4/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix} \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{173}{432} & \frac{259}{432} \\ \frac{259}{648} & \frac{389}{648} \end{bmatrix}$$

Given that may 1 is dry day.

May 3 is a dry day is $5/12$

May 5 is a dry day is $\frac{173}{432}$

Example 2(c)

consider a communication system which transmits the two digit 0 and 1 through several stage. Let $X_n, n \geq 1$ be the digit leaving the n^{th} stage of system and X_0 be the digit entering the 1^{st} stage. At each stage there is a constant probability q that the digit which enters will be transmitted unchanged and probability P otherwise (the digit changes when it leaves) $p+q=1$.

A homogeneous two-state markov chain with unit-step transition matrix

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

$p \rightarrow$ change , $q \rightarrow$ unchange

$$P^2 = P.P$$

$$= \begin{pmatrix} q & p \\ p & q \end{pmatrix} \begin{pmatrix} q & p \\ p & q \end{pmatrix} \Rightarrow \begin{pmatrix} q^2+p^2 & qp+pq \\ pq+qp & p^2+q^2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} q^2+p^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix}$$

$\therefore p+q=1$

square both sides.

$$p^2+q^2=1$$

$$(p+q)^2 = 1^2$$

$$p^2+q^2+2pq = 1 \quad \text{—} (1)$$

Add and sub $2pq$

$$p^2+q^2+2pq+2pq-2pq = 1$$

$$(p^2+q^2-2pq)+4pq = 1$$

$$(q-p)^2+4pq = 1$$

$$4pq = 1-(q-p)^2$$

$$2pq = 1/2 - \frac{(q-p)^2}{2}$$

$$2pq = 1/2 - 1/2\,(q-p)^2$$

from (1) $\Rightarrow$ $p^2+q^2 = 1-2pq$

$$p^2+q^2 = 1-\frac{1}{2}+\frac{1}{2}(q-p)^2$$

$$P^2 = 1/2 + 1/2\,(q-p)^2$$

$$P^m = 1/2 + 1/2\,(q-p)^m$$

$$P^2 = \begin{bmatrix} 1/2 + 1/2(q-p)^2 & 1/2 - 1/2(q-p)^2 \\ 1/2 - 1/2(q-p)^2 & 1/2 + 1/2(q-p)^2 \end{bmatrix}$$

$$\therefore P^m = \begin{array}{c|c} & \begin{array}{cc} P_{00} & P_{01} \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1/2 + 1/2(q-p)^m & 1/2 - 1/2(q-p)^m \\ 1/2 - 1/2(q-p)^m & 1/2 + 1/2(q-p)^m \end{bmatrix} \\ & \begin{array}{cc} P_{10} & P_{11} \end{array} \end{array}$$

$$P_{00}^{(m)} = P_{11}^{(m)} = 1/2 + 1/2\,(q-p)^m$$

$$P_{01}^{(m)} = P_{10}^{(m)} = 1/2 - 1/2\,(q-p)^m$$

$m \to \infty$

$$\lim P_{00}^{(m)} = \lim P_{01}^{(m)} = \lim P_{10}^{(m)} = \lim P_{11}^{(m)} \to 1/2$$

Suppose that the initial distribution is given by

$$Pr\{X_0 = 0\} = a \text{ and } Pr\{X_0 = 1\} = b = 1 - a$$

We have,

$$Pr\{X_m = 0, X_0 = 0\} = Pr\{X_m = 0 \mid X_0 = 0\}\, Pr\{X_0 = 0\}$$

$$\therefore Pr\{X_m = 0 \mid X_0 = 0\} = P$$

$$= a P_{00}^{(m)}$$

$$Pr\{X_m = 0, X_0 = 1\} = Pr\{X_m = 0 \mid X_0 = 1\}\, Pr\{X_0 = 1\}$$

$$= b P_{10}^{(m)}$$

$$Pr\{X_0 = 0 \mid X_m = 0\}$$

$$= \frac{Pr\{X_m = 0 \mid X_0 = 0\}\, Pr\{X_0 = 0\}}{Pr\{X_m = 0 \mid X_0 = 0\}\, Pr\{X_0 = 0\} + Pr\{X_m = 0 \mid X_0 = 1\}\, Pr\{X_0 = 1\}}$$

$$= a P_{00}^{(m)} \Big/ a P_{00}^{(m)} + b P_{10}^{(m)}$$

$$= \frac{a\left[\frac{1}{2}+\frac{1}{2}(q-p)^m\right]}{a\left[\frac{1}{2}+\frac{1}{2}(q-p)^m\right]+b\left[\frac{1}{2}-\frac{1}{2}(q-p)^m\right]}$$

$$= \frac{\frac{1}{2}a\left[1+(q-p)^m\right]}{\frac{1}{2}a\left[1+(q-p)^m\right]+\frac{1}{2}b\left[1-(q-p)^m\right]}$$

$$= \frac{\frac{1}{2}a\left(1+(q-p)^m\right)}{\frac{1}{2}\left[a\left(1+(q-p)^m\right)+b\left(1-(q-p)^m\right)\right]}$$

$$= \frac{a\left[1+(q-p)^m\right]}{a\left[1+(q-p)^m\right]+b\left[1-(q-p)^m\right]}$$

$$= \frac{a+a(q-p)^m}{a+a(q-p)^m+b-b(q-p)^m}$$

$$= \frac{a\left[1+(q-p)^m\right]}{(a+b)+(q-p)^m(a-b)} \Rightarrow \frac{a\left[1+(q-p)^m\right]}{1+(q-p)^m(a-b)}$$

S: 3.3

Markov- Bernoulli chain

~~chans~~ consider a chain having t.p.m

$$\begin{array}{cc} & \begin{array}{cc} & X_n \\ 0 & 1 \end{array} \\ X_{n-1} \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-(1-c)p & (1-c)p \\ (1-c)(1-p) & (1-c)p+c \end{pmatrix} \end{array} \quad 0<p<1,\ 0\leq c\leq 1$$

with initial distribution

$$p_1 = Pr\{X_0=1\} = p = 1-Pr\{X_0=0\}.$$

where $c=0$, than we get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and the chain remains is its initial state for ever with probability 1.

Consider the case $0 < c < 1$

$$p_n = p_{n-1}\{(1-c)p + c\} + q_{n-1}\{(1-c)p\}$$

$$= c\,p_{n-1} + (1-c)p$$

$$p_n = A c^{n-1} p_1 + \frac{(1-c)p}{1-c}$$

$\because A$ is constant

$$= A c^{n-1} p + p, \quad n \geq 1$$

using $p_1 = p$, we get $p = Ap + p$ so that $A = 0$

$$p_n = p = Pr\{X_0 = 1\} \text{ for all } n.$$

The probability that the event occurs is the same at all trials. we get

$$E\{X_n\} = p$$

$$Var\{X_n\} = p - p^2 = p(1-p)$$

$$E\{X_{n-1}, X_n\} = [(1-c)p + c]p$$

$$cov\{X_{n-1}, X_n\} = cp(1-p), \quad n \geq 1$$

$$\text{correlation } \{X_{n-1}, X_n\} = c.$$

It can be easily seen that

$$cov\{X_{n-2}, X_n\} = c^2 p(1-p)$$

generally, $cov\{X_{n-k}, X_n\} = c^k p(1-p)$

$$corr\{X_{n-k}, X_n\} = c^k, \quad k \geq 1$$

$$S_n = X_1 + \cdots + X_n$$

Markov Bernoulli sequence

$E\{S_n\} = np$ and $Var\{S_n\} = \sum_{k=1}^{n} Var\{X_k\} +$

$2\sum_{k<n}$

S_n gives the accumulated number of successes in n trials in the markov - bernoulli sequence.

$$E\{S_n\} = np$$

$$\operatorname{var}\{S_n\} = \sum_{k=1}^{n} \operatorname{var}\{X_k\} + 2\sum_{\substack{j,k \\ j<1 \\ j=1}}^{k=n} \operatorname{cov}\{X_j, X_k\}$$

$$\sum \operatorname{var}(X_k) = np(1-p)$$

$$\frac{\sum \operatorname{cov}(X_j, X_k)}{p(1-p)} = (c + c^2 + \dots + c^{n-1}) + (c + c^2 + \dots + c^{n-2}) + \dots + c$$

$$= \frac{c}{1-c}\left[(n-1) - c\,\frac{(1-c^{n-1})}{1-c}\right]$$

$$\operatorname{var}\{S_n\} = np(1-p) + 2p(1-p)\left[\frac{c(n-1)}{1-c} - c^2\,\frac{(1-c^{n-1})}{(1-c)^2}\right]$$

$n \to \infty$ and $p \to 0$, $np \to \lambda$

$$E\{S_n\} \to \lambda$$

$$\operatorname{var}\{S_n\} \to \lambda + \frac{2\lambda c}{1-c}$$

when $c=0$. the sequence becomes a sequence of independent bernoulli trials and the limit we get Poission distribution with $E\{S_n\} = \operatorname{var}\{S_n\} = \lambda$.

class property:-

A class of states is a subset of the state space such that every state of the class communicates with every other and there is no other state outside the class which communicates with every (all) other states in the class.

A property defined for all states of a chain is a class property if its possession by one state in a class implies its possession by all states of the same class. One such property is the periodicity of a state.

periodicity:

state i is a return state if $P_{ii}^{(n)} > 0$ for some $n \geq 1$. The period d_i of a return to state i is defined as the greatest common divisor of all m such that $P_{ii}^{(m)} > 0$. Thus.

$$d_i = G.C.D \{ m : P_{ii}^{(m)} > 0 \}$$

state i is said to be aperiodic if d

If ~~$d_i > 0$~~ $d_i > 1$, then state i is said t periodic.

state i is aperiodic if $P_{ii} \neq 0$.

classification of chains:-

If C is a set of states such that no

outside C can be reached from any state in C, then C is said to be closed. If C is closed and $j \in C$ while $k \notin C$, then $p_{jk}^{(n)} = 0$, for all, n i.e. C is closed iff $\sum_{j \in C} p_{ij} = 1$ for every $i \in C$. Then the submatrix $P_1 = (p_{ij})$, $i, j \in C$ is also stochastic.

A closed set may contain one or more states. If a closed set contains only one state j then state j is said to be absorbing.

j is absorbing iff $p_{jj} = 1$, $p_{jk} = 0$, $k \neq j$.

Every finite Markov chain contains at least one closed set i.e., the set of all states or the state space. If the chain does not contain any other proper closed subset other than the state space, then the chain is called irreducible.

In an irreducible Markov chain, every state can be reached from every other state.

Reducible (or) non-irreducible:-

Chains which are not irreducible are said to be reducible or non-irreducible.

Theorem. 3.1 First Entrance Theorem.

Whatever be the states j and k.

$$p_{jk} = \sum_{r=0}^{n} f_{jk}^{(r)} \cdot p_{kk}^{(n-r)}, \quad n \geq 1 \rightarrow ①$$

with

$$p_{kk}^{(0)} = 1, \quad f_{jk}^{(0)} = 0, \quad f_{jk}^{(1)} = p_{jk}$$

the probability that starting with j, state k is reached for the first time at the r^{th} step and again after that at $(n-r)$th step is given by $f_{jk}^{(r)} \cdot p_{kk}^{(n-r)}$ for all $r \geq n$. The recursive relation can also be written as.

$$p_{jk}^{(n)} = \sum_{r=1}^{n-1} f_{jk}^{(r)} \, p_{kk}^{(n-r)} + f_{jk}^{(n)}, \quad n>1.$$

3.4.4.1. First Passage Time Distribution.

We have to consider two cases, $F_{jk}=1$ and $F_{jk}<1$.

When $F_{jk}=1$, it is certain that the system starting with state j will reach state k.

In this case, $\{f_{jk}^{(n)}, n=1,2,\ldots\}$ is a proper probability distribution and this gives the first passage time distribution for k given that the system starts with j.

The mean (first passage) time from state j to state k is given by.

$$\mu_{jk} = \sum_{n=1}^{\infty} n \cdot f_{jk}^{(n)}$$

When $k=j$, $\{f_{jj}^{(n)}, n=1,2,\ldots\}$ will represent the distribution of the recurrence times of j;

4 and $F_{jj} = 1$ will imply that the return to the state j is certain.

$$\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$$

is known as the mean recurrence time for the state j.

$$d_i = G.C.D.\{m : P_{ii}^{(m)} > 0\} = G.C.D.\{m : f_{ii}^{(m)} > 0\}$$

Persistent:- A state j is said to be persistent if $F_{jj} = 1$.

Transient:- A state j is said to be transient if $F_{jj} < 1$.

Null Persistent:- A persistent state j is said to be null-persistent if $\mu_{jj} = \infty$.

Non-Null persistent:- A persistent state j is said to be non-null persistent if $\mu_{jj} < \infty$.

Ergodic:-
A persistent non-null and aperiodic state of a Markov chain is said to be ergodic.

Example:- 4(a)
Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{1, 2, 3, 4\}$ and transition matrix.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

Solution:-

For state: 3 $f_{33}^{(1)} = \frac{1}{2}$, $f_{33}^{(2)} = 0$

$$f_{33}^{(3)} = \frac{1}{2} \cdot \frac{1}{3} \cdot (0) = 0.$$

so that $F_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)}$.

$F_{33} = \frac{1}{2} + 0 + 0 = \frac{1}{2} < 1$ and state 3 is transient.

For state: 4 $f_{44}^{(1)} = \frac{1}{2}$.

$$f_{44}^{(2)} = \frac{1}{2} \cdot \frac{1}{2} (0) = 0.$$

F_4

$$f_{44}^{(n)} = 0, \; n \geq 2$$

so that $F_{44} = \frac{1}{2} + 0 = \frac{1}{2} < 1$.

and thus state 4 is also transient.

For state: 1 $f_{11}^{(1)} = \frac{1}{3}$; $f_{11}^{(2)} = \left(\frac{2}{3}\right)(1) = \frac{2}{3}$.

$$F_{11} = \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$$

so, state 1 is persisten.

Further we find $\mu_{11} = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = \frac{1}{3} + \frac{4}{3} = \frac{5}{3} < \infty$

State - 1 is non-null persistent.

$$\boxed{d_i = \text{G.C.D} \left\{ m : P_{ii}^{(m)} > 0 \right\}}$$

6

$$d_i = G.C.D\{1,2\}$$

$$d_i = 1.$$

so, the state 1 is aperiodic.

$\therefore$ state 1 is ergodic.

For state 2:-

$$f_{22}^{(1)} = 0, \quad f_{22}^{(2)} = 1 \cdot \frac{2}{3} = \frac{2}{3}$$

$$f_{22}^{(3)} = 1 \cdot \frac{1}{3} \frac{2}{3}$$

$$f_{22}^{(4)} = 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}.$$

$$\cdots\cdots\cdots\cdots$$

$$f_{22}^{(n)} = 1 \cdot \left(\frac{1}{3}\right)^{n-2} \cdot \frac{2}{3}, \quad n \geq 2.$$

$$F_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^{k-2} \cdot \frac{2}{3}$$

consider $\frac{1}{3} = x$,

$$F_{22} = \frac{2}{3} \left[1 + x + x^2 + x^3 + \cdots\right].$$

$$= \frac{2}{3}(1-x)^{-1}$$

$$= \frac{2}{3}\left(1 - \frac{1}{3}\right)^{-1}$$

$$F_{22} = \frac{2}{3}\left(\frac{2}{3}\right)^{-1} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}} = 1.$$

$$F_{22} = 1$$

We have, $\mu_{22} = \sum_{k=1}^{\infty} k \cdot f_{22}^{(k)} = \sum_{k=2}^{\infty} k \cdot \left(\frac{1}{3}\right)^{k-2} \cdot \frac{2}{3}$

$$\mu_{22} = 2 \sum_{k=2}^{\infty} k \cdot \left(\frac{1}{3}\right)^{k-1}.$$

consider $\frac{1}{3} = x$, here

$$\mu_{22} = 2\left[2x + 3x^2 + 4x^3 + \cdots\right].$$

$$\mu_{22} = 2\left[(1-x)^{-2} - 1\right].$$

$$\mu_{22} = 2\left[(1-1/3)^{-2} - 1\right]$$

$$\mu_{22} = 2\left[\left(\frac{2}{3}\right)^{-2} - 1\right]$$

$$\mu_{22} = 2\left[\frac{9}{4} - 1\right].$$

$$\mu_{22} = 2 \times \frac{(9-4)}{4} = \cancel{2} \times \frac{5}{\cancel{4}\,2}$$

$$\mu_{22} = \frac{5}{2} < \infty.$$

so that state 2 is non-null persistent.

It is also aperiodic because $d_i = G.C.D\left\{m : p_{ii}^{(m)} > 0\right\}$

$$d_i = G.C.D\{1, 2, \ldots\}$$

$$d_i = 1.$$

and hence we find it the state 2 is ergodic.

Example 4(b) :- consider a Markov chain with transition matrix.

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 1/4 & 1/8 & 1/8 & 1/2 \end{array}$$

It can be easily seen that the chain is irreducible.

<u>consider state: 4</u>

$$P_{44}^{(1)} = 1/2 > 0$$

state is aperiodic

$$f_{44}^{(1)} = \frac{1}{2}$$

$$f_{44}^{(2)} = \frac{1}{8} \cdot 1 = \frac{1}{8}$$

$$f_{44}^{(3)} = \frac{1}{8} \cdot 1 \cdot 1 = \frac{1}{8}$$

$$f_{44}^{(4)} = \frac{1}{4} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{4}$$

$$f_{44}^{(n)} = 0, \; n > 4$$

So that $F_{44} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{4+1+1+2}{8} = \frac{\cancel{8}}{\cancel{8}} = 1.$

$F_{44} = 1$ and state 4 is persistent.

$$\mu_{44} = 1 \cdot \frac{1}{2} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{4}$$

$$\mu_{44} = \frac{4+2+3+8}{8} = \frac{17}{8} < \infty$$

Thus state 4 is ergodic.

Hence all the states are ergodic.

Theorem: 3.2 State j is persistent iff

$$\sum^{\infty} p_{jj}^{(n)} = \infty. \longrightarrow (1)$$

8

$$P_{jj}(s) = \sum_{n=0}^{\infty} P_{jj}^{(n)} \cdot s^n = 1 + \sum_{n=1}^{\infty} P_{jj}^{(n)} \cdot s^n, \ |s|<1. \quad [P_{jj}^{(0)} = 1]$$

and $$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} \cdot s^n = \sum_{n=1}^{\infty} f_{jj}^{(n)} \cdot s^n, \ |s|<1. \quad [f_{jj}^{(0)} = 0]$$

be the generating functions of the sequences $\{P_{jj}^{(n)}\}$ and $\{f_{jj}^{(n)}\}$ respectively.

We have from the entrance theorem,

$$P_{jk}^{(n)} = \sum_{r=1}^{n-1} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)} + f_{jk}^{(n)}, \ n>1.$$

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} \cdot P_{jj}^{(n-r)} \longrightarrow ②.$$

Multiplying both sides of eqn ② by s^n and adding for all $n \geq 1$ we get.

$$\cancel{P_{jj}(s) =} \ \frac{1}{\quad}$$

$$\cancel{P_{jj}(s) - 1} =$$

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} \cdot P_{jj}^{(n-r)}.$$

$$P_{jj}^{(n)} \cdot s^n = \sum_{n=1}^{\infty} \left[\sum_{r=0}^{\infty} f_{jj}^{(r)} \cdot P_{jj}^{(n-r)} \cdot s^n \right].$$

$$P_{jj}^{(n)} \cdot s^n = \sum_{r=0}^{\infty} f_{jj}^{(r)} \cdot s^r \sum_{n=1}^{\infty} P_{jj}^{(n-r)} \cdot s^{n-r}$$

$$P_{jj}(s) - 1 = F_{jj}(s) \cdot P_{jj}(s).$$

The r.h.s. of the above immediately obtained by considering the fact that the r.h.s. of eqn ② is a convolution of $\{f_{jj}\}$ and $\{P_{jj}\}$ and that the generating function of the convolution is the product of the two generating function of the convolution is the product of the two generating functions. Thus, we have

$$P_{jj}(s) - F_{jj}(s) \cdot P_{jj}(s) = 1.$$

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}, \quad |s| < 1. \longrightarrow (3)$$

Assume that state j is persistent which implies that $F_{jj} = 1$. Using Abel's Lemma we get

$$\lim_{s \to 1} F_{jj}(s) = 1$$

we get $$\lim_{s \to 1} P_{jj}(s) \Longrightarrow \infty.$$

since the coefficients of $P_{jj}(s)$ are non-negat Abel's Lemma applies and we get $\sum p_{jj}^{(n)} = \infty$.

conversely, if state j is transient then by Abel lemma we get $$\lim_{s \to 1} F_{jj}(s) < 1.$$

and from eqn. (3),

$$\lim_{s \to 1.} P_{jj}(s) < \infty.$$

since the co-efficients $P_{jj}^{(n)} \geq 0$, we get

$$\sum_n P_{jj}^{(n)} < \infty.$$

Example : 4 (c) Consider the Markov chain with t.p.m

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 1 & 0 \end{array}$$

Sol:-

The chain is irreducible as the matrix

$P_{02}^{(2)}$ $0 \to 1 \to 2$.

$P_{04}^{(4)}$ $0 \to 1 \to 0 \to 1 \to 2$.

$P_{01} > 0$, $P_{10}, P_{12}, P_{21} > 0$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$P_{00}^{(2)}, P_{02}^{(2)}, P_{11}^{(2)}, P_{20}^{(2)} > 0$.

$$P^3 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P = P^3 = P^5 = \cdots\cdots = P^{2n+1}, \; n = 0, 1, 2, \ldots$$

$$P_{00}^{2},\; P_{11}^{2},\; P_{22}^{2} > 0.$$

$$P_{00}^{2},\; P_{00}^{4},\; P_{00}^{6} > 0$$

$$P_{11}^{2},\; P_{11}^{4},\; P_{11}^{6} > 0.$$

$$P_{22}^{2},\; P_{22}^{4},\; P_{22}^{6} > 0.$$

So that $P_{ii}^{(2n)} > 0$, $P_{ii}^{(2n+1)} = 0$ for each i

$$\text{period} = G.C.D\{2, 4, 6, \ldots\} = 2 > 1.$$

The states are periodic with period 2.

We find that $f_{11} = 0$

$$f_{22}^{(2)} = 1$$

So that $F_{11} = \sum_n f_{11}^{(n)} = 1$. (i.e) state 1 is persistent and the other states 0 and 2 are also persistent.

$$\mu_{11} = \sum_n n.f_{11}^{(n)} = 1 \times 0 + 2 \times 1 = 2.$$

(i.e) state 1 is non-null. Thus the states of the chains are periodic and non-null persistent.

$$P_{11}^{(2n)} \to \frac{t}{\mu_{11}} = \frac{2}{2} = 1 \text{ for all } n.$$

Lemma: 3.1

Let $\{f_n\}$ be a sequence such that $f_n \geq 0$, $\Sigma f_n = 1$ and $t (\geq 1)$ be the greatest common divisor of those n for which $f_n > 0$.

Proof :- Let $\{u_n\}$ be another sequence such that $u_0 = 1$ and $u_n = \sum_{r=1}^{n} f_r \, u_{n-r}$ $(n \geq 1)$. Then

$$\lim_{n\to\infty} u_{nt} = t/\mu .$$

where $u = \sum_{n=1}^{\infty} n f_n$, the limit being zero when $\mu = \infty$.

and $\lim_{N\to\infty} u_N = 0$ whenever N is not divisible by t.

Theorem : 3.3

If state j is persistent non-null, then as $n \to \infty$,

(i) $p_{jj}^{(nt)} \to t/\mu_{jj} \to ①$.

when state j is periodic with period t.

and (ii) $p_{jj}^{(n)} \to \frac{1}{\mu_{jj}} \to ②$ when state j is aperiodic.

In case state j is persistent null (whenever periodic or aperiodic), then $p_{jj}^{(n)} \to 0$ as $n \to \infty$. $\longrightarrow ③$

Proof:- Let state j be persistent; then

$\mu_{jj} = \sum_n n \cdot f_{jj}^{(n)}$ is defined.

$f_{jj}^{(n)}$ for f_n, $p_{jj}^{(n)}$ u_n, and μ_{jj} for μ.

In the Lemma 3.1, we have to apply this lemma, we get.

$$p_{jj}^{(n)} \to \frac{t}{\mu_{jj}}, \text{ as } n \to \infty.$$

When state j is periodic with period t.

When state j is aperiodic (i.e. $t=1$) then

$$p_{jj}^{(n)} \to \frac{1}{\mu_{jj}} \text{ as } n \to \infty$$

In the case state j is persistent null, $\mu_{jj} = \infty$ and

$$p_{jj}^{(n)} \to 0 \text{ as } n \to \infty.$$

Note:- (1) If state j is persistent non-null, then,

$$\lim_{n\to\infty} p_{jj}^{(n)} > 0.$$

and (2) If j is persistent null or transient then

$$\lim_{n\to\infty} p_{jj}^{(n)} \to 0.$$

Theorem: 3.4 If state k is periodeal null, then for every j.

10M

$$\lim_{n\to\infty} p_{jk}^{(n)} \to 0. \longrightarrow (1)$$

and if state k is aperiodic, persistent non-null then

$$\lim_{n\to\infty} p_{jk}^{(n)} \to \frac{F_{jk}^{l}}{\mu_{kk}}. \longrightarrow (2)$$

proof :-

$$P_{jk}^{(n)} = \sum_{r=1}^{n} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)}$$

Let $n > m$, then

$$P_{jk}^{(n)} = \sum_{r=1}^{m} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)} + \sum_{r=m+1}^{n} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)}$$

$$\leq \sum_{r=1}^{m} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)} + \sum_{r=m+1}^{n} f_{jk}^{(r)} \rightarrow (3)$$

since state k is persistent null,

$$P_{kk}^{(n-r)} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Further since,

$$\sum_{m=1}^{\infty} f_{jk}^{(m)} < \infty, \sum_{r=m+1}^{n} f_{jk}^{(r)} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

Hence as $n \rightarrow \infty$, $P_{jk}^{(n)} \rightarrow 0$.

From (3), $P_{jk}^{(n)} - \sum_{r=1}^{m} f_{jk}^{(r)} \cdot P_{kk}^{(n-r)} \leq \sum_{r=m+1}^{n} f_{jk}^{(r)} \rightarrow (3)(a)$

since j is aperiodic, persistent and non-null then by theorem 3.3.

$$P_{kk}^{(n-r)} \rightarrow \frac{1}{\mu_{kk}} \quad \text{as } n \rightarrow \infty.$$

Hence from eqn (3)(a) we get as $n, m \rightarrow \infty$.

$$P_{jk}^{(n)} \rightarrow \frac{F_{jk}}{\mu_{kk}}$$

Theorem: 3.5. In a irreducible chain, all the states are of the same. They are either transient, all persistent null, or all persistent non-null. All the states are aperiodic and in the latter case they all have the same period.

Proof:-

Since the chain is irreducible, every state can be reached from every other state. If i, j are any two states, then i can be reached from j and j from i.

(i.e) $p_{ij}^{(N)} = a > 0$ for some $N \geq 1$.

$p_{ji}^{(M)} = b > 0$ for some $M \geq 1$.

We have,

$$P_{jk}^{(n+m)} = P_{jk}^{(m+n)} = \sum_r P_{jr}^{(m)} \cdot P_{rk}^{(n)}$$

$$\geq P_{jr}^{(m)} \cdot P_{rk}^{(n)} \text{ for each } r.$$

Hence

$$P_{ii}^{(n+N+M)} \geq P_{ij}^{(N)} \cdot P_{jj}^{(n)} \cdot P_{ji}^{(M)} = ab \cdot P_{jj}^{(n)} \longrightarrow ①$$

and

$$P_{jj}^{(n+N+M)} = P_{ji}^{(M)} \cdot P_{ii}^{(n)} \cdot P_{ij}^{(N)} = ab\, P_{ii}^{(n)} \longrightarrow ②$$

From the above it is clear that the two series $\sum_n P_{ii}^{(n)}$ and $\sum_n P_{jj}^{(n)}$ converge or diverge together. Thus the two states i and j are either both transient or both persistent.

Suppose that i is persistent null, then $P_{ii}^{(n)} \to 0$ as $n \to \infty$; then from ① $P_{jj}^{(n)} \to 0$ as $n \to \infty$, so that j is also persistent null. (i.e) they are both

Suppose that i is persistent non-null and has

period t, then $p_{ii}^{(n)} > 0$ whenever n is a multiple of t.

Now

$$p_{ii}^{(N+M)} \geq p_{ij}^{(N)} \cdot p_{ji}^{(M)} = ab > 0$$

so that $(N+M)$ is a multiple of t from (2),

$$p_{jj}^{(N+M)} \geq ab\, p_{ii}^{(n)} > 0.$$

Thus $(n+N+M)$ is a multiple of t and so t is the period of the state j also.

Corollary :-

In a finite irreducible Markov chain all states are non-null persistent.

Proof :-

Let $S = \{1, 2, \dots, k\}$ be the state space of the chain and P be its t.p.m. Suppose, if possible, that state i is null persistent. Then all other states are null persistent. This implies that

$$\lim_{n \to \infty} p_{ij}^{(n)} = 0 \text{ for all } j \in S.$$

Now,

$$\sum_{j \in S} p_{ij}^{(n)} = 1 \text{ for all } n.$$

Thus since S is finite we are lead to a contradiction. Hence all states must be non-null persistent.

18 Example :- 5(a) consider the two-state Markov chain $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$, $a < a, b < 1$.

Sol:-

The eigen values of P are $t_1 = 1$ and $t_2 = 1-a-$ $|t_2| < 1$. The right eigen vector corresponding to $t_1 = 1$ and $t_2 = 1-a-b$ are ginven below by x_1 and x_2

right eigen vectors.

$$A X_1 = t_1 X_1 .$$

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = 1 \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$$

$$(1-a) x_{11} + a x_{21} = x_{11} .$$

$$b x_{11} + (1-b) x_{21} = x_{21} .$$

$$-a x_{11} + a x_{21} = 0 .$$

$$x_{11} = x_{21} = 1 .$$

$$b x_{11} - b x_{21} = 0 .$$

$$\Rightarrow \quad x_{11} = x_{21} = 1 .$$

$$X_1' = (1,1) .$$

$$A X_2 = t_2 X_2 .$$

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = (1-a-b) \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$a x_{22} = -b x_{21}$.

$x_{21} = -\frac{a}{b} \; x_{22} = 1$.

$x_{21} = 1 \; , \; x_{22} = -b/a$. $\quad X_2' = (1, -b/a)$.

<u>Left Eigen vector.</u>

$Y_1' A = t_1 Y_1'$.

$$(y_{11} \; y_{12}) \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = 1 \; (y_{11} \; y_{12}).$$

$(1-a) y_{11} + b y_{12} = y_{11}$.

$-a y_{11} + b y_{12} = 0$.

$a y_{11} = b y_{12}$

$y_{11} = \frac{b}{a} \; y_{12} = 1$.

$y_{11} = 1 \; , \; y_{12} = \frac{a}{b}$.

$Y_1' = (1, a/b)$

$y_2' A = t_2 y_2'$

$y_2' A = t_2 . y_2'$

$$(y_{21}, y_{22}) \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = (1-a-b) \; (y_{21}, y_{22}).$$

$(1-a) y_{21} + b y_{22} = (1-a-b) y_{21} \longrightarrow ①$

$y_{21} a + (1-b) y_{22} = (1-a-b) y_{22} \longrightarrow ②$

$① \Longrightarrow b y_{22} = -b y_{21} \Longrightarrow \quad y_{22} = -y_{21} = 1$.

$y_{21} = 1$, $y_{22} = -1$.

$$c_1 = \frac{1}{Y_1' X_1} = \frac{1}{X_{11} Y_{11} + X_{12} Y_{12}}$$

$$= \frac{1}{(1 \times 1) + \left(1 \times \frac{a}{b}\right)}.$$

$$= \frac{1}{1 + \frac{a}{b}}$$

$$= \frac{1}{\frac{a+b}{b}}$$

$$\boxed{c_1 = \frac{b}{a+b}}$$

$$c_2 = \frac{1}{Y_2' X_2} = \frac{1}{\sum X_2 Y_2'} = \frac{1}{X_{21} Y_{21} + X_{22} Y_{22}}$$

$$= \frac{1}{1 \times 1 + \left(-\frac{b}{a} \times -1\right)} = \frac{1}{\left(1 + \frac{b}{a}\right)}$$

$$c_2 = \frac{1}{\left(\frac{a+b}{a}\right)} = \frac{a}{a+b}.$$

$$X_1 Y_1' = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} (y_{11} \quad y_{12}).$$

$$= \begin{pmatrix} x_{11} y_{11} & x_{11} y_{12} \\ x_{12} y_{11} & x_{12} y_{12} \end{pmatrix}$$

$$X_1 Y_1' = \begin{pmatrix} 1 & a/b \\ 1 & a/b \end{pmatrix}$$

$$X_2 Y_2' = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} (y_{21} \quad y_{22}) = \begin{pmatrix} x_{21} y_{21} & x_{21} y_{22} \\ x_{22} y_{21} & x_{22} y_{22} \end{pmatrix}$$

$$X_2 Y_2' = \begin{pmatrix} 1 & -1 \\ -b/a & b/a \end{pmatrix}$$

Thus, $P = \sum_{j=1}^{2} t_j c_j x_j y_j' = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{1-a-b}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$

$$P^n = \sum_{j} t_j^n c_j x_j y_j' = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}, n = 0,1,2,\ldots$$

Further.

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}.$$

As $n\to\infty$, $\lim P_{i1} \to b/a+b$ and $\lim P_{i2} \to \frac{a}{a+b}$, $i=1,2$.

Remark:-

$$P_n = P\{X_n = 1\}.$$

$$= \left(\frac{a}{a+b} - \frac{a}{a+b}(1-a-b)^n\right)(1-P_1)$$

$$+ \left(\frac{a}{a+b}\right) + \frac{b}{a+b}(1-a-b)^n \Big] P_1$$

$$= \frac{a}{a+b} + (1-a-b)^n \left[\frac{-a}{a+b} + \frac{a}{a+b}P_1 + \frac{b}{a+b}P_1\right].$$

$$P_n = \frac{a}{a+b} + (1-a-b)^n \left(P_1 - \frac{a}{a+b}\right).$$

Example: 5(b)

Consider the three-state markov chain with t.p.m.

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}.$$

Solution:-

The eigenvalues of P are $t_1 = 1$, $t_2 = 0.1$ and $t_3 = 0.2$.

sum of the eigen values = sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.5 + 0.4 + 0.4.$$

$$1 + \lambda_2 + \lambda_3 = 1.3$$

$$\lambda_2 + \lambda_3 = 1.3 - 1 = 0.3.$$

Product of the eigen values $= |P|$.

$$= -0.5\ (.16 - .2) - 0.3\ (-0.8 - 0.04) + 0.2\ (0.1 - 0.4$$

$$= (-0.5\ (-0.4) - 0.3 \times 0.4 + 0.2 \times 0.6.$$

$$= 0.2 - 0.012 + 0.012.$$

$$= 0.2.$$

$$\lambda_1 \lambda_2 \lambda_3 = .02.$$

$$\lambda_2 \lambda_3 = .02 \qquad [\because \lambda_1 = 1].$$

Right eigen vectors.

$$A X_1 = t_1 X_1$$

$$-0.5 x_{11} + 0.3 x_{12} + 0.2 x_{13} = x_{11}$$

$$0.2 x_{11} + 0.4 x_{12} + 0.4 x_{12} = x_{12} \rightarrow A.$$

$$0.1 x_{11} + 0.5 x_{12} + 0.4 x_{13} = x_{13} \rightarrow B$$

$$\begin{pmatrix} -0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = 1 \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}$$

$$(3 \times 3)\ (3 \times 1) = 3 \times 1 \text{ matrix.}$$

From A and B,

$$0.2 x_{11} - 0.6 x_{12} + 0.4 x_{13} = 0.$$

$$0.1x_{11} + 0.5x_{12} - 0.6.x_{13} = 0.$$

$$\frac{x_{11}}{.36 - .2} = \frac{x_{12}}{-(-0.12 - 0.04)} = \frac{x_{13}}{.1 + .06}$$

$$\frac{x_{11}}{.16} = \frac{x_{12}}{.16} = \frac{x_{13}}{.16}.$$

(i.e) $x_{11} = x_{12} = x_{13} = 1$.

Right Eigen Vector.

$$Y_1'A = t_1 Y_1'.$$

$$-0.5y_{11} + 0.2y_{12} + 0.1y_{13} = 0.$$

$$0.3y_{11} - 0.6y_{12} + 0.5y_{13} = 0.$$

$$\frac{y_{11}}{.1 + .06} = \frac{y_{12}}{-(-0.25 - .03)} = \frac{y_{13}}{.3 - .06}$$

$$\frac{y_{11}}{.16} = \frac{y_{12}}{.28} = \frac{y_{13}}{.24}.$$

$$P_{ij}^{(n)} \rightarrow t_1 y_{ij} \left(\frac{1}{0.68} (0.16, 0.28, 0.24) \right)$$

$$P_{ij}^{(n)} = (0.2353, 0.4118, 0.3529)$$

Example 5 (C):

Let the genotypes AA, Aa, aa be denoted by 1, 2, 3 respectively. Let $X_n, (n \geq 1)$ be the genotype of the offspring of a father of genotype X_{n-1}.

29 $\{X_n, n \geq 0\}$ is a Markov chain with state space $S = \{1,2,3\}$ and transition matrix.

$$P = \begin{array}{c} \\ AA \\ Aa \\ aa \end{array} \begin{array}{c} \begin{array}{ccc} AA & Aa & aa \end{array} \\ \begin{bmatrix} p & q & 0 \\ p/2 & 1/2 & q/2 \\ 0 & p & q \end{bmatrix} \end{array}, \quad p+q=1, \quad 0<p<1.$$

Sol:-

sum of the eigen values = sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = p+q+1/2.$$

$$\lambda_2 + \lambda_3 = \frac{1}{2} \longrightarrow ①.$$

Product of the eigen values $= |P|$.

$$t_2 t_3 = p\left(\frac{q}{2} - \frac{pq}{2}\right) - q\left(\frac{pq}{2}\right).$$

$$= \frac{pq}{2}(1-p-q)$$

$$= \frac{pq}{2}(1-(p+q)).$$

$$= \frac{pq}{2}(1-1)$$

$$t_2 . t_3 = 0.$$

$$\boxed{t_2 . t_3 = 0}$$

Any one of the eigen values $=0$.

(i.e) $t_2 = 0$ sub in ①.

$$t_3 = 1/2.$$

$$A = I - zP = \begin{pmatrix} 1-zp & -zq & 0 \\ -\frac{zp}{2} & -z/2 & -zq/2 \\ 0 & -z/p & 1-zq \end{pmatrix}.$$

The eigen values of P are $0, 1/2$ and 1.

The eigen values of the matrix $I-zP$ are 1, $(1-z/2)$ and $1-z$.

So that, $\det(I-zP) =$ product of E values

$$= (1-z/2)(1-z).$$

$$I-zP = \frac{Adj(I-zP)}{\det(I-zP)\ [ie)\ (1-z/2)(1-z)}$$

Now the co-factor.

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1-z/2 & -zq/2 \\ -zp & 1-zq \end{vmatrix}$$

$$= (1-z/2)(1-zq) - z^2 \frac{pq}{2}.$$

$$(I-zP)^{-1} = Adj \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \frac{1}{(1-z/2)(1-z)}.$$

is $$\frac{A_{11}}{(1-z/2)(1-z)} = \frac{(1-z/2)(1-zq) - z^2\frac{pq}{2}}{(1-z/2)(1-z)}$$

Using partial fractions and generating components,

we gets
$$\frac{A_{11}}{(1-z/2)(1-z)} = \frac{a_1}{1-z} + \frac{a_2}{1-z/2} \rightarrow (A)$$

$$= \frac{a_1(1-z/2) + a_2(1-z)}{(1-z/2)(1-z)}$$

where (equating w.respect to)

$$\frac{a_1}{2} = \lim_{z\to 1} A_{11} = 1-q/2 - pq/2 = p/2\,(1-q) = \frac{p^2}{2}$$

$$-a_2 = \lim_{z\to 2} A_{11} = -2pq.$$

From (A)

(i.e) $$\frac{A_{11}}{(1-z/2)(1-z)} = \frac{p^2}{1-z} + \frac{2pq}{1-z/2}.$$

$$A_{21} = (-1)^{2+1}\begin{pmatrix} -zq & 0 \\ -zp & 1-zq \end{pmatrix}.$$

$$= -(-zq)(1-zq).$$

$$\frac{A_{21}}{(1-z/2)(1-z)} = \frac{a_1}{1-z} + \frac{a_2}{1-z/2}.$$

$$\frac{1}{2}a_1 = \lim_{z\to 1} A_{21} = q(1-q) = pq.$$

$$-a_2 = \lim_{z\to 2} A_{21} = 2q(1-2q).$$

$$= 2q(p-q)$$

$$a_2 = 2q(q-p).$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -zq & 0 \\ 1-z/2 & \frac{-zq}{2} \end{vmatrix}$$

$$= \frac{z^2q^2}{2}$$

$$\frac{1}{2}a_1 = \lim_{z\to 1} A_{31} = \frac{q^2}{2}.$$

$$-a_2 = \lim_{z\to 2} A_{31} = 2q^2.$$

$$(I-zP)^{-1} = \begin{pmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{pmatrix}\frac{1}{1-z} + \frac{2}{1-z/2}$$

$$\begin{pmatrix} pq & q(q-p) & -q^2 \\ \frac{p(q-p)}{2} & \frac{1-4pq}{2} & \frac{q(p-q)}{2} \\ -p^2 & p(p-q) & pq \end{pmatrix}$$

given matrix for $n \geq 1$.

$$= (1+z+z^2+\dots\dots+z^n+\dots) \, e(p^2, 2pq, q^2)$$

$$+ 2\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\dots\dots+\frac{z^n}{2^n}+\dots\right) \begin{pmatrix} \end{pmatrix}$$

$$p^n = \text{coefficient of } z^n \text{ in } (I-zP)^{-1}$$

$$= e(p^2, 2pq, q^2) + \frac{1}{2^{n-1}} \begin{pmatrix} pq & q(q-p) & -q^2 \\ p(q-p)/2 & (1-4pq)/2 & \frac{q(p-q)}{2} \\ -p^2 & p(p-q) & pq \end{pmatrix}$$

$$p^n = \text{co-efficient } \cancel{\text{of } z^n \text{ in}} \; \cancel{(I-zP)^{-1}}.$$

$$\cancel{= e(p^2, 2pq, q^2) + \frac{1}{2^{n-1}} \Big(}$$

As $n \to \infty$

$$p^n \to e(p^2, 2pq, q^2)$$

which is a matrix with identical rows with elements $(p^2, 2pq, q^2)$. This gives the limiting distribution of genotype.