Course : M. Sc., Mathematics. Semester : III

Subject Code & Name : 18KP3MELM4 Operation Research.

Unit I : Methods of Integer Programming, Cutting - Plane Algorithms Branch - and - Bound Method.

Chapter 8 - Integer Programming Sections 8.2 to 8.4

Unit II : Dynamic (Multistage) Programming : Elements of the DP model - The Capital Budgeting Example, More On the definition of the State, Examples of DP models and Computations.

Chapter 9 - Sections 9.1 to 9.3

Unit III : Decision theory and Games : Decisions under Risk - Decision Trees - Decisions under Uncertainty - Game Theory.

Chapter 11 : Sections 11.1 to 11.4

Unit IV : Inventory Models : A Generalized Inventory Model - Types of Inventory Models - Deterministic Models.

Chapter 13 : Sections 13.1 to 13.3

Unit V : Non-linear Programming Algorithms : unConstrained Non-linear Algorithms – Constrained Non-linear Algorithms.

Chapter 19 : Sections 19.1 and 19.2.4

Text Book : Operation Research by Hamdy A. Taha (3rd Edition)

References : Prem Kumar Gupta & D.S. Hira Operation Research An Introduction, S. Chand and Co., Ltd. New Delhi.

S.S. Rao, Optimization Theory & Applications. Wiley Eastern Ltd. New Delhi.

Question Pattern.

Section A : 10×2 = 20 Marks, 2 Questions from each unit.

Section B : 5×5 = 25 Marks Either OR Pattern, One Question from each unit.

Section C : 3×10 = 30 Marks 3 out of 5 One Question From each Unit.

Integer Programming : Integer Programming deals with the Solution of Mathematical Programming Problems in which Some or all the Variables Can assume nonnegati integer Values only. An integer Program is Called mixed or Pure depending on whether Some or all the Variables are restricted to integer Values. If in the absence of the integrality Conditions the objective and Constraint functions are linear the resulting Model is Called an integer linear Program.

Methods of Integer Programming :

The name of two Integer Programming Methods are

(i) Cutting Methods

(ii) Search Methods (iii)

Cutting Methods : Cutting Plane Algorithm, The method Consists in first Solving the integer Programming Problems as ordinary Continuous L.P Problem and then introducing additional Constraints one after the other to Cut (eliminate) Certain Parts of the Solution Space until an integral Solution is obtained.

Search Methods: Branch and Bound algorithm is more efficient and is more widely used for solving all integer and mixed integer Programming Problem. In this method, the Problem is solved as ordinary continuous L.P Problem and then the solution space is systematically Partitioned into subproblems by deleting Parts that contain no feasible integer solutions.

Note: ① The Cutting Methods developed by R.E Gomory include the fractional algorithm which applies to the Pure integer Problem, and the mixed algorithm which is applied for the mixed integer Problem.

② Branch and Bound algorithm was developed by A.H Land & A.G Doig

③ Additive algorithm applies to the Pure zero-one Problem.

Concept of Cutting Plane Algorithms

To illustrate the Concept of Cutting Plane

Let us Consider the Problem

$$\text{Max } Z = 5x_1 + 7x_2$$

Subject to $-2x_1 + 3x_2 \leq 6$

$6x_1 + x_2 \leq 30$, $x_1, x_2 \geq 0$ and integer.

Now find the optimum Solution using graphical Method.

Let $-2x_1 + 3x_2 = 6$

At $x_1 = 0 \Rightarrow 3x_2 = 6 \Rightarrow x_2 = 2$

At $x_2 = 0 \Rightarrow -2x_1 = 6 \Rightarrow x_1 = -3$

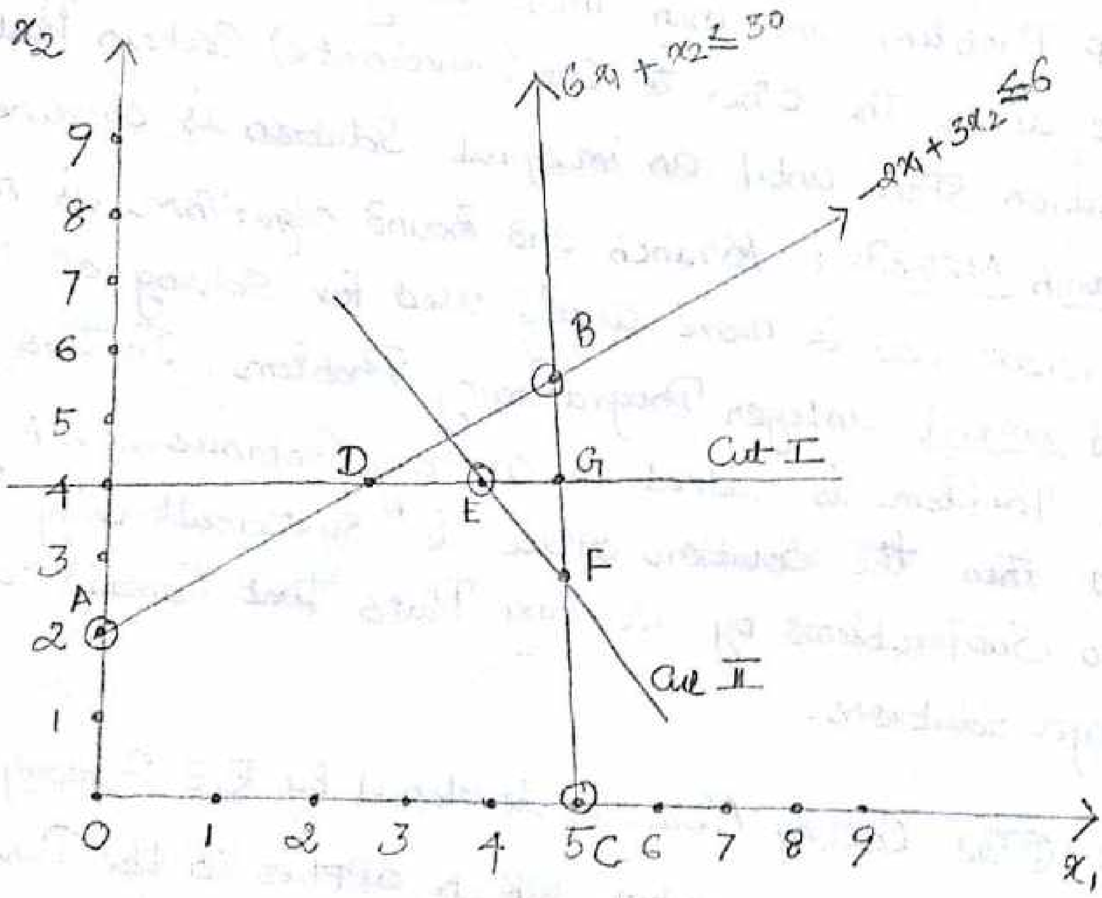
$\therefore$ The Points are $(0,2)$ and $(-3,0)$

Let $6x_1 + x_2 = 30$

At $x_1 = 0 \Rightarrow x_2 = 30$

At $x_2 = 0 \Rightarrow 6x_1 = 30 \Rightarrow x_1 = 5$

$\therefore$ The Points are $(0, 30)$ & $(5, 0)$.



The feasible region is OABC

At O (0,0) then $Z = 0$

At A (5,0) then $Z = 25$

At B $\left(\frac{21}{5}, \frac{24}{5}\right)$ then $Z = \frac{273}{5} = 54.6$

At C (0,2) then $Z = 14$

The Max $Z = 54.6$ at $\left(\frac{21}{5}, \frac{24}{5}\right)$

$\therefore$ The non integer optimum Solution is

$$\boxed{\text{Max } Z = 54.6 \quad \text{at} \quad x_1 = \frac{21}{5} \; ; \; x_2 = \frac{24}{5}}$$

Using Cutting Plane algorithm

The Cutting Plane algorithm modifies the solution space by adding cuts that produce an optimal integer extreme Point.

Cut I : The max non integer ... is $x_2 = 4.8$

Now the fractional Cut at $x_2 = 4$

At the Point $D(3,4)$ then $Z = 5(3) + 7(4) = 43$.

At the Point $G\left(\frac{13}{3}, 4\right)$ then $Z = 5\left(\frac{13}{3}\right) + 28 = \frac{149}{3} = 49.67$

The optimum Solution is $\boxed{\text{Max } Z = 49.67 \text{ at } x_1 = \frac{13}{3}, x_2 = 4}$

This Solution is a non integer optimum Solution.

Again using Fractional Cut

Cut II : The fractional Cut at $x_1 = 4$, $x_2 = 4$

At the Point E (4, 4) then $z = 5(4) + 7(4) = 20 + 28 = 48$

The optimum Solution is $\boxed{\text{Max } z = 48 \text{ at } x_1 = 4 ; x_2 = 4}$

Which is an integer Solution. The new Feasible Convex region reduces to OADEFC.

The method developed by Gomory is used to develop integer Programming Constraints for all integer and mixed integer Problems.

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Ex: Consider the integer Linear Programming Problem

$$\text{Max } Z = 7x_1 + 9x_2$$

Subject to $\quad -x_1 + 3x_2 \leq 6$

$\qquad 7x_1 + x_2 \leq 35$

Solve the L.P Problem by graphical Method.

The optimal Continuous Solution (ignoring the integrality Condition) is shown in fig.

Let $-x_1 + 3x_2 = 6$

At $x_1 = 0 \Rightarrow x_2 = 2$

At $x_2 = 0 \Rightarrow x_1 = -6$ $\quad \therefore$ The Points are $(0,2)$ & $(-6,0)$

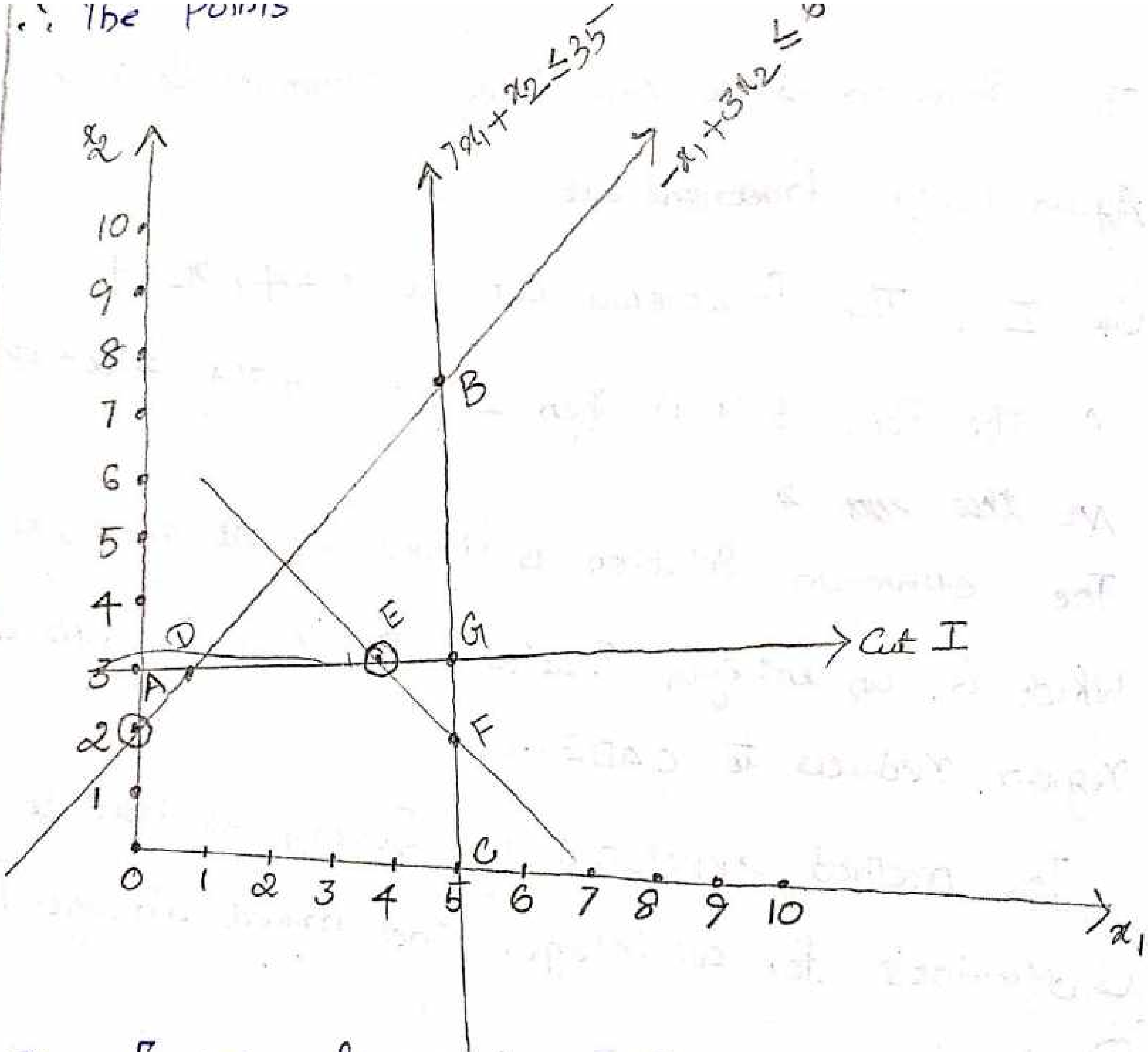
Let $7x_1 + x_2 = 35$

At $x_1 = 0 \Rightarrow x_2 = 35$

At $x_2 = 0 \Rightarrow x_1 = 5$

$\therefore$ The Points are $(0, 35)$ & $(5, 0)$

∴ The points



The feasible region is OABC

At the O (0,0) then $Z = 0$

At the A (0,2) then $Z = 7(0) + 9(2) = 12$

At the B $(9/2, 7/2)$ then $Z = 7\left(\frac{9}{2}\right) + 9\left(\frac{7}{2}\right) = \frac{63+63}{2} = 63$

At the C (5,0) then $Z = 35$.

The non integer optimum Solution is

$$\boxed{Max\ Z = 63 \text{ at } x_1 = \frac{9}{2}\ ;\ x_2 = \frac{7}{2}}$$

The Cutting Plane algorithm modifies the Solution space by adding Cuts that Produce an optimal integer Extreme Point.

<u>Cut I:</u> When added Produces the L.P optimum at

At D(3, 3) then $Z = 7(3) + 9(3) = 21 + 27 = 48$

At $G\left(\frac{32}{7}, 3\right)$ Then $Z = 7\left(\frac{32}{7}\right) + 9(3) = 32 + 27 = 59$.

The non integer optimum Solution is

$$\boxed{Max\ Z = 59 \text{ at } x_1 = \frac{32}{7} \text{ and } x_2 = 3}$$

Cut II : We add Cut II which together with Cut I and the Original Constraints Produces the L.P optimum at E

At (4, 3) then $z = 7(4) + 9(3) = 28 + 27 = 55$

which is all integer.

The new feasible Convex region reduces to OADEFC.

Hence the Solution is

$$\boxed{Max\ z = 55 \text{ at } x_1 = 4 ;\ x_2 = 3}$$

Gomory Fractional (All integer) Algorithm :

Gomory Cutting Plane algorithm starts by solving the Continuous L.P Problem. From the optimum L.P. table is selected a row called the source row for which the basic variable is non integer. The desired cut is then constructed from the fractional components of the coefficients of the source row. For this reason it is referred to as the fractional cut.

The various steps involved in solving an all integer programming problem by the Gomory's cutting plane method are summarized below.

(1) Integerise the Constraints : Transform the Constraints so that all the coefficients are whole numbers.

For example the constraint equation

$$\frac{7}{4}x_1 + \frac{1}{5}x_2 + \frac{3}{4}x_3 = \frac{17}{5}$$

Can be expressed as

$$35x_1 + 4x_2 + 15x_3 = 68$$

(ii) Solve using the Simplex method : Ignoring integrality restriction. Find the optimal solution to the Problem using Simplex method. If the solution is all integer it is an optimal basic feasible integer solution. If not proceed to step (iii) Ignore non-integer values for slack variables since they represent unused resources only.

(iii) Develop a Cutting Plane : From the final Simplex table select the constraint with largest fractional cut. In case of a tie choose the constraint having the lowe contribution (maximisation Problem) or the highest cost (minimization Problem). Alternatively select the constraint with Max $\frac{f_i}{\sum_{j=1}^{n} f_{ij}}$

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If the Coefficient is negative, express it as the Sum of a negative integer and a non-negative fraction.

Construct the Gomory's Constraint

$$S_i = \sum_{j=1}^{n} f_{ij} y_j - f_i$$

and add it to the final Simplex table.

Add an additional Column for S_i also.

(iv) Solve using the dual Simplex method: Solve the augmented I.P.P obtained above by the dual Simplex method. So that the outgoing Variable is S_i. If the optimal Solution is obtained has all integral Values, it is an optimal feasible Solution for the given I.P.P. If not repeat step (iii) until an optimal feasible integer Solution is obtained.

Note: The optimum Solution is all integer (ie) Values of decision Variables, slacks and objective function are integer.

Examples:

① Solve: Maximize $Z = 5x_1 + 7x_2$

Subject to $-2x_1 + 3x_2 \leq 6$

$6x_1 + x_2 \leq 30$

$x_1, x_2 \geq 0$ and integer.

Solution: Ignoring integrality restriction, the Problem can be expressed in standard form as

$$\text{Max } Z = 5x_1 + 7x_2 + 0S_1 + 0S_2$$

Subject to $-2x_1 + 3x_2 + S_1 = 6$

$6x_1 + x_2 + S_2 = 30$

$x_1, x_2, S_1, S_2 \geq 0$

The initial basic feasible Solution is $S_1 = 6$; $S_2 = 30$

To Form a Simplex table.

C_B	c_j Basis	5 x_1	7 x_2	0 S_1	0 S_2	b	θ	
0	S_1	-2	3	1	0	6	2	$\rightarrow S_1$ is a L.V
0	S_2	6	1	0	1	30	30	
$Z_j - c_j$		-5	-7	0	0			

$\uparrow x_2$ is a E.V

C_B	c_j Basis	5 x_1	7 x_2	0 S_1	0 S_2	b	θ	
7	x_2	$-2/3$	1	$1/3$	0	2	-3	
0	S_2	$20/3$	0	$-1/3$	1	28	$\frac{21}{5}$	$\rightarrow S_2$ is a L.V
	$Z_j - c_j$	$-\frac{29}{3}$	0	$7/3$	0			

$\uparrow x_1$ is a E.V

$\hat{R}_1 \rightarrow \frac{R_1}{3}$

$\hat{R}_2 \rightarrow R_2 - \hat{R}_1$

C_B	c_j Basis	5 x_1	7 x_2	0 S_1	0 S_2	b
7	x_2	0	1	$3/10$	$1/10$	$24/5$
5	x_1	1	0	$-1/20$	$\frac{3}{20}$	$\frac{21}{5}$
	$Z_j - C_j$	0	0	$\frac{37}{20}$	$29/20$	$273/5$

All $Z_j - C_j \geq 0$.

$\hat{R}_2 \to \frac{3}{20} R_2$

$\hat{R}_1 \to R_1 + \frac{2}{3} \hat{R}_2$

$\therefore$ The optimum Solution is

$x_1 = \frac{21}{5}$; $x_2 = \frac{24}{5}$, $Max\ Z = \frac{273}{5}$

To Construct Gomory's Constraint. Select x_2 - row which has the largest fractional Part $\frac{4}{5}$.
Each of non integer Coefficients are factored into integer and fractional Components, the fractional Components are strictly Positive.

The x_2-row can be written as

$$(1+0)\,x_2 + \left(0+\frac{3}{10}\right)s_1 + \left(0+\frac{1}{10}\right)s_2 = 4 + 4/5 \rightarrow (1)$$

The Gomory's constraint to be added.

$$S_i = \sum_{j=1}^{n} f_{ij}\, y_j - f_i$$

Accordingly the following equation is generated

$$x_2 + 0s_1 + 0s_2 + s' = 4 \rightarrow (2)$$

$$(2)-(1) \Rightarrow -\frac{3}{10}s_1 - \frac{1}{10}s_2 + s' = -\frac{4}{5}$$

The Modified Simplex table after inserting the Gomory's constraint is

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	C_j	5	7	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S'	b	θ
7	x_2	0	1	$3/10$	$1/10$	0	$24/5$	16
5	x_1	1	0	$-1/20$	$3/20$	0	$21/5$	
0	S'	0	0	$-3/10$	$-1/10$	1	$-4/5$	$\Rightarrow S'$ is a L.V.
	$Z_j - C_j$	0	0	$37/20$	$29/20$	0		
	Ratio			$\frac{-37}{6}\uparrow$	$\frac{-29}{2}$			

S_1 is a E.V

Since $Z_j - C_j$ are either Zero or Positive. it is feasible for the dual and dual Simplex method is used to find the optimum Solution.

Select ~~most~~ +ve Value of $Z_j - C_j \Rightarrow S_1$ is a E.V

Select -ve Value of b is $-4/5 \Rightarrow S'$ is a L.V

Now Regular Simplex method is applied to find an New optimal Solution.

C_B	C_j Basis	5 x_1	7 x_2	0 S_1	0 S_2	0 S'	b
7	x_2	0	1	0	0	1	4
5	x_1	1	0	0	$1/6$	$-1/6$	$13/3$
0	S_1	0	0	1	$1/3$	$-10/3$	$8/3$
	$Z_j - C_j$	0	0	0	$5/6$	$37/6$	

$\hat{R}_3 = -\frac{10}{3} R_3$

$\hat{R}_1 = R_1 - \frac{3}{10}\hat{R}_3$

$\hat{R}_2 = R_2 + \frac{1}{20}\hat{R}_3$

All $Z_j - C_j \geq 0$

The optimal feasible non integer solution is $x_2 = 4$; $x_1 = \frac{13}{3}$. Since x_1 has non integer value.

Now construct the 2^{nd} Gomory constraint.

$$x_1 + \frac{1}{6}S_2 - \frac{1}{6}S' = \frac{13}{3} \Rightarrow (1+0)x_1 + \left(0+\frac{1}{6}\right)S_2 + \left(-1+\frac{5}{6}\right)S'$$

$$= 4 + \frac{1}{3}$$

The Gomory's Constraint to be added. $S'' = \frac{1}{6}S_2 + \frac{5}{6}S' - \frac{1}{3}$

$$-\frac{1}{6}S_2 - \frac{5}{6}S' + S'' = -\frac{1}{3}$$

The modified Simplex table becomes,

C_B	C_j Basis	5 x_1	7 x_2	0 S_1	0 S_2	0 S'	0 S''	b	
7	x_2	0	1	0	0	1	0	4	
5	x_1	1	0	0	$1/6$	$-1/6$	0	$13/3$	
0	S_1	0	0	1	$1/3$	$-10/3$	0	$8/3$	
0	S''	0	0	0	$-1/6$	$-5/6$	1	$-1/3$	→ S'' is a L.V
	$Z_j - C_j$	0	0	0	$5/6$	$\frac{37}{6}$	0		
				Ratio	-5	$-37/5$			

↑ S_2 is a E.V

C_B	C_j Basis	5 x_1	7 x_2	0 s_1	0 s_2	0 s'	0 s''	b
7	x_2	0	1	0	0	1	0	4
5	x_1	1	0	0	0	-1	1	4
0	s_1	0	0	1	0	-5	2	2
0	s_2	0	0	0	1	5	-6	2
	$Z_j - C_j$	0	0	0	0	2	5	48

$\hat{R}_4 = -6R_4$

$\hat{R}_1 = R_1$

$\hat{R}_2 = R_2 + \frac{1}{6}\hat{R}_4$

$\hat{R}_3 = R_3 - \frac{1}{3}\hat{R}_4$

All $Z_j - C_j \geq 0$

Hence the Optimal integer Solution is

$$\boxed{\text{Max } Z = 48 \text{ at } x_1 = 4 \;;\; x_2 = 4}$$

(2) Solve the following integer linear Programming Problem by using Gomory fractional cut.

$$\text{Max } Z = -4x_1 + 5x_2$$

Subject to $-3x_1 + 3x_2 \leq 6$

$2x_1 + 4x_2 \leq 12$

x_1, x_2 are non negative integer.

<u>Solution</u>: Ignoring the Condition of integrality the Problem is Solved by the Simplex method.

C_B	C_j x_B	-4 x_1	5 x_2	0 S_1	0 S_2	b	θ	
0	S_1	-3	3	1	0	6	2	→ S_1 is a L.V
0	S_2	2	4	0	1	12	3	
	$Z_j - C_j$	4	-5	0	0			
			↑ x_2 is a E.V					

5	x_2	-1	1	$1/3$	0	2	$-$	
0	S_2	6	0	$-4/3$	1	4	$\frac{2}{3}$	$\rightarrow S_2$ is a L.V
	$z_j - c_j$	-1	0	$5/3$	0			$\hat{R}_1 = R_1/3$ $\hat{R}_2 = R_2 - 4\hat{R}_1$

$\uparrow x_1$ is a E.V

5	x_2	0	1	$1/9$	$1/6$	$8/3$		
-4	x_1	1	0	$-2/9$	$1/6$	$2/3$		
	$z_j - c_j$	0	0	$13/9$	$1/6$	$32/3$		$\hat{R}_2 = R_2/6$ $\hat{R}_1 = R_1 + \hat{R}_2$

Since All $z_j - c_j \geq 0$

Hence the optimal non integer Solution is

$$\boxed{\text{Max } Z = 32/3 \text{ at } x_1 = 2/3 \;;\; x_2 = 8/3}$$

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To Construct Gomory's Constraint.

Since fractional Part of x_2 and x_1 are equal

(ie) $f_1 = 2/3$; $f_2 = 2/3$.

$\frac{f_i}{\Sigma f_{ij}}$ Values are obtained to find the Source row.

$$\frac{f_i}{\Sigma f_{ij}} \text{ for } x_2 \text{ row } = \frac{2/3}{\frac{1}{9}+\frac{1}{6}} = \frac{12}{5}$$

$$\text{and for } x_1 \text{ row } = \frac{2/3}{7/9+\frac{1}{6}} = \frac{12}{17}$$

$\therefore$ x_2 - row is Selected as the Source row.

(ie) $x_2 + \frac{1}{9} S_1 + \frac{1}{6} S_2 = 2 + \frac{2}{3}$

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The Corresponding Secondary Constraint is

$$S' - \frac{1}{9}S_1 - \frac{1}{6}S_2 = -2/3$$

Adding this Secondary Constraint to the Simplex table We get.

C_B	C_j / x_B	-4 / x_1	5 / x_2	0 / S_1	0 / S_2	0 / S'	b	
5	x_2	0	1	$1/9$	$1/6$	0	$8/3$	
-4	x_1	1	0	$-2/9$	$1/6$	0	$2/3$	
0	S'	0	0	$-1/9$	$-1/6$	1	$-2/3$	→ S' is a L.V
	$Z_j - C_j$	0	0	$13/9$	$1/6$	0		
	Ratio			-13	-1			

↑ S_2 is a E.V

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C_B	$\overset{c_j}{x_B}$	$\overset{-4}{x_1}$	$\overset{5}{x_2}$	$\overset{0}{S_1}$	$\overset{0}{S_2}$	$\overset{0}{S'}$	b
5	x_2	0	1	0	0	1	2
−4	x_1	1	0	$-3/9$	0	1	0
0	S_2	0	0	$6/9$	1	−6	4
	$Z_j - C_j$	0	0	$12/9$	0	1	10

$\hat{R}_3 \to -6R_3$

$\hat{R}_1 \to R_1 - \frac{1}{6}\hat{R}_3$

$\hat{R}_2 \to R_2 - \frac{1}{6}\hat{R}_3$

All $Z_j - C_j \geq 0$

Hence the optimal integer solution is

$\boxed{\text{Max } z = 10 \text{ at } x_1 = 0 \,;\, x_2 = 2}$

(3) Solve the following integer linear Programming Problem by using Gomory fractional Cut.

$$\text{Max } Z = 7x_1 + 9x_2$$

Subject to $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

x_1, x_2 are non negative integers.

<u>Sol:</u> Ignoring the Condition of integrality the Problem is Solved by the Simplex method.

	C_j	7	9	0	0			
C_B	x_B	x_1	x_2	S_1	S_2	b	θ	
0	S_1	-1	3	1	0	6	2	$\rightarrow S_1$ is a L.V
0	S_2	7	1	0	1	35	35	
	$Z_j - C_j$	-7	-9	0	0			

$\uparrow x_2$ is a E.V

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C_B	C_j / x_B	7 / x_1	9 / x_2	0 / S_1	0 / S_2	b	θ	
9	x_2	$-1/3$	1	$1/3$	0	2	-	
0	S_2	$22/3$	0	$-1/3$	1	33	$9/2$	$\rightarrow S_2$ is a L.V $\hat{R}_1 = R_1/3$ $\hat{R}_2 = R_2 - \hat{R}_1$
	$Z_j - C_j$	-10	0	$9/3$	0			

$\uparrow x_1$ is a E.V

C_B	x_B	7 / x_1	9 / x_2	0 / S_1	0 / S_2	0 / b		
9	x_2	0	1	$7/22$	$1/22$	$7/2$		
7	x_1	1	0	$-1/22$	$3/22$	$9/2$		$\hat{R}_2 = \frac{3}{22} R_2$
	$Z_j - C_j$	0	0	$56/22$	$72/22$	63		$\hat{R}_1 = R_1 + \frac{1}{3}\hat{R}_2$

Here All $Z_j - C_j \geq 0$.

$\therefore$ The optimal non integer solution is

$$\boxed{\text{Max } Z = 63 \text{ at } x_1 = 9/2 \;;\; x_2 = 7/2}$$

To Construct Gomory's Constraint.

The Fractional Part of x_1 and x_2 are $\frac{1}{2}$. Since both are equal.

To find the Source row.

$$\frac{f_i}{\Sigma f_{ij}} \text{ for } x_2 \text{ row} = \frac{1/2}{\frac{7}{22}+\frac{1}{22}} = \frac{1}{2}\cdot\frac{22}{8} = \frac{11}{8}$$

$$\text{,, for } x_1 \text{ row} = \frac{\frac{1}{2}}{\frac{3}{22}+\frac{21}{22}} = \frac{1}{2}\times\frac{22}{24} = \frac{11}{24}$$

$\therefore$ x_2 row is Selected as Source row.

(ie) $x_2 + \frac{7}{22}S_1 + \frac{1}{22}S_2 = 3 + \frac{1}{2}$

The Corresponding Secondary Constraint is

$$S' - \frac{7}{22}S_1 - \frac{1}{22}S_2 = -\frac{1}{2}$$

Adding the Secondary Constraint to the Simplex table we get

C_B	C_j / x_B	7 / x_1	9 / x_2	0 / s_1	0 / s_2	0 / s'	b	
9	x_2	0	1	$7/22$	$1/22$	0	$7/2$	
7	x_1	1	0	$-1/22$	$3/22$	0	$9/2$	
0	s'	0	0	$-7/22$	$-1/22$	1	$-3/2$	$\rightarrow s'$ is a L.V
	$Z_j - C_j$	0	0	$56/22$	$30/22$	0		

Ratio $\uparrow$ $-\frac{56}{7}$ -30 s_1 is a E.V

C_B	C_j / x_B	7 / x_1	9 / x_2	0 / s_1	0 / s_2	0 / s'	b
9	x_2	0	1	0	0	1	$3/2$
7	x_1	1	0	0	$1/7$	$-1/7$	$\frac{32}{7}$
0	s_1	0	0	1	$\frac{1}{7}$	$\frac{-22}{7}$	$33/7$
	$Z_j - C_j$	0	0	0	1	8	

$\hat{R}_3 = -\frac{22}{7} R_3$

$\hat{R}_1 = R_1 - \frac{7}{22}\hat{R}_3$

$\hat{R}_2 = R_2 + \frac{1}{22}\hat{R}_3$

All $z_j - c_j \geq 0$ but the solution is an non integer solution.

Now Construct the 2nd Gomory's Constraint.

$$x_1 + \frac{1}{7}s_2 + \frac{6}{7}s' = 4 + \frac{4}{7}$$

The Corresponding Secondary Constraint is

$$-\frac{1}{7}s_2 - \frac{6}{7}s' + s'' = -\frac{4}{7}.$$

Adding This Constraint to The Simplex table, we get

	c_j	7	9	0	0	0	0		
C_B	x_B	x_1	x_2	S_1	S_2	S'	S''	b	
9	x_2	0	1	0	0	1	0	3	
7	x_1	1	0	0	$1/7$	$-1/7$	0	$\frac{32}{7}$	
0	S_1	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	0	$\frac{11}{7}$	
0	S''	0	0	0	$\frac{-1}{7}$	$-\frac{6}{7}$	1	$-\frac{4}{7}$	$\rightarrow$ s'' is a L.V
	$Z_j - C_j$	0	0	0	1	8	0		
				Ratio	-7 $\uparrow$	[illegible]	[illegible] E.V		

	c_j	7	9	0	0	0	0	
C_B	x_B	x_1	x_2	S_1	S_2	S'	S''	b
9	x_2	0	1	0	0	1	0	3
7	x_1	1	0	0	0	-1	1	4
0	S_1	0	0	1	0	-3	1	1
0	S_2	0	0	0	1	6	-7	4
	$Z_j - C_j$	0	0	0	0	2	7	55

All $Z_j - C_j \geq 0$

Hence the optimum integer solution is

$$\boxed{Max\ Z = 55 \text{ at } x_1 = 4;\ x_2 = 3.}$$

$\hat{R}_4 = -7R_4$

$\hat{R}_1 = R_1$

$\hat{R}_2 = R_2 - \frac{1}{7}\hat{R}_4$

$\hat{R}_3 = R_3 - \frac{1}{7}\hat{R}_4$

(4) A Company manufacturing three Products P_1, P_2, P_3 Which yield Per unit Profit of Rs.200, Rs.400 and Rs.300 respectively. Each of these Products is Processed on three different machines The time required on each machine Per unit of the Product is given below. How many Products of each type should be Produced to maximize the Profit?

Product	Time required (hours/unit) Machine I	Machine II	Machine III
P_1	30	20	10
P_2	40	10	30
P_3	20	20	20
Time Available (hours)	600	400	800

Soln: Let x_1, x_2 and x_3 be the number of units of Products P_1, P_2 & P_3 respectively. Then the Problem is the form of Mathematical model is

$$\text{Max } Z = 200x_1 + 400x_2 + 300x_3$$

Subject to

$$30x_1 + 40x_2 + 20x_3 \leq 600$$

$$20x_1 + 10x_2 + 20x_3 \leq 400$$

$$10x_1 + 30x_2 + 20x_3 \leq 800$$

Model can be written as

$$\text{Max } Z' = Z/100 = 2x_1 + 4x_2 + 3x_3$$

Subject to $3x_1 + 4x_2 + 2x_3 \leq 60$

$$2x_1 + x_2 + 2x_3 \leq 40$$

$$x_1 + 3x_2 + 2x_3 \leq 80$$

The optimum non integer solution is

$$\text{Max } Z = 7666.67 \text{ at } x_1 = 0,\ x_2 = \frac{20}{3},\ x_3 = \frac{50}{3}$$

The optimum integer solution is

$$\boxed{\text{Max } Z = 7600 \text{ at } x_1 = 0,\ x_2 = 7,\ x_3 = 16}$$

(5) Find the non-integer optimum solution of

$$\text{Max } z = 3x_1 + x_2 + 3x_3$$

Subject to
$$-x_1 + 2x_2 + x_3 \le 4$$
$$2x_2 - \frac{3}{2}x_3 \le 1$$
$$x_1 - 3x_2 + 2x_3 \le 3$$
$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution:

The given L.P Problem can be written as

$$\text{Max } z = 3x_1 + x_2 + 3x_3$$

Subject to
$$-x_1 + 2x_2 + x_3 \le 4$$
$$4x_2 - 3x_3 \le 2$$
$$x_1 - 3x_2 + 2x_3 \le 3$$
$$x_1, x_2, x_3 \geq 0.$$

The optimum non integer solution is
$\text{Max } z = 29$ at $x_1 = 16/3$, $x_2 = 3$, $x_3 = 10/3$

The optimum integer solution is
$\text{Max } z = 23$ at $x_1 = 5$, $x_2 = 2$, $x_3 = 2$

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(6) Consider the Problem

$$\text{Max } Z = 2x_1 + 20x_2 - 10x_3$$

Subject to $2x_1 + 20x_2 + 4x_3 \leq 15$

$$6x_1 + 20x_2 + 4x_3 = 20$$

$x_1, x_2, x_3 \geq 0$ and integer.

Solve the Problem as a Continuous linear Program Then show that it is impossible to obtain feasible integer solution by using simple rounding. Solve the Problem using any integer Problem algorithm.

Soln! Ignoring the integrality restriction, The Problem can be expressed in standard form as

$$\text{Max } Z = 2x_1 + 20x_2 - 10x_3 + 0S_1 - MA_1$$

Subject to $2x_1 + 20x_2 + 4x_3 + S_1 = 15$

$$6x_1 + 20x_2 + 4x_3 + A_1 = 20$$

$$x_1, x_2, x_3, S_1, A_1 \geq 0$$

The intial basic feasible solution is $[S_1, A_1] = [15, 20]$

C_B	c_j / x_B	2 / x_1	20 / x_2	−10 / x_3	0 / S_1	−M / A_1	b	θ	
0	S_1	2	20	4	1	0	15	3/4	→ S_1 is a L.V
−M	A_1	6	20	4	0	1	20	1	
	$Z_j - C_j$	−6M −2	−20M −20	−4M +10	0	0			

↑ x_2 is a E.V

C_B	c_j / x_B	2 / x_1	20 / x_2	−10 / x_3	0 / S_1	−M / A_1	b	θ	
20	x_2	1/10	1	1/5	1/20	0	3/4	15/2	
−M	A_1	4	0	0	−1	1	5	5/4	→ A_1 is a L.V; $\hat{R}_1 = R_1/20$, $\hat{R}_2 = R_2 - 20\hat{R}_1$
	$Z_j - C_j$	−4M	0	−M +14	M+1	0			

↑ x_1 is a E.V

20	x_2	0	1	1/5	3/40	−1/40	5/8		
2	x_1	1	0	0	−1/4	1/4	5/4		$\hat{R}_2 = R_2/4$; $\hat{R}_1 = R_1 - \frac{1}{10}\hat{R}_2$
	$Z_j - C_j$	0	0	14	1	M	15		

All $Z_j - C_j \geq 0$

$\therefore$ The optimum non-integer solution is

Max $Z = 15$ at $x_1 = \frac{5}{4}$; $x_2 = 5/8$; $x_3 = 0$

The round solution is $x_1 = 1$; $x_2 = 1$; $x_3 = 0$ x is not a feasible sol.

To Construct the Gomory's Constraint :

The largest fractional part is $5/8$.

Select x_2-row the equation is

$$x_2 + \frac{1}{5} x_3 + \frac{3}{40} S_1 = \frac{5}{8}$$

The corresponding secondary constraint is

$$-\frac{1}{5} x_3 - \frac{3}{40} S_1 + S' = -\frac{5}{8}.$$

The Modified Simplex table is

	C_j	2	20	-10	0	0		
C_B	x_B	x_1	x_2	x_3	S_1	S'	b	
20	x_2	0	1	$1/5$	$3/40$	0	$5/8$	
2	x_1	1	0	0	$-1/4$	0	$5/4$	
0	S'	0	0	$-1/5$	$-3/40$	1	$-5/8$	$\leftrightarrow$ S' is a L.V
	$Z_j - C_j$	0	0	14	1	0		
	Ratio			-60	$-\frac{40}{3}$			

$\uparrow$ S_1 is a E.V

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C_B	C_j x_B	2 x_1	20 x_2	-10 x_3	0 S_1	0 S'	b
20	x_2	0	1	0	0	1	0
2	x_1	1	0	$2/3$	0	$-10/3$	$10/3$
0	S_1	0	0	$8/3$	1	$-\frac{40}{3}$	$\frac{25}{3}$
	$Z_j - C_j$	0	0	$34/3$	0	$40/3$	

The optimum solution is non integer Solution.

Add the 2nd Gomory's Constraint.

$$\hat{R}_3 = -\frac{40}{3} R_3$$

$$\hat{R}_2 = R_2 + \frac{1}{4} \hat{R}_3$$

$$\hat{R}_1 = R_1 - \frac{8}{40} \hat{R}_3$$

The fraction Part of x_1 - row, s_1 row have the Same $\frac{1}{3}$

x_1 has a Contribution of 2, while s_1 has a zero Contribution.

Select s_1 row

The equation is

$$\frac{8}{3}x_3 + s_1 - \frac{40}{3}s' = \frac{25}{3}$$

(ie) $\left(2+\frac{2}{3}\right) x_3 + s_1 + \left(-14+\frac{2}{3}\right)s' = \left(8+\frac{1}{3}\right)$

The Corresponding fractional Cut is

$$-\frac{2}{3}x_3 - \frac{2}{3}s' + s'' = \frac{-1}{3}.$$

The modified Simplex table becomes.

	c_j	2	20	-10	0	0	0	
C_B	x_B	x_1	x_2	x_3	s_1	s'	s''	b
20	x_2	0	1	0	0	1	0	0
2	x_1	1	0	2/3	0	-10/3	0	10/3
0	s_1	0	0	8/3	1	-40/3	0	25/3
0	s''	0	0	-2/3	0	-2/3	1	-1/3 → s'' is a L.V
	$Z_j - C_j$	0	0	34/3	0	40/3	0	
				↑	x_3 is	a E.V		

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c_B	x_B	$\overset{2}{x_1}$	$\overset{20}{x_2}$	$\overset{-10}{x_3}$	$\overset{0}{S_1}$	$\overset{0}{S'}$	$\overset{0}{S''}$	b
20	x_2	0	1	0	0	1	0	0
2	x_1	1	0	0	0	-4	1	3
0	S_1	0	0	0	1	-16	4	7
-10	x_3	0	0	1	0	1	$-3/2$	$1/2$
	$Z_j - C_j$	0	0	0	0	2	17	

Since the solution is a non-integer.

$\hat{R}_2 = R_2 - \frac{2}{3}\hat{R}_4$ $\quad \hat{R}_4 = -\frac{3}{2}R_4$

$\hat{R}_1 = R_1$ $\quad \hat{R}_3 = R_3 - \frac{8}{3}R_4$

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Add the 3rd Gomory's Constraint.

The fractional Part is $\frac{1}{2}$ Corresponding to the row is x_3

Select x_3-row the equation is

$$x_3 + S' - \frac{3}{2} S'' = \frac{1}{2}$$

(ie) $(1+0)x_3 + (1+0)S' + \left(-2+\frac{1}{2}\right) S'' = \frac{1}{2}$.

The Corresponding fractional Cut is

$$-\frac{1}{2} S'' + S''' = \frac{-1}{2}$$

The modified Simplex table becomes.

	c_j	2	20	-10	0	0	0	0	
c_B	x_B	x_1	x_2	x_3	S_1	S'	S''	S'''	b
20	x_2	0	1	0	0	1	0	0	0
2	x_1	1	0	0	0	-4	1	0	3
0	S_1	0	0	0	1	-16	4	0	7
-10	x_3	0	0	1	0	1	$-3/2$	0	$1/2$
0	S'''	0	0	0	0	0	$-\frac{1}{2}$	1	$-1/2$
	Z_j-C_j	0	0	0	0	2	17	0	
		Ratio				1	-34		

$\rightarrow S'''$ is a L.V

$\uparrow$ S'' is a E.V

	c_j	2	20	-10	0	0	0	0	
C_B	x_B	x_1	x_2	x_3	S_1	S'	S''	S'''	b
20	x_2	0	1	0	0	1	0	0	0
2	x_1	1	0	0	0	-4	0	2	2
0	S_1	0	0	0	1	-16	0	8	8
-10	x_3	0	0	1	0	1	0	-3	2
0	S''	0	0	0	0	0	1	-2	1
	$Z_j - c_j$	0	0	0	0	2	0	34	-16

The optimum feasible integer Solution is $\text{Max } Z = -16$ at $x_1 = 2$; $x_2 = 0$; $x_3 = 2$

$\hat{R}_3 = R_3 - 4\hat{R}_5 \qquad \hat{R}_5 = -2R_5$

$\hat{R}_2 = R_2 - \hat{R}_5 \qquad \hat{R}_4 = R_4 + \frac{3}{2}\hat{R}_5$

$\hat{R}_1 = R_1$

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Strength of the Fractional Cut

Consider the two inequalities

$$\sum_{j=1}^{n} f_{ij} w_j \geq f_i \rightarrow (1)$$

and

$$\sum_{j=1}^{n} f_{kj} w_j \geq f_k \rightarrow (2)$$

Cut (1) is said to be stronger than (2) if $f_i \geq f_k$ and $f_{ij} \leq f_{kj}$ for all j, with the strict inequality holding atleast once.

This definition of strength is difficult to implement computationally. Thus empirical rules reflecting this definition are devised.

Two such rules call for generating the cut from the source row that has (1) $\max_i\{f_i\}$ or (2) $\max_i\{f_i / \sum_{j=1}^{n} f_{ij}\}$.

The 2nd is more effective, since it more closely represents the definition of strength [given in Taha (1975) pp. 184-185]

Example:

Let us Consider the example 3

W.K.T optimum Continuous Solution is

$$\text{Max } z = 63, \; x_1 = 9/2 \text{ and } x_2 = 7/2$$

Since Z is already integer. its equation Cannot be taken as a Source row.

According to the empirical rules given Since $f_1 = f_2 = 1/2$. the rule is non conclusive about Which Source row may be be better. But to apply the Second rule, it is necessary to develop all the Coefficients of the respective fractional Cuts from each row.

x_1 row : $\frac{21}{22} S_1 + \frac{3}{22} S_2 \geq \frac{1}{2}$

$$\frac{7}{22} S_1 + \frac{1}{22} S_2 \geq \frac{1}{2}$$

Since $\dfrac{\frac{1}{2}}{\frac{7}{22} + \frac{1}{22}} > \dfrac{\frac{1}{2}}{\frac{3}{22} + \frac{21}{22}} \Rightarrow \frac{11}{8} > \frac{11}{24}$

x_2 equation is Selected as a Source row.

The two Cuts from the x_1-row and x_2-row are Compar

The Cut from the x_2 - row expressed in terms of

x_1 and x_2 is given by $S_1 + x_2 = 3$

(ie) $x_2 \leq 3$

x_1 - row is expressed as ~~x_2~~

$$x_1 - S_1 + S_2 = 4$$

$$x_1 + x_2 - 3 + S_2 = 4$$

$$x_1 + x_2 + S_2 = 7 \Rightarrow x_1 + x_2 \leq 7 \Rightarrow x_2 \leq \frac{10}{3}$$

The First Cut is more restrictive and Stronger than Second Cut.

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The Mixed Algorithm

Let x_K be an integer Variable of the mixed Problem.

Pure integer Case, Consider x_K equation in The optimal Solution.

The equation x_K is given by

$$x_K = \beta_K - \sum_{j=1}^{n} \alpha_K^j w_j = [\beta_K] + f_K - \sum_{j=1}^{n} \alpha_K^j w_j$$

(Source row)

$$\text{Or} \quad x_K - [\beta_K] = f_K - \sum_{j=1}^{n} \alpha_K^j w_j$$

Because Som of the w_j Variables may not be restricted to integer Values.

For x_K to be integer either

$x_K \leq [\beta_K]$ or $x_K \geq [\beta_K]+1$ must be Satisfied.

From the Source row

$$\sum_{j=1}^{n} \alpha_K^j w_j \geq f_K \rightarrow (1)$$

$$\sum_{j=1}^{n} \alpha_K^j w_j \leq f_K - 1 \rightarrow (2)$$

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Let J^+ = Set of Subscripts j for which $\alpha_K^j \geq 0$

J^- = Set of Subscripts j for which $\alpha_K^j < 0$

From (1) & (2) We get

$$\sum_{j \in J^+} \alpha_K^j w_j \geq f_K \rightarrow (3)$$

$$\frac{f_K}{f_K - 1} \sum_{j \in J^-} \alpha_K^j w_j \geq f_K \rightarrow (4)$$

The equation (3) & (4) Can be Combined into one Constraint.

of the form

$$S_K - \left\{ \sum_{j \in J^+} \alpha_K^j w_j + \frac{f_K}{f_K - 1} \sum_{j \in J^-} \alpha_K^j w_j \right\} = -f_K \text{ (mixed Cut)}$$

Where $S_K \geq 0$ is a nonnegative Slack Variable.

The last equation is required mixed Cut and it represent a necessary Condition for x_K to be integer. Since all $w_j = 0$ at the Current optimal table it follows that the cut is infeasible.

The Dual Simplex method is used and Clear the infeasibility.

The Stronger Cut is

$$S_K = -f_K + \sum_{j=1}^{n} \lambda_j w_j$$

Where

$$\lambda_j = \begin{cases} \alpha_K^j & \text{if } \alpha_K^j \geq 0 \text{ and } w_j \text{ is nonintegral} \\ \dfrac{f_K}{f_K - 1}\alpha_K^j & \text{if } \alpha_K^j < 0 \text{ '' '' ''} \\ f_{Kj} & \text{if } f_{Kj} \leq f_K \text{ and } w_j \text{ is integral} \\ \dfrac{f_K}{1 - f_K}(1 - f_{Kj}) & \text{if } f_{Kj} > f_K \text{ and } w_j \text{ is integral} \end{cases}$$

Branch and Bound Method :

In this method the Problem is first Solved as a Continuous L.P Problem ignoring the integrality Condition. If in the optimal Solution some Variable Say x_j is not an integer, then

$$x_j^* < x_j < x_j^* + 1$$

Where x_j^* and $x_j^* + 1$ are Consecutive non-negative integer. It follows that any feasible integer Value of x_j must Satisfy one of the two Conditions, namely

$$x_j \leq x_j^* \quad \text{or} \quad x_j \geq x_j^* + 1$$

Since Variable has no integer Value between x_j^* and $x_j^* + 1$. These two Conditions are mutually exclusive and When applied Separately to the Continuous L.P Problem, form two different Sub Problems. Thus the orignal Problem is 'branched' or 'Partitioned' into two SubProblems. Geometrically it means that branching Process eliminate that Portion of feasible region that Contains no feasible integer Solution.

This branching Process yields two SubProblems, one by adding the Constraint $x_j \le x_j^*$ and other by adding the Constraint $x_j \ge x_j^* + 1$ to the original Set of Constraints. Each of these Sub Problem is Solved Separately as a linear Program, using the Same objective Function of the original Problem. If any SubProblem yields an optimal integer Solution, it is not further branched. However, if it yields a non-integer Solution it is further branched into two SubProblems. This branching Process is Continued, until each Problem terminates with either integral Valued optimal Solution or there is evidence that it Cannot yield a better One.

Whenever a better integer solution is found for any
SubProblem, it replaces the one Previously found. The
integer Valued Solution, among all the SubProblems that
gives the most optimal Value of the objective function
is Selected as the optimum Solution.

Main drawback of this algorithm is that it
requires the optimum Solution of each SubProblem and
in large Problems it Could be Very time-Consuming.
However, the Computational efficiency of this algorithm
is increased by applying the Concept of 'bounding'.
According to this Concept, Whenever the Continuous
optimum Solution of a SubProblem yields a Value of the
objective function lower than that of the best available
integer Solution (Maximization Case) it is Useless to
explore the Problem any further. This SubProblem is
fathomed and is dropped from further Consideration.

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fathomed and is ...

Thus one a feasible integer solution is obtained, its associated objective function can be used as a lower bound (Maximization case) to delete inferior subproblems. Hence efficiency of a branch and bound algorithm depends upon how soon the successive subproblems are fathomed.

If the objective function is to be minimized, the procedure remains the same except that upper bounds are used. Thus the value of the first integral solution becomes an upper bound for the problem and the programs are eliminated when their objective function values are greater than the current upper bound.

This algorithm can be extended directly to the mixed integer problems.

Example :

(10) Solve by branch and bound Method.

$$Max\ Z = 2x_1 + 3x_2$$

Subject to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10.$

Solution :

Ignoring the integrality restriction, the Prob can be expressed in Standard form

$$Max\ Z = 2x_1 + 3x_2 + 0S_1 + 0S_2$$

$\Rightarrow 6x_1 + 5x_2 + 0S_1 = 25$

$x_1 + 3x_2 + 0S_2 = 10.$

The initial basic feasible Solu. is $[S_1, S_2] = [25, 10]$

	c_j	2	3	0	0		
c_B	x_B	x_1	x_2	S_1	S_2	b	θ
0	S_1	6	5	1	0	25	5
0	S_2	1	3	0	1	10	10/3
	$Z_j - c_j$	-2	-3	0	0		

$\rightarrow S_2$ is a L.V

$\uparrow x_2$ is an E.V

0	S_1	$13/3$	0	1	$-5/3$	$\frac{25}{3}$	$25/13$	$\rightarrow S_1$ is a L.V
3	x_2	$1/3$	1	0	$1/3$	$10/3$	10	
	$Z_j - C_j$	-1	0	0	1	10		$\hat{R}_2 = R_2/3$; $\hat{R}_1 = R_1 - 5\hat{R}_2$

$\uparrow x_1$ is a E.V

2	x_1	1	0	$3/13$	$-5/13$	$25/13$	
3	x_2	0	1	$-1/13$	$6/13$	$85/13$	
	$Z_j - C_j$	0	0	$3/13$	$8/13$	$155/13$	$\hat{R}_1 = \frac{3}{13} R_1$; $\hat{R}_2 = R_2 - \frac{1}{3}\hat{R}_1$

All $Z_j - C_j \geq 0$.

$\therefore$ The optimal feasible non integer solution is

$$\boxed{\text{Max } z = \frac{\cancel{265}\ 155}{13} \text{ at } x_1 = \frac{25}{13};\ x_2 = \frac{35}{13}}$$

The upperbound of the value of z for the integer problem is 12. Since the solution is non-integer with both x_1 and x_2 having fractional values, any variable may be arbitrarily selected for branching. If x_2 is selected then

$$x_2 = \frac{35}{13} \text{ gives } 2 < x_2 < 3.$$

For an integer solution $x_2 \leq 2$; $x_3 \geq 3$

Add a new constraint either $x_2 \leq 2$ or $x_2 \geq 3$ to the original L.P problem yielding two sub problems.

<u>SubProblem I</u> :

Max $z = 2x_1 + 3x_2$

Subject to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

x_1, x_2 non-negative integers.

		2	3	0	0	0		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	6	5	1	0	0	25	5
0	S_2	1	3	0	1	0	10	10/3
0	S_3	0	1	0	0	1	2	2 $\rightarrow S_3$ is a L.V
	$Z_j - C_j$	-2	-3	0	0	0		

$\uparrow x_2$ is a E.V

0	S_1	6	0	1	0	-5	15	15/6 $\rightarrow S_1$ is a L.V.
0	S_2	1	0	0	1	-3	4	4
3	x_2	0	1	0	0	1	2	–
	$Z_j - C_j$	-2	0	0	0	3		

$\hat{R}_3 = R_3$

$\hat{R}_2 = R_2 - 3\hat{R}_3$

$\hat{R}_1 = R_1 - 5\hat{R}_3$

$\uparrow x_1$ is a E.V

		2	3	0	0	0		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	b	
2	x_1	1	0	$1/6$	0	$-5/6$	$5/2$	
0	S_2	0	0	$-1/6$	1	$-13/6$	$3/2$	$\hat{R}_1 = R_1/6$ $\hat{R}_2 = R_2 - \hat{R}_1$ $\hat{R}_3 = R_3$
3	x_2	0	1	0	0	1	$\cancel{4}$ 2	
	$Z_j - C_j$	0	0	$2/6$	0	$8/6$	11	

All $Z_j - C_j \geq 0$

$\therefore$ The optimal non-integer Solution is

$$\boxed{\text{Max } Z = 11 \text{ at } x_1 = 2.5 \; ; \; x_2 = 2}$$

Sub Problem II :

$$\text{Max } Z = 2x_1 + 3x_2$$

Subject to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \geq 3$ $\quad x_1, x_2$ non negative integer.

x_2 ...

C_B	x_B	$\overset{2}{x_1}$	$\overset{3}{x_2}$	$\overset{0}{s_1}$	$\overset{0}{S_2}$	$\overset{0}{s_3}$	$\overset{-M}{A}$	b	θ
0	s_1	6	5	1	0	0	0	25	5
0	S_2	1	3	0	1	0	0	10	10/3
−M	A	0	1	0	0	−1	1	3	3 → A is a L.V
	$Z_j - C_j$	−2	−M−3	0	0	M	0		

↑ x_2 is an E.V

0	s_1	6	0	1	0	5	−5	10	2
0	s_2	1	0	0	1	3	−3	1	1/3 → S_2 is a L.V
3	x_2	0	1	0	0	−1	1	3	−
	$Z_j - C_j$	−2	0	0	0	−3			

$\hat{R}_3 = R_3$

$\hat{R}_2 = R_2 - 3\hat{R}_3$

$\hat{R}_1 = R_1 - 5\hat{R}_3$

↑ s_3 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	S_3	A	b	θ
0	S_1	$13/3$	0	1	$-5/3$	0	0	$25/3$	$25/13$
0	S_3	$1/3$	0	0	$1/3$	1	-1	$1/3$	1 → S_3 is a L.V
3	x_2	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	0	$10/3$	10 $\hat{R_2} = R_2/3$; $\hat{R_1} = R_1 - 5\hat{R_2}$; $\hat{R_3} = R_3 + \hat{R_2}$
	$z_j - c_j$	-1	0	0	1	0	0		

↑ x_1 is an E.V

C_B	x_B	2 x_1	3 x_2	0 S_1	0 S_2	0 S_3	A		
0	S_1	0	0	1	-6	-13	13	4	
2	x_1	1	0	0	1	3	-3	1	$\hat{R_2} = 3R_2$
3	x_2	0	1	0	0	-1	1	3	$\hat{R_1} = R_1 - \frac{13}{3}\hat{R_2}$; $\hat{R_3} = R_3 - \frac{1}{3}\hat{R_2}$
	$z_j - c_j$	0	0	0	2	3	-	11	

All $z_j - c_j \geq 0$.

The optimal Solution is $\boxed{x_1 = 1 \; ; \; x_2 = 3 \; ; \; \text{Max } z = 11}$

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The optimal solution is

The SubProblem 1 is a non-integer Solution.

Since $x_1 = 2.5 \ni 2 \leq x_1 \leq 3$.

Adding this new Constraint $x_1 \leq 2$ & $x_1 \geq 3$ to the SubProblem 3 and 4.

<u>SubProblem 3 :</u>

$$\text{Max } Z = 2x_1 + 3x_2$$

Subject to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

$x_1 \leq 2$. x_1, x_2 non-negative integer.

		2	3	0	0	0	0		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	S_4	b	θ
0	S_1	6	5	1	0	0	0	25	5
0	S_2	1	3	0	1	0	0	10	10/3
0	S_3	0	1	0	0	1	0	2	2/1 $\rightarrow S_3$ is
0	S_4	1	0	0	0	0	1	2	—
	$Z_j - C_j$	−2	−3	0	0	0	0		

↑ x_2 is a E.V

0	S_1	6	0	1	0	−5	0	15	5/2
0	S_2	1	0	0	1	−3	0	4	4
3	x_2	0	1	0	0	1	0	2	-
0	S_4	1	0	0	0	0	1	2	2
	$Z_j - C_j$	−2	0	0	0	3	0		

$\hat{R}_3 = R_3 \rightarrow S_4$ ...

$\hat{R}_4 = R_4$

$\hat{R}_2 = R_2 - 3\hat{R}_3$

$\hat{R}_1 = R_1 - 5\hat{R}_3$

↑ x_1 is an E.V

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0	s_1	0	0	1	0	-5	-6	3	
0	s_2	0	0	0	1	-3	-1	2	
8	x_2	0	1	0	0	1	0	2	$\hat{R}_4 = R_4$ $\hat{R}_3 = R_3$
2	x_1	1	0	0	0	0	1	2	$\hat{R}_2 = R_2 - \hat{R}_4$ $\hat{R}_1 = R_1 - 6\hat{R}_4$
	$z_j - c_j$	0	0	0	0	3	2	10	

All $z_j - c_j \geq 0$.

The optimal solution is $\boxed{x_1 = 2 \ ; \ x_1 = 2 \quad Max = 10}$

Subproblem 1 :

Max $Z = 2x_1 + 3x_2$

Subject to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

$x_1 \geq 3$.

C_B	x_B	2 x_1	3 x_2	0 S_1	0 S_2	0 S_3	0 S_4	-M A	b	θ
0	S_1	6	5	1	0	0	0	0	25	25/6
0	S_2	1	3	0	1	0	0	0	10	10
0	S_3	0	1	0	0	1	0	0	2	-
-M	A	1	0	0	0	0	-1	1	3	3 → A is a L.V
	$Z_j - C_j$	-M-2	-3	0	0	0	M	0		
		↑ x_1 is a E.V								

0	s_1	0	5	1	0	0	6	–	7	$7/5 \rightarrow s_1$ is a L.V
0	S_2	0	3	0	1	0	1	–	7	$7/3$
0	S_3	0	1	0	0	1	0	–	2	2
2	x_1	1	0	0	0	0	-1	–	3	–
	$Z_j - C_j$	0	-3	0	0	0	-2	–		

$\hat{R}_4 = R_4$
$\hat{R}_3 = R_3$
$\hat{R}_2 = R_2 - \hat{R}_4$
$\hat{R}_1 = R_1 - 6\hat{R}_4$

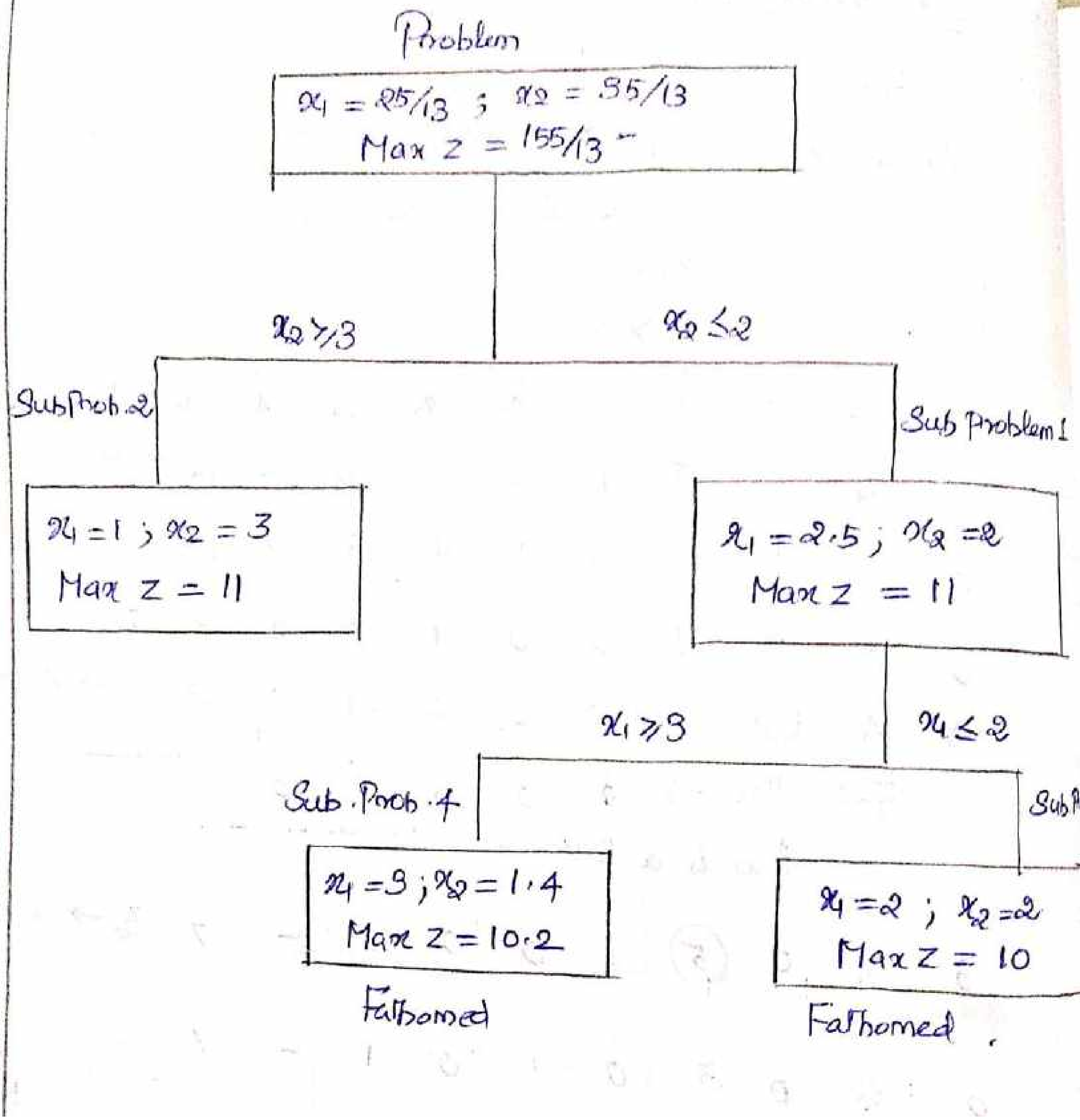
↑ x_2 is an E.V

3	x_2	0	1	$1/5$	0	0	$6/5$	–	$7/5$
0	S_2	0	0	$-3/5$	1	0	$-13/5$	–	$14/5$
0	S_3	0	0	$-1/5$	0	1	$-6/5$	–	$3/5$
2	x_1	1	0	0	0	0	-1	–	3
	$Z_j - C_j$	0	0	$3/5$	0	0	$8/5$		

$\hat{R}_1 = \frac{1}{5} R_1$
$\hat{R}_2 = R_2 - 3\hat{R}_1$
$\hat{R}_3 = R_3 - \hat{R}_1$
$\hat{R}_4 = R_4$

All $Z_j - C_j \geq 0$.

∴ The optimal solution is $\boxed{x_1 = 3;\ x_2 = 7.5 \quad \text{Max } Z = 10.2}$


Problem
$x_1 = 25/13$; $x_2 = 35/13$
Max Z = 155/13
$x_2 \geq 3$
$x_2 \leq 2$
Sub Prob. 2
Sub Problem 1
$x_1 = 1$; $x_2 = 3$
Max Z = 11
$x_1 = 2.5$; $x_2 = 2$
Max Z = 11
$x_1 \geq 3$
$x_1 \leq 2$
Sub. Prob. 4
$x_1 = 3$; $x_2 = 1.4$
Max Z = 10.2
Fathomed
$x_1 = 2$; $x_2 = 2$
Max Z = 10
Fathomed.

SubProblem 4 Can be further branched with x_2 as the branching Variable. But Value of its objective function is inferior to the lower bound and hence This does not Promise a Solution better than the one already obtained.
∴ The Subproblem is also fathomed. Now there is no Subproblem which Can be further branched and the best available Solution Corresponding to Subproblem 2 is the optimal Solution of the Problem.

Hence the optimal non-integer Solution is

$$\boxed{\text{Max } z = 11 \text{ at } x_1 = 1 ; x_2 = 3}$$

1) Solve the following mixed integer Problem by branch and bound technique.

$Max\ Z = x_1 + x_2$

Subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0$ and x_1 is integer.

Solution:

Ignoring the integrality restriction, the given Problem can be expressed in Standard form

$Max\ Z = x_1 + x_2 + 0S_1 + 0S_2$

Subject to $2x_1 + 5x_2 + 0S_1 = 16$

$6x_1 + 5x_2 + 0S_2 = 30$

C_B	C_j / x_B	1 / x_1	1 / x_2	0 / S_1	0 / S_2	b	θ
0	S_1	2	5	1	0	16	8
0	S_2	6	5	0	1	30	5 → S_2 is a L.V
	$Z_j - C_j$	−1	−1	0	0		

↑ x_1 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	b	θ
0	S_1	0	$10/3$	1	$-1/3$	6	$9/5$
1	x_1	1	$5/6$	0	$1/6$	5	6
	Z_j-C_j	0	$-1/6$	0	$1/6$		

$\to S_1$ is a L.V

$\hat{R}_2 = R_2/6$

$\hat{R}_1 = R_1 - 2\hat{R}_2$

$\uparrow x_2$ is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	b
1	x_2	0	1	$3/10$	$-1/10$	$9/5$
1	x_1	1	0	$-1/4$	$1/4$	$7/2$
	Z_j-C_j	0	0	$1/20$	$3/20$	$53/10$

$\hat{R}_1 = \frac{3}{10} R_1$

$\hat{R}_2 = R_2 - 5/6\,\hat{R}_1$

All $Z_j - C_j \geq 0$

The optimal non-integer solution is

$$\boxed{\text{Max } Z = 53/10 \text{ at } x_1 = \frac{7}{2};\ x_2 = \frac{9}{5}}$$

Since x_1 is integer constrained. The problem is branched into two subproblems.

$$x_1 = \frac{7}{2} = 3.5 \quad 3 \le x_1 \le 4$$

(i.e) $x_1 \le 3$; $x_1 \geq 4$

Subproblem 1 :

Max $Z = x_1 + x_2$

Subject to $2x_1 + 5x_2 \le 16$; $6x_1 + 5x_2 \le 30$; $x_1 \le 3$

C_B	x_B	x_1 (1)	x_2 (1)	S_1 (0)	S_2 (0)	S_3 (0)	b	θ
0	S_1	2	5	1	0	0	16	8
0	S_2	6	5	0	1	0	30	5
0	S_3	1	0	0	0	1	3	3 → S_3 is a L.V
	$Z_j - C_j$	−1	−1	0	0	0		

↑ x_1 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	S_3	b	
0	S_1	0	5	1	0	−2	10	2 → S_1 is a L.V
0	S_2	0	5	0	1	−6	12	12/5
1	x_1	1	0	0	0	1	3	–
	$Z_j - C_j$	0	−1	0	0	1		

$\hat{R}_3 = R_3$

$\hat{R}_2 = R_2 - 6\hat{R}_3$

$\hat{R}_1 = R_1 - 2\hat{R}_3$

↑ x_2 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	S_3	b
1	x_2	0	1	$1/5$	0	$-2/5$	2
0	S_2	0	0	-1	1	-4	2
1	x_1	1	0	0	0	1	3
	$Z_j - C_j$	0	0	$1/5$	0	$3/5$	5

$\hat{R}_1 = R_1/5$

$\hat{R}_2 = R_2 - 5\hat{R}_1$

$\hat{R}_3 = \hat{R}_3 \; 6\hat{R}_1$

All $Z_j - C_j \geq 0$ $\therefore$ The Solution is Max $Z = 5$ at $x_1 = 3$; $x_2 = 2$

Sub Problem 2 :

Max $z = x_1 + x_2$

Subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

		1	1	0	0	0	$-M$		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	A	b	θ
0	S_1	2	5	1	0	0	0	16	8
0	S_2	6	5	0	1	0	0	30	5
$-M$	A	1	0	0	0	-1	1	4	4 $\rightarrow$ A is a L.V
	$Z_j - C_j$	$-M-1$	-1	0	0	M	0		

↑ x_1 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	S_3	A	b	
0	S_1	0	5	1	0	2	-2	8	$8/5$
0	S_2	0	5	0	1	6	-6	6	$6/5 \rightarrow$ S_2 is a L.V
1	x_1	1	0	0	0	-1	1	4	4
	$Z_j - C_j$	0	-1	0	0	-1	$-$		

$\hat{R}_3 = R_3$

$\hat{R}_2 = R_2 - 6\hat{R}_3$

$\hat{R}_1 = R_1 - 2\hat{R}_3$

↑ x_2 is an E.V

C_B	x_B	x_1	x_2	S_1	S_2	S_3	A	b
0	S_1	0	0	1	-1	-4	4	2
1	x_2	0	1	0	$1/5$	$6/5$	$-6/5$	$6/5$
1	x_1	1	0	0	0	-1	1	4
	$Z_j - C_j$	0	0	0	$1/5$	$1/5$	$-$	$26/5$

$\hat{R}_2 = R_2/5$

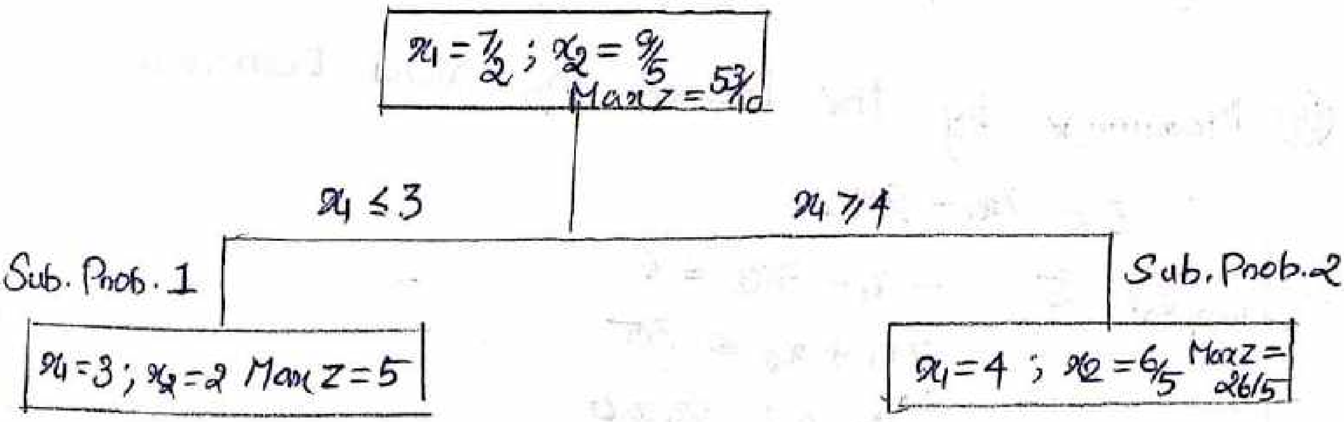
$\hat{R}_3 = R_3$

$\hat{R}_1 = R_1 - 5\hat{R}_2$

All $z_j - c_j \geq 0$ The solution is $x_1 = 4$; $x_2 = 6/5$ $\text{Max } z = \frac{26}{5}$

This Solution also Satisfies the Condition of x_1 being non-negative integer and the Value of $z = 5.2$ is better than the lower bound. Therefore This branch is also fathomed.

Hence the optimal Solution to the given Problem is



$$\boxed{\text{Max } z = 5.2 \text{ at } x_1 = 4 ;\ x_2 = 6/5}$$

Solve the following Problem by branch and bound techniq

$$\text{Max } Z = x_1 + x_2$$

Subject to

$$3x_1 + 2x_2 \leq 12$$

$$x_2 \leq 2$$

$x_1, x_2 \geq 0$ and are integers.

<u>Ans:</u> Max $Z = 4$ at $x_1 = 2$; $x_2 = 2$

Solve by branch and bound Method.

$$\text{Max } Z = 3x_1 + x_2 + 3x_3$$

Subject to $-x_1 + 2x_2 + x_3 \leq 4$

$$4x_2 - \underline{3}x_2 \leq 2$$

$$x_1 - 3x_2 + 2x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

<u>Ans:</u> Max $Z = 23$ at $x_1 = 5$; $x_2 = 2$; $x_3 = 2$

Use I.P branch and bound algorithm to solve the Problem

$$\text{Max } z = 5x_1 + 4x_2$$

Subject to $x_1 + x_2 \leq 5$

$10x_1 + 6x_2 \leq 45$

x_1, x_2 non negative integers.

Use x_1 as branching Variable.

Ans: Max $z = 23$ at $x_1 = 3$; $x_2 = 2$

2) Maximize by the branch and bound . technique.

$$z = 7x_1 + 9x_2$$

Subject to $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

$x_2 \leq 7 \quad x_1 \geq 0$

x_1, x_2 are integers.

Ans: Max $z = 55$ at $x_1 = 4$; $x_2 = 3$.

(16) Solve the Problem by Gomory's Algorithm

$$\text{Max } z = 3x_1 + 4x_2$$

Subject to $x_1 + x_2 \leq 4$

$\frac{3}{5}x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$ and integer.

Ans: Max $z = 13$ at $x_1 = 3$; $x_2 = 1$.

(17) Solve by Cutting Plane Algorithm

$$\text{Max } z = 7x_1 + 10x_2$$

Subject to $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

$x_1, x_2 \geq 0$ and integer.

Ans: Max $z = 58$ at $x_1 = 4$; $x_2 = 3$

(18) Use Gomory's Cutting Plane algorithm

$$\text{Max } z = 2x_1 + x_2$$

Subject to $2x_1 + 5x_2 \leq 17$

$3x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$ and integer.

Ans: Max $z = 6$ at $x_1 = 3$; $x_2 = 0$

Ans. Max z

Solve the following mixed integer Program by Cutting Plane Method.

$$\text{Max } z = x_1 + x_2$$

Subject to $3x_1 + 2x_2 \le 5$

$x_2 \le 2$

$x_1, x_2 \geq 0$ x_1 integer.

<u>Ans:</u> $x_1 = 0$; $x_2 = 2$; $\text{Max } z = 2$

Solve the I.P.P

$$\text{Max } z = 2x_1 + 2x_2$$

Subject to $5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \geq 0$ and integer.

<u>Ans:</u> $\text{Max } z = 4$ at $x_1 = 1$; $x_2 = 1$.

Unit II

Dynamic Programming : Dynamic Programming (D.P) is a mathematical technique which deals with the optimization of Multistage decision Problem. A multistage decision Problem can be separated into a number of sequential steps and stages, which may be accomplished in one or more ways. The solution of each stage is a decision and the sequence of decisions for all the stages constitutes a decision Policy. With each decision is associated some return in the form of costs or benefits. The objective in dynamic Programming is to select a decision Policy.(ie) a sequence of decisions so as to optimize the returns.

Stage : A stage signifies a portion of the total Problem for which a decision can be taken. At each stage there are a number of alternatives, and the best out of those is called the stage decision, which may not be optimal for the stage, but contributes to obtain the optimal decision Policy.

State: The Condition of the decision Process at a Stage is Called its State. The Variables Which Specify the Condition of the decision Process. (ie) describe the Status of the System at a Particular Stage are Called State Variables. The number of state Variables should be as small as Possible, Since larger the number of State Variables, more Complicated is the decision Process.

Principle of optimality: Bellman's Principle of optimality States "An optimal Policy (a Sequence of decisions) has the Property that whatever the initial State and decisions are the remaining decisions must Constitute an optimal Policy with regard to the state resulting from the first decision.

The analysis of dynamic Programming Problems can be summarized as follows:

Step 1 : Define the Problem Variables, determine the objective function and specify the constraints.

Step 2 : Define the stages of the Problem. Determine the state variables whose values constitute the state at each stage and the decision required at each stage. Specify the relationship by which the state at one stage can be expressed as a function of the state and decisions at the next stage.

Step 3 : Develop the recursive relationship for the optimal return function which Permits Computation of the optimal Policy at any stage. Decide whether to follow the forward or the backward method to solve the Problem. Specify the optimal return function at stage 1, since it is generally a bit ~~function~~ different from the general optimal return function for the other stages.

Step 4 : Make a tabular representation to show the required values and calculations for each stage.

Step 5 : Find the optimal decision at each stage and then the overall optimal Policy. There may be more than one such optimal Policy.

DP Model :

In DP, Computations are Carried out in Stages by breaking down the Problem into SubProblems. Each SubProblem is then Considered Separately With The objective of Reducing the Volume of Computations. However, Since the Subproblems are interdependent, a Procedure must be devised to link the Computations in manner that guarantees that a feasible Solution for each Stage is also feasible for the entire Problem.

A Stage in DP is defined as the Portion of the Problem that Possesses a Set of mutually exclusive alternatives from which the best alternative is to be selected.

In terms of the Capital budgeting example, each Plant defines a Stage With 1^{st}, 2^{nd} and 3^{rd} stages having Three, four and two alternatives respectively. These Stages are interdependent because all three Plants must Compete for a limited budget.

Example 1: A manufacturing Company has three Sections
Producing automobile Parts, bicycle Parts and Sewing
machine Parts respectively. The management has allocated
Rs. 20,000 for expanding the Production facilities. In the
auto Parts and bicycle Parts Sections, the Production Can
be increased either by adding new machines or by replacing
Some old inefficient machines by automatic machines.
The Sewing machine Parts Section Was started only a few
years back and thus the additional amount Can be invested
only by adding new machines to the Section. The Cost of
adding and replacing the machines along with the associated
expected returns in the different Sections is given in
table. Select a Set of expansion Plans Which may yield
the maximum return.

Alternatives	Auto Parts Section		Bicycle Parts Section		Sewing machine Section	
	Cost Rs.	Return Rs.	Cost Rs.	Return Rs.	Cost Rs.	Return Rs.
1. No expansion	0	0	0	0	0	0
2. Add new machine	4,000	8,000	8,000	12,000	2,000	8,000
3. Replace old mach.	6,000	10,000	12,000	18,000	—	—

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Solution: Here each Section of the Company is stage.
At each stage there are a no. of alternatives for expansion
Capital represents the state variable. Let us consider the
First stage - the auto Parts Section. There are 3 alternative
no expansion, add new machines and replace old machines.
The amount that may be allocated to stage 1 vary from
0 to Rs.20,000 it will be overspending if it is more than
Rs.6000. The returns of various alternatives is given in table

Stage 1 : Auto Parts Section

State x_1 (000 of Rs)	Evaluation of alternatives (Values in thousand of Rupees) 1: Cost $C_{11} = 0$ Return	2: $C_{12} = 4$ Return	3: $C_{13} = 6$ Return	Optimal Solution: Optimal Return	Deci
0	0	–	–	0	1
2	0	–	–	0	1
4	0	8	–	8	2
6	0	8	10	10	3
8	0	8	10	10	3
10	0	8	10	10	3
12	0	8	10	10	3
14	0	8	10	10	3
16	0	8	10	10	3
18	0	8	10	10	3
20	0	8	10	10	3

When the Capital allocated is Zero or 2000 only 1st alternative is Possible. Return is of Course Zero. When the amount is allocated is Rs.4,000 alternative 1 and 2 are Possible with returns of Rs.0 and Rs.8000 So we Select alternative 2. and When the amount is allocated is Rs.6,000 all the three alternatives are Possible, giving returns of Zero, Rs.8,000 and Rs.10,000 respectively. So we Select alternative 3 with return of Rs. 10,000 and so on.

Stage 2: Let us now move to Stage 2. Here again 3 alternatives are available.

Bicycle Parts Section (+ Auto Parts Section)

Stage x_2	Evaluation of alternatives (Values in Thousands of Rupees) 1	2	3	Optimal Solution	
	Cost $C_{21} = 0$ Return	Cost $C_{22} = 8$ Return	Cost $C_{23} = 12$ Return	Optimal return	Decision
0	0+0 = 0	–	–	0	1
2	0+0 = 0	–	–	0	1
4	0+8 = 8	–	–	8	1
6	0+10 = 10	–	–	10	1
8	0+10 = 10	12+0 = 12	–	12	2
10	0+10 = 10	12+0 = 12	–	12	2
12	0+10 = 10	12+8 = 20	18+0 = 18	20	2
14	0+10 = 0	12+10 = 22	18+0 = 18	22	2
16	0+10 = 0	12+10 = 22	18+8 = 26	26	3
18	0+10 = 0	12+10 = 22	18+10 = 28	28	3
20	0+10 = 0	12+10 = 22	18+10 = 28	28	3

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Here State x_2 represents the total amount allocated to the Current Stage (Stage 2) and the Preceding stage. Similarly, the return also is the Sum of the current Stage and the Preceding Stage. When $x_2 < 8000$ only the 1st alternative is Possible. When $x_2 =$ Rs. 8000 a return of Rs. 12,000 is Possible by Selecting 2nd alternative. When $x_2 = 12,000$ 3 alternatives are Possible. With maximum return of Rs. 20,000 from alternative 2. The optimal Policy Consists of a Set of two decisions adopt alternative 2 at 2nd Stage and again alternative 2 at the 1st stage.

Stage 3 : Sewing machine Parts Section (+ Bicycle Parts Section + Auto Parts Section)

State x_3	Evaluation of alternatives. Cost $C_{31} = 0$ Return.	Cost $C_{32} = 2$ Return.	Optimal Solution Opt. Return	Decision
0	0 + 0 = 0	—	0	1
2	0 + 0 = 0	8 + 0 = 8	8	2
4	0 + 8 = 8	8 + 0 = 8	8	1, 2
6	0 + 10 = 10	8 + 8 = 16	16	2
8	0 + 12 = 12	8 + 10 = 18	18	2
10	0 + 12 = 12	8 + 12 = 20	20	2
12	0 + 20 = 20	8 + 12 = 20	20	1, 2
14	0 + 22 = 22	8 + 20 = 28	28	2
16	0 + 26 = 26	8 + 22 = 30	30	2
18	0 + 28 = 28	8 + 26 = 34	34	2
20	0 + 28 = 28	8 + 28 = 36	36	2

For $x_3 = 20,000$ The optimal decision for stage 3 is alternative 2, Which gives a total Return of Rs. 36,000 This involves Cost of RS. 2000 and leave Rs. 18,000 to be alloted for stages 2 & 1 Combined. from table 2 For allocation of Rs. 18,000 alternative 3 is choosen. Which Cost of 12,000 For Remaining Sum of Rs. 6,000 From table 1 decision alternative 3 is Choosen The optimal Policy of expanding Production facilities 3-3-2.

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(iv) Replace old machines with automatic in auto Parts Section, replace old machines with automatic in bicycle Parts Section, and add new machines to the Sewing machines Parts Section.

This Policy gives the optimal return of 36,000.

Ex2: A Corporation is entertaining Proposal from its 3 Plants for Possible expansion of facilities. The Corporation budgeting \$5 million for allocation to all three Plants. Each plant is requested to submit its Proposals giving total Cost (c) and total revenue (R) for each Proposal. The Cost and total revenue in million of dollars. The zero Cost Proposals are introduced to allow for Possibility of not allocating funds to individual Plants. The goal of the Corporation is to maximize the total revenue resulting from the allocation of the \$5 million to the 3 Plants.

Proposal	Plant 1		Plant 2		Plant 3	
	C_1	R_1	C_2	R_2	C_3	R_3
1	0	0	0	0	0	0
2	1	5	2	8	1	3
3	2	6	3	9	–	–
4	–	–	4	12	–	–

In the Capital budgeting example we define States for Stages 1, 2 and 3 as follows.

x_1 = amount of Capital allocated to stage 1

x_2 = amount of " " to stages 1 & 2

x_3 = " " " " " " 1, 2 and 3.

Now decompose the Capital budgeting Problem into three Computationally Separate SubProblems.
The Values of x_1 and x_2 are not known exactly but must lie Somewhere between 0 and 5.
In fact, because the Costs of the different Proposals are discrete, x_1 and x_2 may assume the Values 0, 1, 2, 3, 4 or 5. On the other hand x_3, which is the total Capital allocated to all three stages, is equal to 5.
We Solve the Problem is to Start with stage (Plant 1)

Stage 1 Plant 1

x_1	Evaluation of alternatives $C_{11}=0$	$C_{12}=1$	$C_{13}=2$	Optimum Return	Decision
0	0	-	-	0	1
1	0	5	-	5	2
2	0	5	6	6	3
3	0	5	6	6	3
4	0	5	6	6	3
5	0	5	6	6	3

Stage 2: Plant 2

x_2	Evaluation of alternatives $C_{21}=0$	$C_{22}=2$	$C_{23}=3$	$C_{24}=4$	Optimum Return	Decision
0	0+0=0	-	-	-	0	1
1	0+5=5	-	-	-	5	1
2	0+6=6	8+0=8	-	-	8	2
3	0+6=6	8+5=13	9+0=0	-	13	2
4	0+6=6	8+6=14	9+5=14	12+0=0	14	2 or 3
5	0+6=6	8+6=14	9+6=15	12+5=17	17	4

Stage 3 : Plant 3

x_3	Evaluation of alternatives		Optimum Return	Decision
	$c_{31} = 0$	$c_{32} = 1$		
0	$0+0=0$	-	0	1
1	$0+5=5$	$3+0=0$	5	1
2	$0+8=8$	$3+5=8$	8	1,2
3	$0+13=13$	$3+8=11$	13	1
4	$0+14=14$	$3+13=16$	16	2
5	$0+17=17$	$3+14=17$	17	1,2

For $x_3 = 5$ The optimal decision for stage 3 is alternative 1 or 2. Which gives the total return 17. This involves cost 0 and leave 5 to be alloted for stages 2 & 1 combined. From the table 2 for allocation stage 1 & 2, $x_2 = 5$ this yields cost 4. For stage I $x_1 = 1$ gives decision 2.

∴ The optimal combination of proposals for stages 1, 2 & 3 is (2, 4, 1).

The optimal policy gives the optimum returns of 17

Forward Recursive equation:

The Computations are Carried out in the order $f_1 \to f_2 \to f_3$. This method of Computations is known as the forward Procedure, because the Computations advance from the first to last stage.

Backward Recursive equation:

The Computations are Carried out in the order $f_3 \to f_2 \to f_1$. This method of Computations is known as the Backward Procedure because the Computations start at the last stage and then Proceed backward to stage 1.

Let us Consider the Capital budgeting example for the backward Procedure, we define the state y_i as

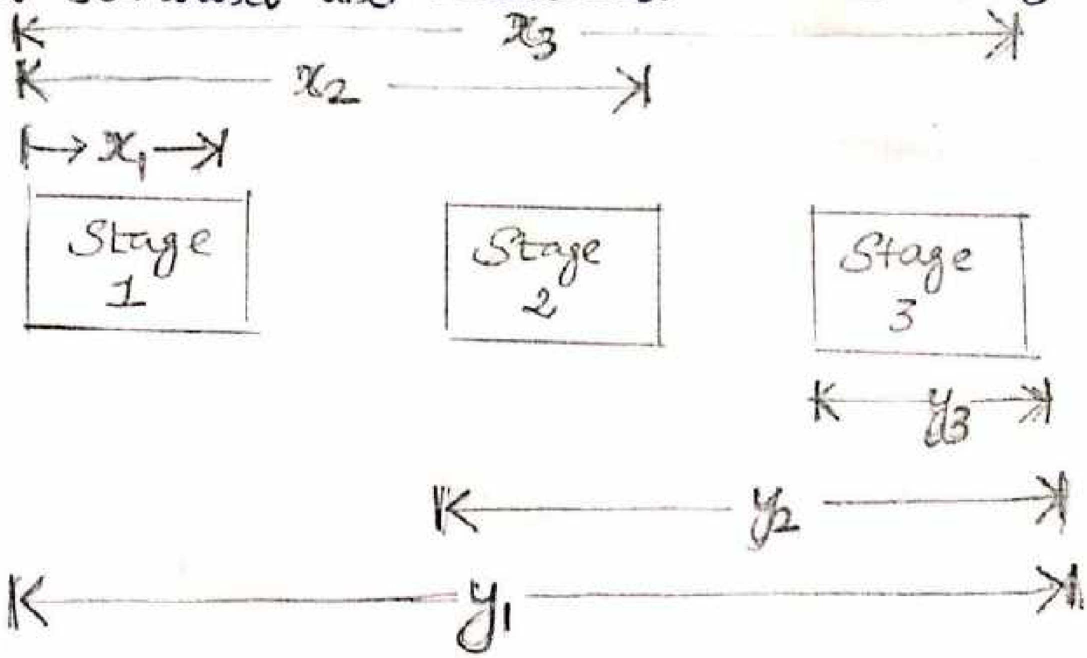
y_1 = amount of Capital allocated to stages 1, 2 and 3

y_2 = amount of Capital allocated to stages 2 and 3

y_3 = amount of Capital allocated to stage 3

y_3 = amount of capital allocated to stage 3

The difference between the definition of states x_j and y_j in the forward and backward methods are given in this f



Now define

$f_3(y_3)$ = Optimum revenue for stage 3 given y_3

$f_2(y_2)$ = optimum revenue for stages 2 & 3 given y_2

$f_1(y_1)$ = " " " " 1, 2 & 3 given y_1

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The Backward recursive equation is

$$f_j(y_j) = \max_{\substack{k_j \\ c_j(k_j) \le y_j}} \{ R_j(k_j) + f_{j+1}[y_j - c(k_j)] \} \quad j = 1, 2$$

$$f_3(y_3) = \max_{\substack{k_3 \\ c_3(k_3) \le y_3}} \{ R_3(k_3) \}$$

The Backward recursive equation is

$$f_1(x_1) = \max_{c_1(k_1) \le x_1} \{ R_1(k_1) \}$$

$$f_j(x_j) = \max_{c_j(k_j) \le x_j} \{ R_j(k_j) + f_{j-1}[x_j - c_j(k_j)] \} \quad \text{for } j = 2, 3.$$

Let us Consider the Ex: 2 to find the optimum Solution Through forward and Backward recursive manner.

Forward Recursive Equation :

Stage 1 : $f_1(x_1) = \max\limits_{\substack{C_1(K_1) \le x_1 \\ K_1 = 1,2,3}} \{R_1(K_1)\}$

	$R_1(K_1)$			Optimal Solution	
x_1	$K_1 = 1$	$K_1 = 2$	$K_1 = 3$	$f_1(x_1)$	K_1^*
0	0	–	–	0	1
1	0	5	–	5	2
2	0	5	6	6	3
3	0	5	6	6	3
4	0	5	6	6	3
5	0	5	6	6	3

Stage 2 :

$$f_2(x_2) = \max_{\substack{c_2(k_2) \leq x_2 \\ k_2 = 1,2,3,4}} \left\{ R_2(k_2) + f_1\left[x_2 - c_2(k_2)\right] \right\}$$

$R_2(k_2) + f_1\left[x_2 - c_2(k_2)\right]$ — Optimal Solution

x_2	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$	$f_2(x_2)$	k_2^*
0	0+0=0	–	–	–	0	1
1	0+5=5	–	–	–	5	1
2	0+6=6	8+0=8	–	–	8	2
3	0+6=6	8+5=13	9+0=9	–	13	2
4	0+6=6	8+6=14	9+5=14	12+0=12	14	2 or 3
5	0+6=6	8+6=14	9+6=15	12+5=17	17	4

Stage 3 : $f_3(x_3) = \underset{\substack{c_3(k_3) \le x_3 \\ k_3 = 1,2}}{\text{Max}} \left\{ R_3(k_3) + f_2[x_3 - c_3(k_3)] \right\}$

$R_3(k_3) + f_2[x_3 - c_3(k_3)]$

x_3	$k_3 = 1$	$k_3 = 2$	$f_3(x_3)$	k_3^*
5	$0+17=17$	$3+14=17$	17	1 or 2

For $x_3 = 5$ the optimal proposal is either $k_3^* = 1$ or $k_3^* = 2$.

Consider $k_3^* = 1$ first since $c_3(1) = 0$ this leaves

$x_2 = x_3 - c_3(1) = 5$ for stages 2 and 1.

Now stage 2 shows that $x_2 = 5$ yields $k_2^* = 4$

Since $c_2(4) = 4$ this leaves $x_1 = 5 - 4 = 1$

Now stage 1 $x_1 = 1$ gives $k_1^* = 2$

$\therefore$ The optimal combination of proposals for stages 1, 2 & 3 is $(2, 4, 1)$

Backward Recursive Equation:

Stage 3: $f_3(y_3) = \underset{\substack{C_3(K_3) \le y_3 \\ K_3 = 1,2}}{\text{Max}} \{R_3(K_3)\}$

y_3	$R_3(K_3)$ $K_3=1$	$K_3=2$	Optimum Solution $f_3(y_3)$	K_3^*
0	0	–	0	1
1	0	3	3	2
2	0	3	3	2
3	0	3	3	2
4	0	3	3	2
5	0	3	3	2

Stage 2: $f_2(y_2) = \underset{\substack{C_2(K_2) \le y_2 \\ K_2 = 1,2,3,4}}{\text{Max}} \{R_2(K_2) + f_3(y_2 - C_2(K_2))\}$

y_2	$R_2(K_2) + f_3(y_2 - C_2(K_2))$ $K_2=1$	$K_2=2$	$K_2=3$	$K_2=4$	Optimum Soln. $f_2(y_2)$	K_2^*
0	0+0=0	–	–	–	0	1
1	0+3=3	–	–	–	3	1
2	0+3=3	8+0=8	–	–	8	2
3	0+3=3	8+3=11	9+0=9	–	11	2
4	0+3=3	8+3=11	9+3=12	12+0=12	12	3 or 4
5	0+3=3	8+3=11	9+3=12	12+3=15	15	4

Stage I : $f_1(y_1) = \max\limits_{\substack{c_1(K_1) \le y_1 \\ K_1 = 1, 2, 3}} \left\{ R_1(K_1) + f_2(y_1 - c_1(K_1)) \right\}$

	$R_1(K_1) + f_2(y_1 - c_1(K_1))$			Optimum Solution	
y_1	$K_1 = 1$	$K_1 = 2$	$K_1 = 3$	$f_1(y_1)$	K_1^*
5	$0 + 15 = 15$	$5 + 12 = 17$	$6 + 11 = 17$	17	2 or 3

The optimum is obtained by Starting with y_1 and Stage I.

For $y_1 = 5$ The optimal Proposal is either 2 or 3.

Consider $k_1^* = 2$ first Since $c_1(2) = 1$

this leaves $y_2 = y_1 - c_1(2) = 5 - 1 = 4$ for stages 2 & 3

Now stage 2 Shows $y_2 = 4$ yields 3 or 4

Consider $k_2^* = 3$ Since $c_2(3) = 3$ This leaves

$y_3 = y_2 - c_2(3) = 4 - 3 = 1$

Now stage I $y_3 = 1$ yields $k_3^* = 2$ is not Possible.

Now stage 2 Consider $k_2^* = 4$ Since $c_2(4) = 4$

This leaves $y_3 = y_2 - c_2(4) = 4 - 4 = 0$.

Now Stage I $y_3 = 0$ yield $k_3^* = 1$

The optimal Combination of Proposal for Stages 1, 2 & 3 is [2, 4, 1].

1.

Cargo Loading Problem:

Consider the general Problem of N items first. If k_i is the number of units of item i, the Problem becomes.

$$\text{Max } v_1 k_1 + v_2 k_2 + \cdots + v_N k_N$$

Subject to

$$w_1 k_1 + \cdots + w_N k_N \le W$$

k_i non negative integer.

The DP model is Constructed by first Considering its three basic elements.

1. Stage j is represented by item j, $j = 1, 2 - - N$
2. State y_j at stage j is the total weight assigned to stages, $j, j+1 - - - N$; $y_1 = W$ and $y_j = 0, 1 - . W$ for $j = 2 - - N$
3. Alternative k_j at stage j is the no. of units of item j. The value of k_j may be as small as zero or large as $[W/w_j]$ where $[W/w_j]$ is the largest integer included in (W/w_j).

The Capital budgeting and Cargo loading model are of the resource allocation type. About the only difference is the the alternatives in the Cargo loading model are not given Directly as in the Capital budgeting model.

Let $f_j(y_j)$ = optimal value of stages $j, j+1, - - N$ given the state y_j then the backward recursive eqn. is

$$f_N(y_N) = \max_{\substack{k_N = 0,1 -- [y_N/w_n] \\ y_N = 0,1 -- W}} \{ v_n k_n \}$$

$$f_j(y_j) = \max_{\substack{k_j = 0,1 -- [y_j/w_j] \\ y_j = 0,1 -- W}} \left[v_j k_j + f_{j+1}(y_j - w_j k_j) \right] \quad j = 1, 2 - N-1.$$

The max. feasible value of k_j is given by $[y_j/w_j]$

Cargo Loading Problem 2:

Consider loading a vessel with stokes of n items. Each unit of item i has a weight w_i and value v_i $(i = 1, 2 \ldots N)$. The maximum cargo weight is W. It is required to determine the most valuable cargo load without exceeding the maximum weight of the vessel. Consider the following special case of three items and assume that $W = 5$

i	w_i	v_i
1	2	65
2	3	80
3	1	30

Stage 3:

$$f_3(y_3) = \underset{k_3}{Max}\left[30\,k_3\right] \quad max\ k_3 = \left[\frac{5}{1}\right] = 5$$

y_3	$k_3 = 0$ $V_3 k_3 = 0$	30 k_3 1 30	2 60	3 90	4 120	5 150	optimum Solution $f_3(y_3)$	k_3^*
0	0	-	-	-	-	-	0	0
1	0	30	-	-	-	-	30	1
2	0	30	60	-	-	-	60	2
3	0	30	60	90	-	-	90	3
4	0	30	60	90	120	-	120	4
5	0	30	60	90	120	150	150	5

Stage 2 :

$$f_2(y_2) = \underset{K_2}{\text{Max}} \{ 80K_2 + f_3(y_2 - 3K_2) \} \quad \max K_2 = [5/3] = 1$$

y_2	$80K_2 + f_3(y_2 - 3K_2)$ $K_2 = 0$ $V_2 K_2 = 0$	1 80	Optimum Solution $f_2(y_2)$	K_2^*
0	0+0 = 0	–	0	0
1	0+30 = 30	–	30	0
2	0+60 = 60	–	60	0
3	0+90 = 90	80+0 = 80	90	0
4	0+120 = 120	80+30 = 110	120	0
5	0+150 = 150	80+60 = 140	150	0

Stage 1 :

$$f_1(y_1) = \underset{K_1}{\text{Max}} \{ 65K_1 + f_2(y_1 - 2K_1) \} \quad \max K_1 = [5/2] = 2$$

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$$65k_1 + f_2(y_1 - 2k_1)$$

y_1	$k_1 = 0$ $v_1 k_1 = 0$	1 65	2 130	Optimum Soln $f_1(y_1)$	k_1^*
0	$0+0=0$	-	-	0	0
1	$0+30=30$	-	-	30	0
2	$0+60=60$	$65+0=65$	-	65	1
3	$0+90=90$	$65+30=95$	-	95	1
4	$0+120=120$	$65+60=125$	$130+0=130$	130	2
5	$0+150=150$	$65+90=155$	$130+30=160$	160	2

Given $y_1 = W = 5$ The associated optimum Solution is $(k_1^*, k_2^*, k_3^*) = (2, 0, 1)$ with a total Value 160.

The allocation is 2 units of item 1 and 1 unit of item 3

Total Load 5 kg, the maximum Value of Cargo item is Rs. 160. $(130 + 30) = (130 + 30) =$ 2 units in item 1 1 units in item

$y_1 = 5$ The optimal Proposal $\underline{k_1^* = 2}$

$y_2 = y_1 - w_2 = 5 - 3 = 2$

For $y_2 = 2$ The Optimal Proposal $\underline{k_2^* = 0}$

$y_3 = y_2 - w_1 = 2 - 1 = 1$

For $y_3 = 1$ The optimal Proposal $k_3^* = 1$

Reliability Problem :

The total reliability R of a device of N Series Components and K_j Parallel units in Component j $(j=1,2 \ldots N)$ is the Product of the individual reliabilities.

The Mathematical Formulation of the Problem is

$$\text{Max } R = \prod_{j=1}^{N} R_j(K_j)$$

Subject to

$$\sum_{j=1}^{N} c_j(K_j) \leq c$$

Where c is the total Capital available.

The reliability Problem is Similar to the Capital budgeting Problem With the expectation that the return function R is the Product, rather than the Sum of the returns of the Individual Components. The recursive equation based on multiplicative rather than additive decomposition.

F ..

The elements of the DP model are defined as follows.

1. Stage j represents main Component j

2. State y_j is the total Capital assined to components $j, j+1 -- N$.

3. Alternative k_j is the no. of 11el units assigned to main Component j.

Let $f_j(y_j)$ be the total optimal reliability of Components $j, j+1 -- N$ given the Capital y_j.

The recussive equations are

$$f_N(y_N) = \max_{\substack{k_N \\ C_N(k_N) \le y_N}} \{R_N(k_N)\}$$

$$f_j(y_j) = \max_{\substack{k_j \\ C_j(k_j) \le y_j}} \{R_j(k_j) \cdot f_{j+1}(y_j - C_j(k_j))\} \quad j=1,2 - N-1.$$

Example :

Consider the design of an electronic device Consisting of three main Components. The three Component are arranged in Series So that the failure of one Component will Cause the failure of the entire device. The reliability of the device Can be improved by installing Standby units in each Component. The designs Calls for using one or two. Standby units, Which means that each main Component may include upto 3 units in Parallel. The total Capital available for the design of the device is \$10,000 The data for the reliability $R_j(k_j)$ and the Cost $C_j(k_j)$ for the j^{th} Component $j=1,2,3$ given k_j Parallel units are given below.

The objective is to determine the no. of $\|^{el}$ units k_j, in Component j that will maximize the Reliability of the device Without exceeding the allocated Capital.

	$j=1$		$j=2$		$j=3$	
k_j	R_1	C_1	R_2	C_2	R_3	C_3
1	.6	1	.7	3	.5	2
2	.8	2	.8	5	.7	4
3	.9	3	.9	6	.9	5

Soln:

Stating with stage 3, since main Component 3 must include atleast one Parallel unit. We find that y_3 must atleast equal $c_3(1)=2$.

y_3 Cannot exceed $10-(3+1)=6$. Otherwise the remaining Capital will not be Sufficient to Provide main Components 1 and 2 with atleast one ||el unit each. For the Same reason $y_2 = 5, 6 \ldots$ or 9 and $y_1 = 6, 7 \ldots$ or 10.

$c_{13} \leq y_3 \leq c - c_{12} - c_{11} \Rightarrow 2 \leq y_3 \leq 10-3-1=6$

$c_{13}+c_{12} \leq y_2 \leq c - c_{11} \Rightarrow 5 \leq y_2 \leq 10-1=9$

$c_{13}+c_{12}+c_{11} \leq y_1 \leq c \Rightarrow 6 \leq y_1 \leq 10$.

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Stage 3: $f_3(y_3) = \max_{K_3=1,2,3} \{R_3(K_3)\}$

y_3	$K_3=1$	$K_3=2$	$K_3=3$	Optimum Solution	
	$R=.5$ $C=2$	$R=.7$ $C=4$	$R=.9$, $C=5$	$f_3(y_3)$	K_3^*
2	.5	-	-	.5	1
3	.5	-	-	.5	1
4	.5	.7	-	.7	2
5	.5	.7	.9	.9	3
6	.5	.7	.9	.9	3

Stage 2:

$$f_2(y_2) = \max_{K_3=1,2,3} \left[R_2(K_2) \cdot f_3[y_2 - C_2(K_2)] \right]$$

y_2	$K_2=1$	$K_2=2$	$K_2=3$	Optimum Solution	
	$R=.7$ $C=3$	$R=.8$ $C=5$	$R=.9$ $C=6$	$f_2(y_2)$	K_2^*
5	$.7 \times .5 = .35$	-	-	.35	1
6	$.7 \times .5 = .35$	-	-	.35	1
7	$.7 \times .7 = .49$	$.8 \times .5 = .40$	-	.49	1
8	$.7 \times .9 = .63$	$.8 \times .5 = .40$	$.9 \times .5 = .45$	.63	1
9	$.7 \times .9 = .63$	$.8 \times .7 = .56$	$.9 \times .5 = .45$	.63	1

Stage I :

$$f_1(y_1) = \max_{K_1=1,2,3} \{ R_1(K_1) \cdot f_2(y_1 - C_1(K_1)) \}$$

$y_1 = 10 \Rightarrow K_1^* = 2$

$y_2 = y_1 - C_1(2) = 10 - 2 = 8$

$y_2 = 8 \Rightarrow K_2^* = 1$

$y_3 = y_2 - C_2(1) = 8 - 3 = 5 \Rightarrow K_3^* = 3$

y_1	$K_1 = 1$	$K_1 = 2$	$K_1 = 3$	optimum solution	
	$R = .6 \quad C = 1$	$R = .8 \quad C = 2$	$R = .9 \quad C = 3$		
6	$.6 \times .35 = .210$	-	-	$.210$	1
7	$.6 \times .35 = .210$	$.8 \times .35 = .280$	-	$.280$	2
8	$.6 \times .49 = .294$	$.8 \times .35 = .280$	$.9 \times .35 = .315$	$.315$	3
9	$.6 \times .63 = .378$	$.8 \times .49 = .392$	$.9 \times .35 = .315$	$.392$	2
10	$.6 \times .63 = .378$	$.8 \times .63 = .504$	$.9 \times .49 = .441$	$.504$	2

The optimum solution given $C = 10$ is $(2, 1, 3)$ with $R = .504$

Optimal Subdivision Problem:

Consider the mathematical Problem of dividing the quantity q into N Parts. The objective is to determine the optimum Subdivision of q that will maximize the Product of the N Parts.

Let z_j be the j^{th} Portion of q $(j = 1, 2 \ldots N)$

The Problem formulation is

$$\text{Max } P = \prod_{j=1}^{N} z_j$$

Subject to

$$\sum_{j=1}^{N} z_j = q \quad z_j \geq 0 \text{ for all } j$$

This Problem is Similar to the reliability Problem. The main difference occurs in that the Variables z_j are Continuous a Condition that requires the use of Calculus for optimizing each Stage's Problem.

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The elements of the DP model are defined as

(i) Stage j represents the j^{th} portion of q.

(ii) State y_j is the portion of q allocated to stages $j, j+1 - - N$

(iii) Alternative z_j is the portion of q allocated to stage j

Let $f_j(y_j)$ be the optimum value of the objective functi

for stages $j, j+1 - - N$ given the state y_j.

The recursive equations are

$$f_N(y_N) = \max_{z_N \le y_N} \{z_N\}$$

$$f_j(y_j) = \max_{z_j \le y_j} \{ z_j \, f_{j+1}(y_j - z_j) \} \quad j = 1, 2 - - N-1$$

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Ex 2 : Determine the value of u_1, u_2, u_3 so as to maximize $(u_1 u_2 u_3)$ subject to $u_1+u_2+u_3 = 10$ and $u_1, u_2, u_3 \geq 0$

Sol: Let $Q = 10$ is to be divided into 3 parts say u_1, u_2 and u_3 so their product is maximum.

The given problem can be considered as three stage problem with state variables x_1, x_2, x_3 and returns $f_1(x_1)$, $f_2(x_2)$ and $f_3(x_3)$ respectively.

At Stage 3 $\quad x_3 = u_1 + u_2 + u_3$

At Stage 2 $\quad x_2 = u_1 + u_2 = x_3 - u_3$

At Stage 1 $\quad x_1 = u_1 = x_2 - u_2$

$$\therefore \quad f_1(x_1) = u_1 = x_2 - u_2$$

$$f_2(x_2) = \max \{ u_2 \, f_1(x_1) \}, \quad 0 \leq u_2 \leq x_2$$

$$= \max \{ u_2 (x_2 - u_2) \}$$

$$= \max \{ u_2 x_2 - u_2^2 \} \quad 0 \leq u_2 \leq x_2$$

Diff/. w.r.t u_2 and equating the differential to zero.

$$\frac{\partial f_2(x_2)}{\partial u_2} = x_2 - 2u_2 = 0 \quad \text{or} \quad u_2 = \frac{x_2}{2}.$$

$$\therefore \; f_2(x_2) = \frac{x_2}{2}\left(x_2 - \frac{x_2}{2}\right) = \frac{x_2^2}{4}$$

Now $f_3(x_3) = \max\{u_3 \, f_2(x_2)\}$

$$= \max\left\{u_3 \cdot \frac{x_2^2}{4}\right\} = \max\left\{u_3 \cdot \frac{(x_3 - u_3)^2}{4}\right\}$$

Diff w.r.t u_3 and equating to zero.

$$\frac{\partial}{\partial u_3}\left\{\frac{x_3^2 u_3 + u_3^3 - 2u_3^2 x_3}{4}\right\} = 0$$

$$x_3^2 + 3u_3^2 - 4u_3 x_3 = 0$$

$$x_3^2 - 3u_3 x_3 + 3u_3^2 - u_3 x_3 = 0$$

$$x_3(x_3 - 3u_3) - u_3(x_3 - 3u_3) = 0$$

$$(x_3 - 3u_3)(x_3 - 3u_3) = 0$$

$u_3 = x_3$ is trivial since $x_3 = u_1 + u_2 + u_3$

or $u_3 = \frac{x_3}{3} = \frac{10}{3}$

$\therefore x_2 = x_3 - u_3 = 10 - \frac{10}{3} = \frac{20}{3}$

and $u_2 = \frac{x_2}{2} \Rightarrow u_2 = \frac{10}{3}$

$u_1 = x_2 - u_2 = \frac{20}{3} - \frac{10}{3} = \frac{10}{3}$

$\therefore u_1 = u_2 = u_3 = \frac{10}{3}$

Hence the maximum product is

$$u_1 u_2 u_3 = \frac{1000}{27}$$

Ex 2: $\text{Min } z = y_1^2 + y_2^2 + y_3^2$

Subject to $y_1 + y_2 + y_3 \geq 15$

$y_1, y_2, y_3 \geq 0$

Soln Let the state Variables be x_1, x_2 and x_3

$\ni$ $x_3 = y_1 + y_2 + y_3$

$x_2 = y_1 + y_2 = x_3 - y_3$

$x_1 = y_1 = x_2 - y_2$

The recursive equations are

$$f_3(x_3) = \underset{y_3}{\text{Min}} \{ y_3^2 + f_2(x_2) \}$$

$$f_2(x_2) = \underset{y_2}{\text{Min}} \{ y_2^2 + f_1(x_1) \}$$

$$f_1(x_1) = \underset{y_1}{\text{Min}} \{ y_1^2 \} = y_1^2$$

$$f_1^*(x_1) = y_1^2$$

Now $f_2(x_2) = \underset{y_2}{Min} \{ y_2^2 + y_1^2 \}$

$$= \underset{y_2}{Min} \{ y_2^2 + (x_2 - y_2)^2 \}$$

Diff. w.r.t y_2 and equating to zero.

$$2y_2 + 2(x_2 - y_2)(-1) = 0$$

$$2y_2 \cancel{+} - 2x_2 + 2y_2 = 0$$

$$4y_2 - 2x_2 = 0 \Rightarrow y_2 = \frac{x_2}{2}$$

$$f_2^*(x_2) = \left(\frac{x_2}{2}\right)^2 + \left(x_2 - \frac{x_2}{2}\right)^2 = \frac{x_2^2}{2}$$

Now $f_3(x_3) = \underset{y_3}{Min} \{ y_3^2 + f_2(x_2) \}$

$$= \underset{y_3}{Min} \left\{ y_3^2 + \frac{x_2^2}{2} \right\}$$

$$= \underset{y_3}{Min} \left\{ y_3^2 + \frac{(x_3 - y_3)^2}{2} \right\}$$

Diff. w.r.t 'y_3' and Equating to zero.

$$2y_3 - (x_3 - y_3) = 0 \Rightarrow y_3 = \frac{x_3}{3}$$

$$f_3^*(x_3) = \left(\frac{x_3}{3}\right)^2 + \frac{\left(x_3 - \frac{x_3}{3}\right)^2}{2} = \frac{x_3^2}{3}$$

Since $x_3 = y_1 + y_2 + y_3 \geq 15$

Min of $f_3(x_3) \Rightarrow y_1 + y_2 + y_3 = 15$ or $x_3 = 15$

$$f_3^*(x_3) = 75$$

and $y_3 = \frac{x_3}{3} = \frac{15}{3} = 5$

$$y_2 = \frac{x_2}{2} = \frac{x_3 - y_3}{2} = \frac{15-5}{2} = 5$$

$$y_1 = x_1 = x_2 - y_2 = 10 - 5 = 5$$

$\therefore$ Min $Z = 5^2 + 5^2 + 5^2 = 75$ at $y_1 = y_2 = y_3 = 5$

Work Force Size :

The elements of the DP model are given as

(i) Stage j represents the j^{th} week

(ii) State y_{j-1} at stage j is the no. of workers at the end of stage $j-1$.

(iii) Alternative y_j is the no. of workers in week j.

Let $f_j(y_{j-1})$ be the optimal cost for periods (weeks) $j, j+1 \ldots N$ given y_{j-1}. The recursive equations are

$$f_N(y_{N-1}) = \min_{y_N = b_N} \left\{ c_1(y_N - b_N) + c_2(y_N - y_{N-1}) \right\}$$

$$f_j(y_{j-1}) = \min_{y_j \geq b_j} \left\{ c_1(y_j - b_j) + c_2(y_j - y_{j-1}) + f_{j+1}(y_j) \right\}$$

$$j = 1, 2 \ldots N-1$$

Example :

A Contractor needs to decide on the size of his work force over the next 5 weeks. The minimum force size b_i for the 5 weeks to be 5, 7, 8, 4 and 6 workers for $i = 1, 2, 3, 4$ & 5 resp. The Contractor can maintain the required minimum no. of workers by exercising the options of hiring and firing. The additional hiring cost is incurred every time the work force size of current week exceeds that of last week. Let y_j represent the no. of workers for the j^{th} week. Define $C_1(y_j - b_j)$ as the excess cost when y_j exceeds b_j and $C_2(y_j - y_{j-1})$ as the cost of hiring new workers $(y_j > y_{j-1})$.

The Contractor's data are given below:

$$C_1(y_j - b_j) = 3(y_j - b_j) \quad j = 1, 2 \ldots 5$$

$$C_2(y_j - y_{j-1}) = \begin{cases} 4 + 2(y_j - y_{j-1}) & y_j > y_{j-1} \\ 0 & y_j \le y_{j-1} \end{cases}$$

By the Definition of C_2 implies that firing $(y_j \le y_{j-1})$ incurs no additional cost. The initial work force y_0 at the beginning of the first week is 5 workers it is required to determine the optimum size of work force for the 5 week planning horizon.

To define the Possible Values for y_1, y_2, y_3, y_4 and y_5

Since $j=5$ is the last Period and since firing does not incur any cost. y_5 must equal the minimum required no. of workers b_5. (ie) $y_5 = b_5 = 6$. on the other hand

Since $b_4 = 4 < b_5 = 6$ the Contractor may maintain

$y_4 = 4, 5$ or 6. depending on which level will yield lowest cost

|||ly $y_3 = 8$, $y_2 = 7$ or 8 and $y_1 = 5, 6, 7$ or 8

The initial work force size y_0 is 5.

The recursive equations are

<u>Stage 5 :</u> $b_5 = 6$ $\quad f_5(y_4) = \min\limits_{y_5 = b_5} \{c_1(y_5 - b_5) + c_2(y_5 - y_4)\}$

$f_j(y_{j-1}) = \min\limits_{y_j \geq b_j} \{c_1(y_j - b_j) + c_2(y_j - y_{j-1}) + f$

y_4	$c_1(y_5 - b_5) + c_2(y_5 - y_4)$ $y_5 = 6$; $b_5 = 6$	$f_5(y_4)$	y_5^*
4	$3(0) + 4 + 2(6-4) = 8$	8	6
5	$3(0) + 4 + 2(6-5) = 6$	6	6
6	$3(0) + 0 = 0$	0	6

Stage 4: $b_4 = 4$

$c_1(y_4 - b_4) + c_2(y_4 - y_3) + f_5(y_4)$

y_3	$y_4 = 4$	5	6	$f_4(y_3)$	y_4^*
8	$0+0+8=8$	$3+0+6=9$	$3(2)+0+0=6$	6	6

Stage 3: $b_3 = 8$

$c_1(y_3 - b_3) + c_2(y_3 - y_2) + f_4(y_3)$

y_2	$y_3 = 8$	$f_3(y_3)$	y_3^*
7	$0+4+2(1)+6 = 12$	12	8
8	$0+0+6 = 6$	6	8

Stage 2: $b_2 = 7$

$c_1(y_2 - b_2) + c_2(y_2 - y_1) + f_3(y_3)$

y_1	$y_2 = 7$	$y_2 = 8$	$f_2(y_1)$	y_2^*
5	$0+4+2(2)+12=20$	$3(1)+4+2(3)+6=19$	19	8
6	$0+4+2(1)+12=18$	$3(1)+4+2(2)+6=17$	17	8
7	$0+0+12 = 12$	$3(1)+4+2(1)+6=15$	12	7
8	$0+0+12 = 12$	$3(1)+0+6 = 9$	9	8

Stage I : $b_1 = 5$

	$c_1(y_1-b_1)+c_2(y_1-y_0)+f_2(y_1)$					
y_0	$y_1 = 5$	6	7	8	$f_1(y_0)$	y_1^*
5	$0+0+19 = 19$	$3(1)+4+2(1)+17=26$	$3(2)+4+2(2)+12=26$	$3(3)+4+2(3)+9=28$	19	5

The optimum solution is

$y_0 = 5 \rightarrow y_1^* = 5 \rightarrow y_2^* = 8 \rightarrow y_3^* = 8 \rightarrow y_4^* = 6 \rightarrow y_5^* = 6$

Week j	Minimum Requirement b_j	y_j	Decision
1	5	5	No hiring or firing
2	7	8	Hire 3 workers
3	8	8	No hiring or firing
4	4	6	Fire 2 workers.
5	6	6	No hiring or firing